# STOCHASTIC PROCESSES, JANUARY $10^{\text{th}}$ 2022

3 hours. No documents allowed.

Please write the answers to the Course Questions and the solution of two exercises on three different sheets.

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**Course Questions.** Please justify all answers by giving a fair amont of details.

(1) What does it mean that the sequence of random variables $(X_n)$ is tight?
  (a) $(X_n)$ converges in law: does this mean that $(X_n)$ is tight?
  (b) $\lim_n \varphi_{X_n}(t)$ exists for every $t \in \mathbb{R}$: does this mean that $(X_n)$ is tight? Here of course $\varphi_X(t) := \mathbb{E}[\exp(itX)]$.

(2) What does it mean that a family of random variables is Uniformly Integrable (UI)?
  (a) $Y \in \mathbb{L}^1$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$: is the family $\{\mathbb{E}[Y|\mathcal{G}] : \mathcal{G}$ sub-$\sigma$-algebra of $\mathcal{F}\}$ UI?
  (b) $\lim_n X_n$ exists in $L^1$. Is $(X_n)$ UI?
  (c) $(Y_n)$ is UI and $Y_n \to Y$ in law. Is it true that $Y_n \to Y$ in $\mathbb{L}^1$? Is it true that $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$?

(3) $(M_n)$ is a martingale with bounded increments (i.e., $\sup_n \|M_{n+1} - M_n\|_\infty < \infty$) and $\tau$ is a stopping time (we are on a filtered space).
  (a) $\mathbb{P}(\tau < \infty) = 1$: is it true that $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ ?
  (b) $\mathbb{E}[\tau] < \infty$: is it true that $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ ?

(4) $X$ is a $Q$-MC on the state space $E$ and, for $x \in E$, $N_x := \sum_{n=0}^{\infty} \mathbf{1}_{X_n = x}$. Show that either $\mathbb{P}_x(N_x = \infty) = 1$ or there exists $c > 0$ such that $\mathbb{E}_x[\exp(cN_x)] < \infty$.

(5) $(X_n)$ is an irreducible $Q$-MC on $E$. Explain why the existence of a non contant $Q$-superharmonic function $u$ (i.e. $Qu \leq u$) which is bounded below implies that $(X_n)$ is transient.

**Exercise 1.** We work on the state space $E = \mathbb{Z}$ and we consider the stochastic matrix defined for $n \in \mathbb{Z} \setminus \{0\}$ by $Q(n, n+1) = p(n) \in (0,1)$ and $Q(n, n-1) = q(n) = 1 - p(n)$ and $Q(0,0) = r \in (0,1)$, $Q(0,+1) = Q(0,-1) = (1-r)/2$. We assume that $p(n) = q(-n)$ for every $n \neq 0$ and that there exists $\beta \in \mathbb{R}$ and $c > 1$ such that for $n \to \infty$

$$\frac{p(n)}{q(n)} = 1 - \frac{\beta}{n} + O\left(\frac{1}{n^c}\right).$$

$X := (X_j)_{j=0,1,\dots}$ is a $Q$-MC.

(1) Show that $X$ is irreducible an aperiodic.

(2) Show that a bounded non constant $Q$-harmonic function $u$ (i.e., $Qu = u$) exists if and only if $\beta < -1$.
*Obs.: note that* $u(n+1) - u(n) = (u(1) - u(0)) \prod_{j=1}^{n} (q(j)/p(j))$ *for* $n = 1, 2, \dots$

(3) Consider, for $l < r$, the set $D_{l,r} := \{l, l+1, \dots, r\}$ and $\tau_{l,r} := \inf\{j = 0, 1, \dots : X_j \notin D_{l,r}\}$. Express $\mathbb{P}_n(X_{\tau_{l,r}} = l - 1)$ in terms of a $Q$-harmonic function $u$.
*Obs.: this is almost a course question, please give details for each step.*

(4) Note that, by the course question (5), (2) implies that $X$ is transient for $\beta < -1$. Show that $X$ is transient if and only if $\beta < -1$.
*Hint: this is of course a matter of showing recurrence for* $\beta \geq -1$. *And for this it may be of help to exploit the answer to the previous question.*

(5) Remark that for $n = 1, 2, \ldots$

$$v(n) := \frac{p(0)p(1)\ldots p(n-1)}{q(1)q(2)\ldots q(n)},$$

satisfies $vQ(n) = v(n)$ for $n = 1, 2, \ldots$ and build an invariant measure $\mu$ for the $Q$-MC.

(6) Show that the chain is positive recurrent if and only if $\beta > 1$ (equivalently, that the chain is null recurrent if and only if $\beta \in [-1, 1]$).
*Obs.: Hence, by the course question (5), there cannot exist bounded (sub/super) harmonic functions for $\beta \geq -1$.*

(7) Show that for $\beta > 1$ there does not exist a non constant function $u$ which is subharmonic (or superharmonic) for $Q$ and that satisfies $\sum_{x \in E} |u(x)|\mu(x) < \infty$.

(8) Nevertheless, build a $Q$-harmonic function for $\beta \geq -1$.
*Obs.: of course this should be unbounded. And particularly so if $\beta > 1$.*

**Exercise 2.** In this exercise $(X_n)_{n=1,2,\ldots}$ is a sequence of IID random variables. We use the standard notation $S_n = X_1 + \ldots + X_n$. The aim is showing that

$$\mathbb{E}\left[\sup_n \frac{|S_n|}{n}\right] < \infty \quad \Longleftrightarrow \quad \mathbb{E}\left[|X_1| \log_+ |X_1|\right] < \infty . \qquad (\star)$$

In what follows you can use without proof the following generalization of a result that we have proven in the course: if $X$ and $Y$ are non negative random variables defined on the same probability space then if $\lambda \mathbb{P}(Y \geq \lambda) \leq \mathbb{E}[X; Y \geq \lambda]$ for every positive $\lambda$, then $\mathbb{E}[Y] \leq 2(1 + \mathbb{E}[X \log_+ X])$.

Without loss of generality we make the assumption that $\mathbb{P}(|X_1| \leq 1) > 0$ (this will be used only at the very end of the exercise).

(1) (Course question). Show that if $(Z_n)$ is a submartingale with $Z_n \geq 0$ for every $n$, then for every $\lambda > 0$ we have

$$\lambda\, \mathbb{P}\left(\sup_{k \in \{0,\ldots n\}} Z_k \geq \lambda\right) \leq \mathbb{E}\left[Z_n;\ \sup_{k \in \{0,\ldots n\}} Z_k \geq \lambda\right].$$

(2) Show that

$$\mathbb{E}\left[\sup_{k \in \{0,\ldots n\}} Z_k\right] \leq 2\left(1 + \mathbb{E}\left[Z_n \log_+ Z_n\right)\right].$$

*Obs.: this of course is the generalization of Doob's $\mathbb{L}^p$, $p > 1$, inequality to $p = 1$.*

(3) (Course question). Explain in detail why, if $X_1 \in \mathbb{L}^1$, we have $\frac{S_n}{n} = \mathbb{E}[X_1 | S_n, S_{n+1}, \ldots]$ for every $n = 1, 2, \ldots$.

(4) Show that, for every $n$, the process

$$\frac{S_n}{n}, \frac{S_{n-1}}{n-1}, \ldots, \frac{S_2}{2}, X_1, X_1, X_1, \ldots$$

is a martingale for an adequate choice of the filtration (please, give it explicitly).

(5) Show the $\Longleftarrow$ part of (1).

(6) We set now

$$\tau := \inf\left\{n = 1, 2, \ldots : |X_n| > n\right\}.$$

Show that

$$\mathbb{E}\left[\sup_n \frac{|X_n|}{n}\right] \geq \mathbb{E}\left[\frac{|X_\tau|}{\tau}; \tau < \infty\right] = \sum_{n=1}^{\infty} \mathbb{P}(\tau \geqslant n) \frac{1}{n} \mathbb{E}\left[|X_n|; |X_n| > n\right]$$
$$\geq \mathbb{P}(\tau = \infty) \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left[|X_1|; |X_1| > n\right] .$$

*Obs.: three questions.*

(7) Show that $\mathbb{E}[|X_1|] < \infty$ implies $\mathbb{P}(\tau = \infty) > 0$.
*Hint: show first that* $\mathbb{E}[|X_1|] < \infty$ *implies* $\sum_n \mathbb{P}(|X_n| > n) < \infty$.

(8) By putting (6) and (7) together show that $\mathbb{E}[\sup_n |X_n|/n] < \infty$ implies $\mathbb{E}[|X_1| \log_+ |X_1|] < \infty$.

(9) Therefore, to complete the proof of (1) (i.e., to complete the proof of $\Longrightarrow$), it suffices to show that for $X_1 \in \mathbb{L}^1$ we have that

$$\mathbb{E}\left[\frac{1}{\tau} \sum_{k=1}^{\tau-1} |X_k| \,;\, 1 < \tau < \infty\right] \leq \frac{\mathbb{E}[|X_1|]}{\mathbb{P}(|X_1| \leq 1)} . \tag{A}$$

Prove (A).
*Hint: first show that* $\mathbb{E}[|X_k| \,|\, \tau = n] = \mathbb{E}[|X_k| \,|\, |X_k| \leq k]$ *for* $k < n$.

**Solution.**

**Exercise 1.**

(1) The chain is irreducible since one can go from $m$ to $p$ in $m-p$ consecutive steps of positive probability. And from $m$ to $m$ in two steps. The chain is aperiodic since it is irreducible and $Q(0,0)>0$.

(2) Let $u$ be harmonic for $Q$. Then one directly verifies that the formula suggested is correct and $u(n+1)-u(n)=(u(1)-u(0))\prod_{j=1}^n(q(j)/p(j))$ for $n=1,2,\ldots.$ and by symmetry we have

$$u(-n+1)-u(-n)=(u(-1)-u(0))\prod_{j=1}^n(p(-j)/q(-j))=(u(-1)-u(0))\prod_{j=1}^n(q(j)/p(j))$$

for the same values of $n$. So we have functions that are harmonic in the positive and negative semi axes, but the crucial point is to verify that we can match these two functions so that they are harmonic the full line. The harmonicity condition in 0 is $(u(1)+u(-1))(1-r)/2+u(0)r=u(r)$, i.e. $u(1)+u(-1)-2u(r)=0$, i.e. $u(1)-u(0)=-(u(-1)-u(0))$. Therefore every harmonic function is equal to $a+bu$ where $a$ and $b$ are constants and we have chosen $u(1)-u(0)=1$ and $u(0)=0$ so $u(-n)=-u(n)$ for every $n$, $u(1)=1$ and for $n\geq 2$

$$u(n)\;=\;\sum_{j=1}^n\prod_{j=1}^{k-1}\frac{q(j)}{p(j)}\,.$$

Since from the hypothesis we have that (we assume without loss of generality that $c\leq 2$)

$$\frac{q(n)}{p(n)}\;=\;1+\frac{\beta}{n}+O\left(\frac{1}{n^c}\right)\,.$$

$\prod_{j=1}^k(q(j)/p(j))=\exp(\sum_{j=1}^k(\beta/j+O(1/j^c))$ which behaves as $k\to\infty$ as $Ck^\beta$ for a $C>0$. Hence $u$ is bounded if and only if $\beta<-1$.

(3) Since the transition rates are uniformly bounded from below for a finite number of values, the chain exits from this segments almost surely. Thus $\tau_{l,r}<\infty$ almost surely. Choose $u$ a non constant $Q$-harmonic function (we built them in the previous question). We use now the fact that $(u(X_n))$ is a bounded martingale and apply the optional stopping theorem at the first time $X.$ touches $l-1$ or $r+1$ In particular, the optional stopping theorem shows that for $n\in D_{l,r}$

$$u(n)=u(l-1)\mathbb{P}_n(X_{\tau_{l,r}}=l-1)+u(r+1)(1-\mathbb{P}_n(X_{\tau_{l,r}}=l-1)\,.$$

This gives $\mathbb{P}_n(X_{\tau_{l,r}}=l-1)=\frac{u(n)-u(r+1)}{u(l-1)-u(r+1)}$. Obs.: some people decided to show that $n\mapsto\mathbb{P}_n(X_{\tau_{l,r}}=l-1)$ is harmonic inside the interval. This is correct (and accepted as an answer to this question), but it is of no help for the next question. The point was to use one of the non trivial $u$ built in the previous question.

(4) For $\beta\geq-1$ we have that $\lim_n u(n)=\infty$ (we are using here precisely the choice of $u$ made above, in particular $u(0)=0$, $u(1)=1$ and $u$ is increasing). So $\mathbb{P}_1(X_{\tau_{1,r}}=0)=\frac{u(1)-u(r+1)}{u(0)-u(r+1)}\to 1$ as $r\to\infty$. Note also that $\{X_{\tau_{1,r}}=0\}$ is an event that increases in $r$ and converges to the event {there exists $n$ such that $X_n=0$}, i.e. the event $T_0<\infty$ ($T_0$ first hit time to 0 for positive times). Hence $\mathbb{P}_1(T_0=\infty)=0$. But $\mathbb{P}_0(T_0=\infty)=\mathbb{P}_1(T_0=\infty)$ by the strong Markov property (condition of the first time of hitting $\pm1$ and use the symmetry), which means recurrence for $\beta\geq-1$.

(5) By direct checking one sees that if we define $\mu(n)=v(|n|)$ for $n\neq0$ and $\mu(0)=2p(0)/(1-r)$, then $\mu$ is an invariant measure. Once again: one needs to solve $\mu Q(x)=\mu(x)$ for every $x$, included $x=0$!

(6) The chain in recurrent for $\beta \geq -1$, so in this case $\mu$ is the unique invariant measure up to a multiplicative constant. By exploiting the asymptotic behavior of $p(n)/q(n)$ one sees that $\mu$ can be normalized iff $\beta > 1$, hence the chain is positive recurrent iff $\beta > 1$.

(7) Let $(X_n)$ be chosen with $X_0 \sim \mu$ ($\mu$ is now normalized to be a probability. So the law of $u(X_n)$ does not depend on $n$ and therefore $\mathbb{E}[|u(X_n)|] = \mathbb{E}[|u(X_0)|] = \sum_x |u(x)|\mu(x) < \infty$. Note that this implies in particular that $|Qu(x)| < \infty$ for every $x$: in fact $Q|u(x)| < \infty$ for every $x$ because $\sum_x Q|u(x)|\mu(x) = \sum_x |u(x)|\mu(x) < \infty$. With this and the hypothesis on $u$ we verify that $(u(X_n))$ is a submartingale. Since it is bounded in $\mathbb{L}^1$ it converges a.s.. But this is impossible because there exist $x$ and $y$ such that $u(x) \neq u(y)$ and both $x$ and $y$ are visited infinitely often by recurrence.

(8) We have already built explicitly all the harmonic functions in the solution to question (2), for every $\beta$. For $\beta \geq -1$, the non trivial harmonic functions are unbounded and one readily sees also that, for $\beta > 1$, they are not in $\mathbb{L}^1(\mu)$. Note that building explicitly a non trivial harmonic function is not an obvious matter: in particular if we were working on $E = \{0, 1, \ldots\}$ (by choosing the transition probabilities from 0 in such a way one stays on $E$) the harmonicity condition in 0 implies that such an harmonic function is constant.

**Exercise 2.** In this exercise $(X_n)_{n=1,2,\ldots}$ is a sequence of IID random variables. We use the standard notation $S_n = X_1 + \ldots + X_n$. The aim is showing that

$$\mathbb{E}\left[\sup_n \frac{|S_n|}{n}\right] < \infty \qquad \Longleftrightarrow \qquad \mathbb{E}\left[|X_1| \log_+ |X_1|\right] < \infty \,. \tag{1}$$

In what follows you can use without proof the following generalization of a result that we have proven in the course: if $X$ and $Y$ are non negative random variables defined on the same probability space then if $\lambda \mathbb{P}(Y \geq \lambda) \leq \mathbb{E}[X; Y \geq \alpha]$ for every positive $\lambda$, then $\mathbb{E}[Y] \leq 2(1 + \mathbb{E}[X \log_+ X])$.

Without loss of generality we make the assumption that $\mathbb{P}(|X_1| < 1) > 0$ (this will be used only at the very end of the exercise).

(1) See course.
(2) Apply the above given inequality to $Y = \sup_{k \in \{0,\ldots n\}} Z_k$ and $X = Z_n$.
(3) See course.
(4) Fix $n$ and write

$$M_0 := \frac{S_n}{n},\ M_1 := \frac{S_{n-1}}{n-1}, \ldots,\ M_{n-2} := \frac{S_2}{2},\ M_{n-1} := X_1,\ M_n := X_1, M_{n+1} := X_1, \ldots$$

so, by the previous question, $(M_k)$ is a martingale with respect to $(\mathcal{F}_k)$, with $\mathcal{F}_k = \sigma(S_{n-k}, S_{n-k+1}, \ldots)$ for $k \leq n-1$ and $\mathcal{F}_k = \mathcal{F}_{n-1}$ for every larger $k$.

(5) We now apply the result of question (2) for a fixed $n$ to the positive submartingale $(|M_n|)$ and then use (MON) when sending $n$ to infinity.

(6) First and third inequality are immediate. As for the second one, it is enough to see (using independence) that $\mathbb{E}[X_n \mathbf{1}_{\tau=n}] = \mathbb{E}[X_n \mathbf{1}_{X_n > n} \mathbf{1}_{X_1 \leqslant 1} \ldots \mathbf{1}_{X_{n-1} \leqslant n-1}] = \mathbb{E}[X_n \mathbf{1}_{X_n > n}]\mathbb{P}(\tau \geqslant n)$

(7) It is classical that $\mathbb{E}[|X_1|] \leqslant \infty$ iff $\sum \mathbb{P}(|X_1| > n) < \infty$. Then one can write explicitly $\mathbb{P}(\tau = \infty) = \prod_{n=1}^\infty \mathbb{P}(|X_1| \leq n) = \prod_{n=1}^\infty (1 - \mathbb{P}(|X_1| > n)) = \exp(\sum_n \log(1 - \mathbb{P}(|X_1| > n)))$. But the sum in the exponent is finite because $\sum \mathbb{P}(|X_1| > n) < \infty$ so $\mathbb{P}(\tau = \infty) > 0$.

(8) It is enough to note the existence of a universal constant $C$ such that one has $\sum_{n=1}^{\infty} \mathbb{E}[X_1 \frac{1}{n} \mathbf{1}_{X_1 > n}] \geqslant C\mathbb{E}[X_1 \log_+(X_1)]$. This is simply an application of Fubini-Tonelli and of the fact that $\sum_{n<x}(1/n) \sim \log x$ for $x \to \infty$.

(9) The equality $\mathbb{E}[|X_k| \,|\, \tau = n] = \mathbb{E}[|X_k| \,|\, |X_k| \leqslant k]$ comes from the independence of the variables $X_i$. The second bound comes from the fact that $\mathbb{E}[|X_k| | |X_k| \leqslant k] = \mathbb{E}[X_k \mathbf{1}_{X_k \leqslant k}]/\mathbb{P}(|X_k| \leqslant k) \leqslant \mathbb{E}[|X_1|]/\mathbb{P}(|X_1| \leq 1)$. To conclude, write

$\mathbb{E}\left[\frac{1}{\tau}\sum_{k=1}^{\tau-1}|X_k|\,;\,1<\tau<\infty\right]=\mathbb{E}\Big[\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n-1}|X_k|\,\mathbf{1}_{\tau=n}\Big]$ and then use the fact that $\mathbb{E}[\mathbb{E}[|X_k|\,|\,\tau=n]]\leqslant\mathbb{E}[|X_1|]/\mathbb{P}(|X_1|<1)<\infty$.