On Lyapunov Exponent and Sensitivity

Christophe ABRAHAM [a], Gérard BIAU [b,*] and Benoît CADRE [c]

[a] ENSAM-INRA, UMR Biométrie et Analyse des Systèmes,
2, Place Pierre Viala, 34060 Montpellier Cedex 1, France;
[b] Laboratoire de Statistique Théorique et Appliquée,
Université Pierre et Marie Curie – Paris VI,
175, rue du Chevaleret, Boîte 158, 75013 Paris, France;
[c] Laboratoire de Probabilités et Statistique, Université Montpellier II,
Cc 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

Abstract

Sensitive dependence on initial conditions is widely understood as being the central idea of chaos. For a large class of transformations of the interval, we prove that positiveness of the Lyapunov exponent implies the sensitivity property. We also provide bounds for the sensitivity constant.

Index Terms **— Chaos, sensitive dependence on initial conditions, Lyapunov exponent, sensitivity constant.**

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1 Introduction

The notion of chaos has attracted much attention in recent years and several authors have tried to formalize it in various ways, see *e.g.* Berliner [4], Block and Coppel [5], Boyarsky and Góra [6], Chatterjee and Yilmaz [8], Collet [9], Devaney [10], Guégan [12], Lasota and Mackey [13] and the references therein for results and discussions. One such popular attempt uses the definition of *sensitive dependence on initial conditions.* This important notion is actually widely understood as being the central idea of chaos and was popularized by the meteorologist Ed Lorenz through the so-called "butterfly effect". The sensitivity property captures the idea that in a chaotic system a very small change in the initial condition can cause a big change in the

*Corresponding author. Email: biau@ccr.jussieu.fr .

trajectory. In a mathematical setting, let $T : X \to X$ be a map on some metric space (X, d). Then T has sensitive dependence on initial conditions if there exists $\delta > 0$ (a *sensitivity constant*) such that for every point $x \in X$ and every open neighborhood V_x of x, there exists an integer $n \geq 0$ such that $\sup_{y \in V_x} d\big(T^n(x), T^n(y)\big) > \delta$.

In the last decade, several authors proposed sufficient conditions on the transformation T to ensure the sensitive dependence property. Banks, Brooks, Cairns, Davis and Stacey [3] as well as Glasner and Weiss [11] approached the problem using a topological dynamics point of view (Block and Coppel [5]). In particular, these authors showed independently that any continuous and topologically transitive map $T : X \to X$ whose periodic points are dense in X has sensitive dependence on initial conditions. Requiring the existence of a T-invariant probability measure (Petersen [14], Pollicott and Yuri [15]) whose support is all of X, Abraham, Biau and Cadre [1] recently provided sufficient conditions, both topological and ergodic, to force the sensitivity property. However, all these sufficient conditions are mainly qualitative (density of periodic points, mixing, exactness...) and, in most cases, they do not allow to check simply if T is sensitive or not.

A quantitative and easier checkable criterion for sensitivity may be given by the so-called *Lyapunov exponent*. For sake of convenience, we assume from now on that $X = [0, 1]$ and that T is a measurable mapping from $\big([0, 1], \mathcal{B}([0, 1]), \mu\big)$ into itself, where $\mathcal{B}([0, 1])$ denotes the Borel σ-field of $[0, 1]$ and μ is a T-invariant probability measure. Provided the transformation T is ergodic (Petersen [14], Pollicott and Yuri [15]) and μ-a.s. differentiable, the Lyapunov exponent λ_0 associated with T is defined by the formula

$$\lambda_0 = \mathbf{E}_\mu \log |T'| = \int_{[0,1]} \log |T'| \, \mathrm{d}\mu \,, \tag{1.1}$$

assuming that the function $\log |T'|$ is in $L_1(\mu)$. In (1.1), $\mathbf{E}_\mu$ means the expectation with respect to μ. It is commonly accepted (see for example Collet [9]) that positiveness of λ_0 implies the sensitivity property. This argument is essentially heuristic and is based on the following. According to Birkhoff's ergodic theorem (Petersen [14], Pollicott and Yuri [15]), one has

$$\frac{1}{n} \log \big|(T^n)'\big| = \frac{1}{n} \sum_{i=0}^{n-1} \log \big|T'(T^i)\big| \to \lambda_0 \quad \text{as } n \to +\infty, \tag{1.2}$$

μ-a.s. The above formula together with the mean value theorem indicate some exponential rate of divergence of orbits of nearby points provided $\lambda_0 >$

0. Indeed, it is tempting to derive from (1.2) the following heuristic:

$$\big|T^n(x) - T^n(y)\big| \simeq e^{\lambda_0 n}|x-y| \quad \text{for any } x, y \in [0,1],$$

hence the sensitivity of T. Clearly, this approach suffers from a lack of mathematical rigor, at least because the space $[0,1]$ is bounded. In the present paper, we propose to rigorously examine the link between the positiveness of the Lyapunov exponent and the sensitivity property. As a matter of fact, we will focus on a stronger property than the sensitive dependence on initial conditions called *strong sensitive dependence*. This concept was first introduced by the present authors in [1] and reads as follows for any transformation T of $[0,1]$ into itself: there exists $\delta > 0$ (a *strong sensitivity constant*) such that for every point $x \in [0,1]$ and every open neighborhood V_x of x, there exists an integer $N \geq 0$ such that for all $n \geq N$, $\sup_{y \in V_x} \big|T^n(x) - T^n(y)\big| > \delta$. Observe that if δ is a strong sensitivity constant for T, then so is any positive $\delta' \leq \delta$. This leads to consider the following quantity, denoted by $\Delta(T)$:

$$\Delta(T) = \sup\{\delta : \delta \text{ is a strong sensitivity constant for } T\}.$$

We refer to [1] for details and comments about the strong sensitivity property.

The paper is organized as follows. In Section 2, we state the main results and we discuss some examples. Proofs are gathered in Section 3.

2 Main results

2.1 The class of transformations under study

Let T be a measurable mapping from $\big([0,1], \mathcal{B}([0,1]), \mu\big)$ into itself, where μ is a T-invariant probability measure with support $[0,1]$ and density f with respect to (*w.r.t.*) the Lebesgue measure m on $\big([0,1], \mathcal{B}([0,1])\big)$. We equip $[0,1]$ with the standard Euclidean distance $|.|$. Throughout the paper, we assume that any non-empty open interval in $[0,1]$ contains a non-empty open interval on which f is bounded. We also assume that T admits a finite number of discontinuity points as well as a finite number of local extrema. We denote by $\mathcal{P}$ the set $\{0 = a_0 < a_1 < \ldots < a_l < a_{l+1} = 1\} \subset [0,1]$ $(l \geq 0)$ of discontinuity points and local extrema of T. By the very definition of $\mathcal{P}$, the transformation T is *continuous* and *monotone* on each interval $I_j =]a_j, a_{j+1}[$ for all $j \in \{0, \ldots, l\}$. As a matter of fact, T is *strictly* monotone on each interval I_j (see Lemma (3.1)). The above properties imply that T is m-a.s. differentiable and therefore μ-a.s. differentiable (since $\mu \ll m$). From this, one easily deduces that $\forall n \geq 1$, T^n is μ-a.s. differentiable.

Let us now assume that the function $\log|T'|$ is integrable *w.r.t.* μ. Denoting by $\mathcal{J}$ the σ-field of T-invariant Borel subsets of $[0,1]$, *i.e.*, $\mathcal{J} = \{A \in \mathcal{B}([0,1]) : T^{-1}(A) = A\}$ and by λ the conditional expectation of $\log|T'|$ given $\mathcal{J}$, *i.e.*, $\lambda = \mathbf{E}_\mu[\log|T'| \,/\, \mathcal{J}]$, we have, according to Birkhoff's ergodic theorem (Breiman [7], Petersen [14], Pollicott and Yuri [15]):

$$\frac{1}{n}\log\left|(T^n)'\right| \to \lambda \quad \text{as } n \to +\infty, \tag{2.1}$$

this being true both μ-a.s. and in $L_1(\mu)$. It is worth pointing out that in the important case where T is ergodic, then $\mathcal{J}$ is trivial (*i.e.*, $\mathcal{J} = \{\emptyset, [0,1]\}$), and in this case the definition of λ reduces to the Lyapunov exponent λ_0 given in (1.1). Denote by $g(x^-)$ (*resp.* $g(x^+)$) the left-hand (*resp.* right-hand) limit of $g : [0,1] \to [0,1]$ at the point $x \in [0,1]$ (if such a limit does exist). We finally assume that for all $i \in \{0, \ldots, l+1\}$:

$$T(a_i) \in \mathcal{P}, \quad T(a_i^-) \in \mathcal{P} \quad \text{and} \quad T(a_i^+) \in \mathcal{P}. \tag{2.2}$$

Observe that (2.2) forces the transformation to be Markov (see Pollicott and Yuri [15]). We insist on the fact that all the above mentioned conditions are satisfied by classical dynamical systems such as r-adic maps, quadratic map, tent maps, etc. (see for example Boyarsky and Góra [6] or Lasota and Mackey [13]).

2.2 Statement of the results

Theorem 2.1 *If $\lambda > 0$ μ-a.s., then T has strong sensitive dependence on initial conditions and*

$$\Delta(T) \geq \frac{1}{2} \min_{j \in \{0,\ldots,l\}} |a_{j+1} - a_j|.$$

As enlightened by the next theorem, it is sometimes possible to improve the lower bound of $\Delta(T)$ and to provide an upper bound.

Theorem 2.2 *Assume that $\lambda > 0$ μ-a.s. Then*

(i) *If $T\,(]0,1[) \supset]0,1[$ and for all $j \in \{0,\ldots,l\}$, there exists $p \geq 0$ such that $T^p(I_j) \supset]0,1[$, then $\Delta(T) \geq 1/2$.*

(ii) *If T is ergodic, then $0 < \Delta(T) \leq 1/2$.*

2.3 Remarks and examples

• Theorem 2.1 provides us with a nice criterion to show that a transformation T has strong sensitive dependence on initial conditions, whereas Theorem 2.2 allows to bound $\Delta(T)$. Indeed, to show directly that T is strongly sensitive is generally a difficult task. On the other hand, it is often easier to show that $\lambda > 0$ μ-a.s. For example, if $|T'| > 1$ μ-a.s. then $\lambda > 0$ μ-a.s. In particular, under the assumptions of the Folklore theorem (see Adler and Flatto [2] and Boyarsky and Góra [6]) the piecewise expanding transformation T verifies $\lambda > 0$ μ-a.s., with μ an invariant probability measure absolutely continuous *w.r.t.* the Lebesgue measure. Moreover, the density of μ is positive and upper bounded. Therefore, under the additional condition (2.2), T can be dealt with Theorem 2.1 and Theorem 2.2.

• Of course, the transformation T may also not be piecewise expanding but strongly sensitive. As an illustration, let us consider the map

$$T(x) = \begin{cases} x + 1/2 & \text{if } x \in [0, 1/2] \\ 2x - 1 & \text{if } x \in]1/2, 1]. \end{cases}$$

Observation shows that T^2 is Markov and piecewise expanding. According to the Folklore theorem, there exists a T^2-invariant and ergodic probability measure ν, with density h *w.r.t.* m. Moreover, there exists $D \geq 1$ such that for all x in $[0, 1]$:

$$\frac{1}{D} \leq h(x) \leq D. \tag{2.3}$$

It is straightforward to show that the probability measure $\mu = (\nu + \nu T)/2$ is T-invariant and ergodic. In addition, one easily verifies that μ admits a density f *w.r.t.* m, namely, for all x in $[0, 1]$:

$$f(x) = \frac{1}{2}\, h(x) + \frac{1}{4}\, h\Big(\frac{x+1}{2}\Big) + \frac{1}{2}\, h\Big(x - \frac{1}{2}\Big) \mathbf{1}_{[1/2,1]}(x).$$

Thus, using the ergodicity of T:

$$\lambda = \mathbf{E}_\mu\big[\log |T'|\big] = \mu\big([1/2, 1]\big) \log 2 > 0.$$

Finally, since $T(0) = 1/2$, $T(1/2^-) = 1$, $T(1/2^+) = 0$, $T(1) = 1$, and since f is positive and upper bounded by (2.3), T has strong sensitive dependence on initial conditions according to Theorem 2.1. One also has $\Delta(T) = 1/2$ according to Theorem 2.2 since

$$T^2\,\big(]0, 1/2[\big) \supset\,]0, 1[\quad \text{and} \quad T\,\big(]1/2, 1[\big) \supset\,]0, 1[\,.$$

- Another example is given by the *quadratic transformation* $T(x) = 4x(1-x)$, $x \in [0,1]$, which is clearly not piecewise expanding (however, as observed by a referee, it is conjugated to the piecewise expanding tent map). It is well known $\big($Lasota and Mackey [13]$\big)$ that T is an ergodic mapping on $\big([0,1], \mathcal{B}\big([0,1]\big), \mu\big)$, where μ is the T-invariant probability measure with density f *w.r.t.* m defined by

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad x \in]0,1[\,.$$

The support of μ is $[0,1]$ and any non-empty open interval contains a non-empty open interval on which f is bounded. Further, a computation leads to $\lambda = \log 2 > 0$ and thus, according to Theorem 2.1 and Theorem 2.2, T exhibits strong sensitive dependence with $\Delta(T) = 1/2$. This result was previously obtained by the present authors in [1], merely using the fact that the quadratic transformation is also *exact* $\big($Lasota and Mackey [13]$\big)$.

- Let us finally close this paragraph by observing that the set of conditions (2.2) may be slightly relaxed. It is indeed enough to assume that there exists a finite set $\overline{\mathcal{P}} = \{0 = \overline{a}_0 < \overline{a}_1 < \ldots < \overline{a}_q < \overline{a}_{q+1} = 1\}$ $(q \geq 0)$ containing $\mathcal{P}$ such that:

$$T(\overline{a}_i) \in \overline{\mathcal{P}}, \quad T(\overline{a}_i^-) \in \overline{\mathcal{P}} \quad \text{and} \quad T(\overline{a}_i^+) \in \overline{\mathcal{P}}.$$

Then, provided $\lambda > 0$ μ-a.s., T has strong sensitive dependence on initial conditions with

$$\Delta(T) \geq \frac{1}{2} \min_{j \in \{0,\ldots,q\}} |\overline{a}_{j+1} - \overline{a}_j|.$$

3 Proofs

Before proving Theorem 2.1 and Theorem 2.2, we provide three fundamental properties about the iterates of T (Propositions 3.1 – 3.3). For the sake of clarity, proofs of these propositions are deferred to the end of the section.

Proposition 3.1 *Let $n \geq 1$. Then T^n admits a finite number $l_n \geq 0$ of discontinuity points and local extrema, denoted by $0 = a_0^n < a_1^n < \ldots < a_{l_n}^n < a_{l_n+1}^n = 1$. Moreover, on each interval $I_j^n =]a_j^n, a_{j+1}^n[$, $j \in \{0,\ldots,l_n\}$, T^n is continuous and strictly monotone.*

Remark With these notations (that will be used throughout), $l_1 = l$ and $I_j^1 = I_j$.

In the sequel, we denote by diam A the diameter of any set $A \subset [0,1]$ and by $B(x,\varepsilon)$ the open ball in $[0,1]$ with center at x and radius $\varepsilon > 0$.

Proposition 3.2 *Let $n \geq 1$ and $j \in \{0, \dots, l_n\}$. Then there exists $m \in \{0, \dots, l\}$ such that $T^n(I_j^n) \supset I_m$.*

Proposition 3.3 *If $\lambda > 0$ μ-a.s., then*

$$\max_{j \in \{0,\dots,l_n\}} \operatorname{diam} I_j^n \to 0 \quad \text{as } n \to +\infty.$$

We are now ready to prove Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1 Let $x \in [0,1]$ and $\varepsilon > 0$. According to Proposition 3.3, there exists $N \geq 1$ such that $\forall n \geq N$, $\exists j \in \{0, \dots, l_n\}$ with $I_j^n \subset B(x, \varepsilon)$. Let $n \geq N$. According to Proposition 3.2, there exists $m \in \{0, \dots, l\}$ such that

$$T^n\big(B(x,\varepsilon)\big) \supset T^n\big(I_j^n\big) \supset I_m. \tag{3.1}$$

Consequently,

$$\operatorname{diam} T^n\big(B(x,\varepsilon)\big) \geq \operatorname{diam} I_m \geq \min_{j \in \{0,\dots,l\}} |a_{j+1} - a_j|.$$

Thus, $\forall\, 0 < \alpha < 1/2$, $\forall n \geq N$,

$$\sup_{y \in B(x,\varepsilon)} \big|T^n(x) - T^n(y)\big| > \alpha \min_{j \in \{0,\dots,l\}} |a_{j+1} - a_j|.$$

It follows that T has strong sensitive dependence on initial conditions with

$$\Delta(T) \geq \frac{1}{2} \min_{j \in \{0,\dots,l\}} |a_{j+1} - a_j|.$$

■

Proof of Theorem 2.2

(i) Let $x \in [0,1]$ and $\varepsilon > 0$. As in (3.1), one shows that there exists $p \geq 1$ and $m \in \{0, \dots, l\}$ with $T^p\big(B(x,\varepsilon)\big) \supset I_m$. From this and the assumptions, one easily deduces that for all n large enough, $T^n\big(B(x,\varepsilon)\big) \supset]0,1[$ and, consequently, that $\forall\, 0 < \alpha < 1/2$, $\sup_{y \in B(x,\varepsilon)} \big|T^n(x) - T^n(y)\big| > \alpha$. Therefore $\Delta(T) \geq 1/2$.

(ii) According to Theorem 2.1, T has strong sensitive dependence on initial conditions and thus $\Delta(T) > 0$. Since T is ergodic, one also has $\Delta(T) \leq 1/2$ according to Abraham, Biau and Cadre [1]. ■

It remains to prove Propositions 3.1 – 3.3. We first prove two technical lemmas.

Lemma 3.1 *Let $n \geq 1$ and J a non-empty open interval. Then T^n is not constant over J.*

Proof of Lemma 3.1 Assume that T^n is constant on J. Then $T^n(J) = \{\alpha\}$, where $\alpha \in [0, 1]$. Therefore, since $J \subset T^{-n}\big(T^n(J)\big)$,

$$\mu(J) \leq \mu\big(T^{-n}\big(T^n(J)\big)\big) = \mu\big(T^{-n}\big(\{\alpha\}\big)\big) = \mu\big(\{\alpha\}\big) = 0,$$

where the last equality follows from the absolute continuity of μ *w.r.t.* m. But $\mu(J) > 0$ since the support of μ is $[0, 1]$. Contradiction. ■

Lemma 3.2 *Let $n \geq 1$ and $x \in [0, 1]$. If x is a discontinuity point or a local extremum for T^n, then*

$$x \in \bigcup_{i=0}^{n-1} T^{-i}(\mathcal{P}).$$

Proof of Lemma 3.2 Clearly, if x is a discontinuity point for T^n, then $x \in \cup_{i=0}^{n-1} T^{-i}(\mathcal{P})$, since $\mathcal{P}$ contains all discontinuity points of T. Let us now assume that x is a continuity point and a local extremum for T^n, say a maximum. According to Lemma 3.1, x is then a *strict* local maximum for T^n. Let us assume that $x \in \cap_{i=0}^{n-1} T^{-i}(\mathcal{P}^c)$. Since x is a strict local maximum for T^n, there exists $r > 0$ such that $\forall y \in B(x, r) \setminus \{x\}$:

$$T\big(T^{n-1}(y)\big) = T^n(y) < T^n(x) = T\big(T^{n-1}(x)\big). \tag{3.2}$$

Since $T^{n-1}(x) \in \mathcal{P}^c$, there exists $j \in \{0, \ldots, l\}$ such that $T^{n-1}(x) \in I_j$. Following the argument of the beginning of the proof, x is also a continuity point for T^{n-1}. Thus, assuming that r is small enough, we have $T^{n-1}\big(B(x, r)\big) \subset I_j$.

But, I_j is an interval on which T is strictly monotone. This leads with (3.2) either to $T^{n-1}(y) < T^{n-1}(x)$ for all $y \in B(x, r) \setminus \{x\}$, or to $T^{n-1}(y) > T^{n-1}(x)$ for all $y \in B(x, r) \setminus \{x\}$. In both cases, x is a strict local extremum for T^{n-1}. We just have shown that if x is a continuity point for T^n which is also a local extremum, then either $x \in \cup_{i=0}^{n-1} T^{-i}(\mathcal{P})$ or x is a continuity point and a local extremum for T^{n-1}. We then easily deduce that $x \in \cup_{i=0}^{n-1} T^{-i}(\mathcal{P})$, hence the lemma. ■

We are now in position to prove Proposition 3.1.

Proof of Proposition 3.1 For all $n \geq 1$, we denote by $\mathcal{P}_n$ the set of discontinuity points and local extrema of T^n. It is enough to show that for all $n \geq 1$, $\mathcal{P}_n$ is finite. Indeed, if this is true, we will have

$$\mathcal{P}_n = \big\{0 = a_0^n < a_1^n < \ldots < a_{l_n}^n < a_{l_n+1}^n = 1\big\},$$

for some $l_n \geq 0$. By construction, and according to Lemma 3.1, T^n will be continuous and strictly monotone on each interval $]a_j^n, a_{j+1}^n[$.

Let us prove that for all $n \geq 1$, $\mathcal{P}_n$ is finite. It is easy to show by induction that for all $i \geq 0$, $T^{-i}(\mathcal{P})$ is finite (use the fact that $\mathcal{P}$ is finite and that T is strictly monotone on each interval I_j). We conclude by noting that, by Lemma 3.2,

$$\mathcal{P}_n \subset \bigcup_{i=0}^{n-1} T^{-i}(\mathcal{P}).$$

■

The following two lemmas will be useful to prove Proposition 3.2. Proof of Lemma 3.3 is easy and is left to the reader.

Lemma 3.3 *Let $g : [0,1] \to [0,1]$ be a function with left and right-hand limits at any point, and let $(\alpha_p)_{p\geq 0}$ be a sequence in $[0,1]$ converging towards α and such that $\big(g(\alpha_p)\big)_{p\geq 0}$ converges towards l. Then $l \in \big\{g(\alpha), g(\alpha^-), g(\alpha^+)\big\}$.*

According to Proposition 3.1, T^n admits left and right-hand limits at any point. This allows us to state the lemma below.

Lemma 3.4 *Let $n \geq 1$ and $s \in [0,1]$. If T^n is* not *left* (resp. *right*) *continuous at the point s, then $T^n(s^-) \in \mathcal{P}$* (resp. *$T^n(s^+) \in \mathcal{P}$*).

Proof of Lemma 3.4 Assume for example that T^n is not left continuous at the point s. The proof will be divided into two steps.

STEP 1 We show that there exists $i \in \{0, \ldots, n-1\}$ such that $T^i(s^-) \in \mathcal{P}$.

Suppose, contrary to our claim, that for all $i \in \{0, \ldots, n-1\}$, $T^i(s^-) \in \mathcal{P}^c$. In particular, this implies that s is a continuity point for T and thus, that

$$T(s^-) = \lim_{\varepsilon \searrow 0} T(s - \varepsilon) = T(s).$$

But $T(s^-)$ $\big(= T(s)\big)$ is a continuity point for T as well, and thus

$$T^2(s^-) = \lim_{\varepsilon \searrow 0} T^2(s - \varepsilon) = \lim_{\varepsilon \searrow 0} T\big(T(s - \varepsilon)\big) = T^2(s).$$

Iterating this process, one finds that $T^n(s^-) = T^n(s)$. Consequently, T^n would be left continuous at the point s. Contradiction.

STEP 2 Conclusion.

Let $i \in \{0, \ldots, n-1\}$ such that $T^i(s^-) \in \mathcal{P}$ (STEP 1). By definition,

$$T^{i+1}(s^-) = \lim_{p \to +\infty} T\big(T^i(s - 1/p)\big).$$

But, $T^i(s - 1/p) \to T^i(s^-)$ as $p \to +\infty$ so that, according to Lemma 3.3,

$$T^{i+1}(s^-) \in \Big\{T\big(T^i(s^-)\big), T\big(\big(T^i(s^-)\big)^-\big), T\big(\big(T^i(s^-)\big)^+\big)\Big\}.$$

Using the set of assumptions (2.2), we deduce from above that $T^{i+1}(s^-) \in \mathcal{P}$ since $T^i(s^-) \in \mathcal{P}$ according to STEP 1. Iterating this proof, one finally obtains that $T^n(s^-) \in \mathcal{P}$. ■

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2 Recall that $I_j^n =]s_1, s_2[$, where $0 \le s_1 < s_2 \le 1$ are discontinuity points or local extrema for T^n (Proposition 3.1). Since T^n is continuous and strictly monotone on I_j^n (Proposition 3.1) we have, assuming for instance that T^n is strictly increasing,

$$T^n(I_j^n) = \big]T^n(s_1^+), T^n(s_2^-)\big[.$$

We claim that $T^n(s_2^-) \in \mathcal{P}$. To verify this assertion, two cases have to be considered.

CASE 1 T^n is left continuous at the point s_2.

In this case, $T^n(s_2^-) = T^n(s_2)$. In addition, according to Lemma 3.2, $s_2 \in \cup_{i=0}^{n-1} T^{-i}(\mathcal{P})$. Since $T(\mathcal{P}) \subset \mathcal{P}$, we obtain $T^n(s_2^-) \in \mathcal{P}$.

CASE 2 T^n is not left continuous at the point s_2.

According to Lemma 3.4, one has directly $T^n(s_2^-) \in \mathcal{P}$.

Thus, in both cases, $T^n(s_2^-) \in \mathcal{P}$. Similarly, one shows that $T^n(s_1^+) \in \mathcal{P}$. Consequently, there exists $m \in \{0, \ldots, l\}$ such that

$$T^n(I_j^n) = \big]T^n(s_1^+), T^n(s_2^-)\big[\supset I_m.$$

This completes the proof of the proposition. ■

Proof of Proposition 3.3 Let us assume that

$$\limsup_{n\to+\infty} \max_{j\in\{0,\dots,l_n\}} \operatorname{diam} I_j^n > 0.$$

If this is the case, there exists $\varepsilon > 0$ and an increasing sequence $(n_p)_{p\geq 0}$ such that for all $p \geq 0$, $\exists j_p \in \{0, \dots, l_{n_p}\}$ with $\operatorname{diam} I_{j_p}^{n_p} \geq \varepsilon$. By compactness of $[0,1]$, we can assume without loss of generality that there exists a non-empty open interval J such that for all $p \geq 0$, $I_{j_p}^{n_p} \supset J$. Now, according to Birkhoff's ergodic theorem (2.1):

$$\frac{1}{n_p}\int_J \log\left|(T^{n_p})'\right| \mathrm{d}\mu \to \int_J \lambda \, \mathrm{d}\mu \quad \text{as } p \to +\infty.$$

By Jensen's concave inequality and the fact that $\mu(J) > 0$ (since J is a non-empty open interval and the support of μ is $[0,1]$):

$$\liminf_{p\to+\infty} \frac{1}{n_p} \log \int_J \left|(T^{n_p})'\right| \frac{\mathrm{d}\mu}{\mu(J)} \geq \frac{1}{\mu(J)} \int_J \lambda \, \mathrm{d}\mu. \tag{3.3}$$

By assumption on f, there exists a positive real number M such that $M = \sup_{x\in J} f(x)$. Moreover, for all $p \geq 0$, T^{n_p} is monotone on J according to Proposition 3.1. Consequently,

$$\begin{aligned}
\int_J \left|(T^{n_p})'\right| \mathrm{d}\mu &= \int_J \left|(T^{n_p})'(x)\right| f(x) \, \mathrm{d}x \\
&\leq M \int_J \left|(T^{n_p})'(x)\right| \mathrm{d}x \\
&= M \left| \int_J (T^{n_p})'(x) \, \mathrm{d}x \right| \\
&= M \left|T^{n_p}(s_2) - T^{n_p}(s_1)\right| \quad \text{with } J =]s_1, s_2[\\
&\leq M.
\end{aligned}$$

With (3.3), one deduces that

$$\int_J \lambda \, \mathrm{d}\mu \leq 0\,,$$

and, since $\mu(J) > 0$, that

$$\mu\big(\{x \in [0,1] : \lambda(x) > 0\}\big) < 1.$$

This concludes the proof of the proposition. ∎

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