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Stationary measures for the Porous Medium Model

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Abstract. We study the stationary measures for variants of the Porous Medium Model in dimension 1. These are exclusion processes that belong to the class of kinetically constrained models, in which an exchange can occur between x and $x+1$ only if there is a particle either at $x-1$ or $x+2$. We show that any stationary probability measure can be decomposed into a frozen part and a mixture of product measures (although there exist invariant sets which have zero probability under these measures). The proof adapts entropy arguments from Holley and Stroock (1977); Liggett (2005).

1. Introduction

The Porous Medium Model (PMM), is the process on $\Omega := \{0,1\}^{\mathbb{Z}}$ in which a particle jumps between x and $x+1$ or $x+1$ and x at rate $\eta(x-1)+\eta(x+2)$. In particular, isolated particles (at distance at least 3 from any other) are frozen, and this means the process admits infinitely stationary measures. This model has been introduced in Bertini and Toninelli (2004). It takes its name from the fact (established in Gonçalves et al. (2009)) that in the hydrodynamic limit, for initial densities bounded away from 0 and 1, the density of the system converges to the solution of the Porous Medium Equation:

$$\partial_t \rho = \Delta(\rho^2). \tag{1.1}$$

We are interested in the study of the stationary probability measures for this model. Apart from this being a very natural question, such knowledge can be instrumental in the study of the hydrodynamic limit of the system Funaki (1999); Blondel et al. (2021).

It is well known (e.g. Bertini and Toninelli (2004)) and easily checked by detailed balance that for any $\rho \in [0,1]$, $\mu_\rho := \bigotimes_{x\in\mathbb{Z}} \mathrm{Ber}(\rho)$ is a reversible measure for this dynamics. We also noticed already that there are infinitely stationary measures concentrated on frozen configurations. Our main result is that, although there exist invariant sets that contain no frozen configuration and have μ_ρ–probability 0 for any ρ (see (2.3)–(2.5)), the only extremal stationary probability measures are either concentrated on frozen configurations, or product.

The strategy of the proof is to use entropy arguments introduced in Holley and Stroock (1977) to argue that any stationary measure concentrated on non-frozen configurations is in fact reversible w.r.t. allowed jumps. That is the content of Lemma 4.1, which is proved in Section 5. The rest of

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the proof consists in exploiting this result to show that any putative stationary measure gives the same probability to certain sets of sub-configurations. That is the content of Section 4. In turn, these constructions rely on the identification of allowed transformations of configurations, collected in Section 3.

2. Model and result

2.1. *Notations and definitions.* We consider interacting particle systems with state space $\Omega := \{0,1\}^{\mathbb{Z}}$. As usual, for $\eta \in \Omega$, we say there is a particle at x if $\eta(x) = 1$, and that x is empty if $\eta(x) = 0$. We will also need to consider configurations restricted to subsets of $\mathbb{Z}$. For Λ a subset of $\mathbb{Z}$, we define $\Omega_\Lambda = \{0,1\}^\Lambda$. For $\eta \in \Omega$, $\eta_{|\Lambda}$ is the element of Ω_Λ such that for all $x \in \Lambda$, $\eta_{|\Lambda}(x) = \eta(x)$. Recall that a function $f : \Omega \to \mathbb{R}$ is *local* if there exists Λ finite subset of $\mathbb{Z}$ such that: $\forall \eta, \eta' \in \Omega$, $\eta_{|\Lambda} = \eta'_{|\Lambda} \Rightarrow f(\eta) = f(\eta')$. The support of a local function is the smallest such Λ. For $\sigma \in \Omega_\Lambda$ and ν a measure on Ω, we denote abusively $\mathbf{1}_\sigma$ the indicator function of $\{\eta \in \Omega : \ \eta_{|\Lambda} = \sigma\}$ and we write $\nu(\sigma) = \nu(\mathbf{1}_\sigma)$. If a function f has support in Λ and $\sigma \in \Omega_\Lambda$, we write $f(\sigma)$ for the common value of $f(\eta)$, with $\eta \in \Omega$ such that $\eta_{|\Lambda} = \sigma$. Also, for $\sigma \in \Omega_\Lambda$, we write $|\sigma| = \sum_{x\in\Lambda} \sigma(x)$. For $n \in \mathbb{Z}_+$, $\Lambda_n := [-n, n] \cap \mathbb{Z}$. For $\rho \in (0,1)$, $\mu_\rho = \bigotimes_{x\in\mathbb{Z}} \mathrm{Ber}(\rho)$ is the product probability measure on Ω with homogeneous density ρ.

To define the models we consider, we consider a family of constraints $(c_x(\eta))_{x\in\mathbb{Z},\eta\in\Omega}$ that satisfies the following assumption.

Assumption 2.1. The family $(c_x(\eta))_{x\in\mathbb{Z},\eta\in\Omega}$ takes values in $\mathbb{R}_+$ and has the following properties:

- translation invariance: for any $x \in \mathbb{Z}$, $\eta \in \Omega$, $c_x(\eta) = c_0(\eta(x + \cdot))$;
- c_0 is a local function;
- for all $\eta \in \Omega$, $c_0(\eta) = c_0(\eta^{0,1})$;
- for all $\eta \in \Omega$, $c_0(\eta) > 0$ if and only if $\eta(-1) + \eta(2) > 0$.

Typical choices for c_0 are $c_0(\sigma) = \eta(-1) + \eta(2)$ or $c_0(\sigma) = \eta(-1) \vee \eta(2)$.

For f local function, $\eta \in \Omega$, let us define

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}} c_x(\eta) \left[f(\eta^{x,x+1}) - f(\eta) \right], \tag{2.1}$$

where $\eta^{x,x+1}(y) = \eta(y)$ if $y \notin \{x, x+1\}$, $\eta^{x,x+1}(x) = \eta(x+1)$, $\eta^{x,x+1}(x+1) = \eta(x)$.

Let us introduce some additional terminology.

Definition 2.2. Let Λ be an interval of $\mathbb{Z}$.

1. For $\eta \in \Omega$, the transformation $\eta \mapsto \eta^{x,x+1}$ is an *allowed jump* if $c_x(\eta) > 0$.
2. For $\{x, x+1\} \subset \Lambda$, $\sigma \in \Omega_\Lambda$, $\sigma \mapsto \sigma^{x,x+1}$ is an *allowed jump inside* Λ if $c_x(0 \cdot \sigma) > 0$, where $0 \cdot \sigma$ is the configuration equal to 0 on Λ^c and to σ on Λ.
3. For $\eta \in \Omega$, $x \in \mathbb{Z}$ such that $\eta(x) = 1$, the particle at x is *active* if $c_x(\eta) + c_{x-1}(\eta) > 0$ (equivalently, if there is another particle within distance 2 of x).
4. A pair of particles within distance 2 of each other an *mobile cluster*.
5. A configuration without active particles is called *frozen*.

The reason for the name “mobile cluster” is that (as one can easily check), such a pair can move through space autonomously: no matter what the rest of the configuration is, for any target pair of neighbor or next-to-neighbor positions, there exists a sequence of allowed jumps that transports the pair from its initial position to the target (see Figure 2.1). Note that the notions in this definition do not depend on the specific choice of the constraints satisfying the last condition in Assumption 2.1.

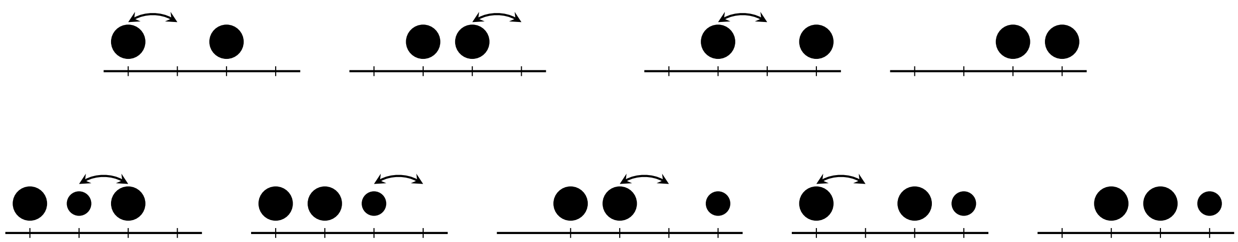

Figure 2.1. On the first line, a strategy for moving a mobile cluster toward the right. The rest of the configuration is represented as empty but is in fact irrelevant. On the second line, steps 2–5, the beginning of the procedure that can be used to transport to the right an extra particle (depicted with smaller size) with a mobile cluster. At the end of the procedure a particle has been pushed from inside the interval to its right side. The first step shows that the extra particle can be moved to the right of the mobile cluster.

We are interested in the stationary measures of the process defined by (2.1), that is in the set of probability measures ν on Ω such that $\nu(\mathcal{L}f) = 0$ for all local functions f. Note that the third condition in Assumption 2.1 ensures that this process is reversible w.r.t. μ_ρ for any $\rho \in (0,1)$.

We first identify the invariant sets of the dynamics defined by (2.1). Notice that the number of particles is preserved by this dynamics. We define $\mathcal{F}$ the set of frozen configurations, $\mathcal{F}'$ the set of non-frozen configurations with finitely many particles, $\mathcal{F}''$ the set of non-frozen configurations with finitely many holes, $\mathcal{E}'$ the set of configurations with finitely many active particles that are in none of the sets $\mathcal{F}, \mathcal{F}', \mathcal{F}''$. Finally, $\mathcal{E}$ is what we call the ergodic set of configurations and is the complement of the previous sets. It contains the configurations that have infinitely many active particles and infinitely many empty sites. It is also a set with full measure under μ_ρ, $\rho \in (0,1)$.

$$\mathcal{F} = \{\eta \in \Omega \,:\, \forall x \in \mathbb{Z},\ \eta(x) = 1 \Rightarrow \eta(x+1) + \eta(x+2) = 0\}, \tag{2.2}$$

$$\mathcal{F}' = \cup_{k=2}^{\infty} \mathcal{F}'_k, \text{ where } \mathcal{F}'_k = \{\eta \in \mathcal{F}^c \,:\, \sum_{x\in\mathbb{Z}} \eta(x) = k\}\ , \tag{2.3}$$

$$\mathcal{F}'' = \cup_{k=0}^{\infty} \mathcal{F}''_k, \text{ where } \mathcal{F}''_k = \{\eta \in \mathcal{F}^c \,:\, \sum_{x\in\mathbb{Z}} (1-\eta(x)) = k\}\ , \tag{2.4}$$

$$\mathcal{E}' = \{\eta \in \Omega \setminus \mathcal{F}' \,:\, 0 < \sum_{x\in\mathbb{Z}} (\eta(x)\eta(x+1) + \eta(x)\eta(x+2)) < \infty\}, \tag{2.5}$$

$$\mathcal{E} = \Omega \setminus (\mathcal{F} \cup \mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{E}'). \tag{2.6}$$

We will also consider G_Λ the set of "good" configurations that contain at least one mobile cluster in Λ, i.e.

$$G_\Lambda = \{\sigma \in \Omega_\Lambda \,:\, \exists \{x,y\} \subset \Lambda \text{ s.t. } |x-y| \in \{1,2\} \text{ and } \sigma(x) + \sigma(y) = 2\}. \tag{2.7}$$

G_Λ is invariant under allowed jumps inside Λ (recall that we assume empty boundary condition). Finally, for $n \in \mathbb{Z}_+^*$ ($\mathbb{Z}_+^*$ is the set of positive integers), let $G_n = G_{\Lambda_n}$.

2.2. *Main result.* Recall that μ_ρ is the product measure on Ω with density ρ. We aim to prove the following.

Theorem 2.3. *Under Assumption 2.1, any stationary measure ν for the process with generator $\mathcal{L}$ can be decomposed into*

$$\nu = \alpha_{\mathcal{F}} \nu_{\mathcal{F}} + \alpha_{\mathcal{E}} \nu_{\mathcal{E}}, \tag{2.8}$$

with ν_X a stationary measure such that $\nu_X(X) = 1$, $\alpha_X = \nu(X)$ for $X \in \{\mathcal{F}, \mathcal{E}\}$. Moreover, if $\nu(\mathcal{E}) > 0$, then there exists λ a probability measure on $(0,1)$ such that

$$\nu_{\mathcal{E}} = \int_0^1 \mu_\rho d\lambda(\rho). \tag{2.9}$$

Remark 2.4. It may seem surprising that there is no stationary probability measure concentrated on the invariant sets $\mathcal{F}', \mathcal{F}'', \mathcal{E}'$. This should be interpreted as a manifestation of the fact that abnormalities (e.g. finitely many active particles in a mostly frozen configuration) escape to infinity in large time and cannot be seen in a stationary regime.

3. Facts about invariant sets and connections

We start by stating a few facts that follow from definitions (2.2)–(2.6).

Definition 3.1. Let Λ be an interval of $\mathbb{Z}$ and let $\sigma, \sigma' \in \Omega_\Lambda$ two configurations. We say that σ and σ' are connected inside Λ, and we write $\sigma \overset{\Lambda}{\leftrightarrow} \sigma'$ if it is possible to reach σ' from σ by performing a finite number of allowed jumps inside Λ. When $\Lambda = \mathbb{Z}$ we omit the superscript.

Lemma 3.2. *We have the following facts.*

(1) *$\mathcal{F}$, the $\mathcal{F}'_k$ with $k \geq 2$, the $\mathcal{F}''_k$ with $k \in \mathbb{Z}_+$, $\mathcal{E}'$ and $\mathcal{E}$ are disjoint and invariant sets for the dynamics with generator* (2.1) *(in fact they are invariant under allowed jumps).*

(2) *For any Λ interval of $\mathbb{Z}$, $\sigma, \sigma' \in G_\Lambda$, $|\sigma| = |\sigma'|$ implies $\sigma \overset{\Lambda}{\leftrightarrow} \sigma'$.*

(3) *For all $\sigma, \sigma' \in \mathcal{F}'_k$ (resp. $\sigma, \sigma' \in \mathcal{F}''_k$), for all $n \in \mathbb{Z}_+^*$ such that $\{x \in \mathbb{Z} : \sigma(x) + \sigma'(x) > 0\} \subset \Lambda_n$ (resp. $\{x \in \mathbb{Z} : \sigma(x)\sigma'(x) = 0\} \subset \Lambda_{n-2}$), $\sigma \overset{\Lambda_n}{\leftrightarrow} \sigma'$.*

Proof: The first point should be clear from the definitions.

The second point is a crucial consequence of the existence of a mobile cluster inside Λ. If $|\sigma| = 2$, this is a restatement of the fact that a mobile cluster can transport itself to any other location through allowed jumps (see Figure 2.1). For the case $|\sigma| \geqslant 3$, the crucial argument is that a mobile cluster can also take an additional particle along with it (see the second half of Figure 2.1). It is enough to show that for $\sigma \in G_\Lambda$, $\sigma \overset{\Lambda}{\leftrightarrow} \sigma'$, where σ' is the configuration with exactly $|\sigma|$ particles occupying the rightmost sites of Λ. We can show this by using $|\sigma|-2$ times the procedure suggested in Figure 2.1: move a mobile cluster next to a particle which is not yet in position (as in the first half of Figure 2.1) and use the mobile cluster to move it to the rightmost empty site (as in the second half of the figure).

The third point is a consequence of the second: if $\sigma, \sigma' \in \mathcal{F}'_k$ (resp. $\mathcal{F}''_k$), n is chosen so that their restriction to Λ_n is in G_n and $|\sigma_{\Lambda_n}| = |\sigma'_{\Lambda_n}|$. Indeed, in the $\mathcal{F}'$ case, Λ_n contains all the particles of σ, σ', and since these configurations are not frozen, there is at least one mobile cluster in Λ_n. In the $\mathcal{F}''$ case, Λ_{n-2} contains all the empty sites of σ, σ', and therefore both configurations have a mobile cluster in the two right-most sites of Λ_n. □

Our first result identifies the configurations that appear with positive probability under ν.

Lemma 3.3. *Fix ν an invariant measure for the PMM.*

(1) *For all $k \in \mathbb{Z}_+ \setminus \{0,1\}$, either $\nu(\mathcal{F}'_k) = 0$ or $\forall \eta \in \mathcal{F}'_k$, $\nu(\eta) > 0$.*
For all $k \in \mathbb{Z}_+$, either $\nu(\mathcal{F}''_k) = 0$ or $\forall \eta \in \mathcal{F}''_k$, $\nu(\eta) > 0$.

(2) *If $\nu(\mathcal{E} \cup \mathcal{E}') > 0$, for all $n \in \mathbb{Z}_+^*$, for all $\sigma \in \Omega_{\Lambda_n}$, $\nu(\sigma) > 0$.*

Proof: The lemma relies on Lemma 3.2 and the following observation: if ν is invariant, for all $\sigma \in \Omega_{\Lambda_n}$, $\nu(\mathcal{L}\mathbf{1}_\sigma) = 0$. This means

$$\sum_{x=-n-1}^{n} \left[\nu(c_x \mathbf{1}_\sigma^{x,x+1}) - \nu(c_x \mathbf{1}_\sigma)\right] = 0,$$

where $f^{x,x+1}(\eta) = f(\eta^{x,x+1})$ for $\eta \in \Omega$. In particular, if $\nu(\sigma) = 0$, the negative part vanishes in every term of the sum, and $\nu(c_x \mathbf{1}_\sigma^{x,x+1}) = 0$ for all $x \in \{-n-1, \dots, n\}$. Equivalently, all configurations that can be obtained from σ by an allowed jump have 0 probability under ν. Iterating the argument yields that all configurations σ' connected to σ inside Λ_n also satisfy $\nu(\sigma') = 0$.

Let us now prove the two points of the lemma.

(1) Fix $k \in \mathbb{Z}_+ \setminus \{0,1\}$ and assume $\nu(\mathcal{F}'_k) > 0$. Assume by contradiction that there exists $\eta \in \mathcal{F}'_k$ such that $\nu(\eta) = 0$. Let us show that for any $\eta' \in \mathcal{F}'_k$, $\nu(\eta') = 0$, which contradicts $\nu(\mathcal{F}'_k) > 0$. Choosing n large enough so that $\{x \in \mathbb{Z} :\ \eta(x) + \eta'(x) > 0\} \subset \Lambda_n$, Lemma 3.2 gives the result.

The proof is similar for $\mathcal{F}''_k$.

(2) Assume by contradiction that there exists $\sigma \in \Omega_{\Lambda_n}$ such that $\nu(\sigma) = 0$. We claim that, if $\nu(\mathcal{E} \cup \mathcal{E}') > 0$, there exists $N \in \mathbb{Z}_+^*$, $\eta, \eta' \in \Omega_{\Lambda_N}$ such that $\eta_{|\Lambda_n} = \sigma$, $\nu(\eta') > 0$ and $\eta \overset{\Lambda_N}{\leftrightarrow} \eta'$. This is a contradiction because $\nu(\sigma) = 0$ implies $\nu(\eta) = 0$, which implies by connection $\nu(\eta') = 0$.

The claim follows if we can show that there exists $N \geq n+2$, $\eta' \in G_N$ such that $|\eta'| \geq |\sigma| + 2$, $|1 - \eta'| \geq |1 - \sigma|$ and $\nu(\eta') > 0$ (note that η' lives in a larger interval than σ, so there is no contradiction in the previous inequalities). Indeed, in that case, it is easy to extend σ into a configuration $\eta \in G_N$ such that $|\eta| = |\eta'|$. Then, by Lemma 3.2, η and η' are connected inside Λ_N.

Let us show that we can find N, η' as above. By definition of $\mathcal{E}, \mathcal{E}'$, for any $\xi \in \mathcal{E} \cup \mathcal{E}'$, there exists $N \in \mathbb{Z}_+^*$ such that $\xi_{|\Lambda_N} \in G_N$, $|\xi_{|\Lambda_N}| \geq |\sigma| + 2$ and $|1 - \xi_{|\Lambda_N}| \geq |1 - \sigma|$ (and these properties hold for any $N' \geq N$). Therefore, since $\nu(\mathcal{E} \cup \mathcal{E}') > 0$, there exists $N \geq n+2$ such that

$$\nu(\{\xi \in \Omega :\ \xi_{|\Lambda_N} \in G_N, |\xi_{|\Lambda_N}| \geq |\sigma| + 2 \text{ and } |1 - \xi_{|\Lambda_N}| \geq |1 - \sigma|\}) > 0,$$

which shows the existence of η' as desired.

□

4. Proof of Theorem 2.3

From the invariance of the sets $\mathcal{F}, \mathcal{F}', \mathcal{F}'', \mathcal{E}, \mathcal{E}'$ under allowed jumps, we deduce immediately the existence of a decomposition $\nu = \alpha_\mathcal{F} \nu_\mathcal{F} + \alpha_{\mathcal{F}'} \nu_{\mathcal{F}'} + \alpha_{\mathcal{F}''} \nu_{\mathcal{F}''} + \alpha_{\mathcal{E}'} \nu_{\mathcal{E}'} + \alpha_\mathcal{E} \nu_\mathcal{E}$, with ν_X stationary, $\nu_X(X) = 1$ and $\alpha_X = \nu(X)$ for $X \in \{\mathcal{F}, \mathcal{F}', \mathcal{F}'', \mathcal{E}, \mathcal{E}'\}$, letting $\nu_X = \nu(\cdot | X)$ if $\nu(X) > 0$. Indeed in that case, for any f local function $\nu(\mathcal{L}f | X) = \nu(\mathbf{1}_X \mathcal{L} f)/\nu(X) = \nu(\mathcal{L}(\mathbf{1}_X f))/\nu(X) = 0$, so that ν_X is stationary.

It remains to show $\alpha_{\mathcal{F}'} = \alpha_{\mathcal{F}''} = \alpha_{\mathcal{E}'} = 0$, and that $\nu_\mathcal{E}$ has the form (2.9). The main ingredient to obtain these results is the following lemma.

Lemma 4.1. *If ν is stationary and $\nu(\mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{E} \cup \mathcal{E}') > 0$, for all $\sigma \in \Omega_{\Lambda_n}$, $x \in \{-n, \dots, n-1\}$,*

$$c_x(\sigma) \left[\nu(\sigma^{x,x+1}) - \nu(\sigma) \right] = 0.$$

Consequently, for any $\Lambda \subset \mathbb{Z}$, $\sigma, \sigma' \in \Omega_\Lambda$, if $\sigma \overset{\Lambda}{\leftrightarrow} \sigma'$, then $\nu(\sigma) = \nu(\sigma')$.

Let us defer the proof of Lemma 4.1 to Section 5 to conclude the proof of Theorem 2.3.

Assume first that $\nu = \nu_\mathcal{E}$. We want to show (2.9). By de Finetti theorem, it is enough to show that ν is exchangeable, i.e. that for $\sigma \in \Omega_{\Lambda_n}$, $\nu(\sigma)$ depends only on $|\sigma|$. Suppose $\Lambda = \Lambda_{n_0}$ and consider $\sigma, \sigma' \in \Omega_\Lambda$ such that $|\sigma| = |\sigma'|$. It is enough to show $\nu(\sigma) = \nu(\sigma')$ in that case. If $\sigma, \sigma' \in G_\Lambda$, Lemma 3.2 and the second part of Lemma 4.1 imply $\nu(\sigma) = \nu(\sigma')$. It remains to treat the case where σ or σ' is not in G_Λ.

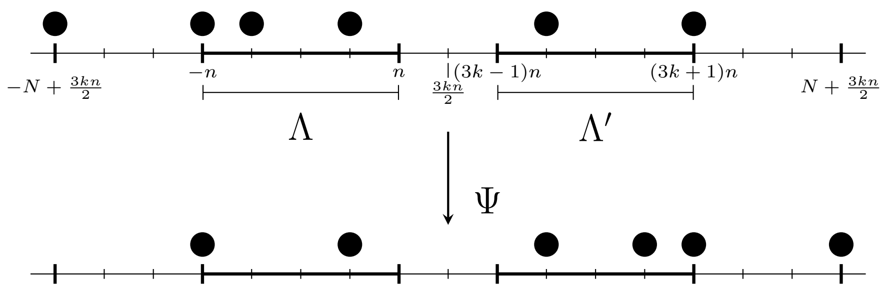

FIGURE 4.2. On the first line are depicted Λ, Λ' appearing in the proof that $\alpha_{\mathcal{E}'} = 0$, with $n = 2, k = 1$. The configuration on top is in B_N, and the bottom part of the picture gives its image under the application Ψ defined in (4.4).

For $n \geq 1$, write $B_n = \Lambda_{n_0+n} \setminus \Lambda$. For $\sigma \in \Omega_\Lambda$, write

$$\nu(\sigma) = \sum_{n\geq 2} \sum_{\zeta \in G_{B_n} \setminus G_{B_{n-1}}} \nu(\sigma \cdot \zeta), \tag{4.1}$$

where $\sigma \cdot \zeta$ denotes the configuration equal to σ in Λ and to ζ in B_n. This equality holds because, thanks to the definition of $\mathcal{E}$, with probability 1 under ν, there exists a pair of active particles outside Λ.

Now fix $\sigma, \sigma' \in \Omega_\Lambda$ such that $|\sigma| = |\sigma'|$, $\zeta \in G_{B_n}$. We clearly have $\sigma \cdot \zeta, \sigma' \cdot \zeta \in G_{\Lambda_{n_0+n}}$ and $|\sigma \cdot \zeta| = |\sigma' \cdot \zeta|$. Therefore, $\nu(\sigma' \cdot \zeta) = \nu(\sigma \cdot \zeta)$. Summing these equalities in the decompositions (4.1) for σ, σ' implies in turn $\nu(\sigma) = \nu(\sigma')$.

Let us now consider the case $\nu = \nu_{\mathcal{F}'}$ or $\nu = \nu_{\mathcal{F}''}$. We want to show that there is a contradiction with ν being stationary. Fix k such that e.g. $\nu(\mathcal{F}'_k) > 0$. By Lemma 3.3, denoting χ_A the indicator function of a set A, $\nu(\{\eta = \chi_{\{1,\dots,k\}}\}) > 0$. Fix now $n \in \mathbb{Z}_+$ and an interval Λ containing $\{1, \dots, k\} \cup \{n+1, \dots, n+k\}$. By Lemma 3.2 (item 2) and Lemma 4.1, $\nu(\{\eta = \chi_{\{n+1,\dots,n+k\}}\})$ does not depend on n. But this means

$$\nu(\mathcal{F}'_k) \geq \sum_{n \in \mathbb{Z}_+} \nu(\{\eta = \chi_{\{n+1,\dots,n+k\}}\}) = \infty. \tag{4.2}$$

With similar arguments in the case $\nu = \nu_{\mathcal{F}''}$, we conclude that $\alpha_{\mathcal{F}'} = \alpha_{\mathcal{F}''} = 0$.

It remains to find a similar contradiction when $\nu = \nu_{\mathcal{E}'}$. For $\Lambda \subset \mathbb{Z}$, consider the event

$$A_\Lambda := \{\eta \in \mathcal{E}' \ : \ \text{all active particles of } \eta \text{ are in } \Lambda\}. \tag{4.3}$$

Then by definition, $\mathcal{E}' = \bigcup_{n=1}^\infty A_{\Lambda_n}$. In particular, there exists $n \in \mathbb{Z}_+^*$ such that $\nu(A_{\Lambda_n}) > 0$. We may and do assume n even. Let us show that Lemma 4.1 implies, for all $k \in \mathbb{Z}_+$, $\nu(A_{\Lambda_n + 3kn}) = \nu(A_{\Lambda_n}) > 0$. This is a contradiction since the $(A_{\Lambda_n+3kn})_k$ are disjoint (because $(\Lambda_n + 3kn) \cap (\Lambda_n + 3k'n) = \emptyset$ if $k \neq k'$) and $\nu(\mathcal{E}') < \infty$. Fix $k \in \mathbb{Z}_+$ and let $\Lambda = \Lambda_n$, $\Lambda' = \Lambda_n + 3kn$. For $N \geq (3k+1)N$, we let (see Figure 4.2) $\Lambda_{N,1} = [-N, -n-1] \cap \mathbb{Z}$, $\Lambda_{N,2} = [n+1, N]$, $\Lambda'_{N,1} = [(3k+1)n+1, N]$, $\Lambda'_{N,2}$. We have that $A_\Lambda = \cap_{N \geq (3k+1)n} B_N$, $A_{\Lambda'} = \cap_{N \geq (3k+1)n} B'_N$, where

$B_N := \{\eta \in \Omega : \text{there are active particles in } \Lambda_N + (3kn)/2 \text{ with empty boundary condition and they are in } \Lambda\}$,

$B'_N := \{\eta \in \Omega : \text{there are active particles in } \Lambda_N + (3kn)/2 \text{ with empty boundary condition and they are in } \Lambda'\}$.

Note that B_N, B'_N are not subsets of $\mathcal{E}'$, and $\mathbf{1}_{B_N}, \mathbf{1}_{B'_N}$ are local functions with support in Λ_N. Moreover, $\nu(A_\Lambda) = \lim_{N\to\infty} \nu(B_N)$ and $\nu(A_{\Lambda'}) = \lim_{N\to\infty} \nu(B'_N)$ since the sequences $(B_N)_N, (B'_N)_N$ are decreasing. Therefore, it is enough to show that $\nu(B_N) = \nu(B'_N)$. This follows from Lemma 4.1 and the following bijection between B_N and B'_N:

$$\Psi(\sigma)(-N + (3kn)/2 + x) = \sigma(N + (3kn)/2 - x), \text{ for all } x \in \Lambda_N + (3kn)/2. \tag{4.4}$$

In words, Ψ performs a mirror reflection on σ inside $\Lambda_N + (3kn)/2$ (see Figure 4.2). It is easy to check that this defines indeed a bijection, and Lemma 4.1 implies $\forall \sigma \in B_N$, $\nu(\sigma) = \nu(\Psi(\sigma))$. It follows that $\alpha_{\mathcal{E}'} = 0$.

5. Proof of the invariance under allowed exchanges (Lemma 3)

The following is an adaptation of the arguments in Holley and Stroock (1977), as well as their reformulations in Liggett (2005, Section IV.4) or Neuhauser and Sudbury (1993) (see also Yaguchi (1990)). For simplicity, we assume here that $c_0(\sigma)$ only depends on $\sigma(-1), \sigma(2)$, but the proof would work similarly for a more general local function.

Let us attempt to give an overview of the strategy. We essentially want to show that our stationary measure is in fact an equilibrium measure (satisfies detailed balance). The basis of the strategy of Holley and Stroock (1977) is to notice that the time-derivative of the entropy (relative to any equilibrium measure) for the stationary process in any finite interval is zero (this yields (5.2) below). This derivative has contributions coming from the transitions in the bulk of the interval and also from those involving the boundary (see I_n, B_n below). Since their sum is null, the two have to be opposite. This seems to be in contradiction with the fact that we expect the boundary contribution to scale like the size of the boundary (i.e., bounded in dimension 1), while the bulk contribution should scale like the volume. If this intuition could be made rigorous, the contradiction could only be resolved by having all contributions be zero, i.e. the stationary state being in fact an equilibrium. The argument in fact does not quite work like this (and in particular it seems to work only up to dimension 2 or for translation invariant stationary states) and the details below are not so pleasant as the previous paragraph might suggest. Showing that the boundary contribution is finite is however an actual key step in the proof (see (5.13)–(5.20)).

Let us first assume $\nu(\mathcal{E} \cup \mathcal{E}') > 0$. Thanks to Lemma 3.3, in that case, the following quantity is well-defined for $\rho \in (0,1)$.

$$H_n(\nu) = \sum_{\sigma \in \Omega_{\Lambda_n}} \nu(\sigma) \log \left[\frac{\nu(\sigma)}{\mu_\rho(\sigma)} \right]. \tag{5.1}$$

Moreover, writing $\nu_t = \nu$ for the distribution of the process at time t starting from ν and with generator (2.1), $\frac{d}{dt} H_n(\nu_t) = 0$. Computing the derivative and evaluating it at time 0, we get the identity

$$\sum_{\sigma \in \Omega_{\Lambda_n}} \log \left(\frac{\nu(\sigma)}{\mu_\rho(\sigma)} \right) \nu(\mathcal{L} \mathbf{1}_\sigma) = 0. \tag{5.2}$$

For $n \in \mathbb{Z}_+^*$, $x \in \{-n, \dots, n-1\}$, $\sigma \in \Omega_{\Lambda_n}$, let us define

$$\Gamma_n(x, \sigma) = \nu(c_x \mathbf{1}_\sigma). \tag{5.3}$$

Note that, for $x \in \{-n+1, \dots, n-2\}$, we simply have $\Gamma_n(x,\sigma) = c_x(\sigma)\nu(\sigma)$. Also, for all $m \geq n$,

$$\Gamma_n(x, \sigma) = \sum_{\zeta \in \Omega_{\Lambda_m}, \zeta_{|\Lambda_n} = \sigma} \Gamma_m(x, \zeta). \tag{5.4}$$

Let us split the left-hand side of (5.2) into $I_n + B_n$, with I_n containing the bulk terms and B_n the boundary terms. Namely, we let

$$I_n = \sum_{\sigma\in\Omega_{\Lambda_n}} \sum_{x=-n+1}^{n-2} \log\left(\frac{\nu(\sigma)}{\mu_\rho(\sigma)}\right)\left[\Gamma_n(x,\sigma^{x,x+1}) - \Gamma_n(x,\sigma)\right], \tag{5.5}$$

$$B_n = \sum_{\sigma\in\Omega_{\Lambda_n}} \log\left(\frac{\nu(\sigma)}{\mu_\rho(\sigma)}\right)\Big\{\Gamma_n(-n,\sigma^{-n,-n+1}) - \Gamma_n(-n,\sigma) \tag{5.6}$$

$$+\ \Gamma_n(n-1,\sigma^{n-1,n}) - \Gamma_n(n,\sigma) \tag{5.7}$$

$$+ \sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma} \left(\Gamma_{n+1}(-n-1,\zeta^{-n-1,-n}) - \Gamma_{n+1}(-n-1,\zeta)\right) \tag{5.8}$$

$$+ \sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma} \left(\Gamma_{n+1}(n,\zeta^{n,n+1}) - \Gamma_{n+1}(n,\zeta)\right)\Big\}. \tag{5.9}$$

For the terms in I_n, $c_x\mathbf{1}_\sigma$ only depends on the configuration inside Λ_n, so that we can simply write $\Gamma_n(x,\sigma) = c_x(\sigma)\nu(\sigma)$ for $x \in \{-n+1,\dots,n-2\}$. By contrast, B_n contains the boundary terms which need to look outside Λ_n to decide the value of c_x and, for the last two lines, that of $\sigma^{x,x+1}$.

Using that $c_x(\sigma) = c_x(\sigma^{x,x+1})$, let us symmetrize I_n into

$$I_n = -\frac{1}{2}\sum_{x=-n+1}^{n-2}\sum_{\sigma\in\Omega_{\Lambda_n}} \log\left(\frac{\nu(\sigma^{x,x+1})\mu_\rho(\sigma)}{\nu(\sigma)\mu_\rho(\sigma^{x,x+1})}\right)\left[\Gamma_n(x,\sigma^{x,x+1}) - \Gamma_n(x,\sigma)\right] \tag{5.10}$$

$$= -\frac{1}{2}\sum_{x=-n+1}^{n-2}\sum_{\sigma\in\Omega_{\Lambda_n}} c_x(\sigma)\log\left(\frac{\nu(\sigma^{x,x+1})}{\nu(\sigma)}\right)\left(\nu(\sigma^{x,x+1}) - \nu(\sigma)\right). \tag{5.11}$$

Define $\Phi(u,v) = \log(u/v)(u-v)$ and notice that it is convex, subadditive[1] and non-negative. Define

$$\alpha_n(x) = \sum_{\sigma\in\Omega_{\Lambda_n}} c_x(\sigma)\Phi(\nu(\sigma^{x,x+1}),\nu(\sigma)), \tag{5.12}$$

and let $N(x) = \inf\{n\colon x, x+1 \in \Lambda_n\}$. The properties of Φ implies that $(\alpha_n(x))_{n\geq N(x)}$ is a non-decreasing sequence of non-negative real numbers[2]. Therefore, $-I_n$ is a non-decreasing and non-negative sequence, so that if $I_n \underset{n\to\infty}{\longrightarrow} 0$, it is actually null and $\alpha_n(x) = 0$ for all $n \geq N(x)$. Since $\Phi(u,v) = 0$ iff $u = v$, this is enough for our purposes and we only need to show that $B_n \underset{n\to\infty}{\longrightarrow} 0$.

We first show that $\sup_n B_n < \infty$. Since $B_n = -I_n$ is non-negative, this implies that $(B_n)_n$ is bounded. Also, since

$$-2I_n = \sum_{x=-n+1}^{n-2} \alpha_n(x)$$

[1] Because convex and $\Phi(\lambda a, \lambda b) = \lambda\Phi(a,b)$. Indeed, by homogeneity $\Phi(a+a', b+b') = 2\Phi((a+a')/2,(b+b')/2) \leq \Phi(a,b) + \Phi(a',b')$ by convexity.

[2] For the monotonicity, use that for $\sigma \in \Omega_{\Lambda_n}$, $\nu(\sigma) = \sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma}\nu(\zeta)$, the same identity applied to $\sigma^{x,x+1}$, and subadditivity of Φ.

$-2(I_{n+1} - I_n) \geq \alpha_{n+1}(-n) \geq 0$, $(I_n)_n$ is non-increasing, and $(B_n)_n$ non-decreasing, so we will also deduce that $(B_n)_n$ converges. Let us symmetrize B_n:

$$2B_n = -\sum_{\sigma\in\Omega_{\Lambda_n}} \log\left(\frac{\nu(\sigma^{-n,-n+1})\mu_p(\sigma)}{\nu(\sigma)\mu_p(\sigma^{-n,-n+1})}\right)\left[\Gamma_n(-n,\sigma^{-n,-n+1}) - \Gamma_n(-n,\sigma)\right] \tag{5.13}$$

$$-\sum_{\sigma\in\Omega_{\Lambda_n}} \log\left(\frac{\nu(\sigma^{n-1,n})\mu_p(\sigma)}{\nu(\sigma)\mu_p(\sigma^{n-1,n})}\right)\left[\Gamma_n(n-1,\sigma^{n-1,n}) - \Gamma_n(n-1,\sigma)\right] \tag{5.14}$$

$$-\sum_{\sigma\in\Omega_{\Lambda_{n+1}}} \log\left(\frac{\nu(\sigma_{|\Lambda_n}^{-n-1,-n})\mu_p(\sigma_{|\Lambda_n})}{\nu(\sigma_{|\Lambda_n})\mu_p(\sigma_{|\Lambda_n}^{-n-1,-n})}\right)\left[\Gamma_{n+1}(-n-1,\sigma^{-n-1,-n}) - \Gamma_{n+1}(-n-1,\sigma)\right] \tag{5.15}$$

$$-\sum_{\sigma\in\Omega_{\Lambda_{n+1}}} \log\left(\frac{\nu(\sigma_{|\Lambda_n}^{n,n+1})\mu_p(\sigma_{|\Lambda_n})}{\nu(\sigma_{|\Lambda_n})\mu_p(\sigma_{|\Lambda_n}^{n,n+1})}\right)\left[\Gamma_{n+1}(n,\sigma^{n,n+1}) - \Gamma_{n+1}(n,\sigma)\right]. \tag{5.16}$$

Let us bound (5.13) and (5.16). (5.14) and (5.15) are bounded in similar ways.

$$(5.13) = -\sum_{\sigma\in\Omega_{\Lambda_n}} \log\left(\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)}\right)\left[\Gamma_n(-n,\sigma^{-n,-n+1}) - \Gamma_n(-n,\sigma)\right] \tag{5.17}$$

$$\overset{(5.4)}{=} -\sum_{\sigma\in\Omega_{\Lambda_n}}\sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma} \log\left(\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)}\right)\left[\Gamma_{n+1}(-n,\zeta^{-n,-n+1}) - \Gamma_{n+1}(-n,\zeta)\right] \tag{5.18}$$

$$= \sum_{\sigma\in\Omega_{\Lambda_n}}\sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma} \log\left(\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)}\right)c_{-n}(\zeta)\left[\nu(\zeta) - \nu(\zeta^{-n,-n+1})\right]. \tag{5.19}$$

Let us see how we can bound (5.19) by restricting the sum over σ to well-chosen subsets of Ω_{Λ_n}. Fix $K' \geq K \geq 1$ and let us consider the subset of configurations $\sigma \in \Omega_{\Lambda_n}$ such that $\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)} \in [K, K']$. For each such σ, we upper bound the terms appearing after the second sum in (5.19), with a procedure depending on the sign of $\nu(\zeta^{-n,-n+1}) - \nu(\zeta)$. If $\nu(\zeta^{-n,-n+1}) \geq \nu(\zeta)$, we discard the corresponding non-positive term. If $\nu(\zeta^{-n,-n+1}) \leq \nu(\zeta)$, we discard the non-positive term $-\nu(\zeta^{-n-n+1})$ and use $\sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{\Lambda_n}=\sigma,\nu(\zeta^{-n,-n+1})\leq\nu(\zeta)} \nu(\zeta) \leq \nu(\sigma)$ to bound

$$\sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma} \log\left(\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)}\right)c_{-n}(\zeta)\left[\nu(\zeta) - \nu(\zeta^{-n,-n+1})\right] \leq \frac{\log K'}{K}\nu(\sigma^{-n,-n+1}).$$

This procedure yields that the sum in (5.19) restricted to configurations $\sigma \in \Omega_{\Lambda_n}$ such that $\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)} \in [K, K']$ is bounded by $\frac{\log K'}{K}$. Similarly, for fixed $0 < \delta' < \delta \leq 1$, the sum in (5.19) restricted to configurations $\sigma \in \Omega_{\Lambda_n}$ such that $\frac{\nu(\sigma^{-n,-n+1})}{\nu(\sigma)} \in [\delta', \delta]$ is bounded by $|\log(\delta')|\delta$.

We now split (5.19) into such restricted sums, using a well-chosen decomposition of $\mathbb{R}_+$ into intervals. Let us fix $K > 1$ and define $K_k = Ke^k$, $\delta_k = K^{-1}e^{-k}$. We can deduce from the bounds we just established (using them with intervals of the form $[K_k, K_{k+1}]$ or $[\delta_{k+1}, \delta_k]$) that

$$(5.13) \leq \log K \sum_{\zeta\in\Omega_{\Lambda_{n+1}}} c_{-n}(\zeta)|\nu(\zeta) - \nu(\zeta^{-n,-n+1})| + 2\sum_{k\geq 0}\frac{\log K + k + 1}{Ke^k} < \infty. \tag{5.20}$$

Let us now focus on (5.16). For $\sigma \in \Omega_{\Lambda_n}$, $\zeta \in \Omega_{\Lambda_{n+2}}, \zeta_{|\Lambda_n} = \sigma$, denote $\sigma \underset{n}{\leftarrow} \zeta$ the configuration in Ω_{Λ_n} equal to σ on $\Lambda_n \setminus \{n\}$ and $\zeta(n+1)$ on n.

$$(5.16) = -\sum_{\sigma\in\Omega_{\Lambda_n}} \sum_{\zeta\in\Omega_{\Lambda_{n+2}},\zeta_{|\Lambda_n}=\sigma} \log\left(\frac{\nu(\sigma \underset{n}{\leftarrow} \zeta)\mu_p(\sigma)}{\nu(\sigma)\mu_p(\sigma \underset{n}{\leftarrow} \zeta)}\right)\left[\Gamma_{n+2}(n,\zeta^{n,n+1}) - \Gamma_{n+2}(n,\zeta)\right] \tag{5.21}$$

$$= \sum_{\sigma\in\Omega_{\Lambda_n}} \sum_{\zeta\in\Omega_{\Lambda_{n+2}},\zeta_{|\Lambda_n}=\sigma} \left[\log\frac{\nu(\sigma \underset{n}{\leftarrow} \zeta)}{\nu(\sigma)} + \log\frac{\mu_p(\sigma)}{\mu_p(\sigma \underset{n}{\leftarrow} \zeta)}\right] c_n(\zeta)\left[\nu(\zeta) - \nu(\zeta^{n,n+1})\right]. \tag{5.22}$$

$|\log\frac{\mu_p(\sigma)}{\mu_p(\sigma \underset{n}{\leftarrow} \zeta)}|$ is clearly bounded by $\log\frac{p\vee(1-p)}{p\wedge(1-p)}$, which takes care of this extra term. Other than that, we perform almost the same analysis as for (5.13), but we further need to split the sum over ζ into $\zeta(n+1) = \sigma(n)$ and $\zeta(n+1) = 1-\sigma(n)$. The first one does not contribute, and the second can be bounded by the analog of the middle term in (5.20), and the conclusion $(5.16) < \infty$ still holds.

Let us collect that we have established

$$B_n \le 8\sum_{k\ge 0}\frac{\log K + k + 1}{Ke^k} + \log K\left[\beta_{n+1}(n-1) + \beta_{n+1}(-n) + \beta_{n+2}(n) + \beta_{n+2}(-n-1)\right], \tag{5.23}$$

for any $K > 1$, where $\beta_m(x) = \sum_{\zeta\in\Omega_{\Lambda_m}} c_x(\zeta)|\nu(\zeta) - \nu(\zeta^{x,x+1})|$ for $\{x, x+1\} \subset \Lambda_{m-1}$.

We have shown that $(B_n)_n$, and therefore $(I_n)_n$, converges. Moreover, recall that $0 \le \alpha_{n+1}(-n) \le -2(I_{n+1}-I_n)$. In particular, $\alpha_{n+1}(-n)$ vanishes as $n \to \infty$. Similarly, the same holds for $\alpha_{n+1}(n-1)$. It now remains to notice that the following holds (Liggett (2005, Lemma IV.5.8 (b)), with adapted notations): for x, m such that $\{x, x+1\} \subset \Lambda_m$,

$$\beta_{m+1}(x) \le \sqrt{2\left[\sup_\sigma c_0(\sigma)\right]\alpha_{m+1}(x)}. \tag{5.24}$$

The proof can be found in Liggett (2005) and relies on the fact that one can write $\beta_{m+1}(x) = M - m$, where $M = \sum_{\zeta\in\Omega_{\Lambda_m}} c_x(\zeta)\max(\nu(\zeta), \nu(\zeta^{x,x+1}))$, and m is defined similarly with a min instead of a max. One can then use $M - m \le M\log(M/m)$, the subadditivity and homogeneity of Φ, and the fact that $M \le 2$ to conclude. Therefore the second term in (5.23) vanishes. This concludes the proof in the case $\nu(\mathcal{E}\cup\mathcal{E}') > 0$.

Let us now assume that $\nu(\mathcal{F}'_k) > 0$ for some $k \geqslant 2$. Lemma 3.3 then implies the following modified entropy is well-defined:

$$\tilde{H}_n(\nu) = \sum_{\sigma\in\Omega_{\Lambda_n},|\sigma|\leqslant k} \nu(\sigma)\log\left[\frac{\nu(\sigma)}{\mu_\rho(\sigma)}\right]. \tag{5.25}$$

The rest of the proof can be carried as above, with minor points of attention:

- one needs to define $\Phi(0,0) = 0$; this preserves non-negativity of Φ, positivity outside $\{(u,u), u \in \mathbb{R}_+\}$, convexity, homogeneity and subadditivity.
- with this convention, the sums over ζ appearing in the proof can be taken as sums over ζ with $|\zeta| \leqslant k$.

The case $\nu(\mathcal{F}''_k) > 0$ for some $k \geqslant 1$ is similar.

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