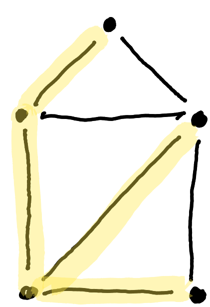
Temperley's bijection

I spanning trees

1. definitions

Let $G = (V, E)$ be a finite graph (not necessarily planar).
A <u>spanning tree</u> T of G is a subset of edges, such that (V, T) is a cycle-free, connected graph.
[By a slight abuse of notation, we will call also this subgraph of G with the letter T]

Example:



A graph G with 5 vertices and 7 edges (black), and a spanning tree (highlighted)

Lemma: if G has n vertices, then every spanning tree T has $n-1$ edges

proof: In a tree, any two vertices are connected by a unique minimal path. Pick a vertex $v_0 \in V$, called the root. Orient the edges of the spanning tree toward the root. Then every vertex of $V \setminus \{v_0\}$ is connected in a bijective way to a unique edge of T exiting from that vertex.

Remark: spanning trees are in terms of number of edges:

- minimal among connected subgraphs of G (with vertex set V)
- maximal among the cycle-free subgraph of G

2. The Laplacian matrix of G

The Laplacian of G, denoted by Δ, is a matrix with rows and colums indexed by V whose coefficients are :

$$\Delta_{v,w} = \begin{cases} -1 & \text{if } v \sim w \\ \deg v & \text{if } v = w \quad (\deg = \#\text{ of outgoing edges}) \\ 0 & \text{otherwise} \end{cases}$$

Rem: • there is a generalisation for multigraphs (changing -1 into the $(-)$ number of edges connecting v and w and a weighted version, where we put conductances (positive real numbers) on edges.

Δ can be seen as a linear map on functions defined on V:
if $f \in \mathbb{R}^V$, $\forall v \in V \quad (\Delta f)(v) = \sum \Delta_{vw} f(w)$.

Δ is a positive symmetric operator on $\mathbb{R}^V$ endowed with its canonical scalar product, and its kernel is the space of constant functions since G is connected (more generally, functions in the kernel of Δ are constant on each connected component of G if it was not connected). So here, the rank of Δ is $n-1$ (with $n = |V|$), [so Δ has at least a non-zero cofactor].

We now write Δ as the product of a rectangular matrix and its transpose: pick an orientation ε of edges of G so that now every $e \in E$ has a source vertex $s(e)$ and a target vertex $t(e)$

$$\underset{s(e)}{\bullet} \xrightarrow{e} \underset{t(e)}{\bullet}$$

Define the incidence matrix of G (relative to ε) M^{ε} to be the rectangular matrix with rows indexed by V and columns by E.

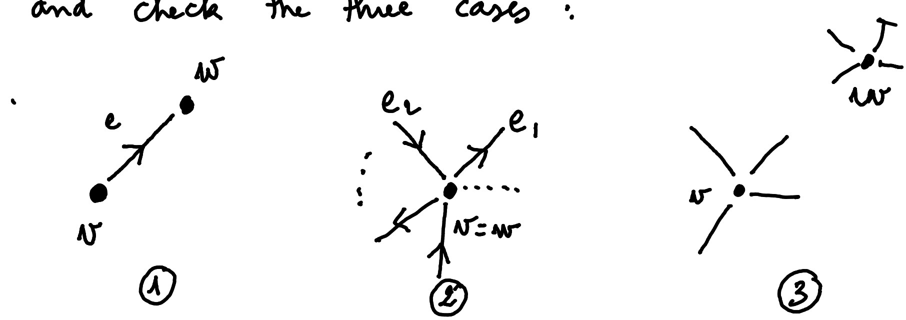
Its coefficients are for $v \in V$ and $e \in E$

$$M^{\varepsilon}_{v,e} = \begin{cases} +1 & \text{if } v = t(e) \\ -1 & \text{if } v = s(e) \\ 0 & \text{otherwise} \end{cases}$$

<u>Lemma:</u> $\Delta = M^{\varepsilon} \cdot (M^{\varepsilon})^T$

<u>proof:</u> Write $\Delta_{vw} = \sum_e (M^{\varepsilon})_{ve} (M^{\varepsilon})_{we}$

and check the three cases:



In particular, it does not depend on ε. Changing ε ammounts to post-multiply M^{ε} by a diagonal matrix with $+1$ (resp. -1) on the diag if the corresponding edge as the same (resp. different) orientation as in ε.

Notation: If A is a matrix, I a subset of row indices and J a subset of column indices, we write A_I^J for the submatrix of A obtained by keeping rows indexed by I and columns indexed by J

Example: $\Delta_{V\setminus\{v_0\}}^{V\setminus\{v_1\}}$ is the matrix obtained from Δ by removing row (resp. column) indexed by v_0 (resp. v_1)

Theorem (matrix-tree theorem, Kirschhoff 1890's)

Let G a simple connected finite graph with n vertices
Let v_0 a vertex of G.
Then the number of spanning trees of G is equal to $\det \Delta_{V\setminus\{v_0\}}^{V\setminus\{v_0\}}$.

proof: there are many different proof for this result. A classical one relies on Cauchy-Binet formula for the determinant of a product of two rectangular matrices:

<u>Lemma</u> (Cauchy-Binet)

if A is a $k \times \ell$ matrix, and B a $\ell \times k$ matrix,

$$\det(AB) = \sum_{\substack{S \subset \{1,\dots,\ell\} \\ |S| = k}} \det(A^S) \cdot \det(B_S)$$

(if $k = \ell$, then is only one $S = \{1,\dots k\}$ in the sum, and this is the usual formula for the determinant of a product)

After noticing that $\Delta^{V\setminus\{v_0\}}_{V\setminus\{v_0\}} = \underbrace{(M^{\mathcal{E}})_{V\setminus\{v_0\}}}_{A} \cdot \left[(M^{\mathcal{E}})_{V\setminus\{v_0\}}\right]^T$

apply Cauchy-Binet lemma to A and $B = A^T$, so that

$$\det \Delta^{V\setminus\{v_0\}}_{V\setminus\{v_0\}} = \sum_{\substack{S \subset \mathcal{E} \\ |S| = n-1}} \det(A^S)^2 = \sum_{\substack{S \subset \mathcal{E} \\ |S| = n-1}} \det\left((M^{\mathcal{E}})^S_{V\setminus\{v_0\}}\right)^2$$

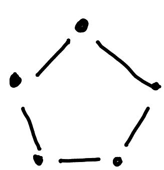
$k = n-1$

S corresponds to a subgraph with $n-1$ edges. By the remarks above, either S is a spanning tree, or it has at least two connected components, and a cycle.

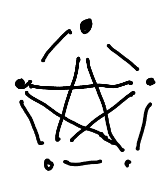
<u>Exercise</u>:
- show that if S is a spanning tree, $$\det \left(M^{\varepsilon}\right)^{S}_{V\setminus\{v_0\}} = \pm 1$$
- show that otherwise $\det \left(M^{\varepsilon}\right)^{S}_{V\setminus\{v_0\}} = 0$ (by exhibiting a function in the kernel of $\left(M^{\varepsilon}\right)^{S}_{V\setminus\{v_0\}}$)

This concludes the proof of the matrix tree theorem.

<u>Exercises</u>:
- compute the number of spanning trees of the cycle graph C_n on n vertices and the complete graph K_n on n vertices



C_5



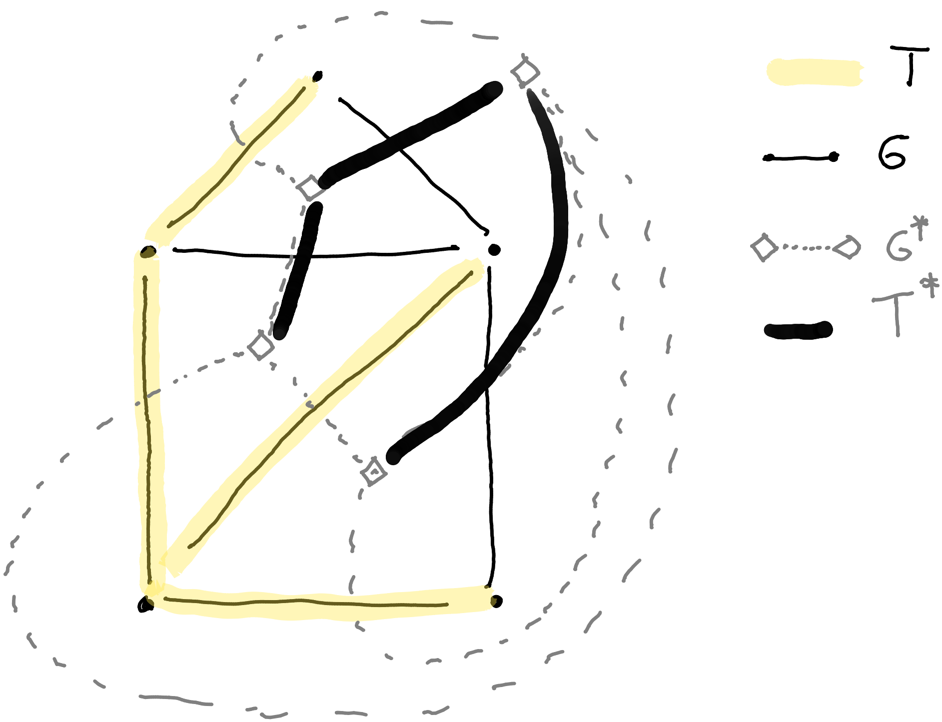
K_5

<u>Rem/Exercise</u>: All the principal cofactors are equal (to the number of sp. trees) their sum is the penultimate coefficient of the charact. polynomial of Δ so that $\#$ spanning trees $= \frac{1}{n} \prod_{\lambda \in \mathrm{Spec}(\Delta)\setminus\{0\}} \lambda$

II Spanning trees on planar graphs

if G is a finite connected planar graph, and T is a spanning tree, then we can define a natural object T^* as a subset of dual edges (or a subgraph of the dual G^*) as follows

$$e^* \in T^* \iff e \notin T$$



if we do this construction for a generic subgraph of G, we see that :

$$T \text{ is connected} \Leftrightarrow T^* \text{ cycle-free}$$

$$T \text{ cycle-free} \Leftrightarrow T^* \text{ connected}$$

so that T spanning tree of G $\Leftrightarrow$ T^* spanning tree of G^*

One can also check that the number of edges of T + number of edges of T^* is equal to $|E|$. By Euler's formula $|V| - |E| + |F| = 2$ for a connected planar graph

recalling that # edges of $T = |V| - 1$, one has.

$$\# \text{ edges of } T' = |E| - (|V| - 1) = |V| + |F| - 2 - (|V| - 1)$$

$$= |F| - 1$$

which is expected for a spanning tree of G^*

III Temperley's bijection

In 1972 Temperley noticed that the "entropy" per vertex of spanning trees on the square grid

$\lim_{n \to +\infty} \frac{1}{n^2} \log \#$ spanning trees of a rectangle $n \times n$ (or torus),

given by

$$\frac{1}{4\pi^2} \iint \log\left(4 - \frac{1}{z} - z - \frac{1}{w} - w\right) \frac{dz}{iz} \frac{dw}{iw} = \frac{4G}{\pi}$$

, where $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} \cdots$ is the Catalan constant

(Check using formula at the end of page 8, using the same strategy as for the free energy of dimers)

which is 4 times the entropy per vertex for the dimer model.

A couple of years later, he gave a bijection between almost these sets of configurations with n fixed.

Temperley's bijection

Theorem: There is a bijection between spanning trees of a $n \times n$ rectangle, and dimer configurations of a $(2n-1) \times (2n-1)$ rectangle with a corner removed.

Note: this result extends to other subgraphs of $\mathbb{Z}^2$ as follows:

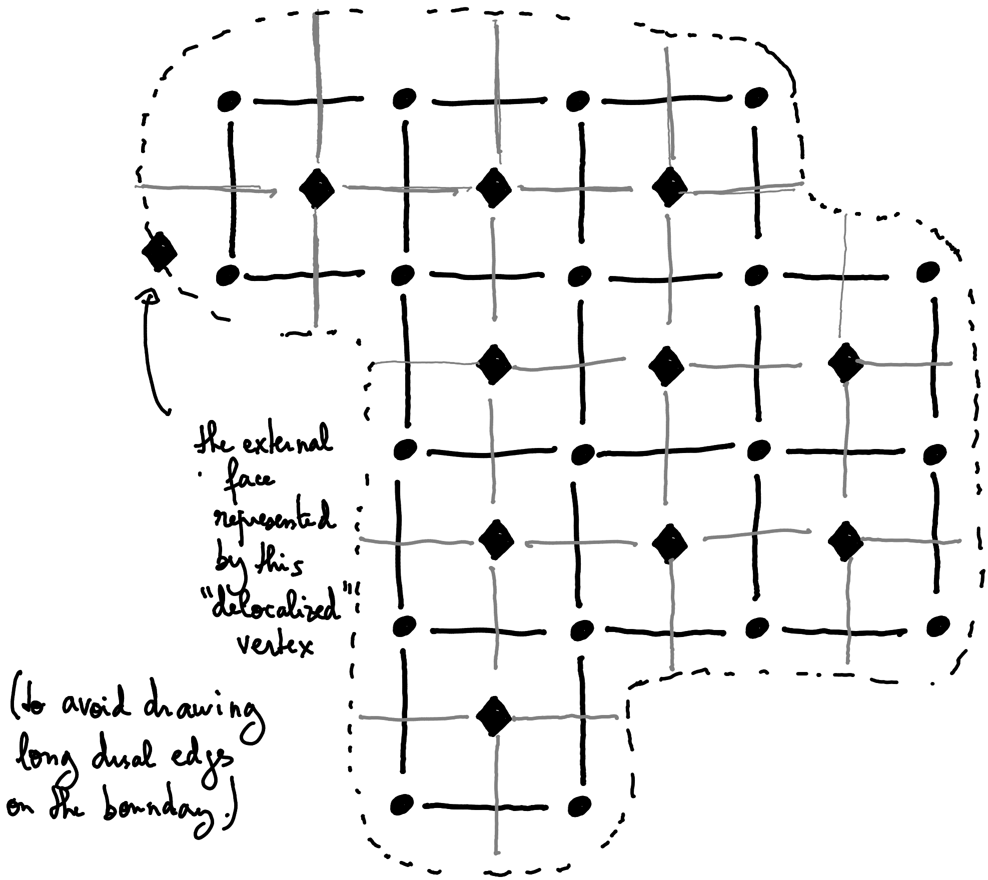
Let G be a finite connected, "singly-connected" subgraph of $\mathbb{Z}^2$ (meaning that if we glue unit "dual" squares on each vertex of G, the domain obtained is singly connected)

The dual graph of G^* is composed of:

- internal faces of G
- the outer face of G

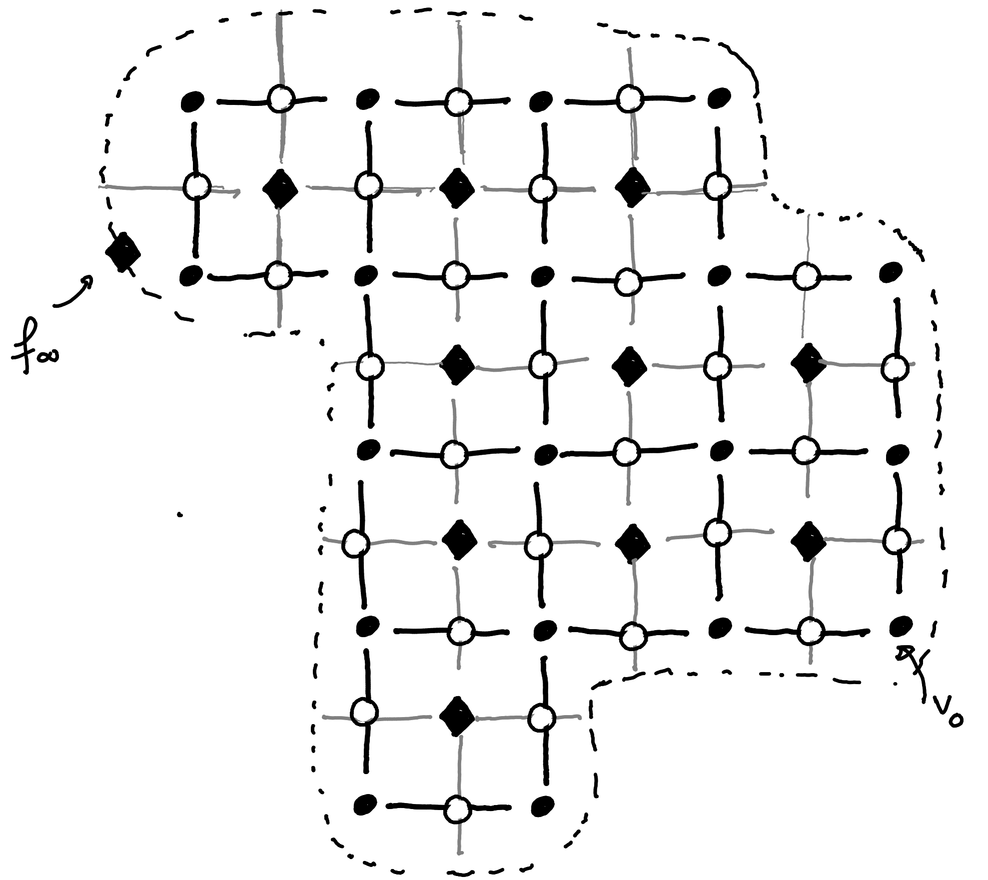
See the example on the next page with ● as vertices of G and ◆ vertices of G^*

(See Kenyon-Wilson: Trees and Matchings for generalization to other planar graphs)



from G and G^*, create a bipartite graph G_{double}:

- black vertices are vertices of G and G^*
- white vertices are edges of G/G^*, drawn at their intersection.



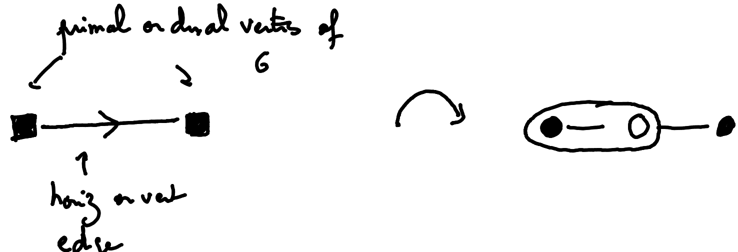
The Euler formula applied to G as an embedded graph tells us that $\underbrace{|V|+|F|}_{\#\text{ of black vertices}} - \underbrace{|E|}_{\#\text{ of white vertices}} = 2$. G_{double} is

unbalanced and has thus no dimer configuration.

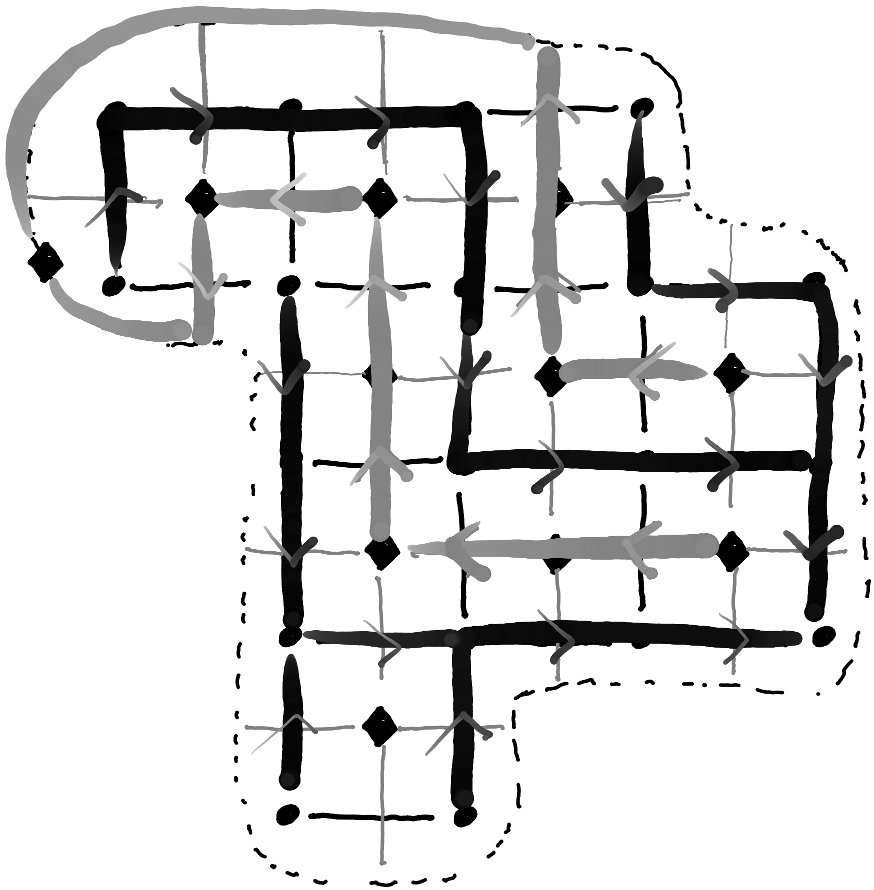
Let G_D the graph obtained by removing the vertices corresponding to the unbounded face f_∞ and a vertex on the boundary v_0

The bijection between spanning trees of G and dimer configurations of G_D is local and works as follows:

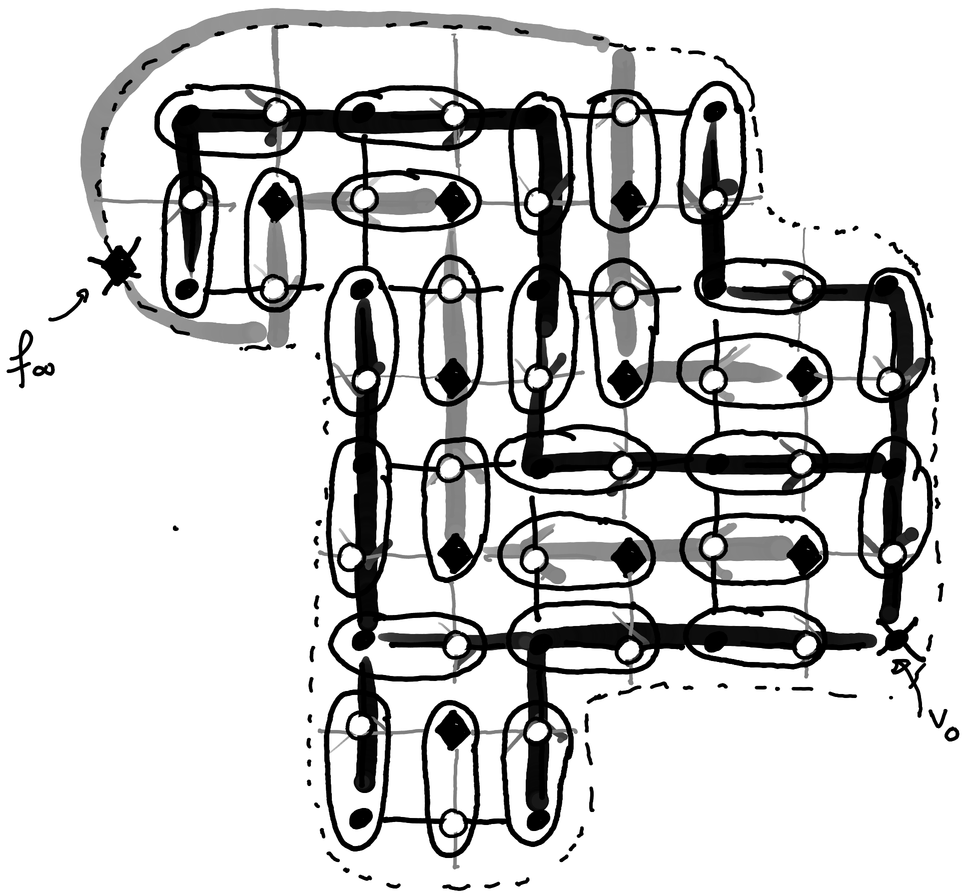
- take a spanning tree T of G
- draw its dual T^*
- orient all edges of T (resp. T^*) toward v_0 (resp. f_∞)
- Apply the following transformation on edges.



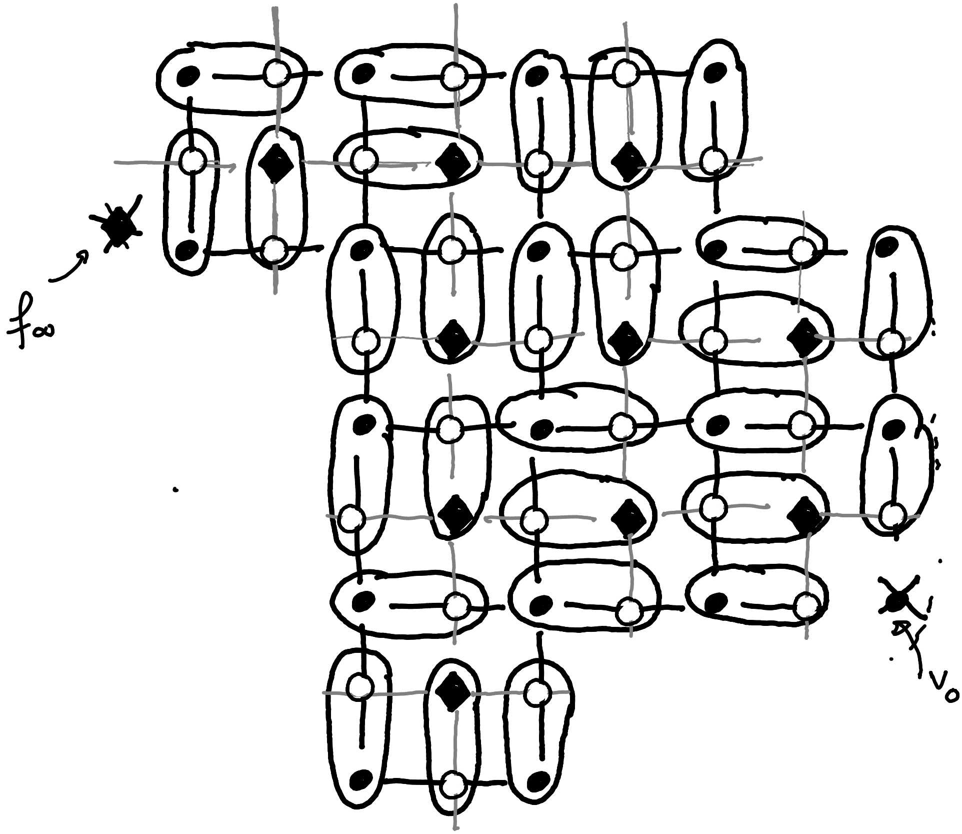
(We put a dimer on the first half of the oriented edges)



spanning tree on G and its dual



superimposition with the dimer configuration on G_D



dimer configuration on G_D

from T (and T^*) we indeed obtain a dimer configuration of G_D:

- every vertex of G (except v_0) is attached to an edge of T
- every dual vertex (except f_∞) is attached to an edge of T^*

this is thus a pairing between all black and white vertices of G_D among neighbors.

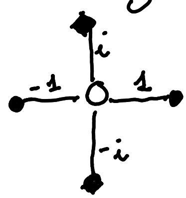
Reciprocally, from a dimer configuration of G_D we can construct from the local bijection T and T^*

Let us check that T is indeed a spanning tree:

- the number of edges in T is $|V|-1$ (one per vertex of G except v_0).
- T is cycle-free. Indeed, if it had a cycle, the interior would have to be filled with dimers. However, an application of Euler's formula to the

subgraph of G inside. The cycle shows that the corresponding subgraph of G_D would be unbalanced... so we reached a contradiction.

There is an operator side to this story: defining the Kasteleyn operator not with ± 1 entries but:



- this operator satisfies Kasteleyn condition
- if we split the black vertices of G_D as vertices of G | vertices of G^* then K can be written as two blocks which are related to the incidence matrix of G / G^*

$$K^* K = \left(\begin{array}{c:c} \Delta_G & \\ \hdashline & \Delta_{G^*} \end{array}\right)$$

where Δ^G and Δ^{G^*} are the Laplacians on G / G^* with special boundary conditions.

details would require another episode.... FIN