LIMITS OF MULTIPLICATIVE INHOMOGENEOUS RANDOM GRAPHS AND LÉVY TREES.

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Abstract

We consider a natural model of inhomogeneous random graphs that extends the classical Erdős–Rényi graphs and shares a close connection with the multiplicative coalescence, as pointed out by Aldous [*Ann. Probab.*, vol. 25, pp. 812–854, 1997]. In this model, the vertices are assigned weights that govern their tendency to form edges. It is by looking at the asymptotic distributions of the masses (sum of the weights) of the connected components of these graphs that Aldous and Limic [*Electron. J. Probab.*, vol. 3, pp. 1–59, 1998] have identified the entrance boundary of the multiplicative coalescence, which is intimately related to the excursion lengths of certain Lévy-type processes. We, instead, look at the metric structure of these components and prove their Gromov–Hausdorff–Prokhorov convergence to a class of (random) compact measured metric spaces. Our asymptotic regimes relate directly to the general convergence condition appearing in the work of Aldous and Limic. Our techniques provide a unified approach for this general "critical" regime, and relies upon two key ingredients: an encoding of the graph by some Lévy process as well as an embedding of its connected components into Galton–Watson forests. This embedding transfers asymptotically into an embedding of the limit objects into a forest of Lévy trees, which allows us to give an explicit construction of the limit objects from the excursions of the Lévy-type process. As a consequence of our construction, we give a transparent and explicit condition for the compactness of the limit objects and determine their fractal dimensions. These results extend and complement several previous results that had obtained via model- or regime-specific proofs, for instance: the case of Erdos-Renyi random graphs obtained by Addario-Berry, Goldschmidt and B. [*Probab. Theory Rel. Fields*, vol. 153, pp. 367–406, 2012], the *asymptotic homogeneous* case as studied by Bhamidi, Sen and Wang [*Probab Theory Rel. Fields*, vol. 169, pp. 565–641, 2017], or the *power-law* case as considered by Bhamidi, Sen and van der Hofstad [*Probab. Theory Rel. Fields*, vol. 170, pp. 387–474, 2018].

1 Introduction

Motivation and model. Random graphs have generated a large amount of literature. This is even the case for one single model: the Erdős–Rényi graph $G(n, p)$ (graph with n vertices connected pairwise in an i.i.d. way with probability $p \in [0, 1]$). Since its introduction by Erdős and Rényi [24] more than fifty years ago, and the discovery of a phase transition where a "giant connected component" gets born, the pursuit of a deeper understanding of its structure has never stopped. Many landmark results by Bollobás [17], Łuczak [34], Janson, Knuth, Łuczak and Pittel [32] have shaped our grasp of this phase transition. From the point of view of precise asymptotics, one of the most important papers is certainly the contribution of Aldous [3], who introduced a stochastic process point of view and paved the way towards the study of scaling limits of critical random graphs. In

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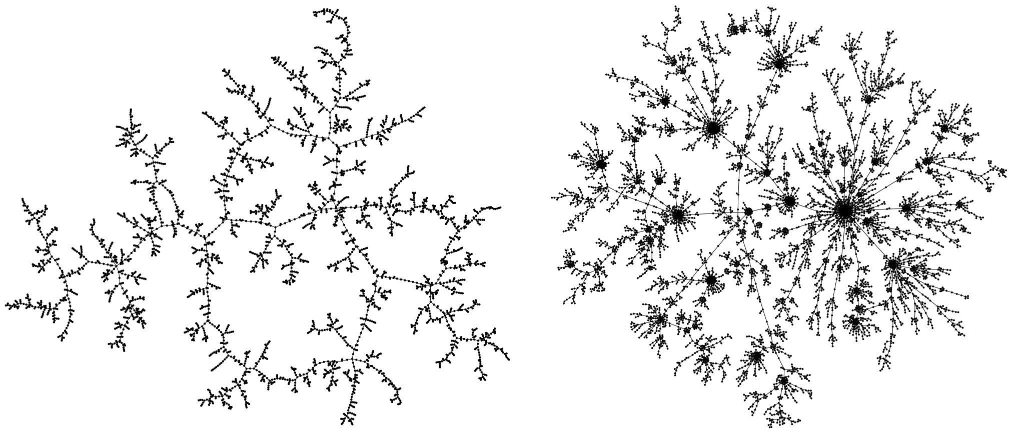

Figure 1: *Left: a picture of a large connected component of* $G(n,p)$*. Right: a picture of a large connected component of* $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$*. Observe the presence of "hubs" (nodes of high degrees) in the latter.*

that paper, he obtained the asymptotics for the sequence of sizes of the connected components of $G(n,p)$ in the so-called critical window where the phase transition actually occurs. His work made possible the construction by Addario-Berry, Goldschmidt and B. [2] of the scaling limits of these connected components, seen as metric spaces, which also confirmed the limiting fractal (Brownian) nature.

Following [2], the question of identifying the scaling limits has been investigated for more general models of random graphs. Particular attention has been paid to the so-called *inhomogeneous random graphs*, which exhibit heterogeneity in the node degrees and whose behaviours are often quite different from the Erdős–Rényi graph. (See Fig. 1 for an illustration of this difference). Besides being a theoretic object with intriguing properties, these graphs are also commonly believed to offer more realistic modelling for the complex real-world networks [see, e.g. 35].

In the present work, we consider such an inhomogeneous random graph model, defined as follows. Let $\mathtt{w} = (w_1, w_2, \dots, w_n)$ be a sequence of n positive real numbers sorted in non-increasing order. Interpreting w_i as the propensity of vertex i to form edges, we define a random graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ as follows: the set of its vertices is $\{1, 2, \dots, n\}$, the events $\{\{i,j\} \text{ is an edge of } \boldsymbol{\mathcal{G}}_{\mathtt{w}}\}$, $1 \le i < j \le n$, are independent and

$$\mathbf{P}\big(\{i,j\} \text{ is an edge of } \boldsymbol{\mathcal{G}}_{\mathtt{w}}\big) = 1 - \exp\big(-w_i w_j/\sigma_1(\mathtt{w})\big), \quad \text{where} \quad \sigma_1(\mathtt{w}) = w_1 + \ldots + w_n.$$

The graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ extends the classical Erdős–Rényi random graph in allowing edges to be drawn with non uniform probabilities, while keeping the independence among edges.

The graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ has come under different names in the literature, for instance, Poisson random graph in [9, 36], the Norros–Reittu graph in [9] or rank-1 model in [12, 13, 18, 40, 41]. Here, we will refer to it as the *multiplicative graph* to emphasise its close connection with the *multiplicative coalescent* as pointed out by Aldous in [3]. This connection is the starting point of the work [4] of Aldous & Limic who identify the entrance boundary of multiplicative coalescent by looking at the asymptotic distributions of the *sizes* of the connected components found in $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$. The asymptotic regime and the limiting processes found in Aldous & Limic [4] lie at the heart of this paper. Namely, we extend this result to the *geometry* of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ by proving the weak convergence of the connected components of multiplicative graphs as it has been done by Addario-Berry, Goldschmidt and B. [2] for the critical Erdős–Rényi graphs.

More precisely, we equip $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ with the graph distance d_{gr} and we introduce the weight measure $\mathbf{m}_{\mathtt{w}} = \sum_{1 \le i \le n} w_i \delta_i$ on $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$. The goal of our article can be roughly rephrased as follows: we construct a class of (pointed and measured) compact random metric spaces $(\mathbf{G}, d, \mathbf{m})$ such that $(\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}, \varepsilon_n d_{\mathrm{gr}}, \varepsilon'_n \mathbf{m}_{w_n}) \to (\mathbf{G}, d, \mathbf{m})$ weakly along suitable subsequences $(\mathtt{w}_n, \varepsilon_n, \varepsilon'_n)$. Of course, here the scaling parameters, ε_n and ε'_n go to 0, so that $\mathbf{G}$ is not discrete. The limits we consider hold in the sense of the weak convergence corresponding to Gromov–Hausdorff–Prokhorov topology on the space of (isometry classes of) compact metric spaces. To achieve the construction of the possible limiting graphs and to prove the convergence of rescaled multiplicative graphs, we rely on two main new ideas: (1) we code multiplicative graphs by processes derived from a LIFO-queue; (2) we embed multiplicative graphs into Galton–Watson trees whose scaling limits are well-understood. Before discussing further the connections with previous works and in order to explain the advantages of our approach, let us give a brief but precise overview of results and of the two above mentioned ideas.

Overview of the results. Our approach relies first on a specific coding of $\mathtt{w}$-multiplicative graphs $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ via a *LIFO-queue* and a related stochastic process; the queue actually yields an exploration of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ and a spanning tree that encompasses almost all the metric structure of the graph. The LIFO-queue is defined as follows: a single server is visited by n clients labelled by $1, \ldots, n$; Client j arrives at time E_j and she/he requests an amount of time of service w_j; the E_j are independent exponentially distributed r.v. such that $\mathbf{E}[E_j] = \sigma_1(\mathtt{w})/w_j$; a LIFO (last in first out) policy applies: whenever a new client arrives, the server interrupts the service of the current client (if any) and serves the newcomer; when the latter leaves the queue, the server resumes the previous service. As mentioned above, the LIFO-queue yields a tree $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ whose vertices are the clients: the server is the root (Client 0) and Client j is a child of Client i in $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ if and only if Client j interrupts the service of Client i (or arrives when the server is idle if $i = 0$). Note that the LIFO-queue is coded by either of the two following processes defined for all $t \in [0, \infty)$ by:

$$Y^{\mathtt{w}}_t = -t + \sum_{1 \leq i \leq n} w_i \mathbf{1}_{\{E_i \leq t\}} \quad \text{and} \quad \mathcal{H}^{\mathtt{w}}_t = \#\Big\{ s \in [0, t] : \inf_{r \in [s,t]} Y^{\mathtt{w}}_r > Y^{\mathtt{w}}_{s-} \Big\} .$$

The quantity $Y^{\mathtt{w}}_t$ is the (algebraic) load of the server, i.e., the amount of service due at time t and $\mathcal{H}^{\mathtt{w}}_t$ is the number of clients waiting in the queue at time t. We easily see that $\mathcal{H}^{\mathtt{w}}$ is the contour (or the depth-first exploration) of $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$; this entails that the graph-metric of $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ is entirely encoded by $\mathcal{H}^{\mathtt{w}}$: namely, the distance between the vertices/clients served at times s and t in $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ is $\mathcal{H}^{\mathtt{w}}_t + \mathcal{H}^{\mathtt{w}}_s - 2\min_{r \in [s \wedge t, s \vee t]} \mathcal{H}^{\mathtt{w}}_r$.

The tree $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ contains most of the metric information of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$, but not all. Surplus edges are added to $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ to obtain $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ as follows: conditionally on $Y^{\mathtt{w}}$, let $\sum_{1 \leq p \leq \mathbf{p}_{\mathtt{w}}} \delta_{(t_p, y_p)}$ be a Poisson point measure on $[0, \infty) \times [0, \infty)$ with intensity $\frac{1}{\sigma_1(\mathtt{w})} \mathbf{1}_{\{0 < y < Y^{\mathtt{w}}_t - \inf_{[0,t]} Y^{\mathtt{w}}\}} \, dt \, dy$ and set

$$s_p = \inf \Big\{ s \in [0, t_p] : \inf_{u \in [s, t_p]} \big(Y^{\mathtt{w}}_u - \inf_{[0,u]} Y^{\mathtt{w}} \big) > y_p \Big\} .$$

Then we define the set of additional edges $\mathcal{S}_{\mathtt{w}}$ as the set of the edges connecting the clients served at times s_p and t_p, for all $1 \leq p \leq \mathbf{p}_{\mathtt{w}}$. Theorem 2.1 asserts that the graph obtained by removing the root 0 from $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ and adding the edges $\mathcal{S}_{\mathtt{w}}$ is distributed as $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$, a $\mathtt{w}$-multiplicative graph. Namely,

$$\boldsymbol{\mathcal{G}}_{\mathtt{w}} \overset{(\mathrm{d})}{=} (\boldsymbol{\mathcal{T}}_{\mathtt{w}} \backslash \{0\}) \cup \mathcal{S}_{\mathtt{w}} .$$

From this representation of the discrete graphs, one expects that if $Y^{\mathtt{w}}$ converges, then the graph should also converge, at least in weak sense. However, since $Y^{\mathtt{w}}$ is not Markovian, it is difficult to obtain a limit for the local-time functional $\mathcal{H}^{\mathtt{w}}$, which is the function that encodes the metric. To circumvent this technical difficulty, we embed the non-Markovian LIFO-queue governed by $Y^{\mathtt{w}}$ into a Markovian one that is defined as follows: a single server receives an infinite number of clients; a LIFO policy applies; clients arrive at unit rate; each client has a type that is an integer ranging in $\{1, \ldots, n\}$; the amount of service required by a client of type j is w_j; types are i.i.d. with law $\nu_{\mathtt{w}} = \frac{1}{\sigma_1(\mathtt{w})} \sum_{1 \leq j \leq n} w_j \delta_j$. Namely, let τ_k be the arrival-time of the k-th client and let $\mathtt{J}_k$ be the type of the k-th client; then, the Markovian LIFO queueing system is entirely characterised by

$\sum_{k\geq 1}\delta_{(\tau_k,\mathtt{J}_k)}$ that is a Poisson point measure on $[0,\infty)\times\{1,\ldots,n\}$ with intensity $\ell\otimes\nu_{\mathtt{w}}$, where ℓ stands for the Lebesgue measure on $[0,\infty)$. The Markovian queue also yields a tree $\mathbf{T}_{\mathtt{w}}$ defined as follows: the server is the root $\mathbf{T}_{\mathtt{w}}$ and the k-th client to enter the queue is a child of the l-th one if the k-th client enters when the l-th client is served. One easily checks that $\mathbf{T}_{\mathtt{w}}$ is a sequence of i.i.d. Galton–Watson trees glued at their root and that their common offspring distribution is $\mu_{\mathtt{w}}(k)=\sum_{1\leq j\leq n}(\sigma_1(\mathtt{w})k!)^{-1}w_j^{k+1}e^{-w_j}$, $k\in\mathbb{N}$. Here, we restrict to (sub)critical GW-trees: namely, we assume that $\sum_{k\in\mathbf{N}}k\mu_{\mathtt{w}}(k)=\sigma_2(\mathtt{w})/\sigma_1(\mathtt{w})\leq 1$, where for all $r\in(0,\infty)$, we use the notation $\sigma_r(\mathtt{w})=\sum_{1\leq j\leq n}w_j^r$. This tree is coded by its contour process $(H^{\mathtt{w}}_t)_{t\in[0,\infty)}$: namely, $H^{\mathtt{w}}_t$ stands for the number of clients waiting in the Markovian queue at time t and it is given by

$$H^{\mathtt{w}}_t=\#\Big\{s\in[0,t]:\inf_{r\in[s,t]}X^{\mathtt{w}}_r>X^{\mathtt{w}}_{s-}\Big\}\quad\text{where}\quad X^{\mathtt{w}}_t=-t+\sum_{k\geq 1}w_{\mathtt{J}_k}\mathbf{1}_{[0,t]}(\tau_k),\ t\in[0,\infty),$$

is the (algebraic) load of the Markovian server. Note that $X^{\mathtt{w}}$ is a spectrally positive Lévy process with initial value 0; it is characterised by its Laplace exponent defined by $\mathbf{E}[e^{-\lambda X^{\mathtt{w}}_t}]=e^{t\psi_{\mathtt{w}}(\lambda)}$, $t,\lambda\in[0,\infty)$, that is explicitly given by:

$$\psi_{\mathtt{w}}(\lambda)=\alpha_{\mathtt{w}}\lambda+\sum_{1\leq j\leq n}\tfrac{w_j}{\sigma_1(\mathtt{w})}\big(e^{-\lambda w_j}-1+\lambda w_j\big)\quad\text{and}\quad\alpha_{\mathtt{w}}:=1-\tfrac{\sigma_2(\mathtt{w})}{\sigma_1(\mathtt{w})}\,.$$

From this tractable model, we derive the LIFO-queue and the tree $\mathcal{T}_{\mathtt{w}}$ governed by $Y^{\mathtt{w}}$ by a time-change that "skips" some time intervals, which is defined as follows. We colour in *blue* or *red* the clients of the Markovian queue in the following recursive way: *(i) if the type* $\mathtt{J}_k$ *of the* k*-th client already appeared among the types of the blue clients who previously entered the queue, then the* k*-th client is red; (ii) otherwise the* k*-th client inherits her/his colour from the colour of the client who is currently served when she/he arrives (and this colour is blue if there is no client served when she/he arrives: namely, we consider that the server is blue).* Note that a client who is the first arriving of her/his type is not necessarily coloured in blue. Check that exactly n clients are coloured in blue and their types are necessarily distinct. Moreover, while a blue client is served, note that the other clients waiting in the line (if any) are blue too. Actually, the sub-queue of blue clients corresponds to the previous LIFO queue governed by $Y^{\mathtt{w}}$. More precisely, we set $\mathtt{Blue}=\{t\in[0,\infty)$: a blue client is served at time $t\}$ and $\theta^{\mathtt{b},\mathtt{w}}_t=\inf\{s\in[0,\infty):\int_0^t\mathbf{1}_{\mathtt{Blue}}(s)ds>t\}$. Then,

$$(Y^{\mathtt{w}}_t,\mathcal{H}^{\mathtt{w}}_t)_{t\in[0,\infty)}\overset{\text{(d)}}{=}\big(X^{\mathtt{w}}_{\theta^{\mathtt{b},\mathtt{w}}_t},H^{\mathtt{w}}_{\theta^{\mathtt{b},\mathtt{w}}_t}\big)_{t\in[0\infty)}\,.$$

This explains how to code $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ in terms of the two tractable processes $X^{\mathtt{w}}$ and $H^{\mathtt{w}}$ derived from the Markovian queue.

Such Markovian queues have analogues in the continous time and space setting. The parameters governing such processes are those identified by Aldous & Limic [4] for the eternal multiplicative coalescent. Namely:

$$\alpha\in\mathbb{R},\ \beta\in[0,\infty),\ \kappa\in(0,\infty)\quad\text{and}\quad\mathbf{c}=(c_j)_{j\geq 1}\text{ decreasing and such that }\sum_{j\geq 1}c_j^3<\infty\,.$$

The load of service of the continuous analogue of the Markovian queue is a spectrally positive Lévy process $(X_t)_{t\in[0,\infty)}$ starting at X_0 whose Laplace exponent ψ is given by

$$\psi(\lambda)=\alpha\lambda+\tfrac{1}{2}\beta\lambda^2+\sum_{j\geq 1}\kappa c_j\big(e^{-\lambda c_j}-1+\lambda c_j\big),\quad\lambda\geq 0.\tag{1}$$

To simplify, we restrict our explanations to the cases where X does not drift to ∞, which is equivalent to assuming that $\alpha\in[0,\infty)$. We explain in Section 2.2.1 how to colour the Markovian queue driven by X: namely, we explain how to define a right-continuous increasing time-change $(\theta^{\mathtt{b}}_t)_{t\in[0,\infty)}$ that is the analogue of the discrete one $\theta^{\mathtt{b},\mathtt{w}}$. Then we define the cadlag process

$Y = X \circ \theta^{\mathtt{b}}$ that represents the load driving the analogue of the LIFO-queue (without repetitions). We prove in Section 2.2.1 that Y can be written under the following form:

$$\forall t \in [0, \infty), \quad Y_t = -\alpha t - \tfrac{1}{2}\kappa\beta t^2 + \sqrt{\beta} B_t + \sum_{j\geq 1} c_j (\mathbf{1}_{\{E_j \leq t\}} - c_j \kappa t), \tag{2}$$

where $(B_t)_{t\in[0,\infty)}$ is a standard linear Brownian motion starting at 0 and where the E_j are independent exponentially distributed r.v. that are independent from B and such that $\mathbf{E}[E_j] = (\kappa c_j)^{-1}$. The sum in (2) has to be understood in the sense of L^2 semimartingales (see Section 2.2.1 for a precise explanation). The latter expression of Y can be found in Aldous & Limic [4] who proved that the lengths of the excursions of Y above its infimum (ranked in decreasing order) are distributed as the marginal laws of multiplicative coalescent, roughly speaking.

The tree corresponding to the clients of the continuous analogue of the Markovian queue, that is driven by X, is actually the Lévy tree yielded by X, which is defined through its contour process as introduced by Le Gall & Le Jan [33]. To that end, we assume that ψ defined in (1) satisfies the following:

$$\int^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty. \tag{3}$$

which implies that either $\sum_j c_j^2 = \infty$ or $\beta \neq 0$ and therefore X has infinite variation sample paths. Under Assumption (3), Le Gall & Le Jan [33] (see also Le Gall & D. [21]) prove that there exists a continuous process $(H_t)_{t\in[0,\infty)}$ such that the following limit holds true for all $t \in [0, \infty)$, in probability

$$H_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s - \inf_{r\in[s,t]} X_r \leq \varepsilon\}} \, ds \; . \tag{4}$$

The process H is called the *height process associated with* X; it is clearly an analogue of $H^{\mathtt{w}}$. The analogue of $\mathcal{H}^{\mathtt{w}}$ is then defined by

$$\forall t \in [0, \infty), \quad \mathcal{H}_t = H_{\theta^{\mathtt{b}}_t}$$

and Proposition 2.7 shows that $\mathcal{H}$ is a.s. a continuous process that is called the *height process associated with* Y.

We show in Lemma 5.6 (see Section 5.2.4) that the excursion intervals of $\mathcal{H}$ above 0 and the excursion intervals of Y above its infimum are the same. Moreover, Proposition 14 in Aldous & Limic [4] (recalled in Proposition 5.8, Section 5.2.4), asserts that these excursions can be indexed in the decreasing order of their lengths. Namely,

$$\big\{t\in[0,\infty) : \mathcal{H}_t > 0\big\} = \Big\{t\in[0,\infty) : Y_t > \inf_{[0,t]} Y\Big\} = \bigcup_{k\geq 1} (l_k, r_k) \tag{5}$$

where the sequence $\zeta_k = l_k - r_k$, decreases.

The continuous analogue of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ is derived from $(Y, \mathcal{H})$ as follows: first, for all $s, t \in [0, \infty)$, we define the usual tree pseudometric associated with $\mathcal{H}$: $d_{\mathcal{H}}(s,t) = \mathcal{H}_s + \mathcal{H}_t - 2\min_{u\in[s\wedge t, s\vee t]} \mathcal{H}_u$. Then, we introduce $\sum_{p\geq 1} \delta_{(t_p, y_p)}$ distributed (conditionnaly on Y) as a Poisson point measure on $[0,\infty) \times [0,\infty)$ with intensity $\kappa \mathbf{1}_{\{0<y<Y_t - \inf_{[0,t]} Y\}} \, dt\, dy$. Next for all $p \geq 1$, set

$$s_p = \inf\Big\{s \in [0, t_p] : \inf_{u\in[s,t_p]} \big(Y_u - \inf_{[0,u]} Y\big) > y_p\Big\} \; .$$

Fix $k \geq 1$. One can prove that if $t_p \in [l_k, r_k]$, then $s_p \in [l_k, r_k]$. Then, we define $\mathbf{G}_k$ as the set $[l_k, r_k]$ where we have identified points $s, t \in [l_k, r_k]$ such that either $d_{\mathcal{H}}(s,t) = 0$ or $(s,t) \in \{(s_p, t_p); p \geq 1 : t_p \in [l_k, r_k]\}$. It actually yields a metric denoted by d_k, on $\mathbf{G}_k$; note that l_k and r_k are identified and we denote by ϱ_k the corresponding point in $\mathbf{G}_k$; we denote by $\mathbf{m}_k$ is the measure induced by the Lebesgue measure on $[l_k, r_k]$. The continuous analogue of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ is then the sequence of pointed measured compact metric spaces $\mathbf{G} = ((\mathbf{G}_k, \mathrm{d}_k, \varrho_k, \mathbf{m}_k)_{k\geq 1}$. We refer to Section 2.2.2 and Section 2.2.3 for a more precise definition.

As already mentioned, the main goal of the paper is to prove that $\mathbf{G}$ is the scaling limit of sequences of rescaled discrete graphs $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ for a suitable sequence of weights $\mathtt{w}_n = (w_j^{(n)})_{j\geq 1}$ (here $\sup\{j \geq 1 : w_j^{(n)} > 0\}$ is not necessarily equal to n but it tends to ∞ as $n \to \infty$). Our main result (Theorem 2.14 in Section 2.3.3) asserts that *if the Markovian processes* $(X^{\mathtt{w}_n}, H^{\mathtt{w}_n})$*, properly rescaled in time and space, weakly converge to* (X, H)*, then* $(Y^{\mathtt{w}_n}, \mathcal{H}^{\mathtt{w}_n})$ *converges weakly to* $(Y, \mathcal{H})$ *within the same scaling.*

More precisely, the graphs $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$, or their coding functions, are rescaled by two factors a_n and b_n tending to ∞; a_n is a weight factor: namely, if the largest weight "persists" in the limit, then $a_n \asymp w_1^{(n)}$ and in general $w_1^{(n)} = O(a_n)$; b_n is a exploration-time factor: namely, $b_n \asymp \mathbf{E}[C_n]$, where C_n stands for the number of clients who are served before the arrival of Client 1 in the $\mathtt{w}_n$-LIFO queue coding $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$. It is also natural to require that counting the number of served clients is roughly the same as counting their time of service, which corresponds to assuming that the $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ are in a critical regime, namely $\sigma_1(\mathtt{w}_n) \asymp \sigma_2(\mathtt{w}_n)$. These constraints amount to assuming the following a priori estimates:

$$(6) \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{b_n}{a_n} = \infty, \quad \lim_{n\to\infty} \frac{b_n}{a_n^2} =: \beta_0, \quad w_1^{(n)} = O(a_n), \quad \lim_{n\to\infty} \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} = \kappa.$$

We refer to Section 2.3.2 for more detailed explanations for such assumptions. A more precise statement of Theorem 2.14 is:

$$(7) \qquad \textit{If} \quad \big(\tfrac{1}{a_n} X^{\mathtt{w}_n}_{b_n \cdot}, \tfrac{a_n}{b_n} H^{\mathtt{w}_n}_{b_n \cdot}\big) \xrightarrow[n\to\infty]{} (X, H)$$

weakly on $\mathbf{D}([0,\infty), \mathbb{R}) \times \mathbf{C}([0,\infty), \mathbb{R})$ *equipped with the product of the Skorokhod and the continuous topologies, then the following joint convergence*

$$(8) \qquad \big(\tfrac{1}{a_n} X^{\mathtt{w}_n}_{b_n \cdot}, \tfrac{a_n}{b_n} H^{\mathtt{w}_n}_{b_n \cdot}, \big(\tfrac{1}{b_n} \theta^{\mathtt{b}, \mathtt{w}_n}_{b_n \cdot}, \tfrac{1}{a_n} Y^{\mathtt{w}_n}_{b_n \cdot}\big), \tfrac{a_n}{b_n} \mathcal{H}^{\mathtt{w}_n}_{b_n \cdot}\big) \xrightarrow[n\to\infty]{} \big(X, H, (\theta^{\mathtt{b}}, Y), \mathcal{H}\big)$$

holds weakly on $\mathbf{D}([0,\infty), \mathbb{R}) \times \mathbf{C}([0,\infty), \mathbb{R}) \times \mathbf{D}([0,\infty), \mathbb{R}^2) \times \mathbf{C}([0,\infty), \mathbb{R})$ *equipped with the product topology.*

Necessary and sufficient conditions on the $(a_n, b_n, \mathtt{w}_n)$ for (7) to hold can be derived from previous results due to Le Gall & D. [21] (let us mention it is not direct: see Proposition 2.12). Namely, (7) holds if and only if the following condition are satisfied

(9)

$$(A): \quad \tfrac{1}{a_n} X^{\mathtt{w}_n}_{b_n} \xrightarrow[n\to\infty]{\text{(weakly)}} X_1 \qquad \text{and} \qquad (B): \quad \exists \delta \in (0,\infty), \quad \liminf_{n\to\infty} \mathbf{P}\big(Z^{\mathtt{w}_n}_{\lfloor b_n \delta / a_n \rfloor} = 0\big) > 0$$

where $(Z_k^{\mathtt{w}_n})_{k\in\mathbb{N}}$ stands for a Galton–Watson Markov chain with offspring distribution $\mu_{\mathtt{w}_n}$ and with initial state $Z_0^{\mathtt{w}_n} = \lfloor a_n \rfloor$. Let us mention that Proposition 2.13 shows that for all $\alpha \in \mathbb{R}$, $\beta \in [0,\infty)$, $\beta_0 \in [0, \beta]$, $\kappa \in (0,\infty)$ and $\mathbf{c}$ such that $\sum_{j\geq 1} c_j^3 < \infty$ and such that Grey's condition (3) is satisfied, there exists a sequence $(a_n, b_n, \mathtt{w}_n)_{n\in\mathbb{N}}$ satisfying (6) and (9), so that (8) holds. Proposition 2.13 also shows that in (9), (A) does not imply necessarily (B). Moreover, Proposition 2.13 also provides a more tractable condition that implies (B) in (9) and that is satisfied in all the examples that have been considered previously.

By soft arguments (see Lemma 2.10), the convergence (8) of the coding functions implies that the rescaled sequence of graphs $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ converges, as random metric spaces. As already mentioned, the convergence holds weakly on the space $\mathbb{G}$ of (pointed and measure preserving) isometry classes of pointed measured compact metric spaces endowed with the Gromov–Hausdorff–Prokhorov distance (whose definition is recalled in Section 2.3.1). Actually, the convergence holds jointly for the connected components of $\mathcal{G}_{\mathtt{w}_n}$: namely, equip $\mathcal{G}_{\mathtt{w}_n}$ with the weight-measure $\mathbf{m}_k^{\mathtt{w}_n} = \sum_{j\geq 1} w_j^{(n)} \delta_j$ and denote by $\mathcal{G}_k^{\mathtt{w}_n}$ the k-largest (with respect to its $\mathbf{m}_k^{\mathtt{w}_n}$-measure) connected component of $\mathcal{G}^{\mathtt{w}_n}$. For the sake of convenience, we complete this finite sequence of connected components by point graphs with null measure to get an infinite sequence of $\mathbb{G}$-valued r.v. $\big((\mathcal{G}_k^{\mathtt{w}_n}, d_k^{\mathtt{w}_n}, \varrho_k^{\mathtt{w}_n}, \mathbf{m}_k^{\mathtt{w}_n})\big)_{k\geq 1}$, where $d_k^{\mathtt{w}_n}$ stands for the graph-metric on $\mathcal{G}_k^{\mathtt{w}_n}$, where $\varrho_k^{\mathtt{w}_n}$ is the first vertex/client of $\mathcal{G}_k^{\mathtt{w}_n}$ who enters the queue and where $\mathbf{m}_k^{\mathtt{w}_n}$ is the restriction of $\mathbf{m}_{\mathtt{w}_n}$ to $\mathcal{G}_k^{\mathtt{w}_n}$. Then, Theorem 2.16 asserts that if (8) holds, then

$$\big((\mathcal{G}_k^{\mathtt{w}_n}, \tfrac{a_n}{b_n} d_k^{\mathtt{w}_n}, \varrho_k^{\mathtt{w}_n}, \tfrac{1}{b_n}\mathbf{m}_k^{\mathtt{w}_n})\big)_{k\geq 1} \xrightarrow[n\to\infty]{} \big((\mathbf{G}_k, \mathrm{d}_k, \varrho_k, \mathbf{m}_k)\big)_{k\geq 1} \tag{10}$$

holds weakly on $\mathbb{G}^{\mathbb{N}^*}$ equipped with the product topology.

Discussion

A unified and exhaustive treatment of the limiting regimes: While important progress has been made on the Gromov–Hausdorff scaling limits of the multiplicative graphs, notably in [11, 12], previous works have distinguished two seemingly orthogonal cases depending on whether the inhomogeneity is mild enough to be washed away in the limit [2, 7, 11], or strong enough to persist asymptotically [12, 14]: the so-called asymptotic (Brownian) homogeneous case and the *power-law* case. The proof strategies greatly differ in these two cases. On the other hand, the remarkable work of Aldous and Limic [4] about the weights of large critical connected components deals with the inhomogeneity in a transparent way. We provide here such a unified approach for the geometry, which works not only for both cases but also for graphs which can be seen as a mixture of the two cases.

Furthermore, an easy correspondence (see (79) below) allows us to link our parameters $(\alpha, \beta, \kappa, \mathbf{c})$ for the limit objects to the ones parametrising all the extremal eternal multiplicative coalescents, as identified by Aldous & Limic in [4]. We note that our limit theorems are valid in the Gromov–Hausdorff–Prokhorov topology, which controls the distances between all pairs of points, and not just in the Gromov–Prokhorov topology where only distances between finitely many typical points are controlled. (A general result has already been proved by Bhamidi, van der Hofstad & Sen [12] for the Gromov–Prokhorov topology in the special case when $\beta=0$.) In light of this, we believe our work contains an exhaustive treatment of all the possible limits related to those multiplicative coalescents. In the mean time, we remove some technical conditions that had been imposed on the weight sequences in some of the previous works.

Compactness and fractal dimensions of the limit object : The "homogeneous" scaling limit of the classical Erdős–Rényi random graphs is compact. One of the hurdles that one faces when dealing with genuinely inhomogeneous limits is that the limit objects may not be compact anymore. For genuinely inhomogeneous graphs, the construction of the scaling limit by Bhamidi, Sen & van der Hofstad [12] (see also [8]) was only proved to yield compact objects in some very specific cases (regular power-tailed weight sequences). The link with Galton–Watson forests in our approach allows to rely on the litterature about these well-studied objects, and to deduce almost directly a compactness criterion. We also argue that the criterion is tight, but the proof of this fact is left for some future work. The same is also true for the fractal dimensions (Hausdorff, Minkowski, packing), which are deduced from the corresponding values for limits of Galton–Watson forests. In some specific cases, namely where the weight sequence has a power law, this confirms conjectures of Bhamidi, Sen and van der Hofstad [12] who had obtained Minkowski dimensions.

Avoiding to compute the law of connected components: The connected components the random graphs may be described as the result of the addition of "shortcut edges" to a tree; this picture is useful both for the discrete models and the limit metric spaces. The work of Bhamidi, Sen & X. Wang and Bhamidi, van der Hofstad & Sen [11, 12] yields an explicit description of the law of the random tree to which one should add shortcuts in order connected components with the correct distribution. As in the case of classical random graphs treated in [2], this law involves a change of measure from one of the "classical" random trees, whose behaviour is in general difficult to control asymptotically. Our connected components are described as the metric induced on a subset of a Galton–Watson tree; the bias of the law of the underlying tree is somewhat transparently handled by the procedure that extracts the relevant subset.

More general models of random graphs. While we focus on the model of the multiplicative graphs, the theorems of Janson [31] on asymptotic equivalent models (see Section 2.3.4) and the expected universality of the limits confers on the results obtained here potential implications that go beyond the realm of this specific model: for instance, random graphs constructed by the celebrated configuration model where the sequence of degrees has asymptotic properties similar to the weight sequence of the present paper are believed to exhibit similar scaling limits; see Section 3.1 in [12] for a related discussion.

Upcoming work. As indicated before, here we have restricted ourselves to the case of *subcritical* weight sequences, for which $\sigma_2(\mathtt{w}_n) \leq \sigma_1(\mathtt{w}_n)$, for a technical reason: as we embed the graph into a Galton–Watson tree with offspring distribution $\mu_{\mathtt{w}_n}$, under this condition $\mu_{\mathtt{w}_n}$ is subcritical; then the Galton–Watson tree is finite a.s.; thereby we can carry out a complete exploration of the tree, upon which our construction of the graph is based. However, in an upcoming version of the work, we will resolve this issue and address the asypmtotics in full generality. Also, we have equipped the graph $\mathcal{G}_{\mathtt{w}_n}$ with an (inhomogeneous) measure induced by $\mathtt{w}_n$. Another natural choice for the measure is the counting measure. The difference between these measures is asymptotically negligible, and we will prove it rigorously in the upcoming version.

Finally, the current version of the linit theorems consider the sequences of connected components in the product topology. The embedding of the graphs in a forest of Galton–Watson forest actually also yields a control the tail of the sequence, which would allow to strengthen the convergence to ℓ^p-like spaces as in [2] or [11]; this will be pursued somewhere else as well.

Organisation of the paper In Section 2.1, we give the construction of the finite graphs and its embedding into Galton–Watson forests, which is based upon a LIFO queueing interpretation. Section 2.2 contains the construction of the continuum graphs. We then state in Section 2.3, our main convergence theorems. The proofs of the results of Section 2 are given in Sections 3-7, while some facts on the Skorokhod topology and branching process are recalled in Appendices.

Contents

2 Main results

2.1 Exploration of discrete multiplicative random graphs.

We briefly describe the model of discrete random graphs that are considered in this paper and we discuss a combinatorial construction thanks to a LIFO-queue. Unless the contrary is specified, all the random variables that we consider are defined on the same probability space $(\Omega, \mathscr{F}, \mathbf{P})$. The graphs $G = (\mathscr{V}(G), \mathscr{E}(G))$ that we consider are *not oriented*, without neither loops nor multiple edges: $\mathscr{E}(G)$ is therefore a set consisting of unordered pairs of distinct vertices.

Let $n \geq 2$ and let $\mathtt{w} = (w_1, \ldots, w_n)$ be a set of *weights*: namely, it is a set of positive real numbers such that $w_1 \geq w_2 \geq \ldots \geq w_n > 0$. We shall use the following notation.

$$\forall r \in (0, \infty), \quad \sigma_r(\mathtt{w}) = w_1^r + \ldots + w_n^r. \tag{11}$$

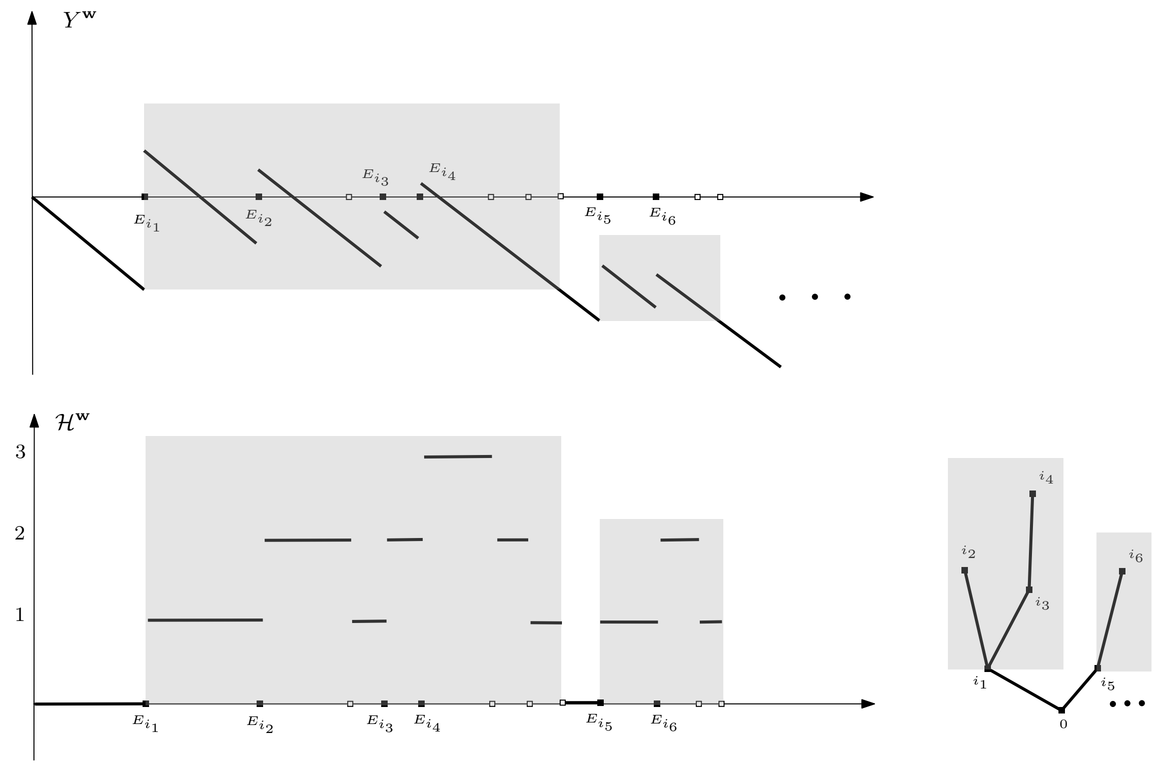

Figure 2: *An example of $Y^{\mathtt{w}}$ and the associated exploration tree. Above, an illustration of $Y^{\mathtt{w}}$. The black squares ■ on the abscissa correspond to the arrivals of clients, namely, the sequence $\{E_i\}$ rearranged in order. The white square □ on the abscissa marks the departures of clients: By the LIFO rule, the client arriving at time E_i leaves at $\inf\{t > E_i : Y^{\mathtt{w}}_t < Y^{\mathtt{w}}_{E_i-}\}$. Below, the exploration tree associated to this queue. Observe that each grey block contains a subtree above the root 0 and is encoded by an excursion of $Y^{\mathtt{w}} - J^{\mathtt{w}}$.* □

The random graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ is said to be $\mathtt{w}$-*multiplicative* if $\mathscr{V}(\boldsymbol{\mathcal{G}}_{\mathtt{w}}) = \{1, \ldots, n\}$ and if

(12) the r.v. $(\mathbf{1}_{\{\{i,j\}\in\mathscr{E}(\boldsymbol{\mathcal{G}}_{\mathtt{w}})\}})_{1\leq i<j\leq n}$, are independent and $\mathbf{P}\big(\{i,j\}\in\mathscr{E}(\boldsymbol{\mathcal{G}}_{\mathtt{w}})\big) = 1-e^{-w_iw_j/\sigma_1(\mathtt{w})}$.

2.1.1 A LIFO queueing system exploring the multiplicative graph.

Let us first explain how to generate a $\mathtt{w}$-multiplicative graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$ thanks to the queueing system that is described as follows: there is a *single server*; at most one client is served at a time; the server applies the *Last In First Out* policy (*LIFO*, for short). Namely, when a client enters the queue, she/he interrupts the service of the previously served client (if any) and the new client is immediately served. When the server completes the service of a client, it comes back to the last arrived client whose service has been interrupted (if there exists such). Exactly n clients will enter the queue; each client is labelled with a distinct integer of $\{1, \ldots, n\}$ and w_i stands for the *total amount of time of service that is needed by Client i* who enters the queue at a time denoted by E_i; we refer to E_i as to the *time of arrival of Client i*; we assume that $E_1, \ldots, E_n \in (0,\infty)$ are distinct. For sake of convenience, we label the server by 0 and we set $w_0 = \infty$. The single-server LIFO queueing system is completely determined by the (always deterministic) times of service $\mathtt{w}$ and the times of arrival $\underline{E} = (E_1, \ldots, E_n)$, that are random variables whose laws are specified below. We introduce the following processes.

$$\forall t\in[0,\infty), \quad Y^{\mathtt{w}}_t = -t + \sum_{1\leq i\leq n} w_i \mathbf{1}_{\{E_i\leq t\}} \quad \text{and} \quad J^{\mathtt{w}}_t = \inf_{s\in[0,t]} Y^{\mathtt{w}}_s \tag{13}$$

The *load* at time t (namely the time of service still due by time t) is then $Y_t^{\mathtt{w}} - J_t^{\mathtt{w}}$. We shall sometimes call $Y^{\mathtt{w}}$ the algebraic load of the queue. LIFO rule implies that Client i arriving at time E_i will leave the queue at the moment $\inf\{t \geq E_i : Y_t^{\mathtt{w}} < Y_{E_i-}^{\mathtt{w}}\}$, namely the first moment when the service load falls back to the level right before her/his arrival. We shall refer to the previous queueing system as to the w-*LIFO queueing system*.

The exploration tree. Denote by $V_t \in \{0, \ldots, n\}$ the label of the client who is served right after time t if there is one; otherwise (namely, if the server is idle right after time t), we set $V_t = 0$ (for a formal definition, see Section 3). First observe that $V_0 = 0$ and that $t \mapsto V_t$ is càdlàg. By convenience, we set $V_{0-} = 0$. Next note that $V_{E_j} = j$ and that V_{E_j-} is the label of the client who was served when Client j entered the queue. Then, the w-LIFO queueing system induces an *exploration tree* $\mathcal{T}_{\mathtt{w}}$ that is defined as follows:

$$(14) \qquad \mathscr{V}(\mathcal{T}_{\mathtt{w}}) = \{0, \ldots, n\} \quad \text{and} \quad \mathscr{E}(\mathcal{T}_{\mathtt{w}}) = \big\{\{V_{E_j-}, j\}; 1 \leq j \leq n\big\}.$$

Namely, $\mathcal{T}_{\mathtt{w}}$ is rooted at 0, which allows to view it as a family tree: the ancestor is 0 (the server) and Client j is a child of Client i if Client j enters the queue while Client i is served. In particular, the ancestors of Client i are those waiting in queue while i is being served. See Figure 2 for an example.

Additional edges. The w-multiplicative graph $\mathcal{G}_{\mathtt{w}}$ is obtained by adding edges to $\mathcal{T}_{\mathtt{w}}$ as follows. Conditionally given $\underline{E}$, let

$$(15) \quad \mathcal{P}_{\mathtt{w}} = \sum_{1 \leq p \leq \mathbf{p}_{\mathtt{w}}} \delta_{(t_p, y_p)} \text{ be a Poisson pt. meas. on } [0,\infty)^2 \text{ with intensity } \frac{1}{\sigma_1(\mathtt{w})} \mathbf{1}_{\{0 < y < Y_t^{\mathtt{w}} - J_t^{\mathtt{w}}\}}\, dt\, dy.$$

Note that a.s. $\mathbf{p}_{\mathtt{w}} < \infty$, since $Y^{\mathtt{w}} - J^{\mathtt{w}}$ is null eventually. We set:

$$(16) \quad \mathbf{\Pi}_{\mathtt{w}} = \big((s_p, t_p)\big)_{1 \leq p \leq \mathbf{p}_{\mathtt{w}}} \quad \text{where} \quad s_p = \inf\Big\{s \in [0, t_p] : \inf_{u \in [s, t_p]} Y_u^{\mathtt{w}} - J_u^{\mathtt{w}} > y_p\Big\},\ 1 \leq p \leq \mathbf{p}_{\mathtt{w}}\ .$$

Note that s_p is well defined, since $y_p < Y_{t_p}^{\mathtt{w}} - J_{t_p}^{\mathtt{w}}$. We then derive $\mathcal{G}_{\mathtt{w}}$ from $\mathcal{T}_{\mathtt{w}}$ and $\mathbf{\Pi}_{\mathtt{w}}$ by setting: $\mathscr{V}(\mathcal{G}_{\mathtt{w}}) = \{1, \ldots, n\}$ and $\mathscr{E}(\mathcal{G}_{\mathtt{w}}) = \mathcal{A} \sqcup \mathcal{S}$, where

$$(17) \qquad \mathcal{A} = \big\{\{i,j\} \in \mathscr{E}(\mathcal{T}_{\mathtt{w}}) : i, j \geq 1\big\} \quad \text{and} \quad \mathcal{S} = \big\{\{V_{s_p}, V_{t_p}\}; 1 \leq p \leq \mathbf{p}_{\mathtt{w}}\big\} \backslash \mathcal{A}.$$

Note that V_{s_p} is necessarily an ancestor of V_{t_p}; in other words, V_{s_p} is in the queue at time t_p. Moreover, we have $V_{s_p} \neq 0$ a.s, since $Y_{s_p}^{\mathtt{w}} - J_{s_p}^{\mathtt{w}} \geq y_p > 0$. It follows that the endpoints of an edge belonging to $\mathcal{S}$ necessarily belong to the same connected component of $\mathcal{T}_{\mathtt{w}} \setminus \{0\}$. Note that 0 is not a vertex of $\mathcal{G}_{\mathtt{w}}$; $\mathcal{S}$ is the set of *surplus edges*. When $\underline{E}$ is suitably distributed, $\mathcal{G}_{\mathtt{w}}$ is distributed as a w-multiplicative graph: this is the content of the following theorem that is the key-point of the paper.

Theorem 2.1 *Keep the previous notation; suppose that $E_1, \ldots, E_n$ are independent exponentially distributed r.v. such that $\mathbf{E}[E_j] = \sigma_1(\mathtt{w})/w_j$, for all $j \in \{1, \ldots, n\}$. Then, $\mathcal{G}_{\mathtt{w}}$ is a w-multiplicative random graph as specified in (12).*

Proof: see Section 3. ■

The connected components of the w-multiplicative graph. The above LIFO-queue construction of the w-multiplicative graph $\mathcal{G}_{\mathtt{w}}$ has the following nice property: the vertex sets of the connected components of $\mathcal{G}_{\mathtt{w}}$ coincide with those of $\mathcal{T}_{\mathtt{w}} \setminus \{0\}$, since surplus edges from $\mathcal{S}$ are only added inside the connected components of the latter. More precisely, we equip $\mathcal{G}_{\mathtt{w}}$ with the measure $\mathbf{m}_{\mathtt{w}} = \sum_{1 \leq j \leq n} w_j \delta_j$ that is the pushforwards measure of the Lebesgue measure via V restricted to the set of times $\{t \in [0,\infty) : V_t \neq 0\}$. Denote by $\mathbf{q}_{\mathtt{w}}$ the number of connected components of $\mathcal{G}_{\mathtt{w}}$ that are denoted by $\mathcal{G}_1^{\mathtt{w}}, \ldots, \mathcal{G}_{\mathbf{q}_{\mathtt{w}}}^{\mathtt{w}}$; here the indexation is such that

$$\mathbf{m}_{\mathtt{w}}(\mathscr{V}(\mathcal{G}_1^{\mathtt{w}})) \geq \ldots \geq \mathbf{m}_{\mathtt{w}}(\mathscr{V}(\mathcal{G}_{\mathbf{q}_{\mathtt{w}}}^{\mathtt{w}}))\ .$$

Note that for all $k \in \{1, \dots, \mathbf{q}_{\mathtt{w}}\}$, $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$ corresponds to a connected component $\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}$ of $\boldsymbol{\mathcal{T}}_{\mathtt{w}} \backslash \{0\}$ such that

$$\mathscr{V}(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}) = \mathscr{V}(\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}) \quad \text{and} \quad \mathcal{E}(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}) = \mathcal{E}(\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}) \sqcup \mathcal{S}_k \quad \text{where} \quad \mathcal{S}_k = \big\{\{i,j\} \in \mathcal{S} : i,j \in \mathscr{V}(\boldsymbol{\mathcal{T}}_k^{\mathtt{w}})\big\}.$$

Indeed, if $\{i,j\} \in \mathcal{S}$, then either i is on the ancestral line of j (namely between 0 and j in $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$), or j is on the ancestral line of i; therefore, i,j are in the same connected component of $\boldsymbol{\mathcal{T}}_{\mathtt{w}} \backslash \{0\}$.

Let $d_{\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}}$ and $d_{\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}}$ be the respective graph-metrics of $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$ and of $\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}$ and denote by $\mathbf{m}_k^{\mathtt{w}}$ the restriction of $\mathbf{m}_{\mathtt{w}}$ to $\mathscr{V}(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}})$. The main purpose of the article is to prove weak limit theorems for the laws of the random measured metric spaces $((\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}, d_{\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}}, \mathbf{m}_k^{\mathtt{w}}))_{1 \le k \le \mathbf{q}_{\mathtt{w}}}$ whose metric and measure are suitably rescaled; the weak limit takes place on the space of sequences of compact measured metric spaces equipped with the Gromov–Hausdorff–Prohorov topology, whose definition is recalled further. In our approach, we actually pass to the limit for the trees $((\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}, d_{\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}}, \mathbf{m}_k^{\mathtt{w}}))_{1 \le k \le \mathbf{q}_{\mathtt{w}}}$ via their coding functions called *height processes* whose definition is recalled next. We then discuss a key ingredient in the proof of the limit theorems: namely, a specific embedding of the exploration tree $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ into a Galton–Watson tree, much easier to analyse; the embedding is carefully explained in the discrete setting, which helps to understand the definition of the continuous analogue of the exploration tree that is discussed hereafter and therefore to understand the definition of the limiting graph.

Height process of the exploration tree. For all $t \in [0,\infty)$, let $\mathcal{H}_t^{\mathtt{w}}$ be the number of clients waiting in the line by time t. Recall that by the LIFO rule, a client entered at time s is still in the queue at time t iff $\inf_{s \le u \le t} Y_u^{\mathtt{w}} > Y_{s-}^{\mathtt{w}}$. In terms of $Y^{\mathtt{w}}$, it is defined by

(18) $$\mathcal{H}_t^{\mathtt{w}} = \# \mathcal{J}_t, \text{ where } \mathcal{J}_t = \big\{s \in [0,t] : J_t^{\mathtt{w},s-} < J_t^{\mathtt{w},s}\big\} \text{ and where } \forall s \in [0,t],\ J_t^{\mathtt{w},s} = \inf_{r \in [s,t]} Y_r^{\mathtt{w}}.$$

We refer to $\mathcal{H}^{\mathtt{w}}$ as to the *height process associated with* $Y^{\mathtt{w}}$. Note that $\mathcal{H}_t^{\mathtt{w}}$ is also the height of the vertex V_t in the exploration tree $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$. Actually, this process is a specific *contour* of the exploration tree $\boldsymbol{\mathcal{T}}_{\mathtt{w}}$ and we easily check that it codes its graph-metric $d_{\boldsymbol{\mathcal{T}}_{\mathtt{w}}}$ as follows:

(19) $$\forall s,t \in [0,\infty), \quad d_{\boldsymbol{\mathcal{T}}_{\mathtt{w}}}(V_s, V_t) = \mathcal{H}_t^{\mathtt{w}} + \mathcal{H}_s^{\mathtt{w}} - 2 \min_{r \in [s \wedge t, s \vee t]} \mathcal{H}_r^{\mathtt{w}}.$$

See Figure 2. Then, $\mathcal{H}^{\mathtt{w}}$ and $\mathbf{\Pi}_{\mathtt{w}}$ completely encode the sequence $((\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}, d_{\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}}, \mathbf{m}_k^{\mathtt{w}}))_{1 \le k \le \mathbf{q}_{\mathtt{w}}}$ of connected components viewed as measured metric spaces. Indeed, each excursion of $\mathcal{H}^{\mathtt{w}}$ above zero corresponds to a connected component $\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}$ of $\boldsymbol{\mathcal{T}}_{\mathtt{w}} \backslash \{0\}$, the length of the excursion interval is $\mathbf{m}_{\mathtt{w}}(\mathscr{V}(\boldsymbol{\mathcal{T}}_k^{\mathtt{w}}))$ and $\mathcal{S}_k$ corresponds to pinching times that fall in this excursion interval. More details are given further.

2.1.2 Embedding the exploration tree into a Galton-Watson tree.

A Markovian LIFO queueing system. We embed the $\mathtt{w}$-LIFO queueing system into the following Markovian LIFO queueing system: *a single server which receives in total an infinite number of clients applying the LIFO policy; clients arrive at unit rate; each client has a* type *that is an integer ranging in* $\{1, \dots, n\}$*; the amount of service required by a client of type* j *is* w_j*; types are i.i.d. with law* $\nu_{\mathtt{w}} = \frac{1}{\sigma_1(\mathtt{w})} \sum_{1 \le j \le n} w_j \delta_j$. Let τ_k be the arrival-time of the k-th client and let $\mathtt{J}_k$ be the type of the k-th client. Then, the Markovian LIFO queueing system is entirely characterised by:

(20) $$\mathscr{X}_{\mathtt{w}} = \sum_{k \ge 1} \delta_{(\tau_k, \mathtt{J}_k)},$$

that is a Poisson point measure on $[0,\infty) \times \{1, \dots, n\}$ whith intensity $\ell \otimes \nu_{\mathtt{w}}$, where ℓ stands for the Lebesgue measure on $[0,\infty)$. We next introduce the following.

(21) $$\forall t \in [0,\infty), \quad X_t^{\mathtt{w}} = -t + \sum_{k \ge 1} w_{\mathtt{J}_k} \mathbf{1}_{[0,t]}(\tau_k) \quad \text{and} \quad I_t^{\mathtt{w}} = \inf_{s \in [0,t]} X_s^{\mathtt{w}}.$$

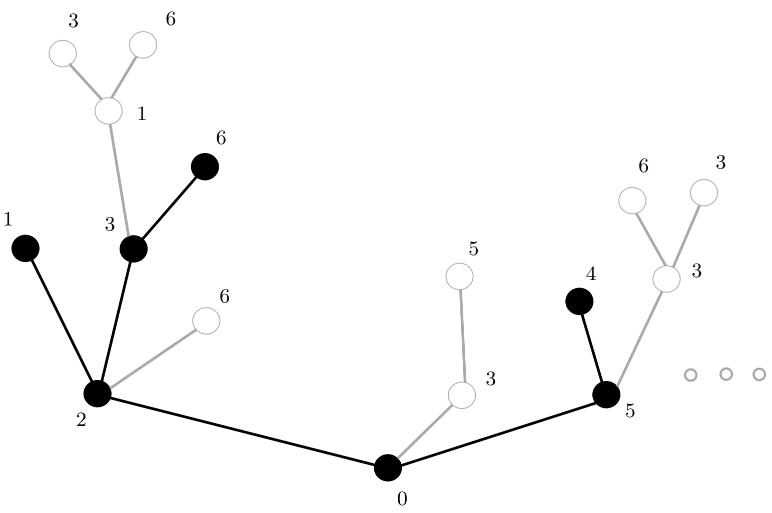

Figure 3: *Colouring the clients of the Markovian LIFO queue. In this example, we use the exploration tree representation of the queue. Clients correspond to nodes in the tree; their types are the numbers next to them. The lexicographic order of the tree (bottom to top, left to right in the picture) corresponds to the arrival orders of the clients. Applying the colouring rules, we color the clients one by one in this order: blue clients are depicted by •, red ones by ◦. Observe that the blue clients form a subtree of the initial tree. Also observe in this example, the first blue client of type* 6 *is not the first type-*6 *client in the queue: there is one previous to it, which has been coloured in red because of a red parent.* □

Then, $X^{\mathtt{w}}_t - I^{\mathtt{w}}_t$ is the load of the Markovian LIFO-queueing system and $X^{\mathtt{w}}$ is called the algebraic load of the queue. Note that $X^{\mathtt{w}}$ is a spectrally positive Lévy process with initial value 0 whose law is given by its Laplace exponent $\psi_{\mathtt{w}}: [0,\infty) \to \mathbb{R}$ given for all $t, \lambda \in [0,\infty)$ by:

$$\mathbf{E}\big[e^{-\lambda X^{\mathtt{w}}_t}\big] = e^{t\psi_{\mathtt{w}}(\lambda)} \quad \text{where} \quad \psi_{\mathtt{w}}(\lambda) = \alpha_{\mathtt{w}}\lambda + \sum_{1\leq j\leq n} \frac{w_j}{\sigma_1(\mathtt{w})}\big(e^{-\lambda w_j} - 1 + \lambda w_j\big) \quad \text{and} \quad \alpha_{\mathtt{w}} := 1 - \frac{\sigma_2(\mathtt{w})}{\sigma_1(\mathtt{w})} .$$

Here, recall from (11) that $\sigma_2(\mathtt{w}) = w_1^2 + \ldots + w_n^2$. We assume that the Markovian LIFO-queueing system is *critical or subcritical*: namely, we assume that $X^{\mathtt{w}}$ does not drift to ∞ which is equivalent to the condition $\alpha_{\mathtt{w}} \geq 0$, that is $\sigma_2(\mathtt{w})/\sigma_1(\mathtt{w}) \leq 1$.

Colouring the clients of the Markovian queueing system. We recover the $\mathtt{w}$-LIFO queueing system from the Markovian one by colouring each client in the following recursive way.

> `Colouring rules.` *Clients are coloured in red or blue. If the type* $\mathtt{J}_k$ *of the* k*-th client already appeared among the types of the blue clients who previously entered the queue, then the* k*-th client is red. Otherwise the* k*-th client inherits her/his colour from the colour of the client who is currently served when she/he arrives (and this colour is blue if there is no client served when she/he arrives: namely, we consider that the server is blue).*

Note that the colour of a client depends in an intricate way on the types of the clients who entered the queue previously. For instance, a client who is the first arriving of her/his type is not necessarily coloured in blue; see Figure 3 for an example. On the other hand, one can check that exactly n clients are coloured in blue and their types are necessarily distinct. While a blue client is served, note that her/his ancestors, namely, the other clients waiting in the line (if any), are blue too. Actually, we will see that the sub-queue constituted by the blue clients corresponds to the previous $\mathtt{w}$-LIFO queue. We next set:

$$\mathtt{Blue} = \big\{ t \in [0,\infty) : \text{a blue client is served at time } t \big\} \quad \text{and} \quad \mathtt{Red} = [0,\infty) \backslash \mathtt{Blue}.$$

Let $\overline{\texttt{Blue}}$ stands for the closure of $\texttt{Blue}$; the set $\overline{\texttt{Blue}} \cap \texttt{Red}$, formed by the left endpoints of the connected components of $\texttt{Red}$, is of particular interest to us. It is the set of times τ_k such that the k-th client is a red one who interrupts the service of a blue one (and possibly the server if it is idle when the k-th client arrives). Note that, by the colouring rule, the type of such a (red) client already appeared among the previous types of the blue clients. A more formal definition of the random subset of times $\texttt{Blue}$ is given in Section 4.2.1. We next define the following change of times. For all $t \in [0, \infty)$, set:

$$\Lambda^{\texttt{b},\texttt{w}}_t = \int_0^t \mathbf{1}_{\texttt{Blue}}(s)\, ds, \quad \Lambda^{\texttt{r},\texttt{w}}_t = t - \Lambda^{\texttt{b},\texttt{w}}_t, \quad \theta^{\texttt{b},\texttt{w}}_t = \inf\{s \in [0,\infty) : \Lambda^{\texttt{b},\texttt{w}}_s > t\} \tag{22}$$

and we also introduce the following:

$$X^{\texttt{b},\texttt{w}}_t = -t + \sum_{k\geq 1} w_{\texttt{J}_k} \mathbf{1}_{\{\tau_k \in \overline{\texttt{Blue}}\,;\, \Lambda^{\texttt{b},\texttt{w}}_{\tau_k} \leq t\}} \quad \text{and} \quad X^{\texttt{r},\texttt{w}}_t = -t + \sum_{k\geq 1} w_{\texttt{J}_k} \mathbf{1}_{\{\tau_k \notin \overline{\texttt{Blue}}\,;\, \Lambda^{\texttt{r},\texttt{w}}_{\tau_k} \leq t\}} \tag{23}$$

Lemma 2.2 *Set $\mathscr{X}^{\texttt{b}}_{\texttt{w}} = \sum_{k\geq 1} \mathbf{1}_{\overline{\texttt{Blue}}}(\tau_k)\, \delta_{(\Lambda^{\texttt{b},\texttt{w}}_{\tau_k}, \texttt{J}_k)}$. Then, $\mathscr{X}^{\texttt{b}}_{\texttt{w}}$ and $X^{\texttt{r},\texttt{w}}$ are independent, $\mathscr{X}^{\texttt{b}}_{\texttt{w}}$ has the same law as $\mathscr{X}_{\texttt{w}}$ and $X^{\texttt{r},\texttt{w}}$ has the same law as $X^{\texttt{w}}$.*

Proof. See Section 4.2.2. ■

By Lemma 2.2, $X^{\texttt{b},\texttt{w}}$ and $X^{\texttt{r},\texttt{w}}$ are independent copies of $X^{\texttt{w}}$. Next observe that (21) immediately implies:

$$\forall t \in [0,\infty), \qquad X^{\texttt{w}}_t = X^{\texttt{b},\texttt{w}}_{\Lambda^{\texttt{b},\texttt{w}}_t} + X^{\texttt{r},\texttt{w}}_{\Lambda^{\texttt{r},\texttt{w}}_t}. \tag{24}$$

See also Figure 4.

As explained before and illustrated by the example in Fig. 3, the colour of a client depends somehow in a complicated way on the types of the previous clients. For this reason, the above colouring procedure of the Markovian queue does not allow for a straightforward generalisation to the limit case, where "clients" will arrive according to a Poisson point process with infinite intensity. Here we explain an alternative construction for the time-change process $\Lambda^{\texttt{b},\texttt{w}}, \theta^{\texttt{b},\texttt{w}}, \Lambda^{\texttt{r},\texttt{w}}$ in terms of the blue and red processes $X^{\texttt{b},\texttt{w}}$ and $X^{\texttt{r},\texttt{w}}$. Namely, to understand the limiting processes, we now explain how to derive the time-change $\theta^{\texttt{b},\texttt{w}}$ directly from $\mathscr{X}^{\texttt{b}}_{\texttt{w}}$ and $X^{\texttt{r},\texttt{w}}$. To that end, for all $j \in \{1, \ldots, n\}$ and all $t \in [0, \infty)$, we set:

$$N^{\texttt{w}}_j(t) = \mathscr{X}^{\texttt{b}}_{\texttt{w}}\big([0,t] \times \{j\}\big) \quad \text{and} \quad E^{\texttt{w}}_j = \inf\big\{t \in [0,\infty) : \mathscr{X}^{\texttt{b}}_{\texttt{w}}([0,t]\times\{j\}) = 1\big\}. \tag{25}$$

Thus, the $N^{\texttt{w}}_j$ are independent homogeneous Poisson processes with jump-rate $w_j/\sigma_1(\texttt{w})$ and the r.v. $(\frac{w_j}{\sigma_1(\texttt{w})} E^{\texttt{w}}_j)_{1\leq j\leq n}$ are i.i.d. exponentially distributed r.v. with unit mean. We also set

$$\mathscr{X}^{\texttt{r}/\texttt{b}}_{\texttt{w}} = \sum_{k\geq 1} \mathbf{1}_{\overline{\texttt{Blue}} \cap \texttt{Red}}(\tau_k)\, \delta_{(\Lambda^{\texttt{b},\texttt{w}}_{\tau_k}, \texttt{J}_k)} \quad \text{and} \quad \mathscr{Y}_{\texttt{w}} = \mathscr{X}^{\texttt{b}}_{\texttt{w}} - \mathscr{X}^{\texttt{r}/\texttt{b}}_{\texttt{w}}. \tag{26}$$

Namely, $\mathscr{Y}_{\texttt{w}}$ and $\mathscr{X}^{\texttt{r}/\texttt{b}}_{\texttt{w}}$ are the empirical measures of the times of arrival (in the blue scale) & the types of resp. the blue clients and the red clients interrupting blue clients. Recall that by the colouring rules, $\tau_k \in \overline{\texttt{Blue}} \cap \texttt{Red}$ iff the client's type $\texttt{J}_k$ has already appeared among the types of the previous blue clients, i.e. $N^{\texttt{w}}_{\texttt{J}_k}(\tau_k) \geq 2$. On the other hand, the (only) blue client of type j corresponds exactly to the first atom of $N^{\texttt{w}}_j$, $1 \leq j \leq n$. Thus,

$$\mathscr{Y}_{\texttt{w}} = \sum_{1\leq j\leq n} \delta_{(E^{\texttt{w}}_j, j)} \quad \text{and} \quad \forall t \in [0,\infty), \forall j \in \{1,\ldots,n\}, \quad \mathscr{X}^{\texttt{r}/\texttt{b}}_{\texttt{w}}\big([0,t]\times\{j\}\big) = (N^{\texttt{w}}_j(t) - 1)_+ . \tag{27}$$

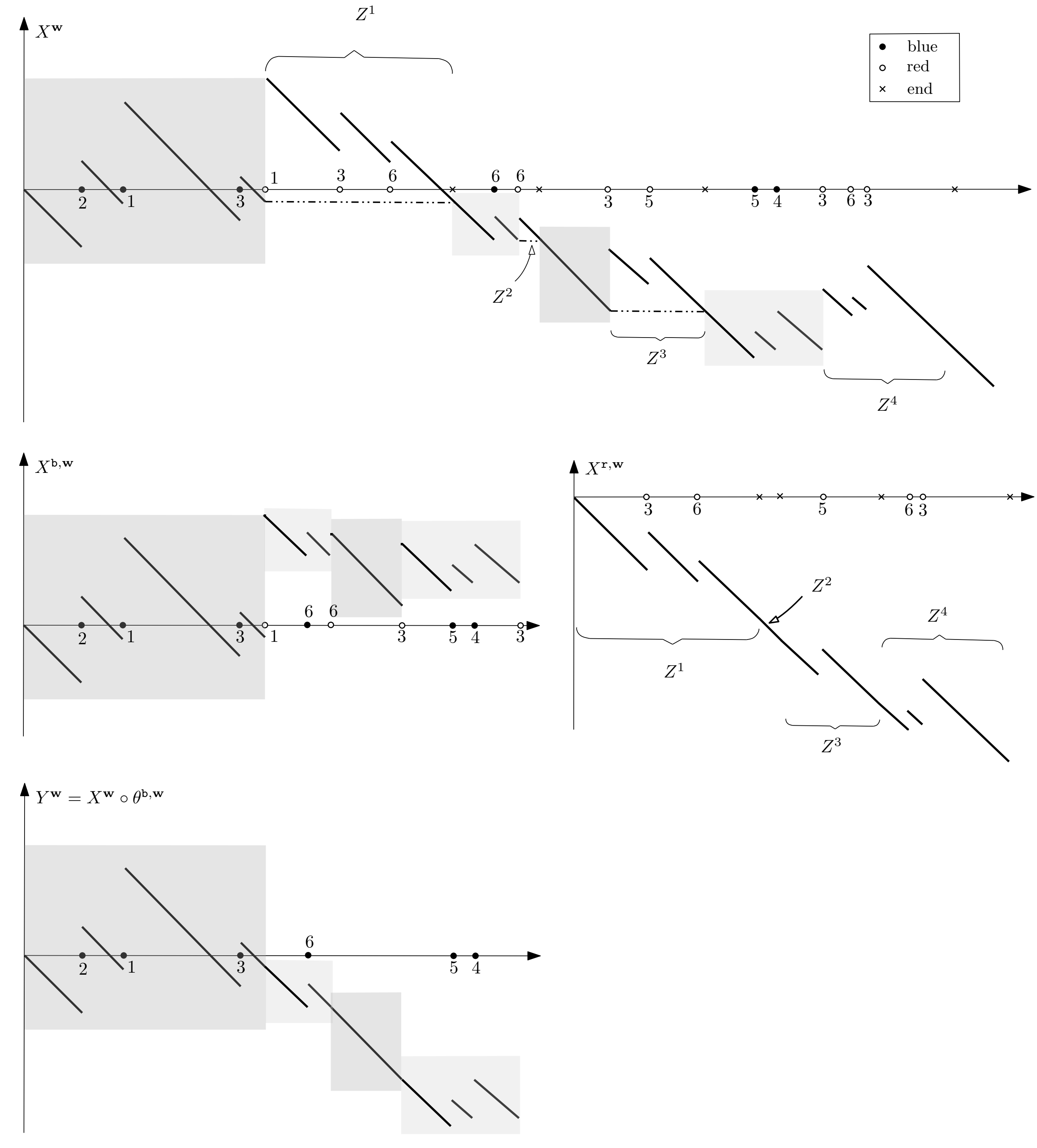

Figure 4: *Decomposition of $X^{\mathtt{w}}$ into $X^{\mathtt{b},\mathtt{w}}$ and $X^{\mathtt{r},\mathtt{w}}$. We take the same example as in Figure 3. Above, the process $X^{\mathtt{w}}$: clients are in bijection with its jumps; their types are the numbers next to the jumps. Blue clients are marked by $\bullet$ and red ones by $\circ$. The restriction of $X^{\mathtt{w}}$ to the set $\overline{\texttt{Blue}}$ corresponds to the grey blocks. Concatenating these blocks yields the blue process $X^{\mathtt{b},\mathtt{w}}$. The remaining pieces of $X^{\mathtt{w}}$, namely, $(Z^i)_{i\geq 1}$, are glued together, producing the red process $X^{\mathtt{r},\mathtt{w}}$. Concatenating the grey blocks but **without** the final jump of each block yields $Y^{\mathtt{w}}$. Alternatively, we can obtain $Y^{\mathtt{w}}$ by erasing in $X^{\mathtt{w}}$ the pieces Z^i, $i \geq 1$ and then removing the temporal gaps between the grey blocks: this is the graphic representation of $Y^{\mathtt{w}} = X^{\mathtt{w}} \circ \theta^{\mathtt{b},\mathtt{w}}$. Observe also that each connected component of* `Red` *begins with the arrival of a red client whose type is a repeat among the types of the previous blue ones, and ends with the departure of this red client, marked by $\times$ on the abscissa.* $\square$

For all $x, t \in [0, \infty)$, we next set

$$(28) \quad Y_t^{\mathtt{w}} = -t + \sum_{1 \le j \le n} w_j \mathbf{1}_{\{E_j^{\mathtt{w}} \le t\}}, \quad A_t^{\mathtt{w}} = \sum_{1 \le j \le n} w_j (N_j^{\mathtt{w}}(t) - 1)_+ \quad \text{and} \quad \gamma_x^{\mathtt{r},\mathtt{w}} = \inf\{t \in [0, \infty) : X_t^{\mathtt{r},\mathtt{w}} < -x\}.$$

Consequently,

$$(29) \qquad A_t^{\mathtt{w}} = \sum_{k \ge 1} w_{\mathtt{J}_k} \mathbf{1}_{\overline{\mathtt{Blue}} \cap \mathtt{Red}}(\tau_k) \, \mathbf{1}_{\{\Lambda_{\tau_k}^{\mathtt{b},\mathtt{w}} \le t\}} \quad \text{and thus} \quad Y_t^{\mathtt{w}} = X_t^{\mathtt{b},\mathtt{w}} - A_t^{\mathtt{w}}.$$

Clearly $(Y^{\mathtt{w}}, A^{\mathtt{w}})$ is independent of the càdlàg subordinator $\gamma^{\mathtt{r},\mathtt{w}}$. Moreover, $Y^{\mathtt{w}}$ has the same law as in (13) when the $w_j E_j / \sigma_1(\mathtt{w})$ are i.i.d. exponential r.v. with unit mean. In the following lemma, we get $\theta^{\mathtt{b},\mathtt{w}}$ in terms of $\mathscr{X}_{\mathtt{w}}^{\mathtt{b}}$ and $X^{\mathtt{r},\mathtt{w}}$.

Lemma 2.3 *A.s. for all* $t \in [0, \infty)$,

$$(30) \qquad \theta_t^{\mathtt{b},\mathtt{w}} = t + \gamma^{\mathtt{r},\mathtt{w}}_{A_t^{\mathtt{w}}} .$$

Then, $\Lambda_t^{\mathtt{b},\mathtt{w}} = \inf\{s \in [0, \infty) : \theta_s^{\mathtt{b},\mathtt{w}} > t\}$, $\Lambda_t^{\mathtt{r},\mathtt{w}} = t - \Lambda_t^{\mathtt{b},\mathtt{w}}$*; we derive* $X^{\mathtt{w}}$ *from* $X^{\mathtt{b},\mathtt{w}}$ *and* $X^{\mathtt{r},\mathtt{w}}$ *thanks to (24). We also get*

$$(31) \qquad a.s. \ \forall t \in [0, \infty), \quad Y_t^{\mathtt{w}} = X^{\mathtt{w}}_{\theta_t^{\mathtt{b},\mathtt{w}}}.$$

Proof. See Section 4.2.3. ■

The red and blue Galton-Watson trees; height processes. Let $Y^{\mathtt{w}}$ be given by (28) or equivalently by (13) with $E_j^{\mathtt{w}} = E_j$, $j \in \{1, \ldots, n\}$. Recall from (14) the definition of the exploration tree $\mathcal{T}_{\mathtt{w}}$ that is derived from the $\mathtt{w}$-LIFO queueing system. Recall from (18) the definition of the height process $\mathcal{H}^{\mathtt{w}}$ in terms of $Y^{\mathtt{w}}$: namely, $\mathcal{H}_t^{\mathtt{w}}$ *is the number of clients waiting on the line at time t in the blue-times scale*. Recall that while a blue client is served, all the other clients in the line are blue too.

Similarly, the Markovian queueing system governed by $\mathscr{X}_{\mathtt{w}}$ as defined in (20) induces an exploration tree denoted by $\mathbf{T}_{\mathtt{w}}$ that is defined as follows: its vertices are the clients (and the server) and its set of edges is specified as follows; we root $\mathbf{T}_{\mathtt{w}}$ at the server, that is viewed as the first "client"; the k-th client to enter the queue is a child of the l-th one if the k-th client enters when the l-th client is served. We next denote by $H_t^{\mathtt{w}}$ the number of clients waiting in the queue right after time t, since both queues are LIFO, the following expression for $H_t^{\mathtt{w}}$ is an analogue of (18):

$$(32) \quad H_t^{\mathtt{w}} = \#\mathcal{K}_t, \ \text{where} \ \mathcal{K}_t = \big\{ s \in [0, t] : I_t^{\mathtt{w},s-} < I_t^{\mathtt{w},s} \big\} \ \text{and where} \ \forall s \in [0, t], \ I_t^{\mathtt{w},s} = \inf_{r \in [s,t]} X_r^{\mathtt{w}}.$$

We shall refer to $H^{\mathtt{w}}$ as to the *height process associated with* $X^{\mathtt{w}}$. One easily checks that $\mathbf{T}_{\mathtt{w}}$ is a sequence of i.i.d. Galton-Watson trees glued at their root and that their common offspring distribution is given by:

$$(33) \qquad \forall k \in \mathbb{N}, \quad \mu_{\mathtt{w}}(k) = \sum_{1 \le j \le n} \frac{w_j^{k+1} e^{-w_j}}{\sigma_1(\mathtt{w}) \, k!} .$$

We refer to Section 4.1.2 for a more formal definition. Observe that

$$\sum_{k \ge 0} k \mu_{\mathtt{w}}(k) = \sum_{1 \le j \le n} w_j^2 / \sigma_1(\mathtt{w}) = \sigma_2(\mathtt{w}) / \sigma_1(\mathtt{w}).$$

Since the Markovian queueing system is assumed to be (sub)critical, we get $\sum_{k \ge 0} k \mu_{\mathtt{w}}(k) \le 1$. Thus, the Galton-Watson trees are a.s. finite and $H^{\mathtt{w}}$ fully explores $\mathbf{T}_{\mathtt{w}}$.

Since the vertices of $\mathbf{T}_{\mathtt{w}}$ are the clients of the Markovian queueing system, the blue clients in $\mathbf{T}_{\mathtt{w}}$ are therefore a subtree tree that is a relabelled version of $\mathcal{T}_{\mathtt{w}}$, the exploration tree of the $\mathtt{w}$-multiplicative graph generated by $Y^{\mathtt{w}}$. Since the order of visit is preserved, we get the following.

Lemma 2.4 *Recall from (31) the joint law of $X^{\mathtt{w}}$ and $Y^{\mathtt{w}}$; recall from (30) the definition of $\theta^{\mathtt{w},\mathtt{b}}$, recall from (18) and (32) the definition of resp. $\mathcal{H}^{\mathtt{w}}$ and $H^{\mathtt{w}}$. Then,*

$$\text{(34)} \qquad a.s. \ \forall t \in [0,\infty), \quad \mathcal{H}^{\mathtt{w}}_t = H^{\mathtt{w}}_{\theta^{\mathtt{b},\mathtt{w}}_t} .$$

Proof. See Section 4.2.4. ∎

Although the law of $\mathcal{T}_{\mathtt{w}}$ is complicated, (34) allows to define its height process in a tractable way to pass to the limit.

Remark 2.1 Note that the height process of $X^{\mathtt{b},\mathtt{w}}$ is actually distinct from $\mathcal{H}^{\mathtt{w}}$. Although related to $\mathcal{H}^{\mathtt{w}}$, the tree coded by $X^{\mathtt{b},\mathtt{w}}$ is not relevant to our purpose. □

Let us summarise the various embeddings introduced in this section. The sub-system formed by the server and the blue clients in the Markovian LIFO queue behave as the $\mathtt{w}$-LIFO queue; the load processes $X^{\mathtt{w}}$, $Y^{\mathtt{w}}$ for the two queues are then related by the time-change relation (31). Since the same genealogical relation is introduced for both queueing systems, the embedding of the queues leads to the embedding of the associated exploration trees, the latter being formalised in (34). Meanwhile, we have the decomposition of $X^{\mathtt{w}}$ into two independent Lévy processes $X^{\mathtt{b},\mathtt{w}}$ and $X^{\mathtt{r},\mathtt{w}}$. This decomposition has great technical significance in our approach. Firstly, it allows for the alternative representation (30) of the time-change, which is later generalised into the continuous setting. Secondly, thanks to this decomposition, we are able to control the pruning as n tends to infinity and derive a joint convergence for $(\theta^{\mathtt{b},\mathtt{w}}, X^{\mathtt{w}})$ after a proper scaling (see in particular Lemma 4.3 and Proposition 6.12), which is an essential ingredient in our proof for the convergence of the graphs.

2.2 The multiplicative graph in the continuous setting.

2.2.1 The continuous exploration tree and its height process.

Notations and conventions. Recall that $\mathbb{N}$ stands for the set of nonnegative integers and that $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We denote by $\ell^{\downarrow}_{\infty} = \big\{(w_j)_{j\geq 1} \in [0,\infty)^{\mathbb{N}^*}\!:\ w_j \geq w_{j+1}\big\}$ the set of *weights*. By an obvious extension of Notation (11), for all $r \in (0,\infty)$ and all $\mathtt{w} = (w_j)_{j\geq 1} \in \ell^{\downarrow}_{\infty}$, we set $\sigma_r(\mathtt{w}) = \sum_{j\geq 1} w_j^r \in [0,\infty]$. We also introduce the following:

$$\ell^{\downarrow}_r = \big\{\mathtt{w} \in \ell^{\downarrow}_{\infty} : \sigma_r(\mathtt{w}) < \infty\big\}, \quad \text{and} \quad \ell^{\downarrow}_f = \big\{\mathtt{w} \in \ell^{\downarrow}_{\infty} : \exists j_0 \geq 1 : w_{j_0} = 0\big\}.$$

Let $(\mathscr{F}_t)_{t\in[0,\infty)}$ be a filtration on $(\Omega, \mathscr{F})$ that is specified further. A process $(Z_t)_{t\in[0,\infty)}$ is said to be $(\mathscr{F}_t)$-Lévy process with initial value 0 if a.s. Z is càdlàg, $Z_0 = 0$ and if for all a.s. finite $(\mathscr{F}_t)$-stopping time T, the process $Z_{T+\,\cdot} - Z_T$ is independent of $\mathscr{F}_T$ and has the same law as Z.

Let $(M_j(\cdot))_{j\geq 1}$ be a sequence of càdlàg $(\mathscr{F}_t)$-martingales that are in L^2 and orthogonal: namely, for all $t \in [0,\infty)$, $\sum_{j\geq 1} \mathbf{E}\big[M_j(t)^2\big] < \infty$ and $\mathbf{E}[M_j(t)M_k(t)] = 0$ if $k > j$. Then $\sum^{\perp}_{j\geq 1} M_j$ stands for the (unique up to indistinguishability) càdlàg $(\mathscr{F}_t)$-martingale $M(\cdot)$ such that for all $j \geq 1$ and all $t \in [0,\infty)$, $\mathbf{E}\big[\sup_{s\in[0,t]} \big|M(s) - \sum_{1\leq k\leq j} M_k(s)\big|^2\big] \leq 4\sum_{l>j} \mathbf{E}[M_l(t)^2]$, by Doob's inequality. Sometimes, we simply write $\sum^{\perp}_{j\geq 1} M_j(t)$ instead of $M(t)$.

Blue processes. We fix the following parameters.

$$\text{(35)} \qquad \alpha, \beta \in [0,\infty), \quad \kappa \in (0,\infty), \quad \mathbf{c} = (c_j)_{j\geq 1} \in \ell^{\downarrow}_3 .$$

These quantites are the *parameters of the continuous multiplicative graph*: $\mathbf{c}$ plays the same role as $\mathtt{w}$ in the discrete setting, α is a drift coefficient similar to $\alpha_{\mathtt{w}}$, β is a Brownian coefficient and the interpretation of κ is explained later. Next, let $(B_t)_{t\in[0,\infty)}$, $(N_j(t))_{t\in[0,\infty)}$, $j \geq 1$ be processes that satisfy the following.

(b_1) *B is a $(\mathscr{F}_t)$-real valued standard Brownian motion with initial value* 0.

(b_2) *For all $j\geq 1$, N_j is a $(\mathscr{F}_t)$-homogeneous Poisson process with jump-rate κc_j.*

(b_3) *The processes B, N_j, $j\geq 1$ are independent.*

The *blue* Lévy process is then defined by

$$\forall t\in[0,\infty), \quad X^{\mathtt{b}}_t = -\alpha t + \sqrt{\beta} B_t + \sum_{j\geq 1}^{\perp} c_j\big(N_j(t)-c_j\kappa t\big). \tag{36}$$

Clearly $X^{\mathtt{b}}$ is a $(\mathscr{F}_t)$-spectrally positive Lévy process with initial value 0 whose law is characterized by the Laplace exponent $\psi:[0,\infty)\to\mathbb{R}$ given for all $t,\lambda\in[0,\infty)$ by:

$$\mathbf{E}\big[e^{-\lambda X^{\mathtt{b}}_t}\big]=e^{t\psi(\lambda)}, \text{ where } \quad \psi(\lambda)=\alpha\lambda+\tfrac{1}{2}\beta\lambda^2+\sum_{j\geq 1}\kappa c_j\big(e^{-\lambda c_j}-1+\lambda c_j\big). \tag{37}$$

Since $\alpha\geq 0$, $X^{\mathtt{b}}$ does not drift to ∞ and we shall refer to it as to the *(sub)criticality assumption* for the continuous multiplicative graph. Most of the time we shall assume that:

$$\text{either } \beta>0 \text{ or } \sigma_2(\mathbf{c})=\infty. \tag{38}$$

This assumption is equivalent to the fact that $X^{\mathtt{b}}$ has infinite variation sample paths.

We next introduce the analogues of $A^{\mathtt{w}}$ and $Y^{\mathtt{w}}$ as in (28) and (29). To that end, note that $\mathbf{E}[c_j(N_j(t)-1)_+]=c_j\big(e^{-c_j\kappa t}-1+c_j\kappa t\big)\leq\frac{1}{2}(\kappa t)^2c_j^3$, which makes sense of the following:

$$\forall t\in[0,\infty), \quad A_t=\tfrac{1}{2}\kappa\beta t^2+\sum_{j\geq 1}c_j\big(N_j(t)-1\big)_+ \quad \text{and} \quad Y_t=X^{\mathtt{b}}_t-A_t. \tag{39}$$

Remark 2.2 To view Y as in (13), set $E_j=\inf\{t\in[0,\infty):N_j(t)=1\}$, note that $c_j(N_j(t)-c_j\kappa t)-c_j(N_j(t)-1)_+=c_j(\mathbf{1}_{\{E_j\leq t\}}-c_j\kappa t)$ and check that $c_j(\mathbf{1}_{\{E_j\leq t\}}-c_j\kappa t)=M_j'(t)-\kappa c_j^2(t-E_j)_+$ where M_j' is a centered $(\mathscr{F}_t)$-martingale such that $\mathbf{E}[M_j'(t)^2]=c_j^2(1-e^{-c_j\kappa t})\leq\kappa t c_j^3$. Since $\mathbf{E}[\kappa c_j^2(t-E_j)_+]\leq\kappa t c_j^2(1-e^{-\kappa c_j t})\leq\kappa^2tc_j^3$, it makes sense to write for all $t\in[0,\infty)$:

$$\begin{aligned} Y_t &= -\alpha t-\tfrac{1}{2}\kappa\beta t^2+\sqrt{\beta}B_t+\sum_{j\geq 1}^{\perp}c_j\big(\mathbf{1}_{\{E_j\leq t\}}-\kappa c_j(t\wedge E_j)\big)-\sum_{j\geq 1}\kappa c_j^2(t-E_j)_+ \\ &\overset{\text{(informal)}}{=} -\alpha t-\tfrac{1}{2}\kappa\beta t^2+\sqrt{\beta}B_t+\sum_{j\geq 1}c_j(\mathbf{1}_{\{E_j\leq t\}}-c_j\kappa t). \end{aligned} \tag{40}$$

Namely the jump-times of Y are the E_j and $\Delta Y_{E_j}=c_j$. □

Lemma 2.5 *We keep the previous notation. We assume (38). Then, a.s. the process A is strictly increasing and the process Y has infinite variation sample paths.*

Proof. See Section 5.2.1. ■

Red and bi-coloured processes. We next introduce the *red process* $X^{\mathtt{r}}$ that satisfies the following.

(r_1) $X^{\mathtt{r}}$ is a $(\mathscr{F}_t)$-spectrally positive Lévy process starting at 0 and whose Laplace exponent is ψ as in (37).

(r_2) $X^{\mathtt{r}}$ is independent of the processes B and $(N_j)_{j\geq 1}$.

To keep the filtration $(\mathscr{F}_t)$ minimal, we may assume that $\mathscr{F}_t$ is the completed sigma-field generated by B_s, $(N_j(s))_{j\geq 1}$ and $X^{\mathtt{r}}_s$, $s\in[0,t]$. We next introduce the following processes:

$$\forall x,t\in[0,\infty), \quad \gamma^{\mathtt{r}}_x=\inf\{s\in[0,\infty):X^{\mathtt{r}}_s<-x\} \quad \text{and} \quad \theta^{\mathtt{b}}_t=t+\gamma^{\mathtt{r}}_{A_t}. \tag{41}$$

Observe that $\gamma^{\mathtt{r}}$ is a subordinator with initial value 0 and Laplace exponent the inverse function ψ^{-1}: see e.g. Bertoin's book, Ch. VII. Then, note that the *blue time-change* $\theta^{\mathtt{b}}$ is strictly increasing and càdlàg; it is the analogue of $\theta^{\mathtt{b},\mathtt{w}}$ by (30) in Lemma 2.3. We next introduce the following.

$$\forall t\in[0,\infty), \quad \Lambda^{\mathtt{b}}_t=\inf\{s\in[0,\infty):\theta^{\mathtt{b}}_s>t\} \quad \text{and} \quad \Lambda^{\mathtt{r}}_t=t-\Lambda^{\mathtt{b}}_t. \tag{42}$$

We prove the following.

Theorem 2.6 *We keep the previous notation. Then, the process* $\Lambda^{\mathtt{r}}$ *is nondecreasing and if we set*

$$\forall t\in[0,\infty), \quad X_t = X^{\mathtt{b}}_{\Lambda^{\mathtt{b}}_t} + X^{\mathtt{r}}_{\Lambda^{\mathtt{r}}_t}, \tag{43}$$

then X *is a spectrally positive Lévy process with initial value* 0 *and Laplace exponent* ψ *as in (37). Namely,* X*,* X^{b} *and* X^{r} *have the same law.*

Proof. See Section 5.2.2. ■

Height processes; pinching points. We next define the analogue of $H^{\mathtt{w}}$. To that end, we assume that ψ defined in (37) satisfies the following:

$$\int^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty. \tag{44}$$

Note that in particular (44) entails (38).

Le Gall & Le Jan [33] (see also Le Gall & D. [21]) prove that there exists a *continuous* process $H=(H_t)_{t\in[0,\infty)}$ such that the following limit holds true for all $t\in[0,\infty)$ in probability :

$$H_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{\{X_s-\inf_{r\in[s,t]} X_r\leq\varepsilon\}}\, ds\ . \tag{45}$$

Note that (45) is a local time version of (32). We refer to H as to the *height process of* X. The analogue of $\mathcal{H}^{\mathtt{w}}$ is then defined as follows.

Proposition 2.7 *For all* $t\in[0,\infty)$*, set* $\mathcal{H}_t=H_{\theta^{\mathtt{b}}_t}$*. Then,* $\mathcal{H}$ *is a.s. a continuous process. We refer to* $\mathcal{H}$ *as to the height process associated with* Y*.*

Proof. See Lemma 5.6, Section 5.2.4. ■

We next define the pinching times as in (47): for all $t\in[0,\infty)$, set $J_t=\inf_{s\in[0,t]} Y_s$ and conditionally given Y, let

$$\mathcal{P}=\sum_{p\geq 1}\delta_{(t_p,y_p)} \text{ be a Poisson pt. meas. on } [0,\infty)^2 \text{ with intensity } \kappa\mathbf{1}_{\{0<y<Y_t-J_t\}}\, dt\, dy. \tag{46}$$

Then, set

$$\mathbf{\Pi}=\big((s_p,t_p)\big)_{p\geq 1} \quad \text{where} \quad s_p=\inf\big\{s\in[0,t_p] : \inf_{u\in[s,t_p]} Y_u-J_u > y_p\big\},\ \ p\geq 1. \tag{47}$$

We claim that the processes $(Y,\mathcal{H},\mathbf{\Pi})$ completely characterise the continuous version of the multiplicative graph as explained in the next section.

2.2.2 Coding graphs.

Coding trees. Let us first briefly recall how functions (not necessarily continuous) code trees. Let $h:[0,\infty)\to[0,\infty)$ be càdlàg and such that

$$h(0)=0 \quad \text{and} \quad \zeta_h=\sup\{t\in[0,\infty):h(t)>0\}<\infty\ . \tag{48}$$

For all $s,t\in[0,\zeta_h)$, we set

$$b_h(s,t)=\inf_{r\in[s\wedge t,s\vee t]} h(r) \qquad \text{and} \qquad d_h(s,t)=h(s)+h(t)-2b_h(s,t). \tag{49}$$

Note that d_h satisfies the four-points inequality: for all $s_1,s_2,s_3,s_4\in[0,\zeta_h)$, $d_h(s_1,s_2)+d_h(s_3,s_4)\leq \big(d_h(s_1,s_3)+d_h(s_2,s_4)\big)\vee\big(d_h(s_1,s_4)+d_h(s_2,s_3)\big)$. Taking $s_3=s_4$ shows that d_h is a pseudometric on $[0,\zeta_h)$. We then denote by $s\sim_h t$ the equivalence relation $d_h(s,t)=0$ and we set

$$T_h=[0,\zeta_h)/\sim_h\ . \tag{50}$$

Then, d_h induces a true metric on the quotient set T_h that we keep denoting by d_h and we denote by $p_h:[0,\zeta_h)\to T_h$ the *canonical projection*. Note that p_h is not necessarily continuous.

Remark 2.3 The metric space (T_h, d_h) is tree-like but in general it is not necessarily connected or compact. However, we shall consider the following cases.

(a) h is pure jump: it takes finitely many values.

(b) h is continuous.

In Case (a), T_h is not connected but it is compact; T_h is in fact formed by a finite number of points. In particular, $\mathcal{H}^{\mathtt{w}}$ is in this case: by (19), the exploration tree $\mathcal{T}_{\mathtt{w}}$ as defined in (14) is actually isometric to $T_{\mathcal{H}^{\mathtt{w}}}$, that is the tree coded by the height process $\mathcal{H}^{\mathtt{w}}$ that is derived from $Y^{\mathtt{w}}$ by (18).

In Case (b), T_h is compact and connected; the metric d_h satisfies the four-points condition: it is therefore a compact real tree, namely a compact metric space such that any pair of points is joined by a unique injective path that turns out to be a geodesic (see Evans [26] for more references on this topic). □

The coding function provide two additional features: a distinguished point $\rho_h = p_h(0)$ that is called the *root* of T_h and the *mass measure* m_h that is the pushforward measure of the Lebesgue measure on $[0, \zeta_h)$ induced by p_h on T_h: for any Borel measurable function $f : T_h \to [0, \infty)$,

$$\int_{T_h} f(\sigma)\, m_h(d\sigma) = \int_0^{\zeta_h} f(p_h(t))\, dt\ . \tag{51}$$

Pinched metric spaces. We next briefly explain how the metric of a graph is modified when several vertices are added. Since we deal with rescaled version of graphs, and continuous limits of such graphs, we discuss this point in a general setting. Our construction of the graphs both in the discrete and the continuous settings consists in creating cycles in the spanning trees of the graphs. In the discrete setting, we do this by adding surplus edges; in the continuous setting, we identify pairs of points in a real tree. In what follows, we provide a unified way to deal with both operations on the metrics.

Let (E, d) be a metric space and let $\mathbf{\Pi} = ((x_i, y_i))_{1\leq i\leq p}$ where $(x_i, y_i) \in E^2$, $1 \leq i \leq p$, are pairs of *pinching points*. Let $\varepsilon \in [0, \infty)$ that is interpreted as the length of the edges that are added to E (if $\varepsilon = 0$, then each x_i is identified with y_i). Set $A_E = \{(x, y); x, y \in E\}$ and for all $e = (x, y) \in A_E$, set $\underline{e} = x$ and $\overline{e} = y$. A path γ joining x to y is a sequence of $e_1, \dots, e_q \in A_E$ such that $\underline{e}_1 = x$, $\overline{e}_q = y$ and $\overline{e}_i = \underline{e}_{i+1}$, for all $1 \leq i < q$. For all $e = (x, y) \in A_E$, we then define its length by $l_e = \varepsilon \wedge d(x_i, y_i)$ if (x, y) or (y, x) is equal to (x_i, y_i); otherwise we set $l_e = d(x, y)$. The length of a path $\gamma = (e_1, \dots, e_q)$ is given by $l(\gamma) = \sum_{1\leq i\leq q} l_{e_i}$, and we set:

$$\forall x, y \in E, \quad d_{\mathbf{\Pi},\varepsilon}(x, y) = \inf \big\{ l(\gamma);\ \gamma \text{ is a path joining } x \text{ to } y \big\}\ . \tag{52}$$

We refer to Section C.1 for more details. Clearly, $d_{\mathbf{\Pi},\varepsilon}$ is a pseudo-metric and we denote the equivalence relation $d_{\mathbf{\Pi},\varepsilon}(x, y) = 0$ by $x \equiv_{\mathbf{\Pi},\varepsilon} y$; the $(\mathbf{\Pi}, \varepsilon)$*-pinched metric space associated with* (E, d) is then the quotient space $E/\equiv_{\mathbf{\Pi},\varepsilon}$ equipped with $d_{\mathbf{\Pi},\varepsilon}$. First note that if (E, d) is compact or connected, so is the associated $(\mathbf{\Pi}, \varepsilon)$-pinched metric space since the canonical projection $\varpi_{\mathbf{\Pi},\varepsilon} : E \to E/\equiv_{\mathbf{\Pi},\varepsilon}$ is 1-Lipschitz. Of course when $\varepsilon > 0$, $d_{\mathbf{\Pi},\varepsilon}$ on E is a true metric, $E = E/\equiv_{\mathbf{\Pi},\varepsilon}$ and $\varpi_{\mathbf{\Pi},\varepsilon}$ is the identity map on E.

Coding pinched trees. Let $h : [0, \infty) \to [0, \infty)$ be a càdlàg function that satisfies (48) and (a) or (b) in Remark 2.3; let $\Pi = ((s_i, t_i))_{1\leq i\leq p}$ where $0 \leq s_i \leq t_i < \zeta_h$, for all $1 \leq i \leq p$ and let $\varepsilon \in [0, \infty)$. Then, the *compact measured metric space coded by* h *and the pinching setup* (Π, ε) is the $(\mathbf{\Pi}, \varepsilon)$-pinched metric space associated with (T_h, d_h) and the pinching points $\mathbf{\Pi} = ((p_h(s_i), p_h(t_i)))_{1\leq i\leq p}$, where $p_h : [0, \zeta_h) \to T_h$ stands for the canonical projection. We shall use the following notation:

$$G(h, \Pi, \varepsilon) = \big(G_{h,\Pi,\varepsilon}, d_{h,\Pi,\varepsilon}, \varrho_{h,\Pi,\varepsilon}, m_{h,\Pi,\varepsilon}\big). \tag{53}$$

We shall denote by $p_{h,\Pi,\varepsilon}$ the composition of the canonical projections $\varpi_{\mathbf{\Pi},\varepsilon} \circ p_h : [0, \zeta_h) \to G_{h,\Pi,\varepsilon}$; then $\varrho_{h,\Pi,\varepsilon} = p_{h,\Pi,\varepsilon}(0)$ and $m_{h,\Pi,\varepsilon}$ stands for the pushforward measure of the Lebesgue on $[0, \zeta_h)$ via $p_{h,\Pi,\varepsilon}$.

Coding $\mathtt{w}$**-multiplicative graphs.** Recall from Paragraph 2.1.1 that $(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}})_{1\leq k\leq \mathbf{q}_{\mathtt{w}}}$, are the connected component of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$. Here, $\mathbf{q}_{\mathtt{w}}$ is the total number of connected components of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$; $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$ is equipped with its graph-metric $d_{\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}}$ and with the restriction $\mathbf{m}_k^{\mathtt{w}}$ of the measure $\mathbf{m}_{\mathtt{w}} = \sum_{1\leq j\leq n} w_j \delta_j$ on $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$; the indexation satisfies $\mathbf{m}_{\mathtt{w}}(\boldsymbol{\mathcal{G}}_1^{\mathtt{w}}) \geq \ldots \geq \mathbf{m}_{\mathtt{w}}(\boldsymbol{\mathcal{G}}_{\mathbf{q}_{\mathtt{w}}}^{\mathtt{w}})$ (with a slight abuse of notation). Let us briefly explain how the excursions of $\mathcal{H}^{\mathtt{w}}$ above 0 code the measured metric spaces $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$.

First, denote by $(l_k^{\mathtt{w}}, r_k^{\mathtt{w}})$, $1\leq k\leq \mathbf{q}_{\mathtt{w}}$, the excursion intervals of $\mathcal{H}^{\mathtt{w}}$ above 0, that are exactly the excursion intervals of $Y^{\mathtt{w}}$ above its infimum process $J_t^{\mathtt{w}} = \inf_{s\in[0,t]} Y_s^{\mathtt{w}}$. Namely,

$$\bigcup_{1\leq k\leq \mathbf{q}_{\mathtt{w}}} (l_k^{\mathtt{w}}, r_k^{\mathtt{w}}) = \big\{t\in[0,\infty) : \mathcal{H}_t^{\mathtt{w}} > 0\big\} = \big\{t\in[0,\infty) : Y_t^{\mathtt{w}} > J_t^{\mathtt{w}}\big\} \tag{54}$$

Here, we set $\zeta_k^{\mathtt{w}} = r_k^{\mathtt{w}} - l_k^{\mathtt{w}} = \mathbf{m}_k^{\mathtt{w}}(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}})$ and thus $\zeta_1^{\mathtt{w}} \geq \ldots \geq \zeta_{\mathbf{q}_{\mathtt{w}}}^{\mathtt{w}}$; moreover, if $\zeta_k^{\mathtt{w}} = \zeta_{k+1}^{\mathtt{w}}$, then we agree on the convention that $l_k^{\mathtt{w}} < l_{k+1}^{\mathtt{w}}$; *excursions processes* are then defined as follows:

$$\forall k\in\{1,\ldots,\mathbf{q}_{\mathtt{w}}\},\ \forall t\in[0,\infty), \qquad \mathtt{H}_k^{\mathtt{w}}(t) = \mathcal{H}^{\mathtt{w}}_{(l_k^{\mathtt{w}}+t)\wedge r_k^{\mathtt{w}}} \quad\text{and}\quad \mathtt{Y}_k^{\mathtt{w}}(t) = Y^{\mathtt{w}}_{(l_k^{\mathtt{w}}+t)\wedge r_k^{\mathtt{w}}} - J^{\mathtt{w}}_{l_k^{\mathtt{w}}}. \tag{55}$$

We next define the sequences of *pinching points of the excursions*: to that end, recall from (15) and (16) the definition of $\boldsymbol{\Pi}_{\mathtt{w}} = \big((s_p, t_p)\big)_{1\leq p\leq \mathbf{p}_{\mathtt{w}}}$ the sequence of pinching points of $\boldsymbol{\mathcal{G}}_{\mathtt{w}}$; observe that if $t_p \in [l_k^{\mathtt{w}}, r_k^{\mathtt{w}}]$, then $s_p \in [l_k^{\mathtt{w}}, r_k^{\mathtt{w}}]$; then, it allows to define the following for all $k\in\{1,\ldots,\mathbf{q}_{\mathtt{w}}\}$:

$$\boldsymbol{\Pi}_k^{\mathtt{w}} = \big((s_p^k, t_p^k)\big)_{1\leq p\leq \mathbf{p}_k^{\mathtt{w}}} \text{ where } (t_p^k)_{1\leq p\leq \mathbf{p}_k^{\mathtt{w}}} \text{ increases and where the } (l_k^{\mathtt{w}} + s_p^k, l_k^{\mathtt{w}} + t_p^k) \text{ are exactly the terms } (s_{p'}, t_{p'}) \text{ of } \boldsymbol{\Pi}_{\mathtt{w}} \text{ such that } t_{p'} \in [l_k^{\mathtt{w}}, r_k^{\mathtt{w}}]. \tag{56}$$

Then, for all $k \in \{1,\ldots,\mathbf{q}_{\mathtt{w}}\}$, we easily see that $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$ is coded by $(\mathtt{H}_k^{\mathtt{w}}, \boldsymbol{\Pi}_k^{\mathtt{w}}, 1)$ as defined in (53). Namely,

$$G(\mathtt{H}_k^{\mathtt{w}}, \boldsymbol{\Pi}_k^{\mathtt{w}}, 1) \text{ is isometric to } \boldsymbol{\mathcal{G}}_k^{\mathtt{w}} . \tag{57}$$

Here, *isometric* means that there is a bijective isometry from $G(\mathtt{H}_k^{\mathtt{w}}, \boldsymbol{\Pi}_k^{\mathtt{w}}, 1)$ onto $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}}$ sending $\mathbf{m}_k^{\mathtt{w}}$ to $\mathbf{m}_{\mathtt{H}_k^{\mathtt{w}}, \boldsymbol{\Pi}_k^{\mathtt{w}}, 1}$.

2.2.3 The continuous multiplicative random graph. Fractal properties.

We fix $\alpha, \beta \in [0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^{\downarrow}$ such that (44) holds true. By analogy with the discrete coding, we now define the $(\alpha,\beta,\kappa,\mathbf{c})$-*continuous multiplicative random graph*, the continuous version of $\mathtt{w}$-multiplicative graph. In this construction, the processes $(Y, \mathcal{H}, \boldsymbol{\Pi})$ defined in Section 2.2.1 plays the role of $(Y^{\mathtt{w}}, \mathcal{H}^{\mathtt{w}}, \boldsymbol{\Pi}_{\mathtt{w}})$.

First, recall from (39) the definition of Y; recall from Proposition 2.7 the definition of $\mathcal{H}$, the height process associated with Y and recall the notation, $J_t = \inf_{s\in[0,t]} Y_s$, $t\in[0,\infty)$. Lemma 5.6 (see further in Section 5.2.4) asserts that the excursion intervals of $\mathcal{H}$ above 0 and the excursion intervals of $Y - J$ above 0 are the same; moreover Proposition 14 in Aldous & Limic [4] (recalled further in Proposition 5.8, Section 5.2.4), asserts that these excursions can be indexed in the decreasing order of their lengths. Namely,

$$\big\{t\in[0,\infty) : \mathcal{H}_t > 0\big\} = \big\{t\in[0,\infty) : Y_t > J_t\big\} = \bigcup_{k\geq 1} (l_k, r_k) \tag{58}$$

where the sequence $\zeta_k = l_k - r_k$, $k\geq 1$ decreases. This proposition also asserts that $\{t\in[0,\infty) : \mathcal{H}_t = 0\}$ has no isolated point, that $\mathbf{P}(\mathcal{H}_t = 0) = 0$ for all $t\in[0,\infty)$ and that the continuous function $t\mapsto -J_t$ can be viewed as a sort of local-time for the set of zeros of $\mathcal{H}$. We refer to Section 5.2.4 for more details. These properties allow to define the *excursion processes* as follows.

$$\forall k\geq 1,\ \forall t\in[0,\infty), \qquad \mathtt{H}_k(t) = \mathcal{H}_{(l_k+t)\wedge r_k} \quad\text{and}\quad \mathtt{Y}_k(t) = Y_{(l_k+t)\wedge r_k} - J_{l_k}. \tag{59}$$

The *pinching times* are defined as follows: recall from (46) and (47) the definition of $\mathbf{\Pi}=\big((s_p,t_p)\big)_{p\geq 1}$. If $t_p\in[l_k,r_k]$, then note that $s_p\in[l_k,r_k]$, by definition of s_p. For all $k\geq 1$, we then define:

(60) $$\mathbf{\Pi}_k=\big((s^k_p,t^k_p)\big)_{1\leq p\leq \mathbf{p}_k} \text{ where } (t^k_p)_{1\leq p\leq \mathbf{p}_k} \text{ increases and where}$$
$$\text{the } (l_k+s^k_p, l_k+t^k_p) \text{ are exactly the terms } (s_{p'},t_{p'}) \text{ of } \mathbf{\Pi} \text{ such that } t_{p'}\in[l_k,r_k].$$

The *connected components of the* $(\alpha,\beta,\kappa,\mathbf{c})$*-continuous multiplicative random graph* are then defined as the sequence of random compact measured metric spaces coded by the excursions $\mathtt{H}_k$ and the pinching setups $(\mathbf{\Pi}_k,0)$. Namely, we shall use the following notation: for all $k\geq 1$,

(61) $$\mathbf{G}_k=\big(\mathbf{G}_k,\mathrm{d}_k,\varrho_k,\mathbf{m}_k\big) \text{ stands for } G(\mathtt{H}_k,\mathbf{\Pi}_k,0) \text{ as defined by (53).}$$

The main purpose of this paper is to get a weak convergence of the sequences of rescaled discrete graphs $(\boldsymbol{\mathcal{G}}^{\mathtt{w}_n}_k)$ to the sequence $(\mathbf{G}_k)$. Before stating such results, let us briefly discuss a geometric property of the random metric spaces $\mathbf{G}_k$. As part of our construction, each component $\mathbf{G}_k$ of the graph is *embedded in a Lévy tree* whose branching mechanism ψ is derived from $(\alpha,\beta,\kappa,\mathbf{c})$ by (37); roughly speaking the measure $\mathbf{m}_k$ is the restriction of the *mass measure* of the Lévy tree; this measure enjoys specific fractal properties and as a consequence of Theorem 5.5 in Le Gall & D. [22], we get the following result.

Proposition 2.8 *Let $\alpha,\beta\in[0,\infty)$, let $\kappa\in(0,\infty)$ and let $\mathbf{c}=(c_j)_{j\geq 1}\in\ell^{\downarrow}_3$ be such that (44) holds true. Let $(\mathbf{G}_k)_{k\geq 1}$ be the connected components of the continuous $(\alpha,\beta,\kappa,\mathbf{c})$-multiplicative random graph as defined in (61). We denote by $\dim_{\mathrm{H}}$ the Hausdorff dimension and by $\dim_{\mathrm{p}}$ the packing dimension. Then, the following assertions hold true a.s. for all $k\geq 1$,*

(i) *If $\beta\neq 0$, then $\dim_{\mathrm{H}}(\mathbf{G}_k)=\dim_{\mathrm{p}}(\mathbf{G}_k)=2$.*

(ii) *Let us assume that $\beta=0$. By (44), we necessarily get $\sigma_2(\mathbf{c})=\infty$. We first introduce the following function:*

$$\forall x\in(0,1),\quad J(x)=\frac{1}{x}\sum_{j:c_j\leq x}\kappa c_j^3+\sum_{j:c_j>x}\kappa c_j^2=\sum_{j\geq 1}\kappa c_j^2\big(1\wedge(c_j/x)\big)$$

that tends to ∞ as $x\downarrow 0$. We next define the following exponents:

(62) $$\gamma=1+\sup\big\{r\in[0,\infty):\ \lim_{x\to 0+}x^rJ(x)=\infty\big\}$$
$$\textit{and}\quad \eta=1+\inf\big\{r\in[1,\infty):\ \lim_{x\to 0+}x^rJ(x)=0\big\}.$$

In particular, if $\mathbf{c}$ varies regularly with index $\rho^{-1}\in(1/3,1/2)$, then $\gamma=\eta=\rho-1$. Then, if $\gamma>1$, we get

$$\dim_{\mathrm{H}}(\mathbf{G}_k)=\frac{\gamma}{\gamma-1}\quad\textit{and}\quad\dim_{\mathrm{p}}(\mathbf{G}_k)=\frac{\eta}{\eta-1}\ .$$

Proof. See Section 5.2.5. ■

2.3 Limit theorems.

Let $\mathtt{w}_n=(w^{(n)}_j)_{j\geq 1}\in\ell^{\downarrow}_f$, $n\in\mathbb{N}$, be a sequence of weights. We want to prove that *rescaled* versions of the connected components $(\boldsymbol{\mathcal{G}}^{\mathtt{w}_n}_k)_{1\leq k\leq q_{\mathtt{w}_n}}$, viewed as random *pointed compact measured metric spaces* as defined in (57) weakly converge to the sequence of connected components $(\mathbf{G}_k)_{k\geq 1}$ of the continuous multiplicative random graphs as defined in (61). To that end, we first recall in Section 2.3.1 the definition of the Gromov–Hausdorff–Prohorov metric on the space of compact measured metric spaces. Since the core of our approach consists in embedding multiplicative graphs into Galton-Watson trees, in Section 2.3.2, we specify the possible asymptotic regimes for such trees and the Markovian processes that generate those trees. Convergence results for multiplicative graphs are stated in Section 2.3.3. A careful discussion about the connections to previous works is given in Section 2.3.4.

2.3.1 Convergence of metric spaces.

Let (G_1, d_1, ρ_1, m_1) and (G_2, d_2, ρ_2, m_2) be two pointed compact measured metric spaces. The pointed *Gromov-Hausdorff-Prohorov distance* of G_1 and G_2 is then defined by

$$\boldsymbol{\delta}_{\mathrm{GHP}}(G_1,G_2)=\inf\Big\{d_E^{\mathrm{Haus}}(\phi_1(G_1),\phi_2(G_2)) + d_E(\phi_1(\rho_1),\phi_2(\rho_2)) + d_E^{\mathrm{Proh}}\big(m_1\circ\phi_1^{-1}, m_2\circ\phi_2^{-1}\big)\Big\}. \tag{63}$$

Here, the infimum is taken over all Polish spaces (E, d_E) and all isometric embeddings $\phi_i : G_i \hookrightarrow E$, $i\in\{1,2\}$; d_E^{Haus} stands for the Hausdorff distance on the space of compact subsets of E, d_E^{Proh} stands for the Prohorov distance on the space of finite Borel measures on E and for all $i\in\{1,2\}$, $m_i\circ\phi_i^{-1}$ stands for the pushforward measure of m_i via ϕ_i.

We recall from Theorem 2.5 in Abraham, Delmas & Hoscheit [1] the following assertions: $\boldsymbol{\delta}_{\mathrm{GHP}}$ is symmetric and it satisfies the triangle inequality; $\boldsymbol{\delta}_{\mathrm{GHP}}(G_1,G_2)=0$ iff G_1 and G_2 are *isometric*, namely iff there exists a bijective isometry $\phi: G_1\to G_2$ such that $\phi(\rho_1)=\rho_2$ and such that $m_2=m_1\circ\phi^{-1}$. Denote by $\mathbb{G}$ the isometry classes of pointed compact measured metric spaces. Then, we recall the following result.

Theorem 2.9 (Theorem 2.5 [1]) *$(\mathbb{G}, \boldsymbol{\delta}_{\mathrm{GHP}})$ is a complete and separable metric space.*

Actually in our paper, weak-limits are proved for coding functions, which entail $\boldsymbol{\delta}_{\mathrm{GHP}}$-limits as asserted by the following lemma.

Lemma 2.10 *Let $h, h' : [0,\infty)\to[0,\infty)$ be two càdlàg functions such that ζ_h and $\zeta_{h'}$ are finite and that satisfy (a) or (b) in Remark 2.3. Let $\Pi=((s_i,t_i))_{1\le i\le p}$ and $\Pi'=((s_i',t_i'))_{1\le i\le p}$ be two sequences such that $0\le s_i\le t_i<\zeta_h$ and $0\le s_i'\le t_i'<\zeta_{h'}$. Let $\varepsilon,\varepsilon'\in[0,\infty)$. Let $\delta\in(0,\infty)$ be such that*

$$\forall i\in\{1,\dots,p\},\quad |s_i-s_i'|\le\delta \quad \textit{and} \quad |t_i-t_i'|\le\delta\ . \tag{64}$$

Recall from (53) the definition of the pointed compact measured metric spaces $G:=G(h,\Pi,\varepsilon)$ and $G':=G(h',\Pi',\varepsilon')$. Then, we get:

$$\boldsymbol{\delta}_{\mathrm{GHP}}(G,G')\le 6(p+1)\big(\|h-h'\|_\infty+\omega_\delta(h)\big)+3p(\varepsilon\vee\varepsilon')+|\zeta_h-\zeta_{h'}|\ , \tag{65}$$

where $\omega_\delta(h)=\max\big\{|h(s)-h(t)|\,;\ s,t\in[0,\infty): |s-t|\le\delta\big\}$ and where $\|\cdot\|_\infty$ stands for the uniform norm on $[0,\infty)$.

Proof. See Appendix Section C. The proof is partly adapted from Theorem 2.1 in Le Gall & D. [22], Proposition 2.4 Abraham, Delmas & Hoscheit [1] and Lemma 21 in Addario-Berry, Goldschmidt & B. in [2]. ∎

2.3.2 Possible asymptotic regimes.

A priori estimates. Let $\mathtt{w}_n=(w_j^{(n)})_{j\ge1}\in\ell_f^\downarrow$, $n\in\mathbb{N}$, be a sequence of weights. We discuss assumptions on $\mathtt{w}_n$ in order to get proper weak-limits of rescaled $\mathtt{w}_n$-multiplicative graphs $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ viewed as random measured metric spaces. Let us first mention that the number of vertices of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ that is $\mathbf{j}_n:=\sup\{j\ge1 : w_j^{(n)}>0\}$, *is not necessarily equal to* n; of course, we want it to tend to ∞ as $n\to\infty$.

We introduce two kinds of scaling factors: *weight factors* a_n that are related to the asymptotics of the large weights, and *exploration-time factors* b_n that take into account the speed of the exploration of the graph. We recall the following convention: if (u_n) and (v_n) are two sequences of non-negative real numbers, then $u_n\asymp v_n$ means that there exists $k\in(1,\infty)$ such that $u_n/k\le v_n\le ku_n$ for all sufficiently large n.

– (A_1) *Large weights are proportional.* Large weights that persist in the limiting graph have to be of the same order of magnitude. Namely, if $w_j^{(n)}$ does not vanish, then $w_j^{(n)} \asymp w_1^{(n)} \asymp a_n$, and more generally $w_1^{(n)} = \mathcal{O}(a_n)$.

– (A_2) *Vertices with large weights tend to be well-separated.* In the limit, if two large weights persist, they cannot fuse and they tend not to be connected by an edge. Namely, if the two largest weights persist, then

$$1 - \exp(-w_1^{(n)} w_2^{(n)}/\sigma_1(\mathtt{w}_n) \longrightarrow 0$$

and since $w_1^{(n)} \asymp w_2^{(n)} \asymp a_n$, it entails $\lim_{n\to\infty} a_n^2/\sigma_1(\mathtt{w}_n) = 0$.

– (A_3) *Exploration time-scale.* We denote by C_n (resp. D_n) the number (resp. the sum of the weights) of the vertices explored before visiting the vertex with the largest weight $w_1^{(n)}$. In terms of the $\mathtt{w}_n$-LIFO-queue coding $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$, C_n is the number of clients who entered the queue before the arrival of Client 1 and D_n is the sum of the times of service of the clients who entered the queue before the arrival of Client 1. We easily check that

$$C_n \overset{\text{(law)}}{=} \sum_{2\leq j\leq \mathbf{j}_n} \mathbf{1}_{\{\mathtt{e}_j < \mathtt{e}_1\}} \quad \text{and} \quad D_n \overset{\text{(law)}}{=} \sum_{2\leq j\leq \mathbf{j}_n} w_j^{(n)} \mathbf{1}_{\{\mathtt{e}_j < \mathtt{e}_1\}},$$

where $(w_j^{(n)} \mathtt{e}_j)_{2\leq j\leq \mathbf{j}_n}$ are i.i.d. exponential r.v. with unit mean. Consequently,

$$\mathbf{E}[C_n] = \sum_{j\geq 2} \frac{w_j^{(n)}}{w_j^{(n)} + w_1^{(n)}} \quad \text{and} \quad \mathbf{E}[D_n] = \sum_{j\geq 2} \frac{(w_j^{(n)})^2}{w_j^{(n)} + w_1^{(n)}}.$$

By (A_1) and (A_2), we get $\sigma_1(\mathtt{w}_n) \asymp a_n \mathbf{E}[C_n]$ and that $\sigma_2(\mathtt{w}_n) \asymp a_n \mathbf{E}[D_n]$. In the asymptotic regime that we consider, we require the two following properties.

– (A_3a) The number of visited vertices has to be of the same order of magnitude as the sum of the corresponding weights: namely $\mathbf{E}[C_n] \asymp \mathbf{E}[D_n]$.

– (A_3b) In the time-scale b_n, the first time of visit of the vertex with the largest weight converges to a non-trivial limit: namely, $b_n \asymp \mathbf{E}[C_n]$.

Assumptions (A_3a) and (A_3b) imply that there exists $K \in (1,\infty)$ such that $K^{-1} \leq \sigma_2(\mathtt{w}_n)/\sigma_1(\mathtt{w}_n) \leq K$ and $K^{-1} \leq a_n b_n/\sigma_1(\mathtt{w}_n) \leq K$. Note that $a_n^2/\sigma_1(\mathtt{w}_n) \to 0$ implies: $a_n/b_n \to 0$. We shall also require the additional technical assumption $b_n = O(a_n^2)$.

To summarise, the previous arguments justify why we restrict to sequences $a_n, b_n \in (0,\infty)$, $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$ satisfying the following *a priori assumptions*:

$$\text{(66)} \qquad a_n \text{ and } \frac{b_n}{a_n} \xrightarrow[n\to\infty]{} \infty, \qquad \frac{b_n}{a_n^2} \xrightarrow[n\to\infty]{} \beta_0 \in [0,\infty),$$
$$\sup_{n\in\mathbb{N}} \frac{w_1^{(n)}}{a_n} < \infty \qquad \text{and} \qquad \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \xrightarrow[n\to\infty]{} \kappa \in (0,\infty).$$

Note that (66) allows β_0 to be null and that we have relaxed the assumption $\inf_{n\in\mathbb{N}} w_1^{(n)}/a_n > 0$.

Remark 2.4 We call (A_3b) the *Condensation Assumption*. If one relaxes this condition, we get interesting asymptotic regimes that have not been considered previously and that we shall study in future works. □

Convergence results for the Markovian queue. As already mentioned the convergence of the graphs $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ is obtained thanks to the convergence of rescaled versions of $Y^{\mathtt{w}_n}$ and $\mathcal{H}^{\mathtt{w}_n}$ and the convergence of these two processes is also obtained by the convergence of the Markovian processes into which they are embedded: namely, the asymptotic regimes of $(Y^{\mathtt{w}_n}, \mathcal{H}^{\mathtt{w}_n})$ and of $(X^{\mathtt{w}_n}, H^{\mathtt{w}_n})$

should be the same. The purpose of this section is to state weak limit-theorems for $X^{\mathtt{w}_n}$ and $H^{\mathtt{w}_n}$. Let us mention that a part of the results of this section rely on standard limit-theorems on random walks, on results due to Grimvall in [27] on branching processes and on results due to Le Gall & D. in [21] on the height processes of Galton–Watson trees. However, the specific form of the jumps and of the offspring distribution of the trees actually requires a careful analysis done in the Proof-Section 6.3.

Recall from (21) the definition of the compensated Poisson process $X^{\mathtt{w}_n}$; recall that the Markovian queueing system induced by $X^{\mathtt{w}_n}$ yields a tree $\mathbf{T}_{\mathtt{w}_n}$ that is an i.i.d. sequence of Galton-Watson trees with offspring distribution $\mu_{\mathtt{w}_n}$ whose definition is given by (33). Denote by $(Z^{\mathtt{w}_n}_k)_{k\in\mathbb{N}}$ a Galton-Watson Markov chain with offspring distribution $\mu_{\mathtt{w}_n}$ and with initial state $Z^{\mathtt{w}_n}_0 = \lfloor a_n \rfloor$. The following proposition is mainly based on Theorem 3.4 in Grimvall [27] (p.1040) that proves weak convergence for Galton-Watson processes to *Continuous States Branching Processes* (CSBP for short). Recall that a (conservative) CSBP is a $[0,\infty)$-valued Markov process obtained from spectrally positive Lévy processes via Lamperti's time-change; the law of the CSBP is completely characterised by the Lévy process and thus by its Laplace exponent that is usually called the *branching mechanism* of the CSBP: we refer to Bingham [15] for more details on CSBP (see also Appendix Section B.2.2 for a very brief account). We denote by $\mathbf{D}([0,\infty),\mathbb{R})$ the space of càdlàg functions from $[0,\infty)$ to $\mathbb{R}$ equipped with Skorokod's topology and we denote by $\mathbf{C}([0,\infty),\mathbb{R})$ the space of continuous functions from $[0,\infty)$ to $\mathbb{R}$, equipped with the topology of uniform convergence on all compact subsets.

Proposition 2.11 *Let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell^{\downarrow}_f$, $n\in\mathbb{N}$, satisfy (66). Recall from above the definition of $X^{\mathtt{w}_n}$ and $Z^{\mathtt{w}_n}$. Let $(X_t)_{t\in[0,\infty)}$ and $(Z_t)_{t\in[0,\infty)}$ be two càdlàg processes such that $X_0 = 0$ and $Z_0 = 1$. Then, the following holds true.*

(i) *The following convergences are equivalent.*

- $(i$-$a)$ *There exists $t\in(0,\infty)$ such that $\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n t} \to X_t$ in law on $\mathbb{R}$.*
- $(i$-$b)$ *$(\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)} \longrightarrow (X_t)_{t\in[0,\infty)}$ in law on $\mathbf{D}([0,\infty),\mathbb{R})$.*
- $(i$-$c)$ *$(\frac{1}{a_n} Z^{\mathtt{w}_n}_{\lfloor b_n t/a_n \rfloor})_{t\in[0,\infty)} \longrightarrow (Z_t)_{t\in[0,\infty)}$ in law on $\mathbf{D}([0,\infty),\mathbb{R})$.*

If any of the three convergences in (i) holds true, then X is a spectrally Lévy process and Z a conservative CSBP; moreover there exist $\alpha\in\mathbb{R}$, $\beta\in[\beta_0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell^{\downarrow}_3$ such that the branching mechanism of Z and the Laplace exponent of X are equal to the same function ψ given by:

$$\forall\lambda\in[0,\infty), \quad \psi(\lambda) = \alpha\lambda + \tfrac{1}{2}\beta\lambda^2 + \sum_{j\geq 1} \kappa c_j \big(e^{-\lambda c_j} - 1 + \lambda c_j\big). \tag{67}$$

(ii) *Any of the three convergences in (i) is equivalent to the following conditions:*

$$(\mathbf{C1}): \quad \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big) \xrightarrow[n\to\infty]{} \alpha \qquad (\mathbf{C2}): \quad \frac{b_n}{a_n^2}\cdot\frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)} \xrightarrow[n\to\infty]{} \beta + \kappa\sigma_3(\mathbf{c}), \tag{68}$$

$$(\mathbf{C3}): \quad \forall j\in\mathbb{N}^*, \quad \frac{w^{(n)}_j}{a_n} \xrightarrow[n\to\infty]{} c_j\,. \tag{69}$$

(iii) *Any of the three convergences of (i) is equivalent to $(\mathbf{C1})$ and the following limit for all $\lambda\in(0,\infty)$:*

$$\frac{a_n b_n}{\sigma_1(w_n)} \sum_{j\geq 1} \frac{w^{(n)}_j}{a_n}\Big(e^{-\lambda w^{(n)}_j/a_n} - 1 + \lambda\, w^{(n)}_j/a_n\Big) \xrightarrow[n\to\infty]{} \psi(\lambda) - \alpha\lambda\,, \tag{70}$$

(*iv*) *For all* $\alpha \in \mathbb{R}$, $\beta \in [0,\infty)$, $\kappa \in (0,\infty)$ *and* $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$, *there are sequences* $a_n, b_n \in (0,\infty)$, $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, *satisfying (66) with* $\beta_0 \in [0,\beta]$, $(\mathbf{C1})$, $(\mathbf{C2})$ *and* $(\mathbf{C3})$.

Proof. See Section 6.3 (and more specifically Section 6.3.2). As already mentioned, Proposition 2.11 (i) strongly relies on Theorem 3.4 in Grimvall [27] (p.1040). However, (ii), (iii) and (iv) require specific arguments. ■

Recall from (32) the definition of $H^{\mathtt{w}_n}$, the height process associated with $X^{\mathtt{w}_n}$.

Proposition 2.12 *Let* $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ *and* $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$ *and let* ψ *be given by (67). We assume that* ψ *satisfies (44): namely,* $\int^{\infty} d\lambda/\psi(\lambda) < \infty$. *Let* X *be a spectrally positive Lévy process with Laplace exponent* ψ. *Let* H *be its height process as defined in (45). Let* $a_n, b_n \in (0,\infty)$, $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, *satisfy (66) with* $\beta_0 \in [0,\beta]$, $(\mathbf{C1})$, $(\mathbf{C2})$ *and* $(\mathbf{C3})$. *Suppose that* $\sigma_2(\mathtt{w}_n) \leq \sigma_1(\mathtt{w}_n)$ *for all* n. *We also assume the following:*

$$(\mathbf{C4}): \quad \exists\, \delta \in (0,\infty), \qquad \liminf_{n\to\infty} \mathbf{P}\big(Z^{\mathtt{w}_n}_{\lfloor b_n\delta/a_n\rfloor} = 0\big) > 0\,. \tag{71}$$

Then, the joint convergence holds true

$$\Big((\tfrac{1}{a_n} X^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)}, (\tfrac{a_n}{b_n} H^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)} \Big) \xrightarrow[n\to\infty]{} (X, H) \tag{72}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}) \times \mathbf{C}([0,\infty),\mathbb{R})$ *equipped with the product topology. We also get:*

$$\forall t \in [0,\infty), \quad \lim_{n\to\infty} \mathbf{P}\big(Z^{\mathtt{w}_n}_{\lfloor b_n t/a_n\rfloor} = 0\big) = e^{-v_\psi(t)} \quad \textit{where} \quad \int_{v_\psi(t)}^{\infty} \frac{d\lambda}{\psi(\lambda)} = t. \tag{73}$$

Proof. See Section 6.3 (and more specifically Section 6.3.2). Proposition 2.12 strongly relies on Theorem 2.3.1 in Le Gall & D. [21]. However, its proof requires more care than expected at first glance because the asymptotic regime is quite restrictive and because $H^{\mathtt{w}_n}$ is not exactly the *height process* as defined in [21] (it is actually a time-changed version of the so-called *contour* process as in Theorem 2.4.1 [21] p. 68). ■

The following proposition provides a pratical criterion to check $(\mathbf{C4})$: in particular, it shows that $(\mathbf{C4})$ *is always true when* $\beta_0 > 0$; it also shows that Proposition 2.12 is never void.

Proposition 2.13 *Let* $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ *and* $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$. *Let* ψ *be given by (67) and assume that* ψ *satisfies (44): namely,* $\int^{\infty} d\lambda/\psi(\lambda) < \infty$. *Then, the following holds true.*

(*i*) *Let* $a_n, b_n \in (0,\infty)$, $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, *satisfy (66),* $(\mathbf{C1})$, $(\mathbf{C2})$ *and* $(\mathbf{C3})$. *Denote by* ψ_n *the Laplace exponent of* $(\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)}$: *namely, for all* $\lambda \in [0,\infty)$,

$$\psi_n(\lambda) = \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big)\lambda + \frac{a_n b_n}{\sigma_1(w_n)} \sum_{j\geq 1} \frac{w_j^{(n)}}{a_n}\Big(e^{-\lambda w_j^{(n)}/a_n} - 1 + \lambda\, w_j^{(n)}/a_n\Big). \tag{74}$$

Then, $(\mathbf{C4})$ *holds true if*

$$\lim_{y\to\infty} \limsup_{n\to\infty} \int_y^{a_n} \frac{d\lambda}{\psi_n(\lambda)} = 0\,. \tag{75}$$

In particular, if $\beta_0 > 0$ *in (66), then (75) is always satisfied and* $(\mathbf{C4})$ *holds true.*

(*ii*) *There are sequences* $a_n, b_n \in (0,\infty)$, $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, *that satisfy (66) with* $\beta_0 = 0$, $(\mathbf{C1})$, $(\mathbf{C2})$ *and* $(\mathbf{C3})$ *but not* $(\mathbf{C4})$.

(*iii*) *There exist* $a_n, b_n \in (0,\infty)$, *and* $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, *that satisfy (66) with any* $\beta_0 \in [0,\beta]$, $(\mathbf{C1}), (\mathbf{C2}), (\mathbf{C3})$ *and* $(\mathbf{C4})$.

Proof. See Sections 6.3.3, 6.3.4 and 6.3.5. ■

2.3.3 Limit theorems for multiplicative random graphs.

A convention. To deal with limits of sequences of pinching times, it is convenient to embed $([0,\infty)^2)^p$ into $(\mathbb{R}^2)^{\mathbb{N}^*}$ by extending any sequence $((s_i,t_i))_{1\le i\le p}\in([0,\infty)^2)^p$ by setting $(s_i,t_i)=(-1,-1)$, for all $i>p$. Here, $(-1,-1)$ plays the role of an unspecific cemetery point. We equip $(\mathbb{R}^2)^{\mathbb{N}^*}$ with the product topology. Then, the main theorem of paper is the following.

Theorem 2.14 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}=(c_j)_{j\ge1}\in\ell_3^{\downarrow}$. Let ψ be given by (67) and assume that ψ satisfies (44): namely, $\int^\infty d\lambda/\psi(\lambda)<\infty$. Recall from (39), the definition of Y. Recall from Proposition 2.7 the definition of $\mathcal{H}$. Recall from (47) the definition of $\mathbf{\Pi}$. Recall from (41) the definition of θ^{b}. Recall from (43) the definition of X. Recall from (45) the definition of H.*

Let $a_n,b_n\in(0,\infty)$, and $\mathtt{w}_n\in\ell_f^{\downarrow}$, $n\in\mathbb{N}$, satisfy (66), $(\mathbf{C1})$, $(\mathbf{C2})$, $(\mathbf{C3})$ and $(\mathbf{C4})$, as specified in (68), (69) and (71). Recall from (13) the definition of $Y^{\mathtt{w}_n}$. Recall from (18) the definition of $\mathcal{H}^{\mathtt{w}_n}$. Recall from (16) the definition of $\mathbf{\Pi}_{\mathtt{w}_n}$. Recall from Lemma 2.2, from (24) and from (30) in Lemma 2.3, the definition of the joint law of $(\theta^{\mathrm{b},\mathtt{w}_n},X^{\mathtt{w}_n})$. Recall from (32) the definition of $H^{\mathtt{w}_n}$.

Then, the joint convergence

$$(76)\qquad \Big(\tfrac{1}{a_n}X^{\mathtt{w}_n}_{b_n\cdot},\ \tfrac{a_n}{b_n}H^{\mathtt{w}_n}_{b_n\cdot},\ \big(\tfrac{1}{b_n}\theta^{\mathrm{b},\mathtt{w}_n}_{b_n\cdot},\ \tfrac{1}{a_n}Y^{\mathtt{w}_n}_{b_n\cdot}\big),\ \tfrac{a_n}{b_n}\mathcal{H}^{\mathtt{w}_n}_{b_n\cdot},\ \tfrac{1}{b_n}\mathbf{\Pi}_{\mathtt{w}_n}\Big)\ \xrightarrow[n\to\infty]{}\ \big(X,H,(\theta^{\mathrm{b}},Y),\mathcal{H},\mathbf{\Pi}\big)$$

holds weakly on $\mathbf{D}([0,\infty),\mathbb{R})\times\mathbf{C}([0,\infty),\mathbb{R})\times\mathbf{D}([0,\infty),\mathbb{R}^2)\times\mathbf{C}([0,\infty),\mathbb{R})\times(\mathbb{R}^2)^{\mathbb{N}^*}$ equipped with the product topology.

Proof. See Section 6.1. ■

Theorem 2.14 implies that rescaled versions of $(\mathcal{H}^{\mathtt{w}_n},\mathbf{\Pi}_{\mathtt{w}_n})$ converge to $(\mathcal{H},\mathbf{\Pi})$. This entails the convergence of the coding processes of the connected components of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$. More precisely, recall from (55) that $(\mathtt{H}^{\mathtt{w}_n}_k(\cdot))_{1\le k\le q_{\mathtt{w}_n}}$ are the excursions of $\mathcal{H}^{\mathtt{w}_n}$ above 0 and recall from (59) that $(\mathtt{H}_k(\cdot))_{k\ge1}$ are the excursions of $\mathcal{H}$ above 0. We also recall that

$$\zeta^{\mathtt{w}_n}_k=\sup\{t\in[0,\infty):\mathtt{H}^{\mathtt{w}_n}_k(t)>0\}\quad\text{and}\quad\zeta_k=\sup\{t\in[0,\infty):\mathtt{H}_k(t)>0\}$$

stand for the respective duration of the excusions $\mathtt{H}^{\mathtt{w}_n}_k$ and $\mathcal{H}$. The indexation is such that the sequences $(\zeta^{\mathtt{w}_n}_k)$ and (ζ_k) are non-increasing. Recall from (56) and from (60) the definition of the respective sequences $(\mathbf{\Pi}^{\mathtt{w}_n}_k)_{1\le k\le q_{\mathtt{w}_n}}$ and $(\mathbf{\Pi}_k)_{k\ge1}$, that are the pinching times. As already specified, we trivially extend each finite sequence $\mathbf{\Pi}^{\mathtt{w}_n}_k$ as a random element of $(\mathbb{R}^2)^{\mathbb{N}^*}$. We pass to the limit for rescaled versions of $((\mathtt{H}^{\mathtt{w}_n}_k,\zeta^{\mathtt{w}_n}_k,\mathbf{\Pi}^{\mathtt{w}_n}_k))_{1\le k\le q_{\mathtt{w}_n}}$. Since $q_{\mathtt{w}_n}$ tends to ∞, it is convenient to extend this sequence by taking for all $k>q_{\mathtt{w}_n}$, $\mathtt{H}^{\mathtt{w}_n}_k$ as the null function, $\zeta^{\mathtt{w}_n}_k=0$ and $\mathbf{\Pi}^{\mathtt{w}_n}_k$ as the sequence constant to $(-1,-1)$. Then the following theorem holds true.

Theorem 2.15 *Under the same assumptions as Theorem 2.14, the following convergence*

$$(77)\qquad \Big(\big((\tfrac{a_n}{b_n}\mathtt{H}^{\mathtt{w}_n}_k(b_nt))_{t\in[0,\infty)},\ \tfrac{1}{b_n}\zeta^{\mathtt{w}_n}_k,\ \tfrac{1}{b_n}\mathbf{\Pi}^{\mathtt{w}_n}_k\big)\Big)_{k\ge1}\ \xrightarrow[n\to\infty]{}\ \big((\mathtt{H}_k,\zeta_k,\mathbf{\Pi}_k)\big)_{k\ge1}$$

holds weakly on $((\mathbf{C}([0,\infty),\mathbb{R})\times[0,\infty)\times(\mathbb{R}^2)^{\mathbb{N}^*})^{\mathbb{N}^*}$ equipped with the product topology.

Proof. See Section 6.2.1. ■

As a consequence of Lemma 2.10, Theorem 2.15 entails the convergence of a rescaled version of the connected component of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$. More precisely, recall from (61) the notation $\big(\mathbf{G}_k,\mathrm{d}_k,\varrho_k,\mathbf{m}_k\big)$ for $G(\mathtt{H}_k,\mathbf{\Pi}_k,0)$, the graph coded by the function $\mathtt{H}_k$ and the pinching setup $(\mathbf{\Pi}_k,0)$: namely, $\mathbf{G}_k$ is the k-th largest connected component of the continuous $(\alpha,\beta,\kappa,\mathbf{c})$-multiplicative graph. Recall that

$$\big(\boldsymbol{\mathcal{G}}^{\mathtt{w}_n}_k,d^{\mathtt{w}_n}_k,\varrho^{\mathtt{w}_n}_k,\mathbf{m}^{\mathtt{w}_n}_k\big),\quad 1\le k\le q_{\mathtt{w}_n}$$

are the connected components of the $\mathtt{w}_n$-multiplicative graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$. Here, $d_k^{\mathtt{w}_n}$ stands for the graph-metric on $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$, $\mathbf{m}_k^{\mathtt{w}_n}$ is the restriction to $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ of the measure $\mathbf{m}_{\mathtt{w}_n} = \sum_{j\geq 1} w_j^{(n)} \delta_j$, $\varrho_k^{\mathtt{w}_n}$ is the first vertex of $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ that is visited during the exploration of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$, and the indexation satisfies:

$$\mathbf{m}_1^{\mathtt{w}_n}\big(\boldsymbol{\mathcal{G}}_1^{\mathtt{w}_n}\big) \geq \ldots \geq \mathbf{m}_{q_{\mathtt{w}_n}}^{\mathtt{w}_n}\big(\boldsymbol{\mathcal{G}}_{q_{\mathtt{w}_n}}^{\mathtt{w}_n}\big).$$

Next, recall from (57) that $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ is isometric to the graph coded by the function $\mathtt{H}_k^{\mathtt{w}_n}$ and the pinching setup $(\boldsymbol{\Pi}_k^{\mathtt{w}_n}, 1)$: thus, they define the same random element in the space $\mathbb{G}$ of the isometry classes of pointed compact measured metric spaces equipped with the Gromov-Hausdorff-Prohorov distance δ_{GHP} defined in (63). Since $q_{\mathtt{w}_n}$ tends to ∞, it is convenient to extend the sequence $(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n})_{1\leq k\leq q_{\mathtt{w}_n}}$ by taking $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ equal to the point space equipped with the null measure for all $k > q_{\mathtt{w}_n}$. Then, Theorem 2.15 and Lemma 2.10 entail the following theorem.

Theorem 2.16 *Under the same assumptions as Theorem 2.14, the following convergence*

$$\Big(\big(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}, \tfrac{a_n}{b_n} d_k^{\mathtt{w}_n}, \varrho_k^{\mathtt{w}_n}, \tfrac{1}{b_n}\mathbf{m}_k^{\mathtt{w}_n}\big)\Big)_{k\geq 1} \xrightarrow[n\to\infty]{} \Big(\big(\mathbf{G}_k, \mathrm{d}_k, \varrho_k, \mathbf{m}_k\big)\Big)_{k\geq 1} \tag{78}$$

holds weakly on $\mathbb{G}^{\mathbb{N}^*}$ equipped with the product topology.

Proof. See Section 6.2.2. ■

2.3.4 Connections with previous results.

Entrance boundary of the multiplicative coalescent. The model of $\mathtt{w}$-multiplicative random graphs appears in the work of Aldous [3] as an extension of Erdős–Rényi random graphs that have close connections with multiplicative coalescent processes. Relying upon this connection, Aldous and Limic determine in [4] the extremal eternal versions of the multiplicative coalescent in terms of the excursion lengths of Lévy-type processes close to Y; to that end, they consider in Proposition 7 [4] asymptotics of the masses of the connected components of sequences of multiplicative random graphs. The asymptotic regime in Proposition 7 [4] is very close to Assumptions (66) and (**C1**) – (**C3**) in our Theorem 2.16.

Let us briefly recall Proposition 7 in [4] since it is used in the proof of Theorem 2.16. Aldous & Limic fix a sequence of weights $\mathtt{x}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, and their notations for multiplicative graphs are the following: let $(\xi_{i,j})_{j>i\geq 1}$ be an array of independent and exponentially distributed r.v. with unit mean; let $N(\mathtt{x}_n) = \max\{j \geq 1 : x_j^{(n)} > 0\}$; then for all $q \in [0,\infty)$, Aldous & Limic consider the random graph $G_n(q)$ whose set of vertices is $\mathscr{V}(G_n(q)) = \{1,\ldots,N(\mathtt{x}_n)\}$ and whose set of edges $\mathscr{E}(G_n(q))$ is such that $\{i,j\} \in \mathscr{E}(G_n(q))$ iff $\xi_{i,j} \leq q x_i^{(n)} x_j^{(n)}$; the graph is equipped with the measure $m_n = \sum_{j\geq 1} x_j^{(n)} \delta_j$; let $\zeta_1(\mathtt{x}_n, q) \geq \ldots \geq \zeta_k(\mathtt{x}_n, q) \geq \ldots$ stand for the (eventually null) sequence of the m_n-masses of the connected components of $G_n(q)$. Then, it is easy to check that $\mathbf{X}_n : q \mapsto (\zeta_k(\mathtt{x}_n, q))_{k\geq 1}$ is a multiplicative coalescent process with finite support. Then, Aldous & Limic describe the limit of the processes $\mathbf{X}_n$ in terms of the excursion-lengths of a process $(W_s^{\kappa_{\mathrm{AL}}, -\tau_{\mathrm{AL}}, \mathbf{c}_{\mathrm{AL}}})_{s\in[0,\infty)}$ whose law is characterized by three parameters: $\kappa_{\mathrm{AL}} \in [0,\infty)$, $\tau_{\mathrm{AL}} \in \mathbb{R}$ and $\mathbf{c}_{\mathrm{AL}} \in \ell_3^{\downarrow}$; this process is connected to the $(\alpha, \beta, \kappa, \mathbf{c})$-process Y defined in (39) as follows:

$$\forall s \in [0,\infty), \quad W_s^{\kappa_{\mathrm{AL}}, -\tau_{\mathrm{AL}}, \mathbf{c}_{\mathrm{AL}}} = Y_{s/\kappa}, \text{ where } \quad \kappa_{\mathrm{AL}} = \frac{\beta}{\kappa}, \quad \tau_{\mathrm{AL}} = \frac{\alpha}{\kappa} \quad \text{and} \quad \mathbf{c}_{\mathrm{AL}} = \mathbf{c}. \tag{79}$$

Proposition 7 [4] assumes the following:

$$\lim_{n\to\infty} \frac{\sigma_3(\mathtt{x}_n)}{(\sigma_2(\mathtt{x}_n))^3} = \kappa_{\mathrm{AL}} + \sigma_3(\mathbf{c}_{\mathrm{AL}}), \quad \forall j \in \mathbb{N}^*, \lim_{n\to\infty} \frac{x_j^{(n)}}{\sigma_2(\mathtt{x}_n)} = c_j^{\mathrm{AL}} \quad \text{and} \quad \lim_{n\to\infty} \sigma_2(\mathtt{x}_n) = 0. \tag{80}$$

and it asserts that for all $\tau_{\mathrm{AL}} \in \mathbb{R}$, $\mathbf{X}_n(\sigma_2(\mathtt{x}_n)^{-1} - \tau_{\mathrm{AL}}) \to (\zeta_k)_{k\geq 1}$, weakly in $\ell_2^{\downarrow}$, where $(\zeta_k)_{k\geq 1}$ are the excursion-lengths of $W^{\kappa_{\mathrm{AL}}, -\tau_{\mathrm{AL}}, \mathbf{c}_{\mathrm{AL}}}$ above its infimum, listed in the decreasing order.

Assumptions (80) are close to $(\mathbf{C2})$ and $(\mathbf{C3})$. More precisely, let $(\alpha, \beta, \kappa, \mathbf{c})$ be connected with κ_{AL}, τ_{AL} and $\mathbf{c}_{\mathrm{AL}}$ as in (79); let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^{\downarrow}$ satisfy (66) and $(\mathbf{C1})$ – $(\mathbf{C3})$; then, set:

$$\forall j \in \mathbb{N}^*, \quad x_j^{(n)} = \frac{\kappa w_j^{(n)}}{b_n} \quad \text{and} \quad \tau_{\mathrm{AL}}^n = \frac{b_n^2}{\kappa^2 \sigma_2(\mathtt{w}_n)}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big) \xrightarrow[n\to\infty]{} \frac{\alpha}{\kappa} = \tau_{\mathrm{AL}}.$$

Recall from (12) the definition of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$, the $\mathtt{w}_n$-multiplicative graph. Recall that $\mathbf{m}_{\mathtt{w}_n} = \sum_{j\geq 1} w_j^{(n)} \delta_j$. Recall from Section 2.1.1 that the $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ are the connected components of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ listed in the non-increasing order of their $\mathbf{m}_{\mathtt{w}_n}$-mass. Then, it is easy to check the following.

$$G_n\big(\sigma_2(\mathrm{x}_n)^{-1} - \tau_{\mathrm{AL}}^n\big) = \boldsymbol{\mathcal{G}}_{\mathtt{w}_n} \quad \text{and} \quad \zeta_k\big(\mathrm{x}_n\,,\, \sigma_2(\mathrm{x}_n)^{-1} - \tau_{\mathrm{AL}}^n\big) = \frac{\kappa}{b_n} \mathbf{m}_{\mathtt{w}_n}\big(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}\big) =: \kappa \zeta_k^n. \tag{81}$$

Note that the ζ_k^n are the excursion-lengths of $(\frac{1}{a_n} Y^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)}$ above its infimum. Since $\tau_{\mathrm{AL}}^n \to \alpha/\kappa$ and since multiplicative coalescent processes have no fixed time-discontinuity, Proposition 7 in [4] immediately entails the following proposition that is used in Section 6.2.1 to prove Theorems 2.15 and 2.16.

Proposition 2.17 (Proposition 7 [4]) *Let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^{\downarrow}$ satisfy (66) and $(\mathbf{C1})$–$(\mathbf{C3})$, with $\alpha, \beta, \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} \in \ell_3^{\downarrow}$. Recall from (28) (resp. from (39)) the definition of $Y^{\mathtt{w}_n}$ (resp. of Y). Let $(\zeta_k^n)_{1\leq k\leq \mathbf{q}_{\mathtt{w}_n}}$ (resp. $(\zeta_k)_{k\geq 1}$) be the excursion-lengths of $(\frac{1}{a_n} Y^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)}$ (resp. of Y) above its infimum. Then,*

$$\big(\zeta_k^n\big)_{1\leq k\leq \mathbf{q}_{\mathtt{w}_n}} \xrightarrow[n\to\infty]{\text{weakly in } \ell_2^{\downarrow}} (\zeta_k)_{k\geq 1}. \tag{82}$$

Gromov–Prokhorov convergence of multiplicative graphs without Brownian component. In light of the above mentioned result of Aldous & Limic [4] on the convergence of the component masses of the multiplicative graph in the asymptotic regime (80), it is natural to expect that the graph itself should also converge in some sense. The first affirmation in this direction is due to Bhamidi, van der Hofstad and Sen who prove the following in [12]: Denote by $\mathscr{C}_i(q)$ the i-largest (in m_n-mass) connected component of $G_n(q)$, that is, $m_n(\mathscr{C}_i(q)) = \zeta_i(\mathrm{x}_n, q)$. Equip each component $\mathscr{C}_i(-\tau_{\mathrm{AL}} + \sigma_2(\mathrm{x}_n))$ with its graph distance rescaled by $\sigma_2(\mathrm{x}_n)$ and with the mass measure m_n, they prove that under (80) with $\tau_{\mathrm{AL}} = 0$, the collection of rescaled metric spaces converge in the sense of Gromov–Prokhorov topology to a collection of measured metric spaces, which are not necessarily compact. They also give an explicit construction of the limiting spaces based upon a model of continuum random tree called ICRT. The Gromov-Prokhorov convergence is equivalent to the convergence of mutual distance of an i.i.d. sequence with law m_n, which is weaker than the Gromov-Hausdorff-Prokhorov.

Limits of Erdős–Rényi graphs in the critical window. The first result proving metric convergence of rescaled Erdős-Rényi graphs and their inhomogeneous extensions is due to Addario-Berry, Goldschmidt & B. in [2]. In this paper, they study the scaling limits of the largest components of the Erdős–Rényi random graph $G(n, p_n)$ in the critical window $p_n = n^{-1} - \alpha n^{-4/3}$, with $\alpha \in \mathbb{R}$. which corresponds to the graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ where $w_j^{(n)} = \mathbf{1}_{\{j\leq n\}} n \log(\frac{1}{1-p_n})$, $j \geq 1$. Taking, $a_n = n^{1/3}$ and $b_n = n^{2/3}$, we immediately see that a_n, b_n and $\mathtt{w}_n$ satisfy (66) with $\kappa = \beta_0 = 1$, $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$, with $\alpha \in [0,\infty)$, $\beta = 1$ and $\mathbf{c} = 0$. Namely, the branching mechanism is $\psi(\lambda) = \alpha\lambda + \frac{1}{2}\lambda^2$. Since $\beta_0 > 0$, Proposition 2.13 (i) implies that $(\mathbf{C4})$ is automatically satisfied and Theorem 2.16 applies: in this case, Theorem 2.16 is a weaker version of Theorem 2 in Addario-Berry, Goldschmidt & B. [2], p. 369: the result in [2] actually holds for α possibly negative and the paper provides more precise estimates on the size of metric components. Let us mention that [2] also contains tail-estimates on the diameters of the small components. Such estimates seem difficult to obtain in the case of general $\mathtt{w}_n$.

Multiplicative graphs in the same bassin of attraction as Erdős-Rényi graphs. Bhamidi, van der Hofstad & van Leeuwaarden in [10], Bhamidi, Sen & X. Wang in [11] and Bhamidi, Sen, X. Wang & B. in [7] investigate asymptotic regimes in the bassin of attraction of Erdős–Rényi graphs in the critical window. More precisely, they consider the cases where $a_n = n^{1/3}$, $b_n = n^{2/3}$ and where $\mathtt{w}_n \in \ell_f^{\downarrow}$ is such that $w_j^{(n)} = 0$ for all $j > n$ and

$$(83)\quad \frac{w_1^{(n)}}{n^{1/3}} \to 0, \quad \exists\, \sigma, \sigma' \in (0,\infty) : \; \sigma_i(\mathtt{w}_n) = n\sigma + o(n^{2/3}), \; i \in \{1,2\} \text{ and } \sigma_3(\mathtt{w}_n) = n\sigma' + o(n).$$

Note that $\sigma' \geq \sigma$ since $\sigma_2(\mathtt{w}_n) \leq \sqrt{\sigma_3(\mathtt{w}_n)}\sqrt{\sigma_1(\mathtt{w}_n)}$ by Cauchy-Schwarz. Another application of Cauchy-Schwarz, $\sigma_1(\mathtt{w}_n) \leq \sqrt{n\sigma_2(\mathtt{w}_n)}$, which implies $\sigma \leq 1$. For all $\alpha \in [0,\infty)$, set

$$\mathtt{w}_n(\alpha) = \big(1 - \alpha n^{-\frac{1}{3}}\big)\mathtt{w}_n = \big(\big(1 - \alpha n^{-\frac{1}{3}}\big) w_j^{(n)}\big)_{j\geq 1} .$$

Then, (83) easily implies that $a_n, b_n, \mathtt{w}_n(\alpha)$ satisfy (66), (**C1**), (**C2**) and (**C3**), with $\alpha \in [0,\infty)$, $\beta_0 = 1$, $\beta = \sigma'/\sigma$, $\kappa = 1/\sigma$ and $\mathbf{c} = 0$. Thus, the branching mechanism is $\psi(\lambda) = \alpha\lambda + \frac{1}{2}\frac{\sigma'}{\sigma}\lambda^2$. Since $\beta_0 = 1$, Proposition 2.13 (i) implies (**C4**). Then, Theorem 2.16 holds true, which extends Theorem 3.3 in Bhamidi, Sen & X. Wang in [11] that has been proved by quite different methods and under two additional technical assumptions (Assumptions 3.1 (c) and (d)). Let us mention that the convergence under the sole assumptions (83), that we proved, has been conjectured in [11], Section 5, part (c).

Power-law cases. We extend the power-law cases investigated in Bhamidi, van der Hofstad & van Leeuwaarden [14] and Bhamidi, van der Hofstad & Sen [12]. Let $W : \Omega \to [0,\infty)$ be a r.v. such that

$$(84)\qquad r = \mathbf{E}[W] = \mathbf{E}[W^2] < \infty \quad \text{and} \quad \mathbf{P}(W \geq x) = x^{-\rho} L(x),$$

where $\rho \in (2,3)$ (in the notations of [12], $\tau = \rho + 1 \in (3,4)$) and where L is slowly varying at ∞. We then set for all $y \in [0,\infty)$,

$$(85)\qquad G(y) = \sup\big\{x \in [0,\infty) : \mathbf{P}(W \geq x) \geq 1 \wedge y\big\}.$$

Note that $G(y) = 0$ for all $y \in [1,\infty)$ and that $G(y) = y^{-1/\rho}\,\ell(y)$, where ℓ is slowly varying at 0. We assume the following.

$$(86)\qquad \forall n \in \mathbb{N}^*, \quad \mathbf{P}\big(W = G(1/n)\big) = 0\,.$$

Let $\kappa, q \in (0,\infty)$ and let $a_n, b_n, \mathtt{w}_n$ be such that

$$(87)\qquad a_n \underset{n\to\infty}{\sim} q^{-1} G(1/n), \quad \forall\, j \geq 1, \;\; w_j^{(n)} = G(j/n), \quad b_n \underset{n\to\infty}{\sim} \kappa\sigma_1(\mathtt{w}_n)/a_n\,.$$

Then, the following lemma holds true.

Lemma 2.18 *We keep the notations from above and we assume (86). Then $a_n \sim q^{-1} n^{\frac{1}{\rho}} \ell(1/n)$, $b_n \sim q\kappa\, n^{1-\frac{1}{\rho}}/\ell(1/n)$ and a_n, b_n and $\mathtt{w}_n$ satisfy (66) with $\beta_0 = 0$.*

Next, let us set for all integers $j \geq 1$ and all $\alpha \in [0,\infty)$,

$$(88)\qquad w_j^{(n)}(\alpha) = \big(1 - \tfrac{a_n}{b_n}(\alpha - \alpha_0)\big) w_j^{(n)}, \;\; \textit{where} \quad \alpha_0 = 2\kappa q^2 \Big(\int_0^1 y\{y^{-\rho}\}\, dy + \tfrac{1}{\rho-2}\Big)$$

and where $\{\cdot\}$ stands for the fractional part function. Then, $a_n, b_n, \mathtt{w}_n(\alpha)$ satisfy (66), (**C1**)*–*(**C4**) *with $\alpha \in [0,\infty)$, $\kappa \in (0,\infty)$, $\beta = \beta_0 = 0$ and $c_j = q\, j^{-\frac{1}{\rho}}$, for all $j \geq 1$.*

Proof. See section 7. ■

Lemma 2.18 implies that Theorem 2.16 applies to a_n, b_n and $\mathtt{w}_n(\alpha)$ as defined above. This extends Theorem 1.2 in Bhamidi, van der Hofstad & Sen [12] (Section 1.1.3) that asserts the same weak-limit under the more restrictive assumption that $L(x) = x^\rho \mathbf{P}(W \geq x) \to c_F \in (0,\infty)$ as $x \to \infty$ (see (1.3) in [12], Section 1.1.1) and where it is also assumed that $\mathbf{P}(W \in dx) = f(x)dx$, where f is a continuous function whose support is of the form $[\varepsilon, \infty)$ with $\varepsilon > 0$, and such that $x \in [\varepsilon, \infty) \mapsto xf(x)$ is non-increasing (see Assumption 1.1 in [12], Section 1.1.3). Let us mention that the proof of Theorem 1.2 [12] is quite different from ours.

Let us also mention that Conjecture 1.3 right after Theorem 1.2 in [12] is solved by our Proposition 2.8 that asserts the following: if $\alpha \in [0,\infty)$, $\kappa \in (0,\infty)$, $\beta = 0$ and $c_j = q\, j^{-1/\rho}$, then $\eta = \gamma = \rho - 1$ (which corresponds to $\tau - 2$ in [12]) and

$$\mathbf{P}\text{-a.s. for all } k \geq 1, \quad \dim_H(\mathbf{G}_k) = \dim_p(\mathbf{G}_k) = \frac{\rho - 1}{\rho - 2},$$

where $\dim_H$ and $\dim_p$ stand respectively for the Hausdorff and for the packing dimensions.

General inhomogeneous Erdős–Rényi graphs that are close to be multiplicative. In [31], Janson investigates strong asymptotic equivalence of general inhomogeneous Erdős–Rényi graphs that are defined as follows: denote by P the set of arrays $\mathbf{p} = (p_{i,j})_{j>i\geq 1}$ of real numbers in $[0,1]$ such that $N_{\mathbf{p}} = \sup\{j \geq 2 : \sum_{1\leq i<j} p_{i,j} > 0\} < \infty$; the $\mathbf{p}$-inhomogeneous Erdős–Rényi graph $G(\mathbf{p})$ is the random graph whose set of vertices is $\{1, \ldots, N(\mathbf{p})\}$ and whose random set of edges $\mathscr{E}(G(\mathbf{p}))$ is such that the r.v. $(\mathbf{1}_{\{\{i,j\}\in\mathscr{E}(G(\mathbf{p}))\}})_{1\leq i<j\leq N(\mathbf{p})}$ are independent and such that $\mathbf{P}(\{i,j\} \in \mathscr{E}(G(\mathbf{p}))) = p_{i,j}$. The asymptotic equivalence is measured through the following function ρ that is defined for all $p, q \in [0,1]$, by $\rho(p,q) = (\sqrt{p} - \sqrt{q})^2 + (\sqrt{1-p} - \sqrt{1-q})^2$. More precisely, let $\mathbf{p}_n, \mathbf{q}_n \in P$, $n \in \mathbb{N}$; then Theorem 2.2 in Janson [31] implies that there are couplings of $G(\mathbf{p}_n)$ and $G(\mathbf{q}_n)$ such that $\lim_{n\to\infty} \mathbf{P}(G(\mathbf{p}_n) \neq G(\mathbf{q}_n)) = 0$ if and only if

$$\lim_{n\to\infty} \sum_{j>i\geq 1} \rho(p_{i,j}^{(n)}, q_{i,j}^{(n)}) = 0\,. \tag{89}$$

We then apply this result as follows: let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^\downarrow$, $n \in \mathbb{N}$, satisfy the assumptions of Theorem 2.16; we set

$$\forall j > i \geq 1, \quad p_{i,j}^{(n)} = \frac{w_i^{(n)} w_j^{(n)}}{\sigma_1(\mathtt{w}_n)} \quad \text{and} \quad u_{i,j}^{(n)} = \begin{cases} \frac{q_{i,j}^{(n)}}{p_{i,j}^{(n)}} - 1 & \text{if } p_{i,j}^{(n)} > 0 \\ 0 & \text{if } p_{i,j}^{(n)} > 0. \end{cases} \tag{90}$$

First note that $\max_{j>i\geq 1} p_{i,j}^{(n)} = O((w_1^{(n)}/a_n)^2 a_n/b_n) \to 0$ by (66); next, as proved in Janson [31] (2.5) p. 30, if $p \leq 0.9$, then $\rho(p,q) \asymp |p-q|\big(1 \wedge |q/p - 1|\big)$. Thus, (89) is equivalent to

$$\lim_{n\to\infty} \sum_{j>i\geq 1} p_{i,j}^{(n)} |u_{i,j}^{(n)}| \big(1 \wedge |u_{i,j}^{(n)}|\big) = 0, \quad \text{with the convention } p_{i,j}^{(n)} |u_{i,j}^{(n)}| = q_{i,j}^{(n)} \text{ if } p_{i,j}^{(n)} = 0. \tag{91}$$

In particular, let $h : [0,1] \to [0,1]$ be such that $h(x) = x + O(x^2)$. If we set $q_{i,j}^{(n)} = h(p_{i,j}^{(n)})$, then there exists $C \in (0,\infty)$ such that $|u_{i,j}^{(n)}| \leq C p_{i,j}^{(n)}$. In this case, for all sufficiently large n,

$$\sum_{j>i\geq 1} p_{i,j}^{(n)} |u_{i,j}^{(n)}| \big(1 \wedge |u_{i,j}^{(n)}|\big) \leq C^2 \sum_{j>i\geq 1} (p_{i,j}^{(n)})^3 \leq C^2 \frac{\sigma_3(\mathtt{w}_n)^2}{\sigma_1(\mathtt{w}_n)^3} \sim C'(a_n/b_n)^3 \longrightarrow 0$$

by (**C2**) and (66). Cases where $h(x) = x$ have been considered by van der Esker, van der Hofstad & Hooghiemstra [39], close cases where $h(x) = 1 \wedge x$ have been studied by Chung & Lu [20]; the cases where $h(x) = 1 - e^{-x}$, was first studied by Aldous [3] and Aldous & Limic [4] and the previous cited papers [2, 7, 10–12, 14], including this paper; cases where $h(x) = x/(1+x)$ have been investigated by Britton, Deijfen & Martin-Löf [19]. To summarise, Janson's Theorem 2.2 [31], p. 31 combined with Theorem 2.16 imply the following result.

Theorem 2.19 (Theorem 2.2 in Janson [31]) *Assume that* $a_n, b_n, \mathtt{w}_n$ *satisfy the same assumptions as in Theorem 2.14 (and thus as in Theorem 2.16). We define* $\mathbf{p}_n$ *by (90). Let* $\mathbf{q}_n \in P$*. We define* $(u_{i,j}^{(n)})_{j>i\geq 1}$ *by (90) and we suppose (91). Then, there exist couplings of* $G(\mathbf{q}_n)$ *and* $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$ *such that*

$$\lim_{n\to\infty} \mathbf{P}(\boldsymbol{\mathcal{G}}_{\mathtt{w}_n} \neq G(\mathbf{q}_n)) = 0 \tag{92}$$

and the weak limit (78) in Theorem 2.16 for the metric part holds true in the same scaling for the connected components of $G(\mathbf{q}_n)$*. In particular, its holds true when* $u_{i,j}^{(n)} = h(p_{i,j}^{(n)})$*,* $j > i \geq 1$*, for all functions* $h : [0,1] \to [0,1]$ *such that* $h(x) = x + O(x^2)$*.*

3 Proof of Theorem 2.1.

Let $G = (\mathscr{V}(G), \mathscr{E}(G))$ be a graph with $\mathscr{V}(G) \subset \mathbb{N}\backslash\{0\}$. We suppose that G has q connected components $\mathtt{C}_1^G, \ldots, \mathtt{C}_q^G$ listed in the increasing order of their least vertex: $\min \mathtt{C}_1^G < \ldots < \min \mathtt{C}_q^G$. Let $\mathtt{w} = (w_j)_{j\in\mathscr{V}(G)}$ be a set of strictly positive weights; we $\mathtt{m} = \sum_{j\in\mathscr{V}(G)} w_j \delta_j$ that is a measure on $\mathscr{V}(G)$. We then define a law $\Lambda_{G,\mathtt{m}}$ on $((0,\infty) \in \mathscr{V}(G))^q$ as follows. Let $(\Pi_j)_{j\in\mathscr{V}(G)}$ be independent Poisson random subsets of $(0,\infty)$ with rate $w_j/\sigma_1(\mathtt{w})$ (for convenience, the Π_j are viewed as random countable subset of $(0,\infty)$); for all non-empty subset $S \subset \mathscr{V}(G)$ we set $\Pi(S) := \bigcup_{j\in S} \Pi_j$. Then $\Pi(S)$ is a Poisson random set with rate $\mathtt{m}(S)/\sigma_1(\mathtt{w})$ and $\Pi(\{j\}) = \Pi_j$, for all $j \in \mathscr{V}(G)$. For all $k \in \{1, \ldots, q\}$, we then define $(T_k, U_k) : \Omega \to (0,\infty) \times \mathscr{V}(G)$ by:

$$T_k = \inf \Pi(\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathtt{s(k)}}) = \inf \Pi_{U_k} \textit{ where the permutation } \mathtt{s} \textit{ is such that } \inf \Pi\big(\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathtt{s}(1)}\big) < \ldots < \inf \Pi\big(\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathtt{s}(q)}\big)$$

(here, we slightly abuse notation by writing $\mathtt{C}_k^G$ instead of $\mathscr{V}(\mathtt{C}_k^G)$). Namely, T_k is the kth order statistic of $\big(\inf \Pi(\mathtt{C}_1^G), \ldots, \inf \Pi\big(\mathtt{C}_q^G\big)\big)$. We denote by $\Lambda_{G,\mathtt{m}}$ the joint law of $((T_k, U_k))_{1\leq k\leq q}$. We easily check:

$$\begin{aligned} \Lambda_{G,\mathtt{m}}(dt_1 \ldots dt_q; j_1, \ldots, j_q) &= \mathbf{P}\big(T_1 \in dt_1; \ldots; T_q \in dt_q; U_1 = j_1; \ldots; U_q = j_q\big) \\ &= \frac{w_{j_1}}{\sigma_1(\mathtt{w})} \cdots \frac{w_{j_q}}{\sigma_1(\mathtt{w})} \exp\Big(-\tfrac{1}{\sigma_1(\mathtt{w})} \sum_{1\leq k\leq l\leq q} t_k \mathtt{m}\big(\mathtt{C}^G_{s(l)}\big)\Big) dt_1 \ldots dt_q. \end{aligned} \tag{93}$$

where s is the unique permutation of $\{1, \ldots, q\}$ such that $j_l \in \mathscr{V}(\mathtt{C}^G_{s(l)})$, for all $l \in \{1, \ldots, q\}$. The following lemma, whose elementary proof is left to the reader, provides a description of the law of $((T_k, U_k))_{2\leq k\leq q}$ conditionally given (T_1, U_1). Let us mention that it is formulated within specific notation for further use.

Lemma 3.1 *Let* G^o *be a finite graph with* q^o *connected components; let* $\mathtt{w}^o = (w_j^o)_{j\in\mathscr{V}(G^o)}$ *be strictly positive weights; let* $\mathtt{m}^o = \sum_{j\in\mathscr{V}(G^o)} w_j^o \delta_j$*. We fix* $j^* \in \mathtt{C}^{G^o}_{k^*}$ *and* $a \in (0,\infty)$*. Then, we set* $G' = G^o \backslash \mathtt{C}^{G^o}_{k^*}$*; we equip* G' *with* $w'_j = a w_j^o$*,* $j \in \mathscr{V}(G')$ *and* $\mathtt{m}' = \sum_{j\in\mathscr{V}(G')} w'_j \delta_j$*. Let* T *and* $(T'_k, U'_k)_{1\leq k\leq q^o-1}$ *be independent r.v. such that* T *is exponentially distributed with unit mean and such that* $(T'_k, U'_k)_{1\leq k\leq q^o-1}$ *has law* $\Lambda_{G',\mathtt{m}'}$*. We then set*

$$T_1^o = T \quad \textit{and} \quad \forall k \in \{1, \ldots, q^o - 1\}, \quad T^o_{k+1} = T + \tfrac{1}{a} T'_k \quad U^o_{k+1} = U'_k\ .$$

Then,

$$\frac{w^o_{j^*}}{\sigma_1(\mathtt{w}^o)} \mathbf{P}\big(T_1^o \in dt_1; \ldots; T^o_{q^o} \in dt_{q^o}; U_2^o = j_2; \ldots; U^o_{q^o} = j_{q^o}\big) = \Lambda_{G^o,\mathtt{m}^o}(dt_1 \ldots dt_{q^o}; j^*, j_2, \ldots, j_{q^o}).$$

Next we briefly recall how to derive a graph from the LIFO queue (and an additional point process) as discussed in Section 2.1.1. Let $\mathscr{V} \subset \mathbb{N}\backslash\{0\}$ be the finite set of vertices (or the labels of clients) associated with strictly positive weight $\mathtt{w} = (w_j)_{i\in\mathscr{V}}$ (the total amount of service of Client j is w_j); let $\underline{E} = (E_j)_{j\in\mathscr{V}}$ be the times of arrival of the clients. We assume that the clients arrive at distinct times and that no client enters when another client definitively leaves the queue. These restrictions correspond to a Borel subset of $(0,\infty)^{\#\mathscr{V}}$ for $\underline{E}$ that has a full Lebesgue measure. We next set

$$(94) \qquad Y_t = -t + \sum_{j\in\mathscr{V}} w_j \mathbf{1}_{\{E_j \le t\}} \quad \text{and} \quad J_t = \inf_{r\in[0,t]} Y_r \ .$$

We then define $V:[0,\infty)\to\mathscr{V}$ such that V_t the label of the client who is served right after time t: since Y only increases by jumps, for all $t\in[0,\infty)$, we get the following.

- either $\{s\in[0,t]:Y_{s-}<\inf_{[s,t]} Y\}$ is empty and we set $V_t=0$,
- or there exists $j\in\mathscr{V}$ such that $E_j=\sup\{s\in[0,t]:Y_{s-}<\inf_{[s,t]} Y\}$ and we set $V_t=j$.

Note that $V_t=0$ if the server is idle and that V is càdlàg. As mentioned in Section 2.1.1, the LIFO-queue yields an exploration forest whose set of vertices is $\mathscr{V}$ and whose set of edges are

$$\mathcal{A}=\big\{\{i,j\}:i,j\in\mathscr{V} \text{ and } V_{E_j-}=i\big\} \ .$$

Additional vertices are created thanks to a finite set of points $\Pi=\{(t_p,y_p)\,;1\le p\le \mathbf{p}_{\mathtt{w}}\}$ in $D=\{(t,y)\in(0,\infty)^2:0<y<Y_t-J_t\}$ as follows. For all $(t,y)\in D$, define $\tau(t,y)=\inf\big\{s\in[0,t]:\inf_{u\in[s,t]} Y_u > y+J_t\big\}$. Then the additional set of edges is defined by

$$\mathcal{S}=\big\{\{i,j\}:i,j\in\mathscr{V} \text{ and } \exists(t,y)\in\Pi \text{ such that } V_{\tau(t,y)}=i \text{ and } V_t=j\big\} \ .$$

Then the graph produced by $\underline{E}$, $\mathtt{w}$ and Π is $\boldsymbol{\mathcal{G}}=\big(\mathscr{V}(\boldsymbol{\mathcal{G}})=\mathscr{V}\,;\,\mathscr{E}(\boldsymbol{\mathcal{G}})=\mathcal{A}\cup\mathcal{S}\big)$.

Theorem 2.1 asserts that if $\underline{E}$, $\mathtt{w}$ and Π have the appropriate distribution, then $\boldsymbol{\mathcal{G}}$ is a $\mathtt{w}$-multiplicative graph, whose law is denoted by $M_{\mathscr{V},\mathtt{w}}$ given as follows: for all graphs G such that $\mathscr{V}(G)=\mathscr{V}$,

$$(95) \qquad M_{\mathscr{V},\mathtt{w}}(G)=\prod_{\{i,j\}\in\mathscr{E}(G)}\big(1-e^{w_j,w_j/\sigma_1(\mathtt{w})}\big)\prod_{\{i,j\}\notin\mathscr{E}(G)} e^{w_j,w_j/\sigma_1(\mathtt{w})} \ ,$$

where the second product is taken over all pairs of distinct $i,j\in\mathscr{V}$ such that $\{i,j\}\notin\mathscr{E}(G)$. We actually prove a result that is slightly more general than Theorem 2.1 and that involves additional features derived from the LIFO queue, namely times T_k and vertices U_k that are defined as follows: Denote by $\mathbf{q}$ the number of excursions of Y strictly above its infimum and denote by (l_k,r_k), $k\in\{1,\ldots,\mathbf{q}\}$ the corresponding excursion intervals listed in the increasing order their left endpoints: $l_1<\ldots<l_{\mathbf{q}}$. Then, we set:

$$(96) \qquad \forall k\in\{1,\ldots,\mathbf{q}\}, \quad T_k=-J_{l_k} \quad \text{and} \quad U_k\in\mathscr{V} \text{ is such that } E_{U_k}=l_k \ .$$

From the definition of $\boldsymbol{\mathcal{G}}$ as a determinsitic function of $(\underline{E},\mathtt{w},\Pi)$, we easily check the following: $\boldsymbol{\mathcal{G}}$ has $\mathbf{q}$ connected components $\mathtt{C}_1^{\boldsymbol{\mathcal{G}}},\ldots,\mathtt{C}_{\mathtt{q}}^{\boldsymbol{\mathcal{G}}}$; recall that they are listed in the increasing order of their least vertex: $\min\mathtt{C}_1^{\boldsymbol{\mathcal{G}}}<\ldots<\min\mathtt{C}_{\mathtt{q}}^{\boldsymbol{\mathcal{G}}}$. Then, we define the permutation $\mathtt{s}$ on $\{1,\ldots,\mathbf{q}\}$ that satisfies $U_k\in\mathtt{C}_{\mathtt{s}(k)}^{\boldsymbol{\mathcal{G}}}$ for all $k\in\{1,\ldots,\mathbf{q}\}$. Observe that $r_k-l_k=\mathbf{m}\big(\mathtt{C}_{\mathtt{s}(k)}^{\boldsymbol{\mathcal{G}}}\big)$ and that the excursion $(Y_{t+l_k}-J_{l_k})_{t\in[0,r_k-l_k]}$ codes the connected component $\mathtt{C}_{\mathtt{s}(k)}^{\boldsymbol{\mathcal{G}}}$. The quantity T_k is actually the total amount of time during which the server is idle before the k-th connected component is visited, and U_k is the first visited vertex of the k-th component. We denote by Φ the (deterministic) function that associates $(\underline{E},\mathtt{w},\Pi)$ to $(\boldsymbol{\mathcal{G}},Y,J,(T_k,U_k)_{1\le k\le\mathbf{q}})$:

$$(97) \qquad \Phi\big(\underline{E},\mathtt{w},\Pi\big)=\big(\boldsymbol{\mathcal{G}},Y,J,(T_k,U_k)_{1\le k\le\mathbf{q}}\big) \ .$$

We next prove the following theorem that implies Theorem 2.1.

Theorem 3.2 *We keep the notation from above. We assume that $\underline{E}=(E_j)_{\mathscr{V}}$ are independent exponentially distributed r.v. such that $\mathbf{E}[E_j]=\sigma_1(\mathtt{w})/w_j$, $j\in\mathscr{V}$. We assume that conditionally given $\underline{E}$, Π is a Poisson random subset of $D=\{(t,y)\in(0,\infty)^2:0\le y<Y_t-J_t\}$ with intensity*

$\sigma_1(\mathtt{w})^{-1}\mathbf{1}_D(t,y)\,dtdy$. Let $\big(\boldsymbol{\mathcal{G}},(T_k,U_k)_{1\leq k\leq \mathbf{q}}\big)$ be derived from $\big(\underline{E},\mathtt{w},\Pi\big)$ by (97). Then, for all graphs G whose set of vertices is $\mathscr{V}$ and that have q connected components, we get

$$\mathbf{P}\big(\boldsymbol{\mathcal{G}}=G\,;\,T_1\in dt_1;\ldots;T_q\in dt_q\,;\,U_1=j_1;\ldots;U_q=j_q\big) \tag{98}$$
$$= M_{\mathscr{V},\mathtt{w}}(G)\,\Lambda_{G,\mathtt{m}}(dt_1,?\ldots,dt_q;j_1,\ldots,j_q)\;.$$

where $M_{\mathscr{V},\mathtt{w}}$ is defined by (95) and $\Lambda_{G,\mathtt{m}}$ is defined by (93).

Proof. We proceed by induction on the number of vertices of $\boldsymbol{\mathcal{G}}$. When, $\boldsymbol{\mathcal{G}}$ has only one vertex, then (98) is obvious. We fix an integer $n\geq 1$ and we assume that (98) holds for all $\mathscr{V}\subset\mathbb{N}\backslash\{0\}$ such that $\#\mathscr{V}=n$ and all sets of positive weights $\mathtt{w}=(w_j)_{j\in\mathscr{V}}$.

Then, we fix $\mathscr{V}^o\subset\mathbb{N}\backslash\{0\}$ such that $\#\mathscr{V}^o=n+1$; let $\mathtt{w}^o=(w^o_j)_{j\in\mathscr{V}^o}$ be strictly positive weights. we also fix $\underline{E}^o=(E^o_j)_{j\in\mathscr{V}^o}$ in $(0,\infty)^{n+1}$; we assume that in the corresponding LIFO queue, clients arrive at distinct times and that no client enters when another client definitively leaves the queue. Let $Y^o_t=-t+\sum_{j\in\mathscr{V}^o}w^o_j\mathbf{1}_{\{E^o_j\leq t\}}$ and $J^o_t=\inf_{[0,t]}Y^o$. Let $\mathbf{q}^o$ be the number of excursions of Y^o strictly above its infimum and let $(T^o_k,U^o_k)_{1\leq k\leq \mathbf{q}^o}$ as in (96): namely $T^o_k=-J^o_{l^o_k}$ and $E^o_{U^o_k}=l^o_k$, where (l^o_k,r^o_k) is the k-th excursion interval of Y^o strictly above J^o listed in the increasing order of their left endpoint: $l^o_1<\ldots<l^o_{\mathbf{q}^o}$.

The main idea for the induction is to shift the LIFO queue at the time of arrival T^o_1 of the first client (with label U^o_1) and to consider the resulting graph. More precisely, we set the following.

$$\mathscr{V}:=\mathscr{V}^o\backslash\{U^o_1\},\;\; a:=\frac{\sigma_1(\mathtt{w}^o)-w^o_{U^o_1}}{\sigma_1(\mathtt{w}^o)},\;\; \forall j\in\mathscr{V},\; w_j=aw^o_j\;\text{ and }\;E_j=a(E^o_j-T^o_1)\;. \tag{99}$$

Let Y and J be derived from $\underline{E}:=(E_j)_{j\in\mathscr{V}}$ and $\mathtt{w}:=(w_j)_{j\in\mathscr{V}}$ as in (94). Then observe that

$$Y_t=a\big(Y^o_{a^{-1}t+T^o_1}-Y^o_{T^o_1}\big)=Y_t=-t+\sum_{j\in\mathscr{V}}w_j\mathbf{1}_{\{E_j\leq t\}}\;,\quad t\in[0,\infty)\;. \tag{100}$$

Note that $T^o_1=\min_{j\in\mathscr{V}^o}E^o_j$. Then the alarm clock lemma implies the following.

(I) *If $(E^o_j)_{j\in\mathscr{V}^o}$ are independent exponentially distributed r.v. such that $\mathbf{E}[E^o_j]=w^o_j/\sigma(\mathtt{w}^o)$, then T^o_1 is an exponentially distributed r.v. with unit mean, $\mathbf{P}(U^o_1=j)=w^o_j/\sigma_1(\mathtt{w}^o)$, $j\in\mathscr{V}^o$, T^o_1 and U^o_1 are independent and under $\mathbf{P}(\,\cdot\,|U^o_1=j^*)$, $(E_j)_{j\in\mathscr{V}}$, as defined in (99), are independent exponentially distributed r.v. such that $\mathbf{E}[E_j]=w_j/\sigma(\mathtt{w})$, $j\in\mathscr{V}=\mathscr{V}^o\backslash\{j^*\}$.*

We next introduce $\mathcal{Q}_1$ and $\mathcal{Q}_2$, two discrete (without limit-point) subsets of $[0,\infty)^2$ that yield the additional edges in a specific way that is explained later. We first set:

$$D_1:=\big\{(t,y)\in[0,\infty)^2:0\leq y<Y_t-J_t\big\},\quad \Pi:=\mathcal{Q}_1\cap D_1 \tag{101}$$
$$\text{and}\quad D_2=\big\{(t,y)\in[0,\infty)^2:-J_t<y\leq aw^o_{U^o_1}\big\},\;\; \Pi^2:=\mathcal{Q}_2\cap\mathcal{D}_2.$$

We next define f_1 and f_2 from $[0,\infty)^2$ to $[0,\infty)^2$ and a set of points Π^o by:

$$f_1(t,y)=\big(\tfrac{1}{a}t+T^o_1\,,\,\tfrac{1}{a}y+(w^o_{U^o_1}+J_t)_+\big),\quad f_2(t,y)=\big(\tfrac{1}{a}t+T^o_1\,,\,w^o_{U^o_1}-\tfrac{1}{a}y\big) \tag{102}$$
$$\text{and}\qquad \Pi^o=f_1(\Pi)\cup f_2(\Pi^2)$$

We check the following.

(II) *Fix $\underline{E}^o$; suppose that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two independent Poisson random subsets of $[0,\infty)^2$ with intensity $\frac{1}{\sigma_1(\mathtt{w})}\,dtdy$. Then, Π^o is a Poisson random subset on $D^o=\{(t,y)\in[0,\infty)^2:0\leq y<Y^o_t-J^o_t\}$ with intensity $\frac{1}{\sigma_1(\mathtt{w}^o)}\mathbf{1}_{D^o}(t,y)\,dtdy$.*

Indeed, observe that $f_1(D_1)$ and $f_2(D_2)$ form a partition of D^o. Then, note that f_1 is piecewise affine with slope $1/a$, on the excursion intervals of $Y-J$ strictly above 0. Note that f_2 is affine with slope $1/a$. Standard results on Poisson subsets entail that Π^o is a Poisson random subset on D^o with intensity $\frac{a^2}{\sigma_1(\mathtt{w})}\mathbf{1}_{D^o}(t,y)\,dtdy$ and by (99) $\frac{a^2}{\sigma_1(\mathtt{w})}=\frac{1}{\sigma_1(\mathtt{w}^o)}$, which implies (II). □

Recall notation (97). We next introduce the the two following graphs:

(103)
$$\Phi\big(\underline{E}^o, \mathtt{w}^o, \Pi^o\big) = \big(\boldsymbol{\mathcal{G}}^o, Y^o, J^o, (T^o_k, U^o_k)_{1\leq k\leq \mathbf{q}^o}\big) \;\text{ and }\; \Phi\big(\underline{E}, \mathtt{w}, \Pi\big) = \big(\boldsymbol{\mathcal{G}}, Y, J, (T_k, U_k)_{1\leq k\leq \mathbf{q}}\big).$$

Then, the previous construction of Π^o combined with (100) easily imply the following:

(III) *Fix $\underline{E}^o$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then, $\boldsymbol{\mathcal{G}}$ is obtained by removing the vertex U^o_1 from $\boldsymbol{\mathcal{G}}^o$: namely, $\mathscr{V}(\boldsymbol{\mathcal{G}}) = \mathscr{V} = \mathscr{V}^o\backslash\{U^o_1\}$ and $\mathscr{E}(\boldsymbol{\mathcal{G}}) = \big\{\{i,j\}\in\mathscr{E}(\boldsymbol{\mathcal{G}}^o) : i,j\in\mathscr{V}(\boldsymbol{\mathcal{G}})\big\}$.*

We next consider which connected components of $\boldsymbol{\mathcal{G}}$ are attached to U^o_1 in $\boldsymbol{\mathcal{G}}^o$. To that end, recall that the $\mathtt{C}^{\boldsymbol{\mathcal{G}}^o}_l$ (resp. the $\mathtt{C}^{\boldsymbol{\mathcal{G}}}_l$) are the connected components of $\boldsymbol{\mathcal{G}}^o$ (resp. of $\boldsymbol{\mathcal{G}}$); recall that $\mathbf{s}^o$ (resp. $\mathbf{s}$) is the permutation on $\{1,\ldots,\mathbf{q}^o\}$ (resp. on $\{1,\ldots,\mathbf{q}\}$) such that $U^o_k\in\mathtt{C}^{\boldsymbol{\mathcal{G}}^o}_{\mathbf{s}^o(k)}$ for all $k\in\{1,\ldots,\mathbf{q}^o\}$ (resp. $U_k\in\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(k)}$ for all $k\in\{1,\ldots,\mathbf{q}\}$). We first introduce

$$\text{(104)}\qquad \boldsymbol{\mathcal{G}}' := \boldsymbol{\mathcal{G}}^o\backslash\mathtt{C}^{\boldsymbol{\mathcal{G}}^o}_{\mathbf{s}^o(1)} \quad\text{and}\quad K := \sup\big\{k\in\{1,\ldots,q\} : T_k \leq a w^o_{U^o_1}\big\}.$$

with the convention $\sup\emptyset = 0$. The graph $\boldsymbol{\mathcal{G}}'$ is the graph $\boldsymbol{\mathcal{G}}^o$ where the first (in the order of visit) component has been removed. Note that $\boldsymbol{\mathcal{G}}'$ is possibly empty: it has $\mathbf{q}^o-1$ connected components. We easily check that $\mathbf{q}^o = \mathbf{q}-K+1$. We denote by $\mathbf{s}'$ the permutation of $\{1,\ldots,\mathbf{q}^o-1\}$ such that

$$\text{(105)}\qquad \forall k\in\{1,\ldots,\mathbf{q}^o-1\},\quad \mathtt{C}^{\boldsymbol{\mathcal{G}}'}_{\mathbf{s}'(k)} = \mathtt{C}^{\boldsymbol{\mathcal{G}}^o}_{\mathbf{s}^o(k+1)}$$

We also set:

$$\text{(106)}\qquad \forall k\in\{1,\ldots,\mathbf{q}^o-1\},\quad T'_k = T_{K+k} - a w^o_{U^o_1} \quad\text{and}\quad U'_k = U_{K+k}\,.$$

Suppose that $\underline{E}^o$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ are fixed, then we also check that

$$\text{(107)}\qquad \forall k\in\{1,\ldots,\mathbf{q}^o-1\},\quad T^o_{k+1} = T^o_1 + \tfrac{1}{a}T'_k,\quad U^o_{k+1} = U'_k \quad\text{and}\quad \mathtt{C}^{\boldsymbol{\mathcal{G}}'}_{\mathbf{s}'(k)} = \mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(K+k)}\,.$$

We now explain how additional edges are added to connect $\boldsymbol{\mathcal{G}}$ to U^o_1. For all $j\in\mathscr{V}$, let $I_j = \{t\in[0,\infty) : V_t = j\}$; I_j is the set of times during which Client j is served; we easily check that I_j is a finite union of disjoint intervals of the form $[x,y)$ whose Lebesgue measure is w_j: namely, $\mathtt{Leb}(I_j) = w_j$. We also set:

$$\Pi^*_j = \big\{y\in[0,\infty) : \exists t\in I_j \text{ such that } (t,y)\in\mathcal{Q}_2 \text{ and } y > -J_t\big\}$$

Note that if $j\in\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(k)}$, then $-J_t = T_k$. Combined with elementary results on Poisson random sets, it implies the following.

(IV) *Fix $\underline{E}^o$ and $\mathcal{Q}_1$; suppose that $\mathcal{Q}_2$ is a Poisson random subset with intensity $\frac{1}{\sigma_1(\mathtt{w})}\,dtdy$. Then, the $(\Pi^*_j)_{j\in\mathscr{V}}$ are independent and Π^*_j is a Poisson random subset of (T_k,∞) with rate $w_j/\sigma_1(\mathtt{w})$, where k is such that $j\in\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(k)}$.*

Note that the law of $(\Pi^*_j)_{j\in\mathscr{V}}$ only depends on U^o_1 and $(T_k,U_k)_{1\leq k\leq\mathbf{q}}$.

We next introduce the following.

* We set $\Pi_j := \{T_k\}\cup\Pi^*_j$ if there is $k\in\{1,\ldots,\mathbf{q}\}$ such that $j = U_k$.
* We set $\Pi_j := \Pi^*_j$ if $j\in\mathscr{V}\backslash\{U_1,\ldots,U_{\mathbf{q}}\}$.
* We set $\Pi'_j := \big\{y - a w^o_{U^o_1}\,;\, y\in\Pi_j\cap(a w^o_{U^o_1},\infty)\big\}$.
* For all non-empty $S\subset\mathscr{V}$, we set $\Pi'(S) := \bigcup_{j\in S}\Pi'_j$.

We claim the following.

(V) *Fix $\underline{E}^o$, $\mathcal{Q}_1$, $\mathcal{Q}_2$. Then,* (Va) *for all* $j \in \mathscr{V}$, $\{U_1^o, j\} \in \mathscr{E}(\boldsymbol{\mathcal{G}}^o)$ *iff* $\Pi_j \cap [0, aw^o_{U_1^o}] \neq \emptyset$;

(Vb) *for all* $k \in \{1, \dots, \mathbf{q}^o - 1\}$, $T_{K+k} - aw^o_{U_1^o} = \inf \Pi'(\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(K+k)}) = \Pi'_{U_{K+k}}$; *namely,*

$$\forall k \in \{1, \dots, \mathbf{q}^o - 1\}, \quad T'_k = \inf \Pi'(\mathtt{C}^{\boldsymbol{\mathcal{G}}'}_{\mathbf{s}'(k)}) = \Pi'_{U'_k}, \tag{108}$$

within the notation of (104) (105) (106) and (107).

Proof. Suppose that $\{U_1^o, j\} \in \mathscr{E}(\boldsymbol{\mathcal{G}}^o)$. There are two cases to consider. `(Case 1)`: $\{U_1^o, j\}$ is part of the exploration tree generated by $(\underline{E}^o, \mathtt{w}^o)$ which means that $V^o_{E^o_j -} = U_1^o$. Namely, it means that $T_1^o = \sup\{s \in [0, E_j^o] : Y^o_{s-} < \inf_{[s, E_j^o]} Y^o\}$ because $E^o_{U_1^o} = T_1^o$, by definition of (U_1^o, T_1^o); since $Y^o_{T_1^o -} + w^o_{U_1^o} = Y^o_{T_1^o}$ and by (100), it is equivalent to: $-aw^o_{U_1^o} < J_{E_j} = Y_{E_j -}$. It implies that E_j is the left endpoint of an excursion of Y strictly above its infimum; therefore, there exists $k \in \{1, \dots, \mathbf{q}\}$ such that $j = U_k$ and $-J_{E_j} = T_k$, by (96); thus $T_k \in [0, aw^o_{U_1^o}]$ which implies that $\Pi_j \cap [0, aw^o_{U_1^o}] \neq \emptyset$ (since $T_k \in \Pi_j$ in the case where $j = U_k$).

Conversely, let $j = U_k$ be such that $\Pi_j \cap [0, aw^o_{U_1^o}] \neq \emptyset$. It implies that $T_k \leq aw^o_{U_1^o}$ and the previous arguments can be reversed verbatim to prove that $\{U_1^o, j\}$ is an edge of $\boldsymbol{\mathcal{G}}^o$ that is part of the exploration tree generated by $(\underline{E}^o, \mathtt{w}^o)$.

`(Case 2)`: $\{U_1^o, j\}$ is an additional edge of $\boldsymbol{\mathcal{G}}^o$. Then, there exists $(t', y') \in \Pi^o$ such that $V^o_{t'} = j$ and $V^o_{\tau^o(t', y')} = U_1^o$, where $\tau^o(t', y') = \inf\{s \in [0, t'] : \inf_{[s, t']} Y^o > y' + J^o_{t'}\}$. Note that $V^o_{t'} = j$ implies that $t' > T_1^o$ since $j \neq U_1^o$ and since U_1^o is the first visited vertex (or the first client). Also observe that $V^o_{\tau^o(t', y')} = U_1^o$ implies $\tau^o(t', y') = T_1^o$. It also implies that t' lies in the first excursion interval of Y^o strictly above its infimum, which entails that $J^o_{t'} = J^o_{T_1^o} = -T_1^o$. Then, we set $t = a(t' - T_1^o)$ and, thanks to (100), we rewrite the previous conditions in terms of Y, J and V as follows: $V_t = j$ and $0 = \inf\{s \in [0, t] : aw^o_{U_1^o} + \inf_{[s,t]} Y > ay'\}$, which is equivalent to: $t \in I_j$ and $y := a(w^o_{U_1^o} - y') > -J_t$. This proves that there is $(t, y) \in D_2$ (as defined in (101)) such that $(t', y') = f_2(t, y)$ as defined in (102). Since $f_1(\Pi)$ and $f_2(\Pi^2)$ form a partition of Π^o, $(t, y) \in \Pi^2$ and this proves that there is $(t, y) \in \mathcal{Q}_2$ such that $t \in I_j$ and $aw^o_{U_1^o} \geq y \geq -J_t$ which implies $\Pi_j \cap [0, aw^o_{U_1^o}] \neq \emptyset$.

Conversely, suppose that $\Pi_j \cap [0, aw^o_{U_1^o}] \neq \emptyset$ and that $j \in \mathscr{V} \backslash \{U_1, \dots, U_{\mathbf{q}}\}$. Then, $\Pi^*_j \cap [0, aw^o_{U_1^o}] \neq \emptyset$ and the previous arguments can be reversed verbatim to prove that $\{U_1^o, j\}$ is an (additional) edge of $\boldsymbol{\mathcal{G}}^o$, which completes the proof of (Va).

Let us prove (Vb). Let $k \in \{1, \dots, \mathbf{q}^o - 1\}$. By definition $U_{K+k} \in \mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(K+k)}$. Let $y \in \Pi^*_{U_{K+k}}$: namely, there exists t such that $V_t = U_{K+k}$, $(t, y) \in \mathcal{Q}_2$ and $y > -J_t$. But $V_t = U_{K+k}$ implies that $-J_t = T_{K+k}$. Since $\Pi_{U_{K+k}} = \Pi^*_{U_{K+k}} \cup \{T_{K+k}\}$, we get $\inf \Pi_{U_{K+k}} = T_{K+k}$. By definition of K, $T_{K+k} > aw^o_{U_1^o}$, which entails $\inf \Pi'_{U_{K+k}} = T_{K+k} - aw^o_{U_1^o}$.

Let $j \in \mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(K+k)} \backslash \{U_{K+k}\}$ (if any) and let $y \in \Pi_j$. Necessarily, $j \in \mathscr{V} \backslash \{U_1, \dots, U_{\mathbf{q}}\}$, which entails $\Pi_j = \Pi^*_j$, by definition. Then there exists t such that $V_t = j$, $(t, y) \in \mathcal{Q}_2$ and $y > -J_t$. Note that I_j is included in the excursion interval of Y strictly above its infimum whose left endpoint is $E_{U_{K+k}}$, which implies that $J_t = -T_{K+k}$, for all $t \in I_j$. Thus $y > T_{K+k}$. This proves that $\inf \Pi'(\mathtt{C}^{\boldsymbol{\mathcal{G}}}_{\mathbf{s}(K+k)}) = \inf \Pi'_{U_{K+k}}$, which completes the proof of (Vb). □

We now complete the proof of Theorem 3.2 as follows: let $(E_j^o)_{j \in \mathscr{V}^o}$ be independent exponentially distributed r.v.such that $\mathbf{E}[E_j^o] = w_j^o / \sigma(\mathtt{w}^o)$. We then fix $j^* \in \mathscr{V}^o$ and we work under $\mathbf{P}_{j^*} := \mathbf{P}(\,\cdot\,| U_1^o = j^*)$; under $\mathbf{P}_{j^*}$, by (99), we get $\mathscr{V} = \mathscr{V}^o \backslash \{j^*\}$, $a = (\sigma_1(\mathtt{w}^o) - w^o_{j^*}) / \sigma_1(\mathtt{w}^o)$ and for all $j \in \mathscr{V}$, $w_j = aw_j^o$ and $E_j = a(E_j^o - T_1^o)$. Under $\mathbf{P}_{j^*}$, we take $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as two independent Poisson random subsets of $[0, \infty)^2$ with intensity $\frac{1}{\sigma_1(\mathtt{w})} dt dy$; $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are also supposed independent of $\underline{E}^o$. Recall from (101) and (102) the definition of Π and Π^o and from (103) the definition $(\boldsymbol{\mathcal{G}}, (T_k, U_k)_{1 \leq k \leq \mathbf{q}})$. By (I) and (II) under $\mathbf{P}_{j^*}$, the induction hypothesis applies to $(\boldsymbol{\mathcal{G}}, (T_k, U_k)_{1 \leq k \leq \mathbf{q}})$ since $(\underline{E}, \mathtt{w}, \Pi)$ has the appropriate distribution: namely, under $\mathbf{P}_{j^*}$, $\boldsymbol{\mathcal{G}}$ has law $M_{\mathscr{V}, \mathtt{w}}$ (as defined in (95)) and conditionally given $\boldsymbol{\mathcal{G}}$, $(T_k, U_k)_{1 \leq k \leq \mathbf{q}}$ has law $\Lambda_{\boldsymbol{\mathcal{G}}, \mathtt{m}}$ as defined in (93) (namely, (98) holds).

By (IV), under $\mathbf{P}_{j^*}$ and conditionally given $(T_1^o, \mathcal{G}, (T_k, U_k)_{1\leq k\leq \mathbf{q}})$, the $(\Pi_j^*)_{j\in\mathscr{V}}$ are independent and Π_j^* is a Poisson random subset of (T_k, ∞) with rate $w_j/\sigma_1(\mathtt{w})$, where k is such that $j\in \mathtt{C}^{\mathcal{G}}_{\mathtt{s}(k)}$. Since, conditionnally given $(T_1^o, \mathcal{G})$, the r.v. $(T_k, U_k)_{1\leq k\leq \mathbf{q}}$ has law $\Lambda_{\mathcal{G},\mathtt{m}}$, the definition of the Π_j combined with elementary results on Poisson processes imply the following key point.

(VI) *Under $\mathbf{P}_{j^*}$ and conditionnaly given $(T_1^o, \mathcal{G})$, the $(\Pi_j)_{j\in\mathscr{V}}$ are independent and Π_j is a Poisson random subset of $(0,\infty)$ with rate $w_j/\sigma_1(\mathtt{w})$; therefore, under $\mathbf{P}_{j^*}$, the $(\Pi_j)_{j\in\mathscr{V}}$ are independent of $(T_1^o, \mathcal{G})$ and the very definition of the Π_j implies that for all $\{1,\dots,\mathbf{q}\}$,*

$$T_k = \inf \Pi(\mathtt{C}^{\mathcal{G}}_{\mathtt{s(k)}}) = \inf \Pi_{U_k} \text{ where } \mathtt{s} \text{ is such that } \inf \Pi\big(\mathtt{C}^{\mathcal{G}}_{\mathtt{s}(1)}\big) < \ldots < \inf \Pi\big(\mathtt{C}^{\mathcal{G}}_{\mathtt{s}(q)}\big).$$

Consequently, under $\mathbf{P}_{j^*}$, (Va) and the previous arguments entail that $\mathcal{G}^o$ only depends on $\mathcal{G}$ and on the $\Pi_j\cap[0, aw^o_{j^*}]$, $j\in\mathscr{V}$. Thus, by (VI) combined elementary results on Poisson processes, $\mathcal{G}^o$ is independent from Π'_j, $j\in\mathscr{V}$ and T_1^o; (Va) and (VI) also imply that under $\mathbf{P}_{j^*}$, the events $\{\{j^*, j\}\in\mathscr{E}(\mathcal{G}^o)\}$, $j\in\mathscr{V}$, are independent with respective probability $1-\exp(-w_j a w^o_{j^*}/\sigma_1(\mathtt{w}))$. Then, note that $w_j a w^o_{j^*}/\sigma_1(\mathtt{w}) = w^o_j w^o_{j^*}/\sigma_1(\mathtt{w}^o)$. Thus, under $\mathbf{P}_{j^*}$, $\mathcal{G}^o$ has law $M_{\mathscr{V}^o,\mathtt{w}^o}$ and it is independent from $(T_1^o; \Pi'_j, j\in\mathscr{V})$.

Recall from (104) (105) (106) and (107) notation $\mathcal{G}'$, $\mathtt{s}'$ and $(T'_k, U'_k)_{1\leq k\leq \mathbf{q}^o-1}$. Then, observe that under $\mathbf{P}_{j^*}$ and conditionally given $\mathcal{G}^o$, (Vb) and (VI) imply that $(T'_k, U'_k)_{1\leq k\leq \mathbf{q}^o-1}$ has conditional law $\Lambda_{\mathcal{G}',\mathtt{m}'}$ where $\mathtt{m}'=\sum_{j\in\mathscr{V}(\mathcal{G}')} w_j\delta_j$. Then, under $\mathbf{P}_{j^*}$ and conditionally given $\mathcal{G}^o$, Lemma 3.1 applies and (107) entails that

$$\frac{w^o_{j^*}}{\sigma_1(\mathtt{w}^o)}\mathbf{P}_{j^*}\big(T_1^o\in dt_1;\ldots;T^o_{\mathbf{q}^o}\in dt_{\mathbf{q}^o}\,;U^o_2=j_2;\ldots;U^o_{\mathbf{q}^o}=j_{\mathbf{q}^o}\,\big|\,\mathcal{G}^o\big) \\ = \Lambda_{\mathcal{G}^o,\mathtt{m}^o}(dt_1\ldots dt_{\mathbf{q}^o}; j^*, j_2,\ldots,j_{\mathbf{q}^o}).$$

Since $\mathbf{P}(U_1^o=j^*)=w^o_{j^*}/\sigma_1(\mathtt{w}^o)$, it implies that for all graph G^o whose set of vertice is $\mathscr{V}^o$ and that has q^o connected components, we get

$$\mathbf{P}\big(\mathcal{G}^o=G^o\,;\,T_1^o\in dt_1;\ldots;T^o_{q^o}\in dt_{q^o}\,;U_1^o=j^*;U_2^o=j_2;\ldots;U^o_{q^o}=j_{q^o}\big) \\ = M_{\mathscr{V}^o,\mathtt{w}^o}(G^o)\,\Lambda_{G^o,\mathtt{m}^o}(dt_1\ldots dt_{q^o}; j^*, j_2,\ldots,j_{q^o}).$$

This completes the proof of Theorem 3.2 by induction on the number of vertices. ■

4 Embedding the multiplicative graph in a GW-tree.

4.1 The tree associated with the Markovian queue.

We recall here various codings of the tree generated by a Markovian LIFO queue and we prove or recall easy results on these codings.

4.1.1 Height and contour processes of Galton-Watson trees.

Let us briefly recall basic notation about the coding of trees. First set $\mathbb{U}=\bigcup_{n\in\mathbb{N}}(\mathbb{N}^*)^n$, the set of finite words written with positive integers; here, $(\mathbb{N}^*)^0$ is taken as $\{\varnothing\}$. Let $u=[i_1,\ldots,i_n]\in\mathbb{U}$; we set $\overleftarrow{u}=[i_1,\ldots,i_{n-1}]$ that is interpreted as the *parent of* u; if $n=1$, then $\overleftarrow{u}$ is taken as $\varnothing$; we also set $|u|=n$, the *length of* u, with the convention that $|\varnothing|=0$; let $v=[j_1,\ldots,j_m]\in\mathbb{U}$; then $u*v=[i_1,\ldots,i_n,j_1,\ldots,j_m]$ stands for the concatenation of u with v, with the convention that $\varnothing*u=u*\varnothing=u$. A *rooted ordered tree* can be viewed as a subset $t\subset\mathbb{U}$ such that the following holds true.

(a) : $\varnothing\in t$.

(b) : If $u\in t\backslash\{\varnothing\}$, then $\overleftarrow{u}\in t$.

(c) : For all $u \in t$, there exists $k_u(t) \in \mathbb{N} \cup \{\infty\}$ such that $u * [i] \in t$ iff $1 \leq i \leq k_u(t)$.

Here $k_u(t)$ is interpreted as the *number of children of* u and if $1 \leq i \leq k_u(t)$, then $u * [i]$ is the *i-th child of* u. Implicitely, if $k_u(t)$, then there is no child stemming from u and assertion (c) is empty. Let $u \in t$, we set $\theta_u t = \{v \in \mathbb{U} : u * v \in t\}$ that is also a tree in the previous meaning: it is the family subtree of the descendants stemming from u.

We denote by $\mathbb{T}$ the set of such trees. We equipp $\mathbb{T}$ with the sigma-field $\mathscr{F}(\mathbb{T})$ generated by the sets $\{t \in \mathbb{T} : u \in t\}$, $u \in \mathbb{U}$. Then, a *Galton-Watson tree with offspring distribution* μ (a *GW(μ)-tree*, for short) is a $(\mathscr{F}, \mathscr{F}(\mathbb{T}))$-measurable r.v. $\tau : \Omega \to \mathbb{T}$ that satisfies the following.

(a') : $k_\varnothing(\tau)$ has law μ.

(b') : For all $k \geq 1$ such that $\mu(k) > 0$, the subtrees $\theta_{[1]}\tau, \ldots, \theta_{[k]}\tau$ under $\mathbf{P}(\,\cdot\,|k_\varnothing(\tau) = k)$ are independent with the same law as τ under $\mathbf{P}$.

Recall that τ is a.s. finite iff μ is critical or subcritical: namely, iff $\sum_{k \geq 1} k\mu(k) \leq 1$.

A *Galton-Watson forest with offspring distribution* μ (a *GW(μ)-forest*, for short) is a random tree $\mathbf{T}$ such that $k_\varnothing(\mathbf{T}) = \infty$ and such that the subtrees $(\theta_{[k]}\mathbf{T})_{k \geq 1}$ stemming from $\varnothing$ are i.i.d. GW(μ)-trees.

We next recall how to encode a (sub)critical GW(μ)-forest $\mathbf{T}$ thanks to three processes: its *Lukasiewicz path*, its *height process* and its *contour process*: since μ is subcritical, it is possible to list all the vertices of $\mathbf{T}$ in the lexicographical order on $\mathbb{U}$; we denote by $(u_l)_{l \in \mathbf{N}}$ this sequence. Then, we set:

$$V_0^{\mathbf{T}} = 0, \quad \forall l \in \mathbb{N}, \quad V_{l+1}^{\mathbf{T}} = V_l^{\mathbf{T}} + k_{u_{l+1}}(\mathbf{T}) - 1 \quad \text{and} \quad \mathtt{Hght}(\mathbf{T}, l) = |u_{l+1}| - 1. \tag{109}$$

The process $(V_l^{\mathbf{T}})_{l \in \mathbb{N}}$ is the *Lukasiewicz path associated with* $\mathbf{T}$ and $(\mathtt{Hght}(\mathbf{T}, l))_{l \in \mathbb{N}}$ is the *height process associated with* $\mathbf{T}$; $V_\cdot^{\mathbf{T}}$ is distributed as a random walk starting from 0 and with jump-law $\nu(k) = \mu(k+1)$, $k \in \mathbb{N} \cup \{-1\}$. The height process $\mathtt{Hght}(\mathbf{T}, \cdot)$ is derived from $V_\cdot^{\mathbf{T}}$ by

$$\forall l \in \mathbb{N}, \quad \mathtt{Hght}(\mathbf{T}, l) = \#\Big\{m \in \{0, \ldots, l-1\} : V_m^{\mathbf{T}} = \inf_{m \leq j \leq l} V_j^{\mathbf{T}}\Big\}. \tag{110}$$

We refer to Le Gall & Le Jan [33] for a proof of (110).

The *contour process of* $\mathbf{T}$ is informally defined as follows: suppose that $\mathbf{T}$ is embedded in the half plane in such a way that edges have length one; we think of a particle starting at time 0 from $\varnothing$ and exploring the tree from the left to the right, backtracking as less as possible and moving continuously along the edges at unit speed. It is clear that the particle crosses each edge twice (upwards first and then downwards). Then, for all $t \in [0, \infty)$, we define $C_t^{\mathbf{T}}$ as the distance at time t of the particle from the root $\varnothing$. The contour process is close to the height process: $C_\cdot^{\mathbf{T}}$. The associated distance $d_{C^{\mathbf{T}}}$ as defined in (49) is the graph distance of $\mathbf{T}$. We refer to Le Gall & D. [21] (Section 2.4, Chapter 2, pp. 61-62) for a formal definition and the connection with the height process.

4.1.2 Coding processes related to the Markovian queueing system.

The Markovian LIFO queueing system. We fix the set of weights $\mathtt{w} = (w_1, \ldots, w_n, 0, 0, \ldots) \in \ell_f^\downarrow$ and we consider the Markovian LIFO queueing system that is described as follows: the server is visited be an infinite number of clients; the clients arrive according to a Poisson process with unit rate; each client has a *type* that is a positive integer ranging in $\{1, \ldots, n\}$; the amount of time of service required by a client of type j is w_j; the types are i.i.d. with law

$$\nu_{\mathtt{w}} = \frac{1}{\sigma_1(\mathtt{w})} \sum_{j \geq 1} w_j \delta_j \tag{111}$$

Let τ_k stand for the time of arrival of the k-th client in the queue and let $\mathtt{J}_k$ stand for her/his type; then, the queueing system is entirely characterised by the the following point measure:

$$\mathscr{X}_{\mathtt{w}} = \sum_{k \geq 1} \delta_{(\tau_k, \mathtt{J}_k)}, \tag{112}$$

that is distributed as a Poisson point measure on $[0,\infty)\times\{1,\dots,n\}$ whith intensity $\ell\otimes\nu_{\mathtt{w}}$, where ℓ stands for the Lebesgue measure on $[0,\infty)$. We next introduce the following:

$$\forall t\in[0,\infty),\quad X^{\mathtt{w}}_t=-t+\sum_{k\geq 1}w_{\mathtt{J}_k}\mathbf{1}_{[0,t]}(\tau_k)\quad\text{and}\quad I^{\mathtt{w}}_t=\inf_{s\in[0,t]}X^{\mathtt{w}}_s. \tag{113}$$

Then, $X^{\mathtt{w}}_t-I^{\mathtt{w}}_t$ is interpreted as the load of the Markovian queueing system at time t and $X^{\mathtt{w}}_t$ is the algebraic load of the queue.

We next denote by $\mathscr{G}_t$ the sigma field generated by $\sum_{k\geq 1}\mathbf{1}_{\{\tau_k\leq t\}}\delta_{(\tau_k,\mathtt{J}_k)}$ and completed with the $\mathbf{P}$-negligible events. We recall the following fact: let T be an a.s. finite $(\mathscr{G}_t)$-stopping time; we set

$$\mathscr{X}^T_{\mathtt{w}}=\sum_{k\geq 1}\mathbf{1}_{\{\tau_k>T\}}\delta_{(\tau_k-T,\mathtt{J}_k)}\quad\text{and}\quad X^{\mathtt{w},T}_s=X^{\mathtt{w}}_{T+s}-X^{\mathtt{w}}_T,\ s\in[0,\infty). \tag{114}$$

Then $\mathscr{X}^T_{\mathtt{w}}$ is independent of $\mathscr{G}_T$ and it has the same law as $\mathscr{X}_{\mathtt{w}}$. In particular it implies that $X^{\mathtt{w}}$ is a $(\mathscr{G}_t)$-spectrally positive Lévy process. We shall assume that $X^{\mathtt{w}}$ does not drift to ∞: this is equivalent to assume that $\mathbf{E}[X^{\mathtt{w}}_1]\leq 0$, namely,

$$\frac{\sigma_2(\mathtt{w})}{\sigma_1(\mathtt{w})}\leq 1\ . \tag{115}$$

The tree generated by the Markovian queueing system. The LIFO queueing system governed by $\mathscr{X}_{\mathtt{w}}$ generates a tree $\mathbf{T}_{\mathtt{w}}$ that can be informaly defined as follows.

> *The clients are the vertices and the server is the root (or the ancestor); the j-th client to enter the queue is a child of the i-th one if the j-th client enters when the i-th client is served; among siblings, the clients are ordered according to their time of arrival.*

A more formal definition is given below and we see that $\mathbf{T}_{\mathtt{w}}$ is a GW-forest. We also show below that the Lukasiewicz path $(V^{\mathbf{T}_{\mathtt{w}}}_k)$ and the contour process $(C^{\mathbf{T}_{\mathtt{w}}}_t)$ are close to $X^{\mathtt{w}}$ and $H^{\mathtt{w}}$, as defined (32). We prove estimates that are used in the proof of Proposition 2.12.

First we formally define $\mathbf{T}_{\mathtt{w}}$ as a random element of $\mathbb{T}$, as defined in Section 4.1.1. To that end, recall from (114) that $X^{\mathtt{w},t}_s=X^{\mathtt{w}}_{t+s}-X^{\mathtt{w}}_t$ and set

$$\forall t,x\in[0,\infty),\quad M(t,x)=\#\big\{s\in[0,\infty):\ X^{\mathtt{w},t}_s>X^{\mathtt{w},t}_{s-}=\inf_{r\in[0,s]}X^{\mathtt{w},t}_r\geq -x\big\}. \tag{116}$$

Among the clients waiting in the line by time t, exactly $M(t,x)$ of them will have entered the queue when the load will have decreased of x, after time t. Let T be an a.s. finite $(\mathscr{G}_t)$-stopping time; since we assume that $X^{\mathtt{w}}$ does not drift to ∞, it is easy to check the following: $(M(T,x))_{x\in[0,\infty)}$ *is a homogeneous Poisson process on* $[0,\infty)$ *with unit rate and it is independent of* $\mathscr{G}_T$.

To each client we inductively associate a label in $\mathbb{U}$ in order to define $\mathbf{T}_{\mathtt{w}}$ as a random element of $\mathbb{T}$: by convention, client 0 is the server and we set $u_0=\varnothing$; we denote by $u_k\in\mathbb{U}$ the label associated with the k-th client; we suppose that $u_0,\dots,u_{k-1}$ have been already defined and we denote by $m\in\{0,\dots,k-1\}$ the client that is the direct parent of Client k: namely

$$m=\sup\Big\{l\in\{0,\dots,k-1\}:X^{\mathtt{w}}_{\tau_l-}<\inf_{t\in[\tau_l,\tau_k]}X^{\mathtt{w}}_t\Big\},$$

with the conventions $\tau_0=0$ and $X^{\mathtt{w}}_{0-}=-\infty$. Then, the number of clients who arrived before Client k (including Client k) and whose direct parent is Client m is given by

$$r_k=M\Big(\tau_m\,,-\inf_{s\in[0,\tau_k-\tau_m]}X^{\mathtt{w},\tau_m}_s\Big).$$

We then define u_k by the concatenation of the words u_m and $[r_k]$: namely $u_k=u_m*[r_k]$ (with the notation of Section (4.1.1). This inductively defines the sequence $(u_k)_{k\in\mathbb{N}}$ of the labels of the clients; here, the lexicographical order excatly corresponds to the order of arrival of the clients. The tree generated by the queueing system is then formally given by $\mathbf{T}_{\mathtt{w}} = \{u_k; k \in \mathbb{N}\}$. We let the reader check that $\mathbf{T}_{\mathtt{w}}$ is Galton-Watson forest (as defined in Section 4.1.1) whose offspring distribution $\mu_{\mathtt{w}}$ is defined by

$$\forall k \in \mathbb{N}, \quad \mu_{\mathtt{w}}(k) = \sum_{1 \le j \le n} \frac{w_j^{k+1} e^{-w_j}}{\sigma_1(\mathtt{w})\, k!} \, . \tag{117}$$

Actually, the subtree $\theta_{u_k} \mathbf{T}_{\mathtt{w}}$ stemming from u_k is completely determined by $\Delta X^{\mathtt{w}}_{\tau_k}$ and the path $(X^{\mathtt{w},\tau_k}_s)_{s \in [0,\tau]}$ where $\tau = \inf\{s \in [0,\infty) : X^{\mathtt{w},\tau_k}_s < -\Delta X_{\tau_k}\}$. In particular, we get

$$k_{u_k}(\mathbf{T}_{\mathtt{w}}) = M(\tau_k, \Delta X^{\mathtt{w}}_{\tau_k}). \tag{118}$$

Then, conditionally given $\mathscr{G}_{\tau_k}$, $k_{u_k}(\mathbf{T}_{\mathtt{w}})$ is distributed as a Poisson r.v. with parameter $\Delta X^{\mathtt{w}}_{\tau_k} = \mathtt{w}_{\mathtt{J}_k}$ that has law $\nu_{\mathtt{w}}$, which explains the form of the offspring distribution $\mu_{\mathtt{w}}$. Since $\sum_{k \ge 0} k \mu_k(k) = \sigma_2(\mathtt{w})/\sigma_1(\mathtt{w})$, (115) implies that $\mu_{\mathtt{w}}$ is (sub)critical.

The Lukasiewicz path associated with $\mathbf{T}_{\mathtt{w}}$: estimates. Recall from Section 4.1.1 the definition of $(V^{\mathbf{T}_{\mathtt{w}}}_k)_{k \in \mathbb{N}}$, the Lukasiewicz path of $\mathbf{T}_{\mathtt{w}}$; recall from (113) the definition of $X^{\mathtt{w}}$ and $I^{\mathtt{w}}$; recall from (116) the definition of $M(t,x)$. We set

$$\forall t \in [0,\infty), \quad N^{\mathtt{w}}(t) = \sum_{k \ge 1} \mathbf{1}_{[0,t]}(\tau_k) \, . \tag{119}$$

Clearly $N^{\mathtt{w}}$ is a homogeneous Poisson process with unit rate. The following lemma expresses $V^{\mathbf{T}_{\mathtt{w}}}$ in terms of $X^{\mathtt{w}}$ and compare these two processes.

Lemma 4.1 *We keep the notation from above. Then, $\mathbf{P}$-a.s. for all $t \in [0,\infty)$*

$$V^{\mathbf{T}_{\mathtt{w}}}_{N^{\mathtt{w}}(t)} = M\big(t, X^{\mathtt{w}}_t - I^{\mathtt{w}}_t\big) - M(0, -I^{\mathtt{w}}_t) \tag{120}$$

and for all $a, x \in (0,\infty)$, we get

$$\mathbf{P}\big(\big|V^{\mathbf{T}_{\mathtt{w}}}_{N^{\mathtt{w}}(t)} - X^{\mathtt{w}}_t\big| > 2a\big) \le 1 \wedge (4x/a^2) + \mathbf{P}\big(-I^{\mathtt{w}}_t > x\big) + \mathbf{E}\big[1 \wedge \big((X^{\mathtt{w}}_t - I^{\mathtt{w}}_t)/a^2\big)\big]. \tag{121}$$

Proof. We denote by Q_t the process on the right hand side of (120); the set of its jump-times is included in $\{\tau_l; l \in \mathbb{N}^*\}$ that is the set of jump-times of $N^{\mathtt{w}}$. To prove (120) it is therefore sufficient to prove that

$$Q_{\tau_{k+1}} - Q_{\tau_k} = M(\tau_{k+1}, \Delta X^{\mathtt{w}}_{\tau_{k+1}}) - 1 = k_{u_{k+1}}(\mathbf{T}_{\mathtt{w}}) - 1 = V^{\mathbf{T}_{\mathtt{w}}}_{k+1} - V^{\mathbf{T}_{\mathtt{w}}}_k \, . \tag{122}$$

Actually, one only needs to prove the first equality since the second one is (118) and the third one is (109). From the definition (116) of $M(t,x)$, we easily get that $M(\tau_l, X^{\mathtt{w}}_{\tau_l} - I^{\mathtt{w}}_{\tau_l}) = \#\{s \in (\tau_l, \infty) : X^{\mathtt{w}}_s > X^{\mathtt{w}}_{s-} = \inf_{[\tau_l, s]} X^{\mathtt{w}} > I^{\mathtt{w}}_{\tau_l}\}$. We fix $k \in \mathbb{N}$ and we set $\sigma = \inf\{s > \tau_{k+1} : X^{\mathtt{w}}_s < X^{\mathtt{w}}_{\tau_{k+1}-}\}$, which is well-defined since $X^{\mathtt{w}}$ does not drift to ∞. Then, for all $s > \sigma$, observe that $\inf_{[\tau_k, s]} X^{\mathtt{w}} = \inf_{[\tau_{k+1}, s]} X^{\mathtt{w}}$ and note that τ_{k+1} is the unique jump-time of $(\tau_k, \sigma]$ and we check that $\#\{s \in (\tau_k, \sigma] : X^{\mathtt{w}}_s > X^{\mathtt{w}}_{s-} = \inf_{[\tau_k, s]} X^{\mathtt{w}} > I^{\mathtt{w}}_{\tau_k}\} = \mathbf{1}_{\{I^{\mathtt{w}}(\tau_k) = I^{\mathtt{w}}(\tau_{k+1})\}}$, which is equal to $1 - M(0, -I^{\mathtt{w}}_{\tau_{k+1}}) + M(0, -I^{\mathtt{w}}_{\tau_k})$. Thus,

$$\begin{aligned} Q_{\tau_{k+1}} - Q_{\tau_k} &= M(\tau_{k+1}, X^{\mathtt{w}}_{\tau_{k+1}} - I^{\mathtt{w}}_{\tau_{k+1}}) - M(\tau_k, X^{\mathtt{w}}_{\tau_k} - I^{\mathtt{w}}_{\tau_k}) - M(0, -I^{\mathtt{w}}_{\tau_{k+1}}) + M(0, -I^{\mathtt{w}}_{\tau_k}) \\ &= \#\big\{s \in (\tau_{k+1}, \sigma] : X^{\mathtt{w}}_s > X^{\mathtt{w}}_{s-} = \inf_{[\tau_{k+1}, s]} X^{\mathtt{w}}\big\} - 1 \end{aligned}$$

and the last term equals $M(\tau_{k+1}, \Delta X^{\mathtt{w}}_{\tau_{k+1}}) - 1$, which proves (122).

Let us prove (121). We fix $t \in [0,\infty)$ and to simplify we set $Z = X^{\mathtt{w}}_t - I^{\mathtt{w}}_t$ and $Y = -I^{\mathtt{w}}_t$. Since $M(t,\cdot)$ is independent of $\mathscr{G}_t$ and distributed as a homogeneous Poisson process with unit rate, $\mathbf{E}\big[(M(t,Z)-Z)^2|\mathscr{G}_t\big] = Z$; thus $\mathbf{P}(|M(t,Z)-Z|>a) \leq \mathbf{E}[1\wedge(Z/a^2)]$. For all $x \in (0,\infty)$, we also get $\mathbf{P}(|M(0,Y)-Y|>a) \leq \mathbf{P}(\sup_{y\in[0,x]}|M(0,y)-y|>a) + \mathbf{P}(Y>x) \leq 1\wedge(4x/a^2) + \mathbf{P}(Y>x)$ by Doob L^2 inequality for martingales. This easily completes the proof of(121). ■

The contour of $\mathbf{T}_{\mathtt{w}}$: estimates. Recall from (32) that $H^{\mathtt{w}}_t$ stands for the number of clients waiting in the line right after time t. More formally, for all $s,t \in [0,\infty)$ such that $s \leq t$, we set $I^{\mathtt{w},s}_t = \inf_{r\in[s,t]} X^{\mathtt{w}}_r$ and we set

$$H^{\mathtt{w}}_t = \#\big\{s\in[0,t] \ : \ I^{\mathtt{w},s-}_t < I^{\mathtt{w},s}_t\big\}. \tag{123}$$

The process $H^{\mathtt{w}}$ is called the *height process associated with* $X^{\mathtt{w}}$, by analogy with (110). Actually, $H^{\mathtt{w}}$ is closer to the contour process of $\mathbf{T}_{\mathtt{w}}$. To see this, recall that $(u_k)_{k\in\mathbb{N}}$ stands for the sequence of vertices of $\mathbf{T}_{\mathtt{w}}$ listed in the lexicographical order; we identify u_k with the k-th client to enter the queueing system. For all $t\in[0,\infty)$, we denote by $\mathbf{u}(t)$ the client currently served right after time t: namely $\mathbf{u}(t) = u_k$ where $k = \sup\{l\in\mathbb{N}: \tau_l \leq t \text{ and } X^{\mathtt{w}}_{\tau_l -} < \inf_{s\in[\tau_l,t]} X^{\mathtt{w}}_s\}$. Then, the length of the word $\mathbf{u}(t)$ is the number of clients waiting in the line right after time t: $|\mathbf{u}(t)| = H^{\mathtt{w}}_t$.

We next denote by $(\sigma_m)_{m\geq 1}$ the sequence of jump-times of $H^{\mathtt{w}}$: namely, $\sigma_{m+1} = \inf\{s>\sigma_m : H^{\mathtt{w}}_s \neq H^{\mathtt{w}}_{\sigma_m}\}$, for all $m\in\mathbb{N}$, with the convention $\sigma_0 = 0$. We then set

$$\forall t\in[0,\infty), \quad M^{\mathtt{w}}_t = \sum_{m\geq 1} \mathbf{1}_{[0,t]}(\sigma_m) \ . \tag{124}$$

Note that $(\sigma_m)_{m\geq 1}$ is also the sequence of jump-times of $\mathbf{u}$ and observe that for all $m \geq 1$, $(\mathbf{u}(\sigma_{m-1}), \mathbf{u}_{\sigma_m})$ is necessarily an oriented edge of $\mathbf{T}_{\mathtt{w}}$. We then set $\mathbf{T}_{\mathtt{w}}(t) = \{\mathbf{u}(s); s\in[0,t]\}$, that is a subtree of $\mathbf{T}_{\mathtt{w}}$: it represents the set of the clients who entered the queue before time t; $\mathbf{T}_{\mathtt{w}}(t)$ has $N^{\mathtt{w}}(t)+1$ vertices (including the server represented by $\varnothing$); therefore, $\mathbf{T}_{\mathtt{w}}(t)$ has $2N^{\mathtt{w}}(t)$ oriented edges. Among the $2N^{\mathtt{w}}(t)$ oriented edges of $\mathbf{T}_{\mathtt{w}}(t)$, the $|\mathbf{u}(t)|$ edges going down from $\mathbf{u}(t)$ to $\varnothing$ does not belong to the subset $\{(\mathbf{u}(\sigma_{m-1}), \mathbf{u}(\sigma_m)); m\geq 1: \sigma_m \leq t\}$. Thus, we get

$$\forall t\in[0,\infty), \quad M^{\mathtt{w}}_t = 2N^{\mathtt{w}}(t) - H^{\mathtt{w}}_t \ . \tag{125}$$

Recall from Section 4.1.1 the definition of the contour and the height processes of $\mathbf{T}_{\mathtt{w}}$, denoted resp. by $(C^{\mathbf{T}_{\mathtt{w}}}_t)$ and $(\mathtt{Hght}^{\mathbf{T}_{\mathtt{w}}}_k)$. Then, observe that

$$\forall t\in[0,\infty), \quad C^{\mathbf{T}_{\mathtt{w}}}_{M^{\mathtt{w}}(t)} = H^{\mathtt{w}}_t \quad \text{and} \quad \sup_{s\in[0,t]} H^{\mathtt{w}}_s \leq 1 + \sup_{s\in[0,t]} \mathtt{Hght}^{\mathbf{T}_{\mathtt{w}}}_{N^{\mathtt{w}}_s}. \tag{126}$$

Since $N^{\mathtt{w}}$ is a homogeneous Poisson process with unit rate, Doob's L^2-inequality combined with (125) and (126) imply the following inequality:

$$\forall t,a\in(0,\infty), \quad \mathbf{P}\big(\sup_{s\in[0,t]}|M^{\mathtt{w}}_s - 2s| > 2a\big) \leq 1\wedge(16t/a^2) + \mathbf{P}\big(1 + \sup_{s\in[0,t]} \mathtt{Hght}^{\mathbf{T}_{\mathtt{w}}}_{N^{\mathtt{w}}_s} > a\big). \tag{127}$$

4.2 Colouring the clients of the Markovian queueing system.

4.2.1 Formal definition of the colouring.

We fix the set of weights $\mathtt{w} = (w_1,\ldots,w_n,0,0,\ldots) \in \ell^{\downarrow}_f$. Recall from (20) that $\mathscr{X}_{\mathtt{w}} = \sum_{k\geq 1}\delta_{(\tau_k,\mathtt{J}_k)}$ is a Poisson point measure on $[0,\infty)\times\{1,\ldots,n\}$ with intensity $\ell\otimes\nu_{\mathtt{w}}$ where ℓ stands for the Lebesgue measure on $[0,\infty)$ and where $\nu_{\mathtt{w}} = \frac{1}{\sigma_1(\mathtt{w})}\sum_{1\leq j\leq n} w_j\delta_j$. As explained in Section 4.1.2, the point measure $\mathscr{X}_{\mathtt{w}}$ governs a Markovian single-server LIFO queueing system: τ_k is the time of arrival of the k-th client to enters the queueing system, $\mathtt{J}_k$ stands for his type; recall that the service-time required by a client of type j is w_j; recall from (21) that $X^{\mathtt{w}}_t = -t + \sum_{k\geq 1} w_{\mathtt{J}_k}\mathbf{1}_{[0,t]}(\tau_k)$;

recall that we assume that $X^{\mathtt{w}}$ does not drift to ∞, which is equivalent to assume $\sigma_2(\mathtt{w})/\sigma_1(\mathtt{w}) \leq 1$. Recall that $\mathscr{G}_t$ stands for the sigma field generated by $\sum_{k\geq 1} \mathbf{1}_{\{\tau_k \leq t\}} \delta_{(\tau_k, \mathtt{J}_k)}$ and completed with the $\mathbf{P}$-negligible events. Recall from (114) that $X^{\mathtt{w}}$ is a $(\mathscr{G}_t)$-spectrally positive Lévy process.

To embed the non-Markovian queueing system into the Markovian one, recall that we colour the clients in *blue* or *red* as follows:

> `Colouring rules.` *If the type* $\mathtt{J}_k$ *already appeared among the types of the blue clients who previously entered the queueing system, then the* k*-th client is red; otherwise the* k*-th client inherits his colour from the colour of the client currently served when he arrives (if the server is idle when the kth client arrives, then his colour is blue).*

More precisely, we recursively define an increasing sequence of times $(\mathfrak{s}_l)_{l\geq 0}$ and an associated sequence of marks $\mathfrak{u}_l \in \{\mathtt{blue}, \mathtt{red}, \mathtt{end}\}$ and $\mathfrak{J}_l \in \{0, \ldots, n\}$ that are interpreted as follows:

- $\mathfrak{s}_l$ is either the *time of arrival of a blue client*, $\mathfrak{J}_l$ is her/his type and $\mathfrak{u}_l = \mathtt{blue}$,
- or $\mathfrak{s}_l$ is the *time of arrival of a red client who interrupts the service of a blue client*, $\mathfrak{J}_l$ is her/his type and $\mathfrak{u}_l = \mathtt{red}$,
- or $\mathfrak{s}_l$ is the *time of departure of a red client such that right after time* $\mathfrak{s}_l$ *either a blue client is served or the line is empty*. In that case, $\mathfrak{J}_l = 0$ and $\mathfrak{u}_l = \mathtt{end}$.

Formally, the sequence $(\mathfrak{s}_l, \mathfrak{u}_l, \mathfrak{J}_l)_{l\geq 0}$ is defined by the following induction: we first set $\mathfrak{s}_0 = 0$, $\mathfrak{u}_0 = \mathtt{end}$ and $\mathfrak{J}_0 = 0$; suppose that $(\mathfrak{s}_m, \mathfrak{u}_m, \mathfrak{J}_m)_{0\leq m\leq l}$ are already defined; we first set

$$\overline{S}_l = \big\{\mathfrak{J}_m;\ 1 \leq m \leq l : \mathfrak{u}_m = \mathtt{blue}\big\} \quad \text{and} \quad S_l = \{1, \ldots, n\} \backslash \overline{S}_l$$

and we define $\mathfrak{s}_{l+1}$, $\mathfrak{u}_{l+1}$ and $\mathfrak{J}_{l+1}$ as follows.

(a) If $\mathfrak{u}_l = \mathtt{red}$, then $\mathfrak{u}_{l+1} = \mathtt{end}$, $\mathfrak{J}_{l+1} = 0$ and $\mathfrak{s}_{l+1} = \inf\{t > \mathfrak{s}_l : X^{\mathtt{w}}_t \leq X^{\mathtt{w}}_{\mathfrak{s}_l -}\}$ that is well-defined since $X^{\mathtt{w}}$ does not drift to ∞.

(b) If $\mathfrak{u}_l \neq \mathtt{red}$, then set $M = \min\{k \geq 1 : \tau_k > \mathfrak{s}_l\}$ and $\mathfrak{s}_{l+1} = \tau_M$, $\mathfrak{J}_{l+1} = \mathtt{J}_M$; if $\mathtt{J}_M \in S_l$, we then set $\mathfrak{u}_{l+1} = \mathtt{blue}$ and if $\mathtt{J}_M \notin S_l$, we set $\mathfrak{u}_{l+1} = \mathtt{red}$.

We then set

$$\mathtt{Blue} = \bigcup_{\substack{l\geq 0,\\ \mathfrak{u}_l \neq \mathtt{red}}} [\mathfrak{s}_l, \mathfrak{s}_{l+1}) \quad \text{and} \quad \mathtt{Red} = \bigcup_{\substack{l\geq 0,\\ \mathfrak{u}_l = \mathtt{red}}} [\mathfrak{s}_l, \mathfrak{s}_{l+1}) \ . \tag{128}$$

It is easy to check that for all $l \in \mathbb{N}$, $\mathfrak{s}_l$ is an a.s. finite $(\mathscr{G}_t)$-stopping time and that $(\mathfrak{u}_l, \mathfrak{J}_l)$ is $\mathscr{G}_{\mathfrak{s}_l}$-measurable. We also recall from (22) that $\Lambda^{\mathtt{b},\mathtt{w}}_t = \int_0^t \mathbf{1}_{\mathtt{Blue}}(s)\, ds$. We also set $\Lambda^{\mathtt{r},\mathtt{w}}_t = \int_0^t \mathbf{1}_{\mathtt{Red}}(s)\, ds$. Next, recall from the recursive definition above that if $\mathfrak{u}_l = \mathtt{red}$, then $\mathfrak{u}_{l+1} = \mathtt{end}$. Thus, $\mathtt{Blue} = [0, \infty) \backslash \mathtt{Red}$, which implies that $\Lambda^{\mathtt{b},\mathtt{w}}_t + \Lambda^{\mathtt{r},\mathtt{w}}_t = t$. Note that $\overline{\mathtt{Blue}} \cap \mathtt{Red} = \{\mathfrak{s}_l;\ l \in \mathbb{N} : \mathfrak{u}_l = \mathtt{red}\}$, namely it is the set of arrival-times of red clients interrupting the service of blue clients.

4.2.2 Proof of Lemma 2.2.

Recall that τ_k is the time of arrival of the k-th client. By convenience we set $\tau_0 = 0$. We define the increasing sequences of positives integers $(k(m))_{m\in\mathbb{N}}$ and $(l(m))_{m\in\mathbb{N}}$ by setting

$$\{k(m); m \in \mathbb{N}\} = \{k \in \mathbb{N} : \tau_k \in \overline{\mathtt{Blue}}\} \quad \text{and} \quad \{l(m); m \in \mathbb{N}\} = \{l \in \mathbb{N} : \mathfrak{u}_l \neq \mathtt{end}\}.$$

Note that $l(0) = k(0) = 0$ and observe that for all $l \geq 1$, $\mathfrak{s}_l \in \{\tau_k; k \geq 1\}$ iff $\mathfrak{u}_l \neq \mathtt{end}$. Thus,

$$\{\tau_{k(m)}; m \in \mathbb{N}\} = \{\mathfrak{s}_{l(m)}; m \in \mathbb{N}\} \ .$$

For all $m \in \mathbb{N}$, we define $(\mathtt{e}_{m+1}, \overline{\mathtt{J}}_{m+1})$ by setting

$$(\mathtt{e}_{m+1}, \overline{\mathtt{J}}_{m+1}) = \big(\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m+1)}} - \Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m)}}, \mathtt{J}_{k(m+1)}\big) \ .$$

We consider the two following cases.

(c) Suppose that $\mathtt{u}_{l(m)}=\mathtt{blue}$. Then, $\mathfrak{s}_{l(m)+1}\in\overline{\mathtt{Blue}}$ and $l(m+1)=l(m)+1$ (and $k(m+1)=k(m)+1$). Then,

$$\mathtt{e}_{m+1}=\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m+1)}}-\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m)}}=\mathfrak{s}_{l(m)+1}-\mathfrak{s}_{l(m)}=\min\{\tau_k-\mathfrak{s}_{l(m)};k\geq 1:\tau_k>\mathfrak{s}_{l(m)}\}\ .$$

Thus, $(\mathtt{e}_{m+1},\overline{\mathtt{J}}_{m+1})$ is the first atom of the shifted Poisson point measure $\mathscr{X}_{\mathtt{w}}^{\cdot\mathfrak{s}_{l(m)}}$ as defined by (114); $(\mathtt{e}_{m+1},\overline{\mathtt{J}}_{m+1})$ are independent and they are also independent of $\mathscr{G}_{\mathfrak{s}_{l(m)}}$; moreover, $\mathtt{e}_{m+1}$ has an exponential law with unit mean and $\overline{\mathtt{J}}_{m+1}$ has law $\nu_{\mathtt{w}}$.

(d) Suppose that $\mathtt{u}_{l(m)}=\mathtt{red}$, then

$$[\mathfrak{s}_{l(m)},\mathfrak{s}_{l(m)+1})\subset\mathtt{Red},\ \mathtt{u}_{l(m)+1}=\mathtt{end}\quad\text{and thus}\quad[\mathfrak{s}_{l(m)+1},\mathfrak{s}_{l(m)+2})\subset\mathtt{Blue}\ .$$

It implies that $l(m+1)=l(m)+2$, $\tau_{k(m+1)}=\mathfrak{s}_{l(m)+2}$ and

$$\mathtt{e}_{m+1}=\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m+1)}}-\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m)}}=\mathfrak{s}_{l(m)+2}-\mathfrak{s}_{l(m)+1}.$$

Thus, $(\mathtt{e}_{m+1},\overline{\mathtt{J}}_{m+1})$ is the first atom of the shifted Poisson point measure $\mathscr{X}_{\mathtt{w}}^{\cdot\mathfrak{s}_{l(m)+1}}$: therefore, $(\mathtt{e}_{m+1},\overline{\mathtt{J}}_{m+1})$ are independent, thay are also independent of $\mathscr{G}_{\mathfrak{s}_{l(m)+1}}$; moreover, $\mathtt{e}_{m+1}$ has an exponential law with unit mean and $\overline{\mathtt{J}}_{m+1}$ has law $\nu_{\mathtt{w}}$.

Since $(\mathtt{e}_m,\overline{\mathtt{J}}_m)$ is clearly $\mathscr{G}_{\mathfrak{s}_{l(m)}}$-measurable, this first entails that

$$\mathscr{X}^{\mathtt{b}}_{\mathtt{w}}=\sum_{m\geq 1}\delta_{(\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_{k(m)}},\mathtt{J}_{k(m)})}=\sum_{k\geq 1}\mathbf{1}_{\overline{\mathtt{Blue}}}(\tau_k)\delta_{(\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_k},\mathtt{J}_k)}$$

is a Poisson point measure on $[0,\infty)\times\{1,\dots,n\}$ with intensity $\ell\otimes\nu_{\mathtt{w}}$: $\mathscr{X}^{\mathtt{b}}_{\mathtt{w}}$ has the same law as $\mathscr{X}_{\mathtt{w}}$.

Moreover, Cases (b) implies the following: denote by $\mathbf{1}_{\{\mathtt{u}_{l(m)}=\mathtt{red}\}}\mathscr{X}^{\mathtt{b}}_{\mathtt{w}}$ the measure that is equal to $\mathscr{X}^{\mathtt{b}}_{\mathtt{w}}$ if $\mathtt{u}_{l(m)}=\mathtt{red}$ and that is equal to 0 otherwise. Then, for all $m\geq 1$,

(129) $\mathbf{1}_{\{\mathtt{u}_{l(m)}=\mathtt{red}\}}\mathscr{X}^{\mathtt{b}}_{\mathtt{w}}$ *is measurable with respect to*

the σ-field generated by $\mathscr{G}_{\mathfrak{s}_{l(m)}}$ and the shifted point measure $\mathscr{X}_{\mathtt{w}}^{\cdot\mathfrak{s}_{l(m)+1}}$.

We next define the increasing sequence of integers $(j(p))_{p\geq 1}$ by

(130) $$\{j(p);p\geq 1\}=\{l\in\mathbb{N}:\mathtt{u}_l=\mathtt{red}\}\quad\text{and thus}\quad\mathtt{Red}=\bigcup_{p\geq 1}[\mathfrak{s}_{j(p)},\mathfrak{s}_{j(p)+1}).$$

We then define the "red processes" the following:

(131) $$\forall p\geq 1,\quad Z^p=\big(X^{\mathtt{w}}_{(\mathfrak{s}_{j(p)}+t)\wedge\mathfrak{s}_{j(p)+1}}-X^{\mathtt{w}}_{\mathfrak{s}_{j(p)}}\big)_{t\in[0,\infty)}\quad\text{and}\quad\zeta_p=\mathfrak{s}_{j(p)+1}-\mathfrak{s}_{j(p)}.$$

Note that ζ_p is the duration of Z^p. By (a) in the recursive definition of the $(\mathfrak{s}_l)$, ζ_p is the first time that the process $X^{\mathtt{w}}_{\mathfrak{s}_{j(p)}+\cdot}-X^{\mathtt{w}}_{\mathfrak{s}_{j(p)}}$ has been below $-w_{\mathfrak{J}_{j(p)}}$. Since $X^{\mathtt{w}}$ has no negative jumps, it entails that $Z^p_{\zeta_p}=-w_{\mathfrak{J}_{j(p)}}$. By convenience, we next set

$$\xi_0=0\quad\text{and}\quad\forall p\geq 1,\quad\xi_p=\sum_{1\leq q\leq p}\zeta_q\ .$$

Then, for all $t\in[\mathfrak{s}_{j(p)},\mathfrak{s}_{j(p)+1})$, $\Lambda^{\mathtt{r},\mathtt{w}}_t=t-\mathfrak{s}_{j(p)}+\xi_{p-1}$. Recall from (23) that $X^{\mathtt{r},\mathtt{w}}_t=-t+\sum_{k\geq 0}w_{\mathtt{J}_k}\mathbf{1}_{\{\tau_k\in[0,\infty)\backslash\overline{\mathtt{Blue}}\,;\,\Lambda^{\mathtt{r},\mathtt{w}}_{\tau_k}\leq t\}}$. Then observe that for all $t\in[\xi_{p-1},\xi_p)$,

$$\begin{aligned}X^{\mathtt{r},\mathtt{w}}_t&=\sum_{1\leq q<p}\Big(-\zeta_q+\sum_{k\geq 1}w_{\mathtt{J}_k}\mathbf{1}_{\{\tau_k\in(\mathfrak{s}_{j(q)},\mathfrak{s}_{j(q)+1})\}}\Big)-(t-\xi_{p-1})+\sum_{k\geq 1}w_{\mathtt{J}_k}\mathbf{1}_{\{0<\tau_k-\mathfrak{s}_{j(p)}\leq t-\xi_{p-1}\}}\\&=\sum_{1\leq q<p}\Big(X^{\mathtt{w}}_{\mathfrak{s}_{j(q)+1}}-X^{\mathtt{w}}_{\mathfrak{s}_{j(q)}}\Big)+X^{\mathtt{w}}_{\mathfrak{s}_{j(p)}+t-\xi_{p-1}}-X^{\mathtt{w}}_{\mathfrak{s}_{j(p)}}=Z^p_{t-\xi_{p-1}}+\sum_{1\leq q<p}Z^q_{\zeta_q}.\end{aligned}$$

Thus $X^{\mathtt{r},\mathtt{w}}$ is the concatenation of the paths Z^p, $p\geq 1$.

We moreover get the following: recall from (28) that $\gamma_x^{\mathtt{r},\mathtt{w}}=\inf\{t\in[0,\infty):X_t^{\mathtt{r},\mathtt{w}}<-x\}$, for all $x\in[0,\infty)$; by convenience we set

$$x_0=0 \quad \text{and} \quad \forall p\geq 1, \quad x_p=\sum_{1\leq q\leq p} w_{\mathfrak{J}_{j(q)}} .$$

Then, for all $p\geq 1$,

$$\forall t\in[0,\infty), \quad \big(X^{\mathtt{r},\mathtt{w}}_{(\gamma^{\mathtt{r},\mathtt{w}}_{x_{p-1}}+t)\wedge\gamma^{\mathtt{r},\mathtt{w}}_{x_p}}-X^{\mathtt{r},\mathtt{w}}_{\gamma^{\mathtt{r},\mathtt{w}}_{x_{p-1}}}\big)_{t\in[0,\infty)}=Z^p \tag{132}$$

We then complete the proof of Lemma 2.2 as follows. For all $x\in[0,\infty)$, set $\gamma_x^{\mathtt{w}}=\inf\{t\in[0,\infty):X_t^{\mathtt{w}}<-x\}$ and denote by $P(x)$ the law on $\mathbf{D}([0,\infty),\mathbb{R})$ of the stopped process $(X^{\mathtt{w}}_{t\wedge\gamma^{\mathtt{w}}_x})_{t\in[0,\infty)}$. First observe that the law of Z^p conditionally given $\mathscr{G}_{\mathfrak{s}_{j(p)}}$ is $P(w_{\mathfrak{J}_{j(p)}})$. Next, note that since $\mathscr{X}_{\mathtt{w}}^{\mathfrak{s}_{j(p)+1}}$ is independent of $\mathscr{G}_{\mathfrak{s}_{j(p)+1}}$, $\mathscr{X}_{\mathtt{w}}^{\mathfrak{s}_{j(p)+1}}$ is independent from Z^p. By (129), it implies that the law of Z^p conditionally given $\mathscr{X}_{\mathtt{w}}^{\mathtt{b}}$ is $P(w_{\mathfrak{J}_{j(p)}})$. Moreover, since for all $p'>p$, the paths $Z^{p'}$ depends on $\mathscr{X}_{\mathtt{w}}^{\mathfrak{s}_{j(p)+1}}$ and on $(\mathfrak{J}_{j(q)})_{1\leq q\leq p}$ that are $\mathscr{G}_{\mathfrak{s}_{j(p)}}$-measurable r.v, we have proved finally that conditionally given $\mathscr{X}_{\mathtt{w}}^{\mathtt{b}}$, the paths Z^p, $p\geq 1$ are independent and that the conditional law of Z^p is $P(w_{\mathfrak{J}_{j(p)}})$. This easily entails that the concatenation of the paths Z^p is independent of $\mathscr{X}_{\mathtt{w}}^{\mathtt{b}}$ and that it is distributed as $X^{\mathtt{w}}$: namely, $X^{\mathtt{r},\mathtt{w}}$ is independent of $\mathscr{X}_{\mathtt{w}}^{\mathtt{b}}$ and it is distributed as $X^{\mathtt{w}}$. This proves Lemma 2.2. ■

4.2.3 Proof of Lemma 2.3.

We keep the same notations as in Sections 4.2.1 and 4.2.2. Recall that $x_p=\sum_{1\leq q\leq p} w_{\mathfrak{J}_{j(q)}}$ and observe that (132) implies that

$$\gamma^{\mathtt{r},\mathtt{w}}_{x_p}=\zeta_1+\ldots+\zeta_p=\sum_{1\leq q\leq p}\mathfrak{s}_{j(q)+1}-\mathfrak{s}_{j(q)}.$$

Next observe that $\mathtt{Blue}=[0,\mathfrak{s}_{j(1)})\cup\bigcup_{p\geq 1}[\mathfrak{s}_{j(p)+1},\mathfrak{s}_{j(p+1)})$. Thus, $\Lambda_s^{\mathtt{b},\mathtt{w}}=s$, for all $s\in[0,\mathfrak{s}_{j(1)})$ and for all $p\geq 1$ and all $s\in[\mathfrak{s}_{j(p)+1},\mathfrak{s}_{j(p+1)})$, we get

$$\Lambda_s^{\mathtt{b},\mathtt{w}}=s-\sum_{1\leq q\leq p}\mathfrak{s}_{j(q)+1}-\mathfrak{s}_{j(q)}=s-\gamma^{\mathtt{r},\mathtt{w}}_{x_p} \tag{133}$$

Recall from (22) that $\theta_t^{\mathtt{b},\mathtt{w}}=\inf\{s\in[0,\infty):\Lambda_s^{\mathtt{b},\mathtt{w}}>t\}$. Since $\Lambda^{\mathtt{b},\mathtt{w}}$ is continuous, by definition, $\Lambda^{\mathtt{b},\mathtt{w}}(\theta_t^{\mathtt{b},\mathtt{w}})=t$. Note that $\theta_t^{\mathtt{b},\mathtt{w}}\in\mathtt{Blue}$. Thus, either $\theta_t^{\mathtt{b},\mathtt{w}}\in[0,\mathfrak{s}_{j(1)})$ and obviously $\theta_t^{\mathtt{b},\mathtt{w}}=t$, or there is $p\geq 1$ such that $\theta_t^{\mathtt{b},\mathtt{w}}\in[\mathfrak{s}_{j(p)+1},\mathfrak{s}_{j(p+1)})$, and (133) applied to $s=\theta_t^{\mathtt{b},\mathtt{w}}$ entails $t=\theta_t^{\mathtt{b},\mathtt{w}}-\gamma^{\mathtt{r},\mathtt{w}}_{x_p}$. Next, recall from (29) that $A_t^{\mathtt{w}}=\sum_{k\geq 1}w_{\mathtt{J}_k}\mathbf{1}_{\overline{\mathtt{Blue}}\cap\mathtt{Red}}(\tau_k)\,\mathbf{1}_{\{\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_k}\leq t\}}$. Thus,

$$A^{\mathtt{w}}_{\Lambda_t^{\mathtt{b},\mathtt{w}}}=\sum_{k\geq 1}w_{\mathtt{J}_k}\mathbf{1}_{\overline{\mathtt{Blue}}\cap\mathtt{Red}}(\tau_k)\,\mathbf{1}_{\{\tau_k\leq t\}}=\sum_{q\geq 1}w_{\mathfrak{J}_{j(q)}}\mathbf{1}_{\{\mathfrak{s}_{j(q)}\leq t\}} .$$

Thus, if $\theta_t^{\mathtt{b},\mathtt{w}}\in[\mathfrak{s}_{j(p)+1},\mathfrak{s}_{j(p+1)})$, $A_t^{\mathtt{w}}=\sum_{q\geq 1}w_{\mathfrak{J}_{j(q)}}\mathbf{1}_{\{\mathfrak{s}_{j(q)}\leq\theta_t^{\mathtt{b},\mathtt{w}}\}}=x_p$ and the previous argument entail that $t=\theta_t^{\mathtt{b},\mathtt{w}}-\gamma^{\mathtt{r},\mathtt{w}}_{A_t^{\mathtt{w}}}$. This proves (30).

It remains to prove (31). To that end, recall from (24) that

$$\begin{aligned}X^{\mathtt{w}}(\theta_t^{\mathtt{b},\mathtt{w}})&=X^{\mathtt{b},\mathtt{w}}(\Lambda^{\mathtt{b},\mathtt{w}}_{\theta_t^{\mathtt{b},\mathtt{w}}})+X^{\mathtt{r},\mathtt{w}}(\Lambda^{\mathtt{r},\mathtt{w}}_{\theta_t^{\mathtt{b},\mathtt{w}}})=X_t^{\mathtt{b},\mathtt{w}}+X^{\mathtt{r},\mathtt{w}}(\theta_t^{\mathtt{b},\mathtt{w}}-\Lambda^{\mathtt{b},\mathtt{w}}_{\theta_t^{\mathtt{b},\mathtt{w}}})\\&=X_t^{\mathtt{b},\mathtt{w}}+X^{\mathtt{r},\mathtt{w}}(\theta_t^{\mathtt{b},\mathtt{w}}-t)=X_t^{\mathtt{b},\mathtt{w}}+X^{\mathtt{r},\mathtt{w}}(\gamma^{\mathtt{r},\mathtt{w}}_{A_t^{\mathtt{w}}})=X_t^{\mathtt{b},\mathtt{w}}-A_t^{\mathtt{w}},\end{aligned}$$

which implies (31) by (29). ■

4.2.4 Proof of Lemma 2.4.

We keep the same notations as in Sections 4.2.1 and 4.2.2. Let $t \in \mathtt{Red}$. By (130), there exists $p \geq 1$ such that $t \in [\mathfrak{s}_{j(p)}, \mathfrak{s}_{j(p)+1})$. Note that on $[\mathfrak{s}_{j(p)}, \mathfrak{s}_{j(p)+1})$, the process $s \mapsto A^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_s)$ is constant to $x_p = \sum_{1 \leq q \leq p} w_{\mathfrak{J}_{j(q)}}$ and recall that $\Lambda^{\mathtt{r,w}}_t = t - \mathfrak{s}_{j(p)} + \gamma^{\mathtt{r,w}}_{x_{p-1}}$. Consequently, $A^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_t) + X^{\mathtt{r,w}}(\Lambda^{\mathtt{r,w}}_t) = x_p + X^{\mathtt{r,w}}_{t - \mathfrak{s}_{j(p)} + \gamma^{\mathtt{r,w}}_{x_{p-1}}} > 0$, since $\mathfrak{s}_{j(p)+1} - \mathfrak{s}_{j(p)} + \gamma^{\mathtt{r,w}}_{x_{p-1}} = \gamma^{\mathtt{r,w}}_{x_p}$. This easily entails the following:

$$\big\{ t \in [0, \infty) : A^{\mathtt{w}}_{\Lambda^{\mathtt{b,w}}_t} + X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}_t} > 0 \big\} = \mathtt{Red} \quad \text{and} \quad \big\{ t \in [0, \infty) : A^{\mathtt{w}}_{\Lambda^{\mathtt{b,w}}_t} + X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}_t} = 0 \big\} = \mathtt{Blue} \, .$$

Recall from (24) and from (29) that $X^{\mathtt{w}}_t - Y^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_t) = A^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_t) + X^{\mathtt{r,w}}(\Lambda^{\mathtt{r,w}}_t)$. Thus,

$$\big\{ t \in [0, \infty) : X^{\mathtt{w}}_t > Y^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_t) \big\} = \mathtt{Red} \quad \text{and} \quad \big\{ t \in [0, \infty) : X^{\mathtt{w}}_t = Y^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_t) \big\} = \mathtt{Blue}. \tag{134}$$

Since $\mathtt{Blue} \cup \mathtt{Red} = [0, \infty)$, (134) implies that a.s. $X^{\mathtt{w}}_t \geq Y^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_t)$ for all $t \in [0, \infty)$ and we next claim that:

$$\forall s \leq t \qquad \big([s,t] \cap \mathtt{Blue} \neq \emptyset \big) \Longrightarrow \Big(\inf_{r \in [s,t]} X^{\mathtt{w}}_r = \inf_{r \in [s,t]} Y^{\mathtt{w}}_{\Lambda^{\mathtt{b,w}}_r} \Big). \tag{135}$$

Indeed, suppose that $[s,t] \cap \mathtt{Blue} \neq \emptyset$; since $Y^{\mathtt{w}} \circ \Lambda^{\mathtt{b,w}}$ is constant on $\mathtt{Red}$, since $X^{\mathtt{w}} \geq Y^{\mathtt{w}} \circ \Lambda^{\mathtt{b,w}}$ and since these two processes coincide on $\mathtt{Blue}$, $\inf\{X^{\mathtt{w}}_r ; r \in [s,t]\} = \inf\{X^{\mathtt{w}}_r ; \mathtt{Blue} \cap [s,t]\}$, which easily implies (135).

Recall from (18) that $J^{\mathtt{w},s'}_{t'} = \inf_{[s',t']} Y^{\mathtt{w}}$ and that $\mathcal{H}^{\mathtt{w}}_{t'} = \# \mathcal{J}_{t'}$ where $\mathcal{J}_{t'} = \{ s' \in [0,t'] : J^{\mathtt{w},s'-}_{t'} < J^{\mathtt{w},s'}_{t'} \}$; recall from (32) that $I^{\mathtt{w},s}_t = \inf_{[s,t]} X^{\mathtt{w}}$ and that $H^{\mathtt{w}}_t = \# \mathcal{K}_t$, where $\mathcal{K}_t = \{ s \in [0,t] : I^{\mathtt{w},s-}_t < I^{\mathtt{w},s}_t \}$. Since $\Lambda^{\mathtt{b,w}}$ is continuous and non-decreasing, we get $\inf_{r \in [s,t]} Y^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_r) = \inf_{r' \in [\Lambda^{\mathtt{b,w}}_s, \Lambda^{\mathtt{b,w}}_t]} Y^{\mathtt{w}}_{r'}$. Thus, by (135), for all $t \in \mathtt{Blue}$, we get $\mathcal{J}_{\Lambda^{\mathtt{b,w}}_t} = \{ \Lambda^{\mathtt{b,w}}_s ; s \in \mathcal{K}_t \}$. Then, note that $\mathcal{K}_t \subset \mathtt{Blue}$ and since $\Lambda^{\mathtt{b,w}}$ is increasing on $\mathtt{Blue}$ we get

$$\forall t \in \mathtt{Blue}, \quad H^{\mathtt{w}}_t = \# \mathcal{K}_t = \# \mathcal{J}_{\Lambda^{\mathtt{b,w}}_t} = \mathcal{H}^{\mathtt{w}}_{\Lambda^{\mathtt{b,w}}_t}. \tag{136}$$

Since for all $t \in [0, \infty)$, $\theta^{\mathtt{b,w}}_t \in \mathtt{Blue}$, and since $\Lambda^{\mathtt{b,w}}(\theta^{\mathtt{b,w}}_t) = t$, by definition, (136) entails: $H^{\mathtt{w}}(\theta^{\mathtt{b,w}}_t) = \mathcal{H}^{\mathtt{w}}_t$. This proves (34) and Lemma 2.4. ■

4.3 Estimates on the coloured processes.

We keep the same definition and the same notation as in Section 4.2. In this section, we provide estimates for $A^{\mathtt{w}}$ and $X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}$ and $X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}}$ that are used in the proof of Theorem 2.14.

4.3.1 Increments of $A^{\mathtt{w}}$.

By Lemma 2.2, $\mathscr{X}^{\mathtt{b}}_{\mathtt{w}} = \sum_{k \geq 1} \mathbf{1}_{\overline{\mathtt{Blue}}}(\tau_k) \, \delta_{(\Lambda^{\mathtt{b,w}}_{\tau_k}, J_k)}$ is Poisson point measure on $[0, \infty) \times \{1, \ldots, n\}$ with intensity $\ell \otimes \nu_{\mathtt{w}}$. Recall from (25) that for all $j \in \{1, \ldots, n\}$, and all $t \in [0, \infty)$, $N^{\mathtt{w}}_j(t) = \mathscr{X}^{\mathtt{b}}_{\mathtt{w}}\big([0,t] \times \{j\} \big)$. Then, $N^{\mathtt{w}}_j$ are independent Poisson processes with respective rates $w_j / \sigma_1(\mathtt{w})$. We also recall from (25) that $E^{\mathtt{w}}_j = \inf \big\{ t \in [0, \infty) : \mathscr{X}^{\mathtt{b}}_{\mathtt{w}}([0,t] \times \{j\}) = 1 \big\}$; therefore, the r.v. $(\frac{w_j}{\sigma_1(\mathtt{w})} E_j)_{1 \leq j \leq n}$ are i.i.d. exponential r.v. with unit mean. Next, recall from (26) and from (27) that

$$\mathscr{Y}_{\mathtt{w}} = \sum_{1 \leq j \leq n} \delta_{(E^{\mathtt{w}}_j, j)} \quad \text{and} \quad \mathscr{X}^{\mathtt{r/b}}_{\mathtt{w}} = \mathscr{X}^{\mathtt{b}}_{\mathtt{w}} - \mathscr{Y}_{\mathtt{w}} = \sum_{k \geq 1} \mathbf{1}_{\overline{\mathtt{Blue}} \cap \mathtt{Red}}(\tau_k) \, \delta_{(\Lambda^{\mathtt{b,w}}_{\tau_k}, J_k)}. \tag{137}$$

Therefore, $\mathscr{X}_{\mathtt{w}}^{\mathtt{r}/\mathtt{b}}\big([0,t]\times\{j\}\big)=(N_j^{\mathtt{w}}(t)-1)_+$. We also recall from (28) that

$$Y_t^{\mathtt{w}}=-t+\sum_{1\leq j\leq n} w_j \mathbf{1}_{\{E_j^{\mathtt{w}}\leq t\}} \quad \text{and} \tag{138}$$

$$A_t^{\mathtt{w}}=\sum_{1\leq j\leq n} w_j (N_j^{\mathtt{w}}(t)-1)_+ = \sum_{k\geq 1} w_{\mathtt{J}_k} \mathbf{1}_{\overline{\mathtt{Blue}}\cap\mathtt{Red}}(\tau_k)\, \mathbf{1}_{\{\Lambda^{\mathtt{b},\mathtt{w}}_{\tau_k}\leq t\}}.$$

Let $(\mathscr{F}_t)_{t\in[0,\infty)}$ be a filtration such that for all for all $j\geq 1$, $N_j^{\mathtt{w}}$ is a $(\mathscr{F}_t)$-homogeneous Poisson process: namely, $N_j^{\mathtt{w}}$ is $(\mathscr{F}_t)$-adapted and for all $t\in[0,\infty)$, $N_j(\cdot+t)-N_j(t)$ is independent of $\mathscr{F}_t$. For all a.s. finite $(\mathscr{F}_t)$-stopping time T, we set $N_j^{\mathtt{w},T}(t)=N_j^{\mathtt{w}}(T+t)-N_j^{\mathtt{w}}(T)$. Thus, the $(N_j^{\mathtt{w},T})_{j\geq 1}$ are independent of $\mathscr{F}_T$ and distributed as $(N_j^{\mathtt{w}})_{j\geq 1}$. By convenience we also set $A_t^{\mathtt{w},T}=\sum_{1\leq j\leq n} w_j (N_j^{\mathtt{w},T}(t)-1)_+$. Then, $A^{\mathtt{w},T}$ is independent of $\mathscr{F}_T$ and distributed as $A^{\mathtt{w}}$. It is also easy to observe that

$$A^{\mathtt{w}}_{T+t}-A^{\mathtt{w}}_T=A_t^{\mathtt{w},T}+\sum_{j\geq 1} w_j \mathbf{1}_{\{E_j^{\mathtt{w}}\leq T\}}\mathbf{1}_{\{N_j^{\mathtt{w},T}(t)\geq 1\}}. \tag{139}$$

The Markov inequality and easy calculations combined with (139) immediately entail the following lemma.

Lemma 4.2 *We keep the notation from above. For all $(\mathscr{F}_t)$-stopping time T and all $a,t_0,t\in(0,\infty)$,*

$$a\,\mathbf{P}\big(T\leq t_0\,;\,A^{\mathtt{w}}_{T+t}-A^{\mathtt{w}}_T\geq a\big)\leq \mathbf{E}[A_t^{\mathtt{w}}]+\sum_{j\geq 1} w_j \mathbf{P}(E_j^{\mathtt{w}}\leq t_0)\mathbf{P}(N_j^{\mathtt{w}}(t)\geq 1). \tag{140}$$

Note that $\mathbf{E}[A_t^{\mathtt{w}}]=\sum_{j\geq 1} w_j(e^{-tw_j/\sigma_1(\mathtt{w})}-1+\frac{tw_j}{\sigma_1(\mathtt{w})})$. Thus,

$$a\,\mathbf{P}\big(T\leq t_0\,;\,A^{\mathtt{w}}_{T+t}-A^{\mathtt{w}}_T\geq a\big)\leq t\big(t_0+\tfrac{1}{2}t\big)\frac{\sigma_3(\mathtt{w})}{\sigma_1(\mathtt{w})^2}. \tag{141}$$

4.3.2 Oscillations of $X^{\mathtt{b},\mathtt{w}}_{\Lambda^{\mathtt{b},\mathtt{w}}}$ and $X^{\mathtt{r},\mathtt{w}}_{\Lambda^{\mathtt{r},\mathtt{w}}}$.

Recall that $\mathbf{D}([0,\infty),\mathbb{R})$ stands for the space of $\mathbb{R}$-valued càdlàg functions equipped with Skorokhod's topology. For all $y\in\mathbf{D}([0,\infty),\mathbb{R})$ and for all interval I of $[0,\infty)$, we set

$$\mathtt{ocs}(y,I)=\sup\big\{|y(s)-y(t)|;\ s,t\in I\big\} \tag{142}$$

that is the oscillation of y on I. It is easy to check that

$$\forall\, a<b,\qquad \mathtt{ocs}(y,[a,b)\,)\leq \mathtt{ocs}(y,[a,b])\leq \mathtt{ocs}(y,[a,b)\,)+|\Delta y(b)|\,, \tag{143}$$

where we recall that $\Delta y(b)=y(b)-y(b-)$. We also recall that the definition of the *càdlàg modulus of continuity* of y: let $z,\eta\in(0,\infty)$; then, we set

$$w_z(y,\eta)=\inf\big\{\max_{1\leq i\leq r}\mathtt{osc}(y,[t_{i-1},t_i)\,)\ ;\ 0=t_0<\ldots<t_r=z\ :\ \min_{1\leq i\leq r-1}(t_i-t_{i-1})\geq\eta\ \big\}, \tag{144}$$

Here the infimum is taken on the set of all subdivisions $(t_i)_{0\leq j\leq r}$, of $[0,z]$, r being a positive integer; note that we do not require $t_r-t_{r-1}\geq\eta$. We refer to Jacod & Shiryaev [30] Chapter VI for a general introduction on Skorokod's topology. We first prove the following lemma on the modulus of continuity of $X^{\mathtt{b},\mathtt{w}}\circ\Lambda^{\mathtt{b},\mathtt{w}}$ and $X^{\mathtt{r},\mathtt{w}}\circ\Lambda^{\mathtt{r},\mathtt{w}}$. This technical lemma is a key argument in the proof of Theorem 2.14.

Lemma 4.3 *We keep the notation from above. For all $z_0, z_1, z, \eta \in (0,\infty)$, almost surely on the event $\{\Lambda^{\mathtt{b,w}}_{z_1} \le z_0 < \Lambda^{\mathtt{b,w}}_{z}\}$, the following inequality holds true.*

$$w_{z_1}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, \eta\big) \le w_{z+\eta}\big(X^{\mathtt{w}}, \eta\big) + w_{z_0}\big(X^{\mathtt{b,w}}, \eta\big). \tag{145}$$

Similarly

$$\text{a.s. on } \{z > \Lambda^{\mathtt{r,w}}_{z_1}\}, \qquad w_{z_1}\big(X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}}, \eta\big) \le w_{z}\big(X^{\mathtt{r,w}}, \eta\big). \tag{146}$$

Proof. First note that for all interval I, we get:

$$\mathtt{ocs}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, I\big) = \sup\big\{\big|X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}_t} - X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}_s}\big|; s,t \in I\big\} = \sup\big\{\big|X^{\mathtt{b,w}}_{t} - X^{\mathtt{b,w}}_{s}\big|; s,t \in \big\{\Lambda^{\mathtt{b,w}}_u; u\in I\big\}\big\}.$$

We then fix $\eta, a, b \in (0,\infty)$ such that $b-a \ge \eta$. By (30), $\theta^{\mathtt{b,w}}_{b-} - \theta^{\mathtt{b,w}}_{a} \ge b-a \ge \eta$. Since $\Lambda^{\mathtt{b,w}}$ is non-decrasing and continuous and since $\theta^{\mathtt{b,w}}$ is increasing, we get $\{\Lambda^{\mathtt{b,w}}_t; t\in[\theta^{\mathtt{b,w}}_a, \theta^{\mathtt{b,w}}_{b-})\} = [a,b)$ and

$$\mathtt{ocs}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, [\theta^{\mathtt{b,w}}_a, \theta^{\mathtt{b,w}}_{b-})\big) = \mathtt{ocs}\big(X^{\mathtt{b,w}}, [a,b)\big).$$

We next suppose that $\Delta\theta^{\mathtt{b,w}}_b > 0$. Then, $\{\Lambda^{\mathtt{b,w}}_t; t\in[\theta^{\mathtt{b,w}}_a, \theta^{\mathtt{b,w}}_{b})\} = [a,b]$ and by (143), we get

$$\mathtt{ocs}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, [\theta^{\mathtt{b,w}}_a, \theta^{\mathtt{b,w}}_{b})\big) = \mathtt{ocs}\big(X^{\mathtt{b,w}}, [a,b]\big) \le \mathtt{ocs}\big(X^{\mathtt{b,w}}, [a,b)\big) + |\Delta X^{\mathtt{b,w}}_b|. \tag{147}$$

Since the process $X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}$ is constant on $[\theta^{\mathtt{b,w}}_{b-}, \theta^{\mathtt{b,w}}_{b})$, we also get

$$\max\Big(\mathtt{ocs}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, [\theta^{\mathtt{b,w}}_a, \theta^{\mathtt{b,w}}_{b-})\big), \mathtt{ocs}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, [\theta^{\mathtt{b,w}}_{b-}, \theta^{\mathtt{b,w}}_{b})\big)\Big) = \mathtt{ocs}\big(X^{\mathtt{b,w}}, [a,b)\big). \tag{148}$$

We next assume that $\Delta\theta^{\mathtt{b,w}}_b \in (0,\eta)$ and we want to control $|\Delta X^{\mathtt{b,w}}_b|$ in terms of the càdlàg η-modulus of continuity of $X^{\mathtt{w}}$. To that end, first observe that $\Delta\theta^{\mathtt{b,w}}_b > 0$ implies that $[\theta^{\mathtt{b,w}}_{b-}, \theta^{\mathtt{b,w}}_{b})$ is a connected component of $\mathtt{Red}$; by (130), there exists $p\ge 1$ such that $\theta^{\mathtt{b,w}}_{b-} = \mathfrak{s}_{j(p)}$: namely, $\theta^{\mathtt{b,w}}_{b-}$ is the time of arrival of a red client who interrupts a blue one. By (134), for all $t \in [\mathfrak{s}_{j(p)}, \mathfrak{s}_{j(p)+1})$, $X^{\mathtt{w}}_t \ge X^{\mathtt{w}}_{\mathfrak{s}_{j(p)}-} = Y^{\mathtt{w}}(\Lambda^{\mathtt{b,w}}_{\mathfrak{s}_{j(p)}})$; moreover $\Delta X^{\mathtt{b,w}}_b = \Delta X^{\mathtt{w}}_{\theta^{\mathtt{b,w}}_{b-}}$. To summarize:

$$\forall t\in[\theta^{\mathtt{b,w}}_{b-}, \theta^{\mathtt{b,w}}_{b}), \quad X^{\mathtt{w}}_t > X^{\mathtt{w}}_{(\theta^{\mathtt{b,w}}_{b-})-} = X^{\mathtt{w}}_{\theta^{\mathtt{b,w}}_{b}} \quad \text{and} \quad \Delta X^{\mathtt{w}}_{\theta^{\mathtt{b,w}}_{b-}} = X^{\mathtt{w}}_{\theta^{\mathtt{b,w}}_{b-}} - X^{\mathtt{w}}_{(\theta^{\mathtt{b,w}}_{b-})-} = \Delta X^{\mathtt{b,w}}_b \tag{149}$$

Let $z\in(0,\infty)$ such that $\theta^{\mathtt{b,w}}_{b-} \le z$ and let $0=t_0 < \ldots < t_r = z+\eta$ be such that $\min_{1\le i\le r-1}(t_i - t_{i-1}) \ge \eta$. Then, there exists $i\in\{1,\ldots,r\}$ such that $t_{i-1} \le \theta^{\mathtt{b,w}}_{b-} < t_i$. There are two cases to consider:

- If $t_{i-1} < \theta^{\mathtt{b,w}}_{b-}$, then, by the last point of (149), $\mathtt{osc}(X^{\mathtt{w}}, [t_{i-1}, t_i)) \ge |\Delta X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b-})| = |\Delta X^{\mathtt{b,w}}_b|$.
- If $t_{i-1} = \theta^{\mathtt{b,w}}_{b-}$, since $\Delta\theta^{\mathtt{b,w}}_b \in (0,\eta)$, $\theta^{\mathtt{b,w}}_b < t_i$. Then $\mathtt{osc}(X^{\mathtt{w}}, [t_{i-1}, t_i)) \ge |X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b-}) - X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b})|$. Recall from the first part of (149) that $X^{\mathtt{w}}((\theta^{\mathtt{b,w}}_{b-})-) = X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b})$. Thus, $|X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b-}) - X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b})| = |\Delta X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b-})|$ and we also get $\mathtt{osc}(X^{\mathtt{w}}, [t_{i-1}, t_i)) \ge |\Delta X^{\mathtt{w}}(\theta^{\mathtt{b,w}}_{b-})| = |\Delta X^{\mathtt{b,w}}_b|$.

This proves that if $\Delta\theta^{\mathtt{b,w}}_b \in (0,\eta)$ and if $\theta^{\mathtt{b,w}}_{b-} \le z$, then $|\Delta X^{\mathtt{b,w}}_b| \le \max_{1\le i<r} \mathtt{osc}(X^{\mathtt{w}}, [t_{i-1}, t_i))$ and since it holds true for all subdivisions of $[0, z+\eta]$ satisfying the conditions as above, we get

$$\text{a.s. on } \{\theta^{\mathtt{b,w}}_{b-} \le z\,;\, \Delta\theta^{\mathtt{b,w}}_b \in (0,\eta)\}, \quad |\Delta X^{\mathtt{b,w}}_b| \le w_{z+\eta}\big(X^{\mathtt{w}}, \eta\big). \tag{150}$$

We are now ready to prove (145). Let us fix $z_0, z\in(0,\infty)$ and $0=t_0<\ldots<t_r=z_0$ such that $\min_{1\le i\le r-1}(t_i - t_{i-1}) \ge \eta$. We assume that $\theta^{\mathtt{b,w}}_{z_0} \le z$. For all $i\in\{1,\ldots,r\}$, we set $S_i = \{\theta^{\mathtt{b,w}}_{t_i}\}$ if $\Delta\theta^{\mathtt{b,w}}_{t_i} < \eta$ and $S_i = \{\theta^{\mathtt{b,w}}_{t_i-}, \theta^{\mathtt{b,w}}_{t_i}\}$ if $\Delta\theta^{\mathtt{b,w}}_{t_i} \ge \eta$; we then define $S = \{s_0 = 0 < \ldots < s_{r'} = \theta^{\mathtt{b,w}}_{z_0}\} = \{0\}\cup S_1 \cup \ldots \cup S_r$ that is a subdivision of $[0, \theta^{\mathtt{b,w}}_{z_0}]$ such that $\min_{1\le i\le r'-1}(s_i - s_{i-1}) \ge \eta$ (*indeed,* recall that $\theta^{\mathtt{b,w}}_{t_i-} - \theta^{\mathtt{b,w}}_{t_{i-1}} \geq t_i - t_{i-1}$). By (148) (if S_i has two points) and by (147) and (150) (if S_i reduces to a single point), we get

$$w_{\theta^{\mathtt{b,w}}_{z_0}}(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, \eta) \leq \max_{1 \leq i \leq r'} \Big(\mathtt{osc}\big(X^{\mathtt{b,w}}_{\Lambda^{\mathtt{b,w}}}, [s_{i-1}, s_i)\big)\Big) \leq w_{z+\eta}(X^{\mathtt{w}}, \eta) + \max_{1 \leq i \leq r} \Big(\mathtt{osc}\big(X^{\mathtt{b,w}}, [t_{i-1}, t_i)\big)\Big)$$

Since it holds true for all subdivisions (t_i) and since $z' \mapsto w_{z'}(y(\cdot), \eta)$ is non-decreasing, it easily entails (145) under the assumption that $z_1 \leq \theta^{\mathtt{b,w}}_{z_0} \leq z$. We complete the proof of (145) by observing that $\Lambda^{\mathtt{b,w}}_{z_1} \leq z_0 < \Lambda^{\mathtt{b,w}}_{z}$ implies $z_1 \leq \theta^{\mathtt{b,w}}_{z_0} \leq z$.

The proof (146) is similar: let $\theta^{\mathtt{r,w}}_t = \inf\{s \in [0, \infty) : \Lambda^{\mathtt{r,w}}_s > t\}$. Since $s = \Lambda^{\mathtt{b,w}}_s + \Lambda^{\mathtt{r,w}}_s$, we take $s = \theta^{\mathtt{r,w}}_t$ to get $\theta^{\mathtt{r,w}}_t = \Lambda^{\mathtt{b,w}}(\theta^{\mathtt{r,w}}_t) + t$. Thus, $\theta^{\mathtt{r,w}}$ is strictly increasing and for all $a, b \in (0, \infty)$ such that $b > a$, we get $\theta^{\mathtt{r,w}}_{b-} - \theta^{\mathtt{r,w}}_a \geq b - a$ and $\{\Lambda^{\mathtt{r,w}}_t ; t \in [\theta^{\mathtt{r,w}}_a, \theta^{\mathtt{r,w}}_{b-})\} = [a, b)$. Thus,

$$\mathtt{ocs}\big(X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}}, [\theta^{\mathtt{r,w}}_a, \theta^{\mathtt{r,w}}_{b-})\big) = \mathtt{ocs}\big(X^{\mathtt{r,w}}, [a, b)\big).$$

We next suppose that $\Delta\theta^{\mathtt{r,w}}_b > 0$. Note that $\{\Lambda^{\mathtt{r,w}}_t ; t \in [\theta^{\mathtt{r,w}}_a, \theta^{\mathtt{r,w}}_b)\} = [a, b]$ and by (143), we get

$$\mathtt{ocs}\big(X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}}, [\theta^{\mathtt{r,w}}_a, \theta^{\mathtt{r,w}}_b)\big) = \mathtt{ocs}\big(X^{\mathtt{r,w}}, [a, b]\big) \leq \mathtt{ocs}\big(X^{\mathtt{r,w}}, [a, b)\big) + |\Delta X^{\mathtt{r,w}}_b|.$$

Then, $\theta^{\mathtt{r,w}}_{b-}$ is the departure time of a red client interrupting a blue client: namely, there exists $p \geq 1$ such that $\mathfrak{s}_{j(p)+1} = \theta^{\mathtt{r,w}}_{b-}$. Recall that $x_p = \sum_{1 \leq q \leq p} w_{\mathfrak{J}_{j(q)}}$ and that necessarily, $b = \gamma^{\mathtt{r,w}}_{x_p}$. Thus, $\Delta X^{\mathtt{r,w}}_b = 0$. Thus, for all $b > a > 0$, we have proved that

$$\mathtt{ocs}\big(X^{\mathtt{r,w}}_{\Lambda^{\mathtt{r,w}}}, [\theta^{\mathtt{r,w}}_a, \theta^{\mathtt{r,w}}_b)\big) = \mathtt{ocs}\big(X^{\mathtt{r,w}}, [a, b)\big)$$

and we argue as in the proof of (145) to complete the proof of (146). ∎

5 Properties of the limiting processes.

5.1 The height process of a Lévy tree.

In this section we briefly recall various properties of the height process associated with a Lévy process. To that end, we fix $\alpha, \beta \in [0, \infty)$, $\kappa \in (0, \infty)$, $\mathbf{c} = (c_j)_{j \geq 1} \in \ell_3^{\downarrow}$ and we set

(151) $$\forall \lambda \in [0, \infty), \quad \psi(\lambda) = \alpha\lambda + \tfrac{1}{2}\beta\lambda^2 + \sum_{j \geq 1} \kappa c_j \big(e^{-\lambda c_j} - 1 + \lambda c_j\big).$$

and we assume that

(152) $$\int^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty .$$

Let $(X_t)_{t \in [0,\infty)}$ be a spectrally positive Lévy process with initial state $X_0 = 0$ and with Laplace exponent ψ: namely, $\log \mathbf{E}[\exp(-\lambda X_t)] = t\psi(\lambda)$, for all $t, \lambda \in [0, \infty)$. Then, the Lévy measure of X is $\pi = \sum_{j \geq 1} \kappa c_j \delta_{c_j}$, its Brownian parameter is β and its drift is α. Since $\alpha \geq 0$, X does not drift to ∞: namely a.s. $\liminf_{t \to \infty} X_t = -\infty$. Note that (152) implies that either $\beta > 0$ or $\sigma_2(\mathbf{c}) = \int_{(0,\infty)} r\pi(dr) = \infty$, which also entails that X has infinite variation sample paths.

5.1.1 Local time at the supremum.

For all $t \in [0, \infty)$, we set $S_t = \sup_{s \in [0,t]} X_s$. Basic results of fluctuation theory entail that $S - X$ is a strong Markov process in $[0, \infty)$ and that 0 is regular for $(0, \infty)$ and recurrent with respect to this Markov process (see for instance Bertoin [6] VI.1). We denote by $(L_t)_{t \in [0,\infty)}$ the local time of

X at its supremum (namely, the local time of $S-X$ at 0), whose normalisation is such that for all $t\in[0,\infty)$ the following holds in probability:

$$(153)\qquad L_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{\{S_s-X_s\leq\varepsilon\}}\, ds$$

See Le Gall & D. [21] (Chapter 1, Lemma 1.1.3 p. 21) for more details. If $\beta>0$, then standard results on subordinators imply that a.s. for all $t\in[0,\infty)$, $L_t=\frac{2}{\beta}\ell(\{S_s; s\in[0,t]\})$, where ℓ stands for the Lebesgue measure. When $\sigma_2(\mathbf{c})=\infty$, we also recall the following approximation of L: for all $\varepsilon\in(0,c_1)$, we set

$$(154)\qquad q(\varepsilon)=\int_{(\varepsilon,\infty)} dx\, \pi((x,\infty)) = \sum_{j\geq 1}\kappa c_j(c_j-\varepsilon)_+ \quad \text{and} \quad \mathscr{L}_t^\varepsilon=\big\{s\in(0,t]: S_{s-}+\varepsilon<X_s\big\}\,.$$

If $\sigma_2(\mathbf{c})=\infty$, then the following approximation holds true.

$$(155)\qquad \forall x,t\in(0,\infty),\quad \mathbf{E}\Big[\mathbf{1}_{\{L_t\leq x\}}\sup_{s\in[0,t]}\big|L_s-\tfrac{1}{q(\varepsilon)}\#\mathscr{L}_s^\varepsilon\big|^2\Big]\leq\frac{x}{q(\varepsilon)}.$$

This is a standard consequence of the decomposition of X into excursions under its supremum: see Bertoin [6], Chapter VI.

5.1.2 The height process.

For all $t\in(0,\infty)$, we denote by $\widehat{X}^t=(X_t-X_{(t-s)-})_{s\in[0,t]}$ the process X reversed at time t; recall that $\widehat{X}^t$ has the same law as $(X_s)_{s\in[0,t]}$. Under (152), Le Gall & Le Jan [33] (see also Le Gall & D. [21]) prove that there exists a *continuous* process $H=(H_t)_{t\in[0,\infty)}$ such that for all

$$(156)\qquad \forall t\in[0,\infty),\ \text{a.s.}\quad H_t=L_t(\widehat{X}^t)\,.$$

Namely, H_t is a.s. equal to the local time at of $\widehat{X}^t$ at its supremum evaluated at time t. The previous approximations of L have the following consequences. First introduce the following:

$$(157)\qquad \forall t\geq s\geq 0,\ I_s^t=\inf_{r\in[s,t]}X_r,\ \ I_t=I_t^0=\inf_{r\in[0,t]}X_r \ \text{and}\ \mathscr{H}_t^\varepsilon=\big\{s\in(0,t]: X_{s-}+\varepsilon<I_t^s\big\}$$

Then we easily derive from (156) and (153) that (45) holds true: namely, $H_t=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\mathbf{1}_{\{X_s-I_t^s\leq\varepsilon\}}\,ds$ in probability. Of course,

$$(158)\qquad \text{If } \beta>0, \text{ then a.s. for all } t\in[0,\infty),\quad H_t=\tfrac{2}{\beta}\,\ell(\{I_t^s; s\in[0,t]\})\,.$$

If $\sigma_2(\mathbf{c})=\infty$, then (156) and (155) easily imply that for all $t\in[0,\infty)$, $H_t=\lim_{\varepsilon\to 0}\frac{1}{q(\varepsilon)}\#\mathscr{H}_t^\varepsilon$ in probability. Actually, a closer look at the uniform approximation (155) shows the following.

(159) If $\sigma_2(\mathbf{c})=\infty$, then $\forall t\in[0,\infty)$, $\exists(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to 0 such that:

$$\mathbf{P}\text{-a.s. for all } s\in[0,t] \text{ such that } X_{s-}\leq I_t^s,\quad H_s=\lim_{k\to\infty}\frac{1}{q(\varepsilon_k)}\#\mathscr{H}_s^{\varepsilon_k}.$$

We shall need the following lemma.

Lemma 5.1 *We assume (152). Then $\mathbf{P}$-a.s. for all $t_1>t_0$, if for all $t\in(t_0,t_1)$, $X_t>X_{t_0-}=X_{t_1}$, then for all $t\in(t_0,t_1)$, $H_t\geq H_{t_0}=H_{t_1}$.*

Proof: let $t_1 > t_0$ be such that for all $t \in (t_0, t_1)$, $X_t > X_{t_0-} = X_{t_1}$. Since X has only positive jumps, it implies that $\Delta X_{t_1} = 0$; thus, for all $s \in [t_0, t_1]$, we get $I^s_{t_1} = X_{t_1}$ and for all $s \in [0, t_0)$ and for all $t \in [t_0, t_1]$, we get $I^s_{t_1} = I^s_{t_0} = I^s_t$. It implies for all $t \in [t_0, t_1]$, that $\{I^s_{t_0}; s \in [0, t_0)\} \backslash \{X_{t_0-}\} = \{I^s_{t_1}; s \in [0, t_1)\} \backslash \{X_{t_1}\} \subset \{I^s_t; s \in [0, t)\}$ which entails the desired result when $\beta > 0$ by (158).

Suppose next that $\sigma_2(\mathbf{c}) = \infty$. By a diagonal argument and (159), there is a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to 0 such that $\mathbf{P}$-a.s. for all $t \in [0, \infty) \cap \mathbb{Q}$ and for all $s \in [0, t]$ such that $X_{s-} \leq I^s_t$, $H_s = \lim_{k\to\infty} \frac{1}{q(\varepsilon_k)} \# \mathscr{H}^{\varepsilon_k}_s$. First observe that for all $t \in (t_0, t_1) \cap \mathbb{Q}$, we get $X_{t_0-} \leq I^{t_0}_t$ and that $\#\mathscr{H}^{\varepsilon_k}_{t_0} \subset \#\mathscr{H}^{\varepsilon_k}_t$, for all k. Consequently, $H_{t_0} \leq H_t$, for all $t \in (t_0, t_1) \cap \mathbb{Q}$, and thus for all $t \in [t_0, t_1]$ since H is continuous.

Let $t \in (t_1, \infty) \cap \mathbb{Q}$. Let $s \in [t_1, t]$ be such that $X_{s-} = I^{t_1}_t$. Then observe that $X_{s-} \leq I^s_t$ and that $\#\mathscr{H}^{\varepsilon_k}_s \subset \#\mathscr{H}^{\varepsilon_k}_{t_0}$ for all k. Consequently, $H_s \leq H_{t_0}$. Since s can be arbitrarily close to t_1, the continuity of H entails that $H_{t_1} \leq H_{t_0}$ and the previous inequality implies $H_{t_1} = H_{t_0}$, which completes the proof of the lemma. ■

5.1.3 Excursion of the height process.

Recall that (152) implies that X has unbounded variation sample paths. Then, basic results of fluctuation theory entail that $X - I$ is a strong Markov process in $[0, \infty)$, that 0 is regular for $(0, \infty)$ and recurrent with respect to this Markov process. Moreover, $-I$ is a local time at 0 for $X - I$ (see Bertoin [6], Theorem VII.1). We denote by $\mathbf{N}$ the corresponding excursion measure of $X - I$ above 0. It is not difficult to derive from the previous approximations of H_t, that H_t only depends on the excursion of $X - I$ above 0 that straddles t. Moreover, the following holds true:

$$\mathscr{Z} = \{t \in \mathbb{R}_+ : H_t = 0\} = \{t \in \mathbb{R}_+ : X_t = I_t\} \tag{160}$$

(see Le Gall & D. [21] Chapter 1). Since $-I$ is a local time for $X - I$ at 0, the topological support of the Stieltjes measure $d(-I)$ is $\mathscr{Z}$. Namely,

$$\mathbf{P}\text{-a.s. for all } s, t \in [0, \infty) \text{ such that } s < t, \quad \Big((s, t) \cap \mathscr{Z} \neq \emptyset \Big) \Longleftrightarrow \Big(I_s > I_t \Big) \tag{161}$$

Denote by (a_i, b_i), $i \in \mathcal{I}$, the connected components of the open set $\{t \in [0, \infty) : H_t > 0\}$ and set $H^i_s = H_{(a_i + s) \wedge b_i}$, $s \in \mathbb{R}_+$. Then, the point measure

$$\sum_{i \in \mathcal{I}} \delta_{(-I_{a_i}, H^i)} \tag{162}$$

is a Poisson point measure on $\mathbb{R}_+ \times \mathbf{C}([0, \infty), \mathbb{R})$ with intensity $dx\, \mathbf{N}(dH)$, where, with a slight abuse of notation, $\mathbf{N}(dH)$ stands for the "distribution" of $H(X)$ under $\mathbf{N}(dX)$. In the Brownian case, up to scaling, $\mathbf{N}$ is Itô positive excursion of Brownian motion and the decomposition (162) corresponds to the Poisson decomposition of a reflected Brownian motion above 0. For more details, we refer to Le Gall & D. [21] Chapter 1.

As a consequence of (160), X and H under $\mathbf{N}$ have the same *lifetime* ζ. This lifetime satisfies the following.

$$\mathbf{N}\text{-a.e. } \zeta < \infty, \ \ \forall t \in [\zeta, \infty), \ \ X_0 = H_0 = X_t = H_t = 0 \ \text{ and } \ \forall t \in (0, \zeta), \ \ X_t \text{ and } H_t > 0. \tag{163}$$

We next define for all $x \in [0, \infty)$, $\gamma_x = \inf\{t \in [0, \infty) : X_t < -x\}$. Basic results of fluctuation theory (see e.g. Bertoin [6], Chapter VII) imply that $(\gamma_x)_{x \in [0,\infty)}$ is a subordinator whose Laplace exponent is the inverse function ψ^{-1}. Moreover (152) entails that $\lim_{\lambda\to\infty} \psi^{-1}(\lambda)/\lambda = 0$; consequently, (γ_x) has no drift: it is a pure jump-process; thus, a.s. $\gamma_x = \sum_{i \in \mathcal{I}} \mathbf{1}_{[0,x]}(-I_{a_i}) \zeta_i$ (with an obvious notation) and we get

$$\forall \lambda \in (0, \infty), \quad \mathbf{N}\big[1 - e^{-\lambda\zeta}\big] = \psi^{-1}(\lambda), \tag{164}$$

We next recall from Le Gall & D. [21] (Chapter 1, Corollary 1.4.2, p. 41) the following:

(165) $$\forall a\in(0,\infty),\quad v(a)=\mathbf{N}\Big(\sup_{t\in[0,\zeta]}H_t>a\Big)\quad \text{satisfies}\quad \int_{v(a)}^{\infty}\frac{d\lambda}{\psi(\lambda)}=t\,.$$

Note that $v:(0,\infty)\to(0,\infty)$ is a bijective decreasing C^∞ function. By excursion theory, we then get

(166) $$\forall x,a\in(0,\infty),\quad \mathbf{P}\big(\sup_{t\in[0,\gamma_x]}H_t>a\big)=e^{-xv(a)}\,.$$

5.2 Properties of the coloured processes.

Let $\alpha\in\mathbb{R}$, $\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}=(c_j)_{j\ge1}\in\ell_3^{\downarrow}$. For all $j\ge1$, let $(N_j(t))_{t\in[0,\infty)}$ be a homogeneous Poisson process with jump rate κc_j; let B be a standard Brownian motion with initial value 0. We assume that the processes B and N_j are independent. Let $(B^{\mathfrak{r}};N'_j,j\ge1)$ be independent copies of $(B;N_j,j\ge1)$. Recall from (36) that for all $t\in[0,\infty)$, we have set

$$X^{\mathfrak{b}}_t=-\alpha t+\sqrt{\beta}B_t+\sum_{j\ge1}^{\perp}c_j\big(N_j(t)-c_j\kappa t\big)\quad\text{and}\quad X^{\mathfrak{r}}_t=-\alpha t+\sqrt{\beta}B^{\mathfrak{r}}_t+\sum_{j\ge1}^{\perp}c_j\big(N'_j(t)-c_j\kappa t\big)\,,$$

where $\sum_{j\ge1}^{\perp}$ stands for the sum of orthogonal L^2-martingales. Then $X^{\mathfrak{b}}$ and $X^{\mathfrak{r}}$ are two independent spectrally positive Lévy processes whose Laplace exponent ψ is defined by (151). Recall that (152) implies (38), namely: either $\beta>0$ or $\sigma_2(\mathbf{c})=\infty$. We recall from (39) the definitions of $(A_t)_{t\in[0,\infty)}$ and of $(Y_t)_{t\in[0,\infty)}$:

(167) $$\forall t\in[0,\infty),\quad A_t=\tfrac12\kappa\beta t^2+\sum_{j\ge1}c_j\big(N_j(t)-1\big)_+\quad\text{and}\quad Y_t=X^{\mathfrak{b}}_t-A_t.$$

Recall the notation $\gamma^{\mathfrak{r}}_x=\inf\{s\in[0,\infty):X^{\mathfrak{r}}_s<-x\}$, with the convention that $\inf\emptyset=\infty$. Next, we recall from (41) and (42) the following definitions for all $x,t\in[0,\infty)$:

(168) $$\theta^{\mathfrak{b}}_t=t+\gamma^{\mathfrak{r}}_{A_t},\quad \Lambda^{\mathfrak{b}}_t=\inf\{s\in[0,\infty):\theta^{\mathfrak{b}}_s>t\}\quad\text{and}\quad \Lambda^{\mathfrak{r}}_t=t-\Lambda^{\mathfrak{b}}_t.$$

5.2.1 Properties of A.

Proof of Lemma 2.5. We assume (38). Namely: either $\beta>0$ or $\sigma_2(\mathbf{c})=\infty$. If $\beta>0$, then clearly a.s. A is increasing. Suppose that $\sigma_2(\mathbf{c})=\infty$. With the notation of (167), observe that for all $s,t\in(0,\infty)$,

$$\sum_{j\ge1}\mathbf{1}_{\{N_j(t)\ge1;N_j(t+s)-N_j(t)\ge1\}}\le\#\big\{a\in(t,t+s]:\Delta A_a>0\big\}\,.$$

Note that $\mathbf{P}(N_j(t)\ge1;N_j(t+s)-N_j(t)\ge1)=(1-\exp(-\kappa c_jt))(1-\exp(-\kappa c_js))$. Since there exists $K\in(0,\infty)$, such that $(1-\exp(-\kappa c_jt))(1-\exp(-\kappa c_js))\ge Kc_j^2$ for all $j\ge1$, Borel's Lemma implies that a.s. $\#\big\{a\in(t,t+s]:\Delta A_a>0\big\}=\infty$. This easily implies that A is strictly increasing. To complete the proof of the lemma, observe that under (38), $X^{\mathfrak{b}}$ has infinite variation sample paths. By (167) $Y=X^{\mathfrak{b}}-A$. Since Y has bounded variation sample paths (it is incresing), Y has infinite variation sample paths. ■

We shall need the following estimates on A in the proof of Theorem 2.6.

Lemma 5.2 *We assume (38). For all $t\in[0,\infty)$ we set $A^{-1}_t=\inf\big\{s\in[0,\infty):A_s>t\big\}$, that is well defined. Then, A^{-1} is a continuous process and there exists $a_0,a_1,a_2\in(0,\infty)$ that depend on β,κ and $\mathbf{c}$, such that*

(169) $$\forall t\in[0,\infty),\qquad \mathbf{E}\big[A^{-1}_t\big]\le a_1t+a_0\quad\textit{and}\quad \mathbf{E}\big[(A^{-1}_t)^2\big]\le a_2t^2+a_1t+a_0\,.$$

Proof: first suppose that $c_1 > 0$. Then, by (167) $A_t \geq c_1(N_1(t)-1)_+ \geq c_1 N_1(t) - c_1$. This entails that A tend to ∞ and therefore that A^{-1} is well defined. Moreover, we get $A_t^{-1} \leq N_1^{-1}(1+(t/c_1))$, where: $N_1^{-1}(t) = \inf\{s \in [0,\infty) : N_1(s) > t\}$. Note that $N_1^{-1}(t)$ is the sum of $\lceil t \rceil$ exponentially distributed r.v. with parameter κc_1, which implies that $\mathbf{E}[N_1^{-1}(t)] = \lceil t \rceil/(\kappa c_1)$ and $\mathbf{E}[N_1^{-1}(t)^2] = (\lceil t \rceil^2 + \lceil t \rceil)/(\kappa c_1)^2$. Thus,

$$\mathbf{E}\big[A_t^{-1}\big] \leq \tfrac{1}{\kappa c_1^2} t + \tfrac{2}{\kappa c_1} \quad \text{and} \quad \mathbf{E}\big[(A_t^{-1})^2\big] \leq \tfrac{1}{\kappa^2 c_1^4} t^2 + \tfrac{6}{\kappa^2 c_1^3} t + \tfrac{6}{\kappa^2 c_1^2}.$$

If $\mathbf{c} = 0$, then $\beta > 0$, by (38) and $A_t^{-1} = \sqrt{2t/(\beta\kappa)}$ and it is easy to see that it is possible to choose $a_0, a_1, a_2 \in (0,\infty)$ such that (169). By Lemma 2.5, A is strictly increasing and standard arguments entail that A^{-1} is continuous. ■

5.2.2 Proof of Theorem 2.6.

Let us first introduce notation. We first say that a martingale $(M_t)_{t\in[0,\infty)}$ is of class $(\mathscr{M})$ if

- **(a)** a.s. $M_0 = 0$,
- **(b)** M is càdlàg,
- **(c)** there exists $c \in [0,\infty)$ such that a.s. for all $t \in [0,\infty)$, $0 \leq \Delta M_t \leq c$,
- **(d)** for all $t \in [0,\infty)$, $\mathbf{E}[M_t^2] < \infty$.

Let M be a class $(\mathscr{M})$ martingale relatively to a filtration $(\mathscr{F}_t)_{t\in[0,\infty)}$. Let $\langle M \rangle$ stands for its quadratic variation process that is the continuous process such that $(M_t^2 - \langle M \rangle_t)_{t\in[0,\infty)}$ is a $(\mathscr{F}_t)$-martingale. We shall repeatedly use the following standard optional stopping theorem:

(**Stp**) Let S and T be two $(\mathscr{F}_t)$-stopping times such that a.s. $S \leq T < \infty$ and $\mathbf{E}[\langle M \rangle_T] < \infty$. Then, $\mathbf{E}[M_T^2] = \mathbf{E}[\langle M \rangle_T]$ and a.s. $M_S = \mathbf{E}[M_T | \mathscr{F}_S]$.

Then, the *characteristic measure* of M is a random measure $\mathcal{V}$ on $[0,\infty) \times (0,\infty)$ such that:

- for all $\varepsilon \in (0,\infty)$, the process $t \mapsto \mathcal{V}\big([0,t] \times [\varepsilon,\infty)\big)$ is $(\mathscr{F}_t)$-predictable;
- $t \longmapsto \sum_{s\in[0,t]} \mathbf{1}_{[\varepsilon,\infty)}(\Delta M_s) - \mathcal{V}\big([0,t] \times [\varepsilon,\infty)\big)$ is a $(\mathscr{F}_t)$-martingale.

(See Jacod & Shiryaev [30], Chapter II, Theorem 2.21, p. 80.) The *purely discontinuous part* of M is obtained as the $\mathcal{V}$-compensated sum of its jumps: namely, the L^2-limit as ε goes to 0 of the martingales $t \longmapsto \sum_{s\in[0,t]} \Delta M_s \, \mathbf{1}_{[\varepsilon,\infty)}(\Delta M_s) - \int_{[0,t]\times[\varepsilon,\infty)} r \, \mathcal{V}(dsdr)$. The purely discontinuous part of M is denoted by M^d and it is a $(\mathscr{F}_t)$-martingale in the class $(\mathscr{M})$. The *continuous part* of M is the continuous $(\mathscr{F}_t)$-martingale $M^c = M - M^d$. Note that M^c is also a $(\mathscr{F}_t)$-martingale in the class $(\mathscr{M})$. We call $(\langle M^c \rangle, \mathcal{V})$ the *characteristics* of M. For more details see Jacod & Shiryaev [30] Chapter II, Definition 2.16 p. 76 and §2.d, Theorem 2.34, p. 84, on the canonical representation of semi-martingales.

Let $(\mathscr{F}_t^N)_{t\in[0,\infty)}$ (resp. $(\mathscr{F}_t^B)_{t\in[0,\infty)}$) the right-continuous filtration associated with natural filtration of the process $(N_j(\cdot))_{j\geq 1}$ (resp. B). We also denote $\mathscr{F}_t^0 = \sigma(\mathscr{F}_t^N, \mathscr{F}_t^B)$, $t \in [0,\infty)$. We set

$$\forall t \in [0,\infty), \quad X_t^{*\mathsf{b}} = X_t^{\mathsf{b}} + \alpha t = \sqrt{\beta} B_t + \sum_{j\geq 1}^{\perp} c_j \big(N_j(t) - c_j \kappa t\big) .$$

By standard arguments on Lévy processes, $X^{*\mathsf{b}}$ is a $(\mathscr{F}_t^0)$-martingale. Moreover, set $a_3 = \beta + \kappa\sigma_3(\mathbf{c})$; then, we easily check that

$$t \longmapsto (X_t^{*\mathsf{b}})^2 - a_3 t \ \text{ is a } (\mathscr{F}_t^0)\text{-martingale.} \tag{170}$$

Moreover, we easily check that is $X^{*\mathsf{b}}$ in the class $(\mathscr{M})$ and that its (deterministic) characteristic are the following: its characteristic measure is $dt \otimes \pi(dr)$, where $\pi(dr) = \sum_{j\geq 1} \kappa c_j \delta_{c_j}$; its continuous part is $\sqrt{\beta}.B^{\mathsf{b}}$, whose quadratic variation process is $t \mapsto \beta t$. To prove Theorem 2.6, we shall use the converse of this result: namely, *a martingale whose characteristics are $dt \otimes \pi(dr)$ and $t \mapsto \beta t$ has* *necessarilty the same law as* $X^{*\mathtt{b}}$ (for a proof see Jacod & Shiryaev [30] Chapter II, §4.c, Corollary 4.18, p. 107). To that end, one computes the characteristic of several time-change of $X^{*\mathtt{b}}$ and $X^{\mathtt{r}}$.

First, recall from Lemma 5.2, that A^{-1} is continuous and note that A^{-1}_t is a $(\mathscr{F}^0_r)$-stopping time. We set

$$\forall t \in [0,\infty), \quad M^{(1)}_t = X^{\mathtt{b}}(A^{-1}(t)) \quad \text{and} \quad \mathscr{F}^1_t = \mathscr{F}^0(A^{-1}_t) \; .$$

By (170), $\langle X^{*\mathtt{b}}\rangle_t = a_3 t$ and (169) combined with (**Stp**) imply that $M^{(1)}$ is a square integrable $(\mathscr{F}^1_t)$-martingale and that $\mathbf{E}[(M^{(1)}_t)^2] = a_3 \mathbf{E}[A^{-1}_t]$. Then, set $g(r) = \inf\{s\in[0,\infty) : A^{-1}_s = r\}$, for all $r\in[0,\infty)$. We easily check that:

$$g : \big\{ r\in[0,A^{-1}_t] : \Delta X^{*\mathtt{b}}_r > 0\big\} \longrightarrow \big\{ s\in[0,t] : \Delta M^{(1)}_s > 0\big\} \text{ is one-to-one} \tag{171}$$

and since A^{-1} is continuous, if $\Delta M^{(1)}_s > 0$, then there exists $r \in [0,\infty)$ such that $g(r) = s$ and $\Delta M^{(1)}_s = \Delta X^{*\mathtt{b}}_r$. This implies that $M^{(1)}$ is in the class $(\mathscr{M})$.

For all $\varepsilon\in(0,\infty)$, we next set

$$\forall t\in[0,\infty), \quad J^{\varepsilon}_t = \sum_{r'\in[0,r]} \mathbf{1}_{[\varepsilon,\infty)}(\Delta X^{*\mathtt{b}}_{r'}) - r\pi([\varepsilon,\infty))$$

that is a $(\mathscr{F}^0_r)$-martingale in the class $(\mathscr{M})$ such that $\langle J^{\varepsilon}\rangle_r = \pi([\varepsilon,\infty))r$, and (169) combined with (**Stp**) entails that $J^{\varepsilon}\circ A^{-1}$ is a square integrable $(\mathscr{F}^1_t)$-martingale. Moreover, (171) entails that $J^{\varepsilon}(A^{-1}_t) = \sum_{s\in[0,t]} \mathbf{1}_{[\varepsilon,\infty)}(\Delta M^{(1)}_s) - A^{-1}_t\pi([\varepsilon,\infty))$. Since A^{-1} continuous, it is $(\mathscr{F}^1_t)$-predictable and $dA^{-1}_t \otimes \pi(dr)$ is the characteristic measure of $M^{(1)}$. It easily entails that the continuous part of $M^{(1)}$ is $\sqrt{\beta}.B\circ A^{-1}$. We next set $Q_t = \beta B_t^2 - \beta t$; by Itô's formula $\langle Q\rangle_t = 4\beta^2 \int_0^t B_s^2 ds$ and thus, $\mathbf{E}[\langle Q\rangle_t] = 2\beta^2 t^2$. Since A^{-1} is independent of B, $\mathbf{E}[\langle Q\rangle(A^{-1}_t)] = 2\beta^2\mathbf{E}[(A^{-1}_t)^2]$ that is a finite quantity by (169). Then, by (**Stp**), we see that $\langle \sqrt{\beta}.B\circ A^{-1}\rangle = \beta.A^{-1}$. We have proved that $\beta.A^{-1}$ and $dA^{-1}_t\otimes\pi(dr)$ are the characteristics of $M^{(1)}$. It is easy to realize that $M^{(1)}$ is also a martingale relatively to the natural filtration of $(A^{-1}, M^{(1)})$ with the same characteristics $\beta.A^{-1}$ and $dA^{-1}_t\otimes\pi(dr)$. We next prove the following lemma.

Lemma 5.3 *Let E be a Polish space and let $(Z_t)_{t\in[0,\infty)}$ a E-valued càdlàg process. Let $(M_r)_{r\in[0,\infty)}$ be a càdlàg martingale relatively to the natural filtration of Z. Let $(\phi_t)_{[0,\infty)}$ be a nondecreasing càd process that is adapted to a filtration $(\mathscr{G}_t)_{t\in[0,\infty)}$ we assume that Z and $\mathscr{G}_\infty$ are independent and that for all $t\in[0,\infty)$, $\int \mathbf{P}(\phi_t\in dr)\mathbf{E}[|M_r|] < \infty$. We set $\mathscr{F}_t = \sigma(Z_{\cdot\wedge\phi_t}, \mathscr{G}_t)$, for all $t\in[0,\infty)$. Then, $M\circ\phi$ is a càdlàg $(\mathscr{F}_t)$-martingale.*

Proof: let $t, r_1,\ldots,r_n\in[0,\infty)$ and let $s\in[0,t]$. Let $G : E^n\to[0,\infty)$ be bounded and measurable and let Q be a nonnegative bounded $\mathscr{G}_s$-measurable random variable. Then independence properties imply the following

$$\begin{aligned}
\mathbf{E}\big[M_{\phi_t} Q\, G\big((Z_{r_k\wedge\phi_s})_{1\le k\le n}\big)\big] &= \int \mathbf{P}(\phi_t\in dr'; \phi_s\in dr; Q\in dq)\, q\, \mathbf{E}\big[M_{r'} G\big((Z_{r_k\wedge r})_{1\le k\le n}\big)\big] \\
&= \int \mathbf{P}(\phi_t\in dr'; \phi_s\in dr; Q\in dq)\, q\, \mathbf{E}\big[M_r G\big((Z_{r_k\wedge r})_{1\le k\le n}\big)\big] \\
&= \mathbf{E}\big[M_{\phi_s} Q\, G\big((Z_{r_k\wedge\phi_s})_{1\le k\le n}\big)\big],
\end{aligned}$$

and we completes the proof by use of the monotone class theorem. ■

Recall from the beginning of Section 5.2 the definitions of the processes $B^{\mathtt{r}}$, $(N'_j(\cdot))_{j\ge1}$ and $X^{\mathtt{r}}$; recall that $X^{\mathtt{r}}$ is an independent copy of $X^{\mathtt{b}}$. We next need the following result. For all $t\in[0,\infty)$, we set $I^{\mathtt{r}}_t = \inf_{s\in[0,t]} X^{\mathtt{r}}_s$. Then,

$$\forall p,t\in(0,\infty), \quad \mathbf{E}[(-I^{\mathtt{r}}_t)^p] < \infty \, . \tag{172}$$

Indeed, recall that $\gamma^{\mathtt{r}}_x = \inf\{t\in[0,\infty) X^{\mathtt{r}}_t < -x\}$ and that $x\mapsto \gamma^{\mathtt{r}}_x$ is a (possibly killed) subordinator with Laplace exponent ψ^{-1}. Then for all $\lambda\in(0,\infty)$ we get the following

$$\begin{aligned}\mathbf{E}[(-I^{\mathtt{r}}_t)^p] &= \int_0^\infty px^{p-1}\mathbf{P}(-I^{\mathtt{r}}_t > x)\, dx \le \int_0^\infty px^{p-1}\mathbf{P}(\gamma^{\mathtt{r}}_x \le t)dx \\ &\le \int_0^\infty px^{p-1}e^{\lambda t}\mathbf{E}[e^{-\lambda\gamma^{\mathtt{r}}_x}]\, dx = p\Gamma(p)(\psi^{-1}(\lambda))^{-p}e^{\lambda t} ,\end{aligned}$$

which entails (172). □

We apply Lemma 5.3 to $Z = (A^{-1}, M^{(1)})$, to $\phi_t = -I^{\mathtt{r}}_t$, to $(\mathscr{G}_t)$ that is taken as the right-continuous filtration associated with the natural filtration of $(B^{\mathtt{r}}; N'_j, j\ge 1)$, and to $M = M^{(1)}$, first, and to $M = J^\varepsilon \circ A^{-1}$, next. Recall that $\mathbf{E}\big[(M^{(1)}_t)^2\big] = a_3\mathbf{E}[A^{-1}_t] \le a_3(a_1t + a_0)$ and $\mathbf{E}\big[J^\varepsilon(A^{-1}_t)^2\big] = \pi([\varepsilon,\infty))\mathbf{E}[A^{-1}_t] \le \pi([\varepsilon,\infty))(a_1t + a_0)$, by (169). In both cases ($M = M^{(1)}$or $M = J^\varepsilon\circ A^{-1}$), we get $\int \mathbf{P}(-I^{\mathtt{r}}_t \in dr)\mathbf{E}[M^2_r] < \infty$, by (172). Then, we set for all $t\in[0,\infty)$,

$$M^{(2)}_t = M^{(1)}_{-I^{\mathtt{r}}_t}, \quad J'^{\varepsilon}_t = J^\varepsilon_{A^{-1}(-I^{\mathtt{r}}_t)} \quad \text{and} \quad \mathscr{F}^2_t = \sigma\big(\mathscr{G}_t, A^{-1}_{\cdot\wedge(-I^{\mathtt{r}}_t)}, M^{(1)}_{\cdot\wedge(-I^{\mathtt{r}}_t)}\big) .$$

Lemma 5.3 asserts that $M^{(2)}$ and J'^{ε} are $(\mathscr{F}^2_t)$-square integrable martingales. Since they are càdlàg processes standard arguments entail they are also $(\mathscr{F}^2_{t+})_{t\in[0,\infty)}$-martingales.

Then, set $g'(r) = \inf\{t\in[0,\infty) : -I^{\mathtt{r}}_t = r\}$, for all $r\in[0,\infty)$. It is easy to check that g' is one-to-one correspondence between $\{r\in[0,-I^{\mathtt{r}}_t] : \Delta M^{(1)}_r > 0\}$ and $\{s\in[0,t] : \Delta M^{(2)}_s > 0\}$. This first entails that $M^{(2)}$ is in the class $(\mathscr{M})$. It also implies that

$$J'^{\varepsilon}_t = \sum_{s\in[0,t]} \mathbf{1}_{[\varepsilon,\infty)}(\Delta M^{(2)}_t) - A^{-1}(-I^{\mathtt{r}}_t)\pi([\varepsilon,\infty)) . \tag{173}$$

Since $t\mapsto A^{-1}(-I^{\mathtt{r}}_t)$ is continuous, it is $(\mathscr{F}^2_t)$-predictable and therefore, the characteristic measure of $M^{(2)}$ is $d(A^{-1}\circ(-I^{\mathtt{r}}))(t)\otimes\pi(dr)$.

Consequently, the continuous part of $M^{(2)}$ is $\sqrt{\beta}.B\circ A^{-1}\circ(-I^{\mathtt{r}})$. We then apply Lemma 5.3 to $M = (B\circ A^{-1})^2 - A^{-1}$: note that $\mathbf{E}[|M_t|] \le 2\mathbf{E}[A^{-1}_t] \le 2(a_1t + a_0)$, by (169); thus (172) entails $\int \mathbf{P}(-I^{\mathtt{r}}_t \in dr)\mathbf{E}[|M_r|] < \infty$; thus, Lemma 5.3 applies and asserts that $M\circ(-I^{\mathtt{r}})$ is a $(\mathscr{F}^2_t)$-martigale; by standard arguments, it is also a $(\mathscr{F}^2_{t+})$. This entails that $\beta.A^{-1}\circ(-I^{\mathtt{r}})$ is the quadratic variation of $\sqrt{\beta}.B\circ A^{-1}\circ(-I^{\mathtt{r}})$ that is the continuous part of $M^{(2)}$. Thus, $\sqrt{\beta}.B\circ A^{-1}\circ(-I^{\mathtt{r}})$ and $d(A^{-1}\circ(-I^{\mathtt{r}}))(t)\otimes\pi(dr)$ are the $(\mathscr{F}^2_{t+})$-characteristics of $M^{(2)}$.

Recall from (168) the notations $\theta^{\mathtt{b}}$, $\Lambda^{\mathtt{b}}$ and $\Lambda^{\mathtt{r}}$. Then we check that a.s.

$$\forall t\in[0,\infty), \quad \theta^{\mathtt{r}}_t := \inf\big\{s\in[0,\infty) : \Lambda^{\mathtt{r}}_s > t\big\} = t + A^{-1}(-I^{\mathtt{r}}_t) . \tag{174}$$

Indeed, for all $r < A^{-1}(-I^{\mathtt{r}}_t)$, since A is strictly increasing, we get $A_r < -I^{\mathtt{r}}_t$, which entails $\gamma^{\mathtt{r}}(A_r) < t$. Since $\theta^{\mathtt{b}}_r - r = \gamma^{\mathtt{r}}(A_r)$ and thus $\theta^{\mathtt{b}}_r < t + r < t + A^{-1}(-I^{\mathtt{r}}_t)$. Consequently, $r = \Lambda^{\mathtt{b}}(\theta^{\mathtt{b}}_r) \le \Lambda^{\mathtt{b}}(t + A^{-1}(-I^{\mathtt{r}}_t))$. Since it is true for all $r < A^{-1}(-I^{\mathtt{r}}_t)$, we get $A^{-1}(-I^{\mathtt{r}}_t) \le \Lambda^{\mathtt{b}}(t + A^{-1}(-I^{\mathtt{r}}_t))$.

Similarly, suppose that $r > A^{-1}(-I^{\mathtt{r}}_t)$. Since A is strictly increasing, we get $A_r > -I^{\mathtt{r}}_t$ and thus, $\gamma^{\mathtt{r}}(A_r) > t$. Consequently, $\theta^{\mathtt{b}}_r > t + r > t + A^{-1}(-I^{\mathtt{r}}_t)$ and $r \ge \Lambda^{\mathtt{b}}(t + A^{-1}(-I^{\mathtt{r}}_t))$. Since it holds for all $r > A^{-1}(-I^{\mathtt{r}}_t)$, it entails $A^{-1}(-I^{\mathtt{r}}_t) \ge \Lambda^{\mathtt{b}}(t + A^{-1}(-I^{\mathtt{r}}_t))$. Thus, $A^{-1}(-I^{\mathtt{r}}_t) = \Lambda^{\mathtt{b}}(t + A^{-1}(-I^{\mathtt{r}}_t))$. By (168), $\Lambda^{\mathtt{r}}(t + A^{-1}(-I^{\mathtt{r}}_t)) = t$, which completes the proof. □

We next set for all $t\in[0,\infty)$,

$$X^{*\mathtt{r}}_t = X^{\mathtt{r}}_t + \alpha t \quad \text{and} \quad M_t = X^{*\mathtt{r}}_t + M^{(2)}_t .$$

Clearly, M is a $(\mathscr{F}^2_{t+})$-martingale in the class $(\mathscr{M})$. Note that $X^{*\mathtt{r}}$ and $M^{(2)}$ do not jump simultaneously. Thus, by (173) and (174), we get

$$J''^{\varepsilon}_t = J'^{\varepsilon}_t + \sum_{s\in[0,t]} \mathbf{1}_{[\varepsilon,\infty)}(\Delta X^{*\mathtt{r}}_s) - t\pi([\varepsilon,\infty)) = \sum_{s\in[0,t]} \mathbf{1}_{[\varepsilon,\infty)}(\Delta M_s) - \theta^{\mathtt{r}}_t\pi([\varepsilon,\infty)) .$$

Moreover, J''^{ε} is clearly a $(\mathscr{F}^2_{t+})$-martingale. By (174), $\theta^{\mathtt{r}}$ is continuous and strictly increasing; therefore it is $(\mathscr{F}^2_{t+})$-predictable. This implies that the characteristic measure of M is $d\theta^{\mathtt{r}}_t \otimes \pi(dr)$.

Consequently, the continuous component of M is $M^c_t = \sqrt{\beta}(B^{\mathtt{r}}_t + B(A^{-1}(-I^{\mathtt{r}}_t)))$. The independence of B, A^{-1} and $B^{\mathtt{r}}$ easily entails that $t \mapsto B^{\mathtt{r}}_t + B(A^{-1}(-I^{\mathtt{r}}_t)))$ is a $(\mathscr{F}^2_{t+})$-martingale. Moreover, recall that the quadratic variation of $\sqrt{\beta}.B \circ A^{-1} \circ (-I^{\mathtt{r}})$ is equal to $\beta.A^{-1} \circ (-I^{\mathtt{r}})$. Thus, it is easy to see that the quadratic variation of M^c is equal to $\beta.\theta^{\mathtt{r}}$. We have proved that the characteristics of M are $\beta.\theta^{\mathtt{r}}$ and $d\theta^{\mathtt{r}}_t \otimes \pi(dr)$.

We next recall from (174) that $\Lambda^{\mathtt{r}}$ is the inverse of $\theta^{\mathtt{r}}$ that is also strictly increasing and continuous. We set $X^* = M \circ \Lambda^{\mathtt{r}}$ and we see that a.s. for all $t \in [0, \infty)$,

$$\begin{aligned} X^*_t &= X^{*\mathtt{r}}(\Lambda^{\mathtt{r}}_t) + X^{*\mathtt{b}}(A^{-1}(-I^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t))) \\ &= X^{*\mathtt{r}}(\Lambda^{\mathtt{r}}_t) + X^{*\mathtt{b}}(\Lambda^{\mathtt{b}}_t) \\ &= X^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t) + X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t) + \alpha t \, . \end{aligned} \tag{175}$$

Indeed, the first equality is a direct consequence of the definition; then recall from (174) that $A^{-1}(-I^{\mathtt{r}}_t) = \theta^{\mathtt{r}}_t - t$, thus, $A^{-1}(-I^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t)) = t - \Lambda^{\mathtt{r}}_t = \Lambda^{\mathtt{b}}_t$, which entails the second equality and also (175). □

Observe that for all $t \in [0, \infty)$, $\Lambda^{\mathtt{r}}_t$ is a $(\mathscr{F}^2_{r+})$-stopping time such that $\Lambda^{\mathtt{r}}_t \leq t$. We then set $\mathscr{F}_t = \mathscr{F}^2(\Lambda^{\mathtt{r}}_t +)$. Then, the optional stopping theorem applies to M and J''^{ε} to show that X and $J''^{\varepsilon} \circ \Lambda^{\mathtt{r}}$ are $(\mathscr{F}_t)$-square integrable martingales. Since $\Lambda^{\mathtt{r}}$ is strictly increasing and continuous, X is in the class $(\mathscr{M})$ and $J''^{\varepsilon}(\Lambda^{\mathtt{r}}_t) = \sum_{s \in [0,t]} \mathbf{1}_{[\varepsilon, \infty)}(\Delta M_s) - t\pi([\varepsilon, \infty))$. This proves that $dt \otimes \pi(dr)$ is the characteristic measure of X^*. Consequently, $M^c \circ \Lambda^{\mathtt{r}}$ is the continuous part of X^*. Since $\Lambda^{\mathtt{r}}$ is a bounded stopping-time, the optional stopping theorem applies to the martingale $(M^c)^2 - \beta\theta^{\mathtt{r}}$ and it entails that $\langle M^c \circ \Lambda^{\mathtt{r}} \rangle_t = \beta t$. Thus, the characteristics of X^* are $t \mapsto \beta t$ and $dt \otimes \pi(dr)$. By [30] Corollary 4.18 in Jacod & Shiryaev [30] (Chapter II, §4.c, p. 107), it implies that X^* has the same law as $X^{*\mathtt{b}}$, which completes the proof of Theorem 2.6 by (175). ■

5.2.3 Properties of X and Y.

Next, we recall from (41) and (42) the following definitions for all $x, t \in [0, \infty)$:

$$\gamma^{\mathtt{r}}_x = \inf\{s \in [0, \infty) : X^{\mathtt{r}}_s < -x\}, \quad \theta^{\mathtt{b}}_t = t + \gamma^{\mathtt{r}}_{A_t}, \quad \Lambda^{\mathtt{b}}_t = \inf\{s \in [0, \infty) : \theta^{\mathtt{b}}_s > t\} \quad \text{and} \quad \Lambda^{\mathtt{r}}_t = t - \Lambda^{\mathtt{b}}_t.$$

We also recall from (43) that $X_t = X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t) + X^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t)$ for all $t \in [0, \infty)$. Let us mention that the proof of the following lemma does not use Theorem 2.6.

Lemma 5.4 *Let $\alpha, \beta \in [0, \infty)$, $\kappa \in (0, \infty)$ and $\mathbf{c} = (c_j)_{j \geq 1} \in \ell^{\downarrow}_3$ satisfy (38). Then, $\mathbf{P}$-a.s. the following holds true for all $a \in [0, \infty)$.*

(i) $X_{\theta^{\mathtt{b}}_a} = Y_a$.

(ii) *If $\Delta\theta^{\mathtt{b}}_a = 0$, then $t = \theta^{\mathtt{b}}_a$ is the unique $t \in [0, \infty)$ such that $\Lambda^{\mathtt{b}}_t = a$.*

(iii) *If $\Delta\theta^{\mathtt{b}}_a > 0$, then $\Delta X(\theta^{\mathtt{b}}_{a-}) = \Delta A_a$ and $\Delta Y_a = 0$. Moreover,*

$$\forall t \in \big(\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a\big), \quad X_t \geq X_{t-} > X_{(\theta^{\mathtt{b}}_{a-})-} = X_{\theta^{\mathtt{b}}_a} = Y_a \, .$$

Proof: observe that $\Lambda^{\mathtt{r}}(\theta^{\mathtt{b}}_a) = \theta^{\mathtt{b}}_a - \Lambda^{\mathtt{b}}(\theta^{\mathtt{b}}_a) = \theta^{\mathtt{b}}_a - a = \gamma^{\mathtt{r}}(A_a)$. Thus, $X(\theta^{\mathtt{b}}_a) = X^{\mathtt{b}}_a + X^{\mathtt{r}}(\gamma^{\mathtt{r}}(A_a)) = X^{\mathtt{b}}_a - A_a = Y_a$, which proves (i). We next prove (ii): since $\Lambda^{\mathtt{b}}$ is the pseudo-inverse of $\theta^{\mathtt{b}}$, if $\Lambda^{\mathtt{b}}_t = a$, then $\theta^{\mathtt{b}}_{a-} \leq t \leq \theta^{\mathtt{b}}_a$; this immediately implies (ii).

We next prove (iii): we suppose that $\Delta\theta^{\mathtt{b}}_a > 0$. Observe that $\theta^{\mathtt{b}}$ is strictly increasing. Thus, for all $b < a$, $\theta^{\mathtt{b}}_b < \theta^{\mathtt{b}}_{a-}$ and $Y_{a-} = \lim_{b \to a-} Y_b = \lim_{b \to a-} X(\theta^{\mathtt{b}}_b) = X(\theta^{\mathtt{b}}_{a-}-)$ by (i).

We first assume that $\Delta A_a > 0$. Since the processes $X^{\mathtt{b}}$ and $X^{\mathtt{r}}$ are independent Lévy processes, it is easy to check that a.s. $\{x \in [0, \infty) : \Delta\gamma^{\mathtt{r}}_x > 0\} \cap \{A_{a-}; a \in [0, \infty) : \Delta A_a > 0\} = \emptyset$. Thus,

$\theta^{\mathtt{b}}_{a-} = a + \gamma^{\mathtt{r}}(A_{a-})$, and for all $t \in [\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a]$, $\Lambda^{\mathtt{b}}_t = a$ and $\Lambda^{\mathtt{r}}_t = t - a = t - \theta^{\mathtt{b}}_{a-} + \gamma^{\mathtt{r}}(A_{a-})$. Thus, for all $s \in [0, \Delta\theta^{\mathtt{b}}_a]$,

$$X_{s+\theta^{\mathtt{b}}_{a-}} = X^{\mathtt{b}}_a + X^{\mathtt{r}}_{s+\gamma^{\mathtt{r}}(A_{a-})} = Y_a + A_a + X^{\mathtt{r}}_{s+\gamma^{\mathtt{r}}(A_{a-})}. \tag{176}$$

Taking $s = 0$ in the previous equality first entails $X(\theta^{\mathtt{b}}_{a-}) = Y_a + \Delta A_a$. Recall that $Y_{a-} = X(\theta^{\mathtt{b}}_{a-}-)$. Thus, $\Delta X(\theta^{\mathtt{b}}_{a-}) = \Delta Y_a + \Delta A_a$, but since Y and A have distinct jump-times, and since we have assumed that $\Delta A_a > 0$, we get $\Delta X(\theta^{\mathtt{b}}_{a-}) = \Delta A_a$ and $\Delta Y_a = 0$. Next, observe the following: since $\theta^{\mathtt{b}}_{a-} = a + \gamma^{\mathtt{r}}(A_{a-})$ and $\theta^{\mathtt{b}}_a = a + \gamma^{\mathtt{r}}(A_a)$, then for all $s \in (0, \Delta\theta^{\mathtt{b}}_a)$, $X^{\mathtt{r}}((s + \gamma^{\mathtt{r}}(A_{a-}))-) > -A_a$; moreover by taking $s = \Delta\theta^{\mathtt{b}}_a$ in (176), we see that $X(\theta^{\mathtt{b}}_a) = Y_a$, by (i). Namely, for all $t \in \big(\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a\big)$, we get $X_t \geq X_{t-} > X((\theta^{\mathtt{b}}_{a-})-) = Y_a$, which proves (iii) when $\Delta A_a > 0$.

We next assume that $\Delta\theta^{\mathtt{b}}_a > 0$ and $\Delta A_a = 0$. Consequently, $\theta^{\mathtt{b}}_{a-} = a + \gamma^{\mathtt{r}}((A_a)-)$. Since $X^{\mathtt{b}}$ and $X^{\mathtt{r}}$ are independent, a.s. $\{x \in [0,\infty) : \Delta\gamma^{\mathtt{r}}_x > 0\} \cap \{A_a; a \in [0,\infty) : \Delta Y_a > 0\} = \emptyset$. Therefore, $\Delta Y_a = 0$. We also check that

$$\forall s \in [0, \Delta\theta^{\mathtt{b}}_a], \quad X_{s+\theta^{\mathtt{b}}_{a-}} = Y_a + A_a + X^{\mathtt{r}}_{s+\gamma^{\mathtt{r}}((A_a)-)} \, . \tag{177}$$

Since $\theta^{\mathtt{b}}_a = a + \gamma^{\mathtt{r}}(A_a)$, applying (177) at $s = 0$ and $s = \Delta\theta^{\mathtt{b}}_a$ implies that $X(\theta^{\mathtt{b}}_{a-}) = X(\theta^{\mathtt{b}}_a) = Y_a$. Since $X(\theta^{\mathtt{b}}_{a-}-) = Y_{a-}$, we get $\Delta X(\theta^{\mathtt{b}}_{a-}) = \Delta Y_a = \Delta A_a = 0$. then, observe that for all $s \in (0, \Delta\theta^{\mathtt{b}}_a)$, $X^{\mathtt{r}}((s + \gamma^{\mathtt{r}}(A_a-))-) > -A_a$; thus (177) entails (iii) when $\Delta A_a = 0$, which completes the proof of the lemma. ■

Lemma 5.5 *Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell^{\downarrow}_3$ satisfy (38). Then, the following holds true.*

- (i) $\mathbf{P}$*-a.s. if $(\Delta X^{\mathtt{r}})(\Lambda^{\mathtt{r}}_t) > 0$, then there exists $a \in (0,\infty)$ such that $\theta^{\mathtt{b}}_{a-} < t < \theta^{\mathtt{b}}_a$.*
- (ii) $\mathbf{P}$*-a.s. for all $b \in [0,\infty)$ such that $\Delta X^{\mathtt{r}}_b > 0$, there is a unique $t \in [0,\infty)$ such that $\Lambda^{\mathtt{r}}_t = b$.*
- (iii) *For all $t \in [0,\infty)$, set $Q^{\mathtt{b}}_t = X^{\mathtt{b}}_{\Lambda^{\mathtt{b}}_t}$ and $Q^{\mathtt{r}}_t = X^{\mathtt{r}}_{\Lambda^{\mathtt{r}}_t}$. Then, a.s. for all $t \in [0,\infty)$, $\Delta Q^{\mathtt{b}}_t \Delta Q^{\mathtt{r}}_t = 0$.*

Proof: suppose that $(\Delta X^{\mathtt{r}})(\Lambda^{\mathtt{r}}_t) > 0$. To simplify notation, we set $b = \Lambda^{\mathtt{r}}_t$ and $x = -\inf_{s\in[0,b]} X^{\mathtt{r}}_s$. Since $X^{\mathtt{r}}$ is a spectrally positive Lévy process, $X_b > -x$. Thus, $b < \gamma^{\mathtt{r}}_x$; moreover, since no excursion above the infimum of the spectrally positive Lévy process $X^{\mathtt{r}}$ starts with a jump we also get, $\gamma^{\mathtt{r}}_{x-} < b$. Thus, $\gamma^{\mathtt{r}}_{x-} < b < \gamma^{\mathtt{r}}_x$. We next set $a = \sup\{s \in [0,\infty) : A_s < x\}$. Then, $A_{a-} \leq x \leq A_a$ and we first prove the following:

$$\theta^{\mathtt{b}}_{a-} - a \leq \gamma^{\mathtt{r}}_{x-} < b < \gamma^{\mathtt{r}}_x \leq \theta^{\mathtt{b}}_a - a \, . \tag{178}$$

Let us first supose that $\Delta A_a > 0$, then a.s $\gamma^{\mathtt{r}}(A_s) \to \gamma^{\mathtt{r}}(A_{a-})$ as $s \to a-$, since a.s. $\gamma^{\mathtt{r}}$ has no jump at time A_{a-} because A and $\gamma^{\mathtt{r}}$ are independent. Thus, $\gamma^{\mathtt{r}}(A_{a-}) \leq \gamma^{\mathtt{r}}(x-)$. Similarly, a.s $\gamma^{\mathtt{r}}(A_s) \to \gamma^{\mathtt{r}}(A_a)$ as $s \to a+$, which implies that $\gamma^{\mathtt{r}}(x) \leq \gamma^{\mathtt{r}}(A_a)$. Note that $\gamma^{\mathtt{r}}(A_{a-}) = \theta^{\mathtt{b}}_{a-} - a$; and that $\gamma^{\mathtt{r}}(A_a) = \theta^{\mathtt{b}}_a - a$, by definition. This implies (178). Now suppose that $\Delta A_a = 0$. Then, $A_{a-} = A_a = x$ and $\theta^{\mathtt{b}}_{a-} - a = \gamma^{\mathtt{r}}(A_a-) = \gamma^{\mathtt{r}}(x-)$, which also implies (178).

We next use (178) to prove (i): first observe that it implies that $\theta^{\mathtt{b}}_{a-} < b + a < \theta^{\mathtt{b}}_a$. But for all $s \in (\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$, $\Lambda^{\mathtt{b}}_s = a$ and thus $\Lambda^{\mathtt{r}}_s = s - a$, which shows that on $(\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$, $\Lambda^{\mathtt{r}}$ is strictly increasing: since $b = \Lambda^{\mathtt{r}}_{a+b} = \Lambda^{\mathtt{r}}_t$, we get $t = a + b$ and finally $\theta^{\mathtt{b}}_{a-} < t < \theta^{\mathtt{b}}_a$, which completes the proof of (i).

Let us prove (ii): let $b \in [0,\infty)$ be such that $\Delta X^{\mathtt{r}}_b > 0$. Since $\Lambda^{\mathtt{r}}$ is continous and tends to ∞, there exists at least one time $t \in [0,\infty)$ such that $\Lambda^{\mathtt{r}}_t = b$. By (i), there exists $a \in (0,\infty)$ such that $\theta^{\mathtt{b}}_{a-} < t < \theta^{\mathtt{b}}_a$. But as previously noticed, $\Lambda^{\mathtt{r}}$ on $(\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$ is strictly increasing, which immediately implies (ii).

Let us prove (iii): suppose that $\Delta Q^{\mathtt{r}}_t > 0$. Since $\Lambda^{\mathtt{r}}$ is continous, it implies that $\Delta X^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t) > 0$ and by (i) there exists $a \in (0,\infty)$, such that $\theta^{\mathtt{b}}_{a-} < t < \theta^{\mathtt{b}}_a$; now observe that for all $s \in (\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$, $\Lambda^{\mathtt{b}}_s = a$. Thus, $Q^{\mathtt{b}}$ is constant on this interval and it implies that $\Delta Q^{\mathtt{b}}_t = 0$. This proves (iii). ■

5.2.4 Excursions of Y above its infimum.

By Theorem 2.6, X is a Lévy process whose Laplace exponent is ψ. recall from Section 5.1 that if ψ satisfies (152), the height process H associated with X is a well-defined continuous process. We prove the following lemma that entails Proposition 2.7.

Lemma 5.6 *Let $\alpha, \beta \in [0, \infty)$, $\kappa \in (0, \infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$ satisfy (152). For all $t \in [0, \infty)$, we recall the following notation: $I_t = \inf_{s\in[0,t]} X_s$, $J_t = \inf_{s\in[0,t]} Y_s$ and $\mathcal{H}_t = H(\theta^{\mathsf{b}}_t)$. Then, the following holds true.*

(i) *A.s. for all $t \in [0, \infty)$, $X_t \geq Y(\Lambda^{\mathsf{b}}_t)$ and $I_t = J(\Lambda^{\mathsf{b}}_t)$.*

(ii) *A.s. $\{t \in [0, \infty) : X_t > I_t\} = \{t \in [0, \infty) : Y(\Lambda^{\mathsf{b}}_t) > J(\Lambda^{\mathsf{b}}_t)\}$.*

(iii) *A.s. the set $\mathscr{E} = \{a \in [0, \infty) : Y_a > J_a\}$ is open. Moreover, if (l, r) is a connected component of $\mathscr{E}$, then $Y_l = Y_r = J_l = J_r$ and for all $a \in (l, r)$, we get $J_a = J_l$ and $Y_{a-} \wedge Y_a > J_l$.*

(iv) *Set $\mathscr{Z}^{\mathsf{b}} = \{a \in [0, \infty) : Y_a = J_a\}$. Then, $\mathbf{P}$-a.s.*

$$(179) \qquad \forall a, z \in [0, \infty) \text{ such that } a < z, \quad \Big(\mathscr{Z}^{\mathsf{b}} \cap (a, z) \neq \emptyset\Big) \Longleftrightarrow \Big(J_z < J_a\Big).$$

(v) *A.s. $\mathcal{H}$ is continuous and a.s. $\{a \in [0, \infty) : Y_a > J_a\} = \{a \in [0, \infty) : \mathcal{H}_a > 0\}$.*

Proof: fix $t \in (0, \infty)$ and set $a = \Lambda^{\mathsf{b}}_t$. Thus, $\theta^{\mathsf{b}}_{a-} \leq t \leq \theta^{\mathsf{b}}_a$. If $\Delta\theta^{\mathsf{b}}_a > 0$, then Lemma 5.4 (iii) implies that $X_s \geq Y_a = X(\theta^{\mathsf{b}}_{a-}-)$, for all $s \in [\theta^{\mathsf{b}}_{a-}, \theta^{\mathsf{b}}_a]$. Thus, $X_t \geq Y_a$. If $\Delta\theta^{\mathsf{b}}_a = 0$, $t = \theta^{\mathsf{b}}_a$ and $X_t = Y_a$ by Lemma 5.4 (i). Thus, we have proved that a.s. for all $t \in [0, \infty)$, $X_t \geq Y(\Lambda^{\mathsf{b}}_t)$.

Let $z \in [0, a]$ be such that $Y_{z-} = \inf_{y\in[0,a]} Y_y$. Note that $Y_y = X(\theta^{\mathsf{b}}_y) \to X((\theta^{\mathsf{b}}_{z-})-)$ as $y \to z-$. Thus, $Y_{z-} = X((\theta^{\mathsf{b}}_{z-})-)$ and since $\theta^{\mathsf{b}}_{z-} \leq \theta^{\mathsf{b}}_{a-} \leq t$, we get $\inf_{s\in[0,t]} X_s = I_t \leq J(\Lambda^{\mathsf{b}}_t) = \inf_{y\in[0,a]} Y_y$. But since a.s. for all $s \in [0, \infty)$, $X_s \geq Y(\Lambda^{\mathsf{b}}_s)$, we get $I_t \geq J(\Lambda^{\mathsf{b}}_t)$. Thus, $I_t = J(\Lambda^{\mathsf{b}}_t)$, which completes the proof of (i).

We next fix $t \in (0, \infty)$ such that $X_t > I_t$; we set $g_t = \sup\{s < t : X_s = I_s\}$ and $d_t = \inf\{s > t : X_s = I_s\}$; standard result on the excursion of spectrally positive processes above their infimum entails that $\Delta X(g_t) = \Delta X(d_t) = 0$: consequently, for all $s \in [g_t, d_t]$, $I_s = I_t = X(g_t) = X(d_t)$.

Let us suppose that $Y(\Lambda^{\mathsf{b}}_t) = J(\Lambda^{\mathsf{b}}_t)$; to simplify, we set $a = \Lambda^{\mathsf{b}}_t$, and thus we get $\theta^{\mathsf{b}}_{a-} \leq t \leq \theta^{\mathsf{b}}_a$. If $\Delta\theta^{\mathsf{b}}_a = 0$, then by Lemma 5.4 (i), we get $X_t = X(\theta^{\mathsf{b}}_a) = Y_a = J_a$ but $J_a = I_t$ by (i) which contradicts $X_t > I_t$. Thus, $\Delta\theta^{\mathsf{b}}_a > 0$ and Lemma 5.4 (iii) asserts that for all $s \in (\theta^{\mathsf{b}}_{a-}, \theta^{\mathsf{b}}_a)$, $X_s > X(\theta^{\mathsf{b}}_a) = Y_a = X((\theta^{\mathsf{b}}_{a-})-)$. Recall that we suppose $Y_a = J_a$ and that $I_t = J_a$, by (i); thus, for all $s \in (\theta^{\mathsf{b}}_{a-}, \theta^{\mathsf{b}}_a)$, we get $X_s > I_s = X(\theta^{\mathsf{b}}_a) = X((\theta^{\mathsf{b}}_{a-})-)$. Thus, $g_t = \theta^{\mathsf{b}}_{a-}$ and $d_t = \theta^{\mathsf{b}}_a$, and since $\Delta X(g_t) = 0$, Lemma 5.4 (iii) entails that $\Delta A_a = \Delta X(\theta^{\mathsf{b}}_{a-}) = 0$. Thus we have proved that a.s. for all $t \in (0, \infty)$, if $X_t > I_t$ and if $Y(\Lambda^{\mathsf{b}}_t) = J(\Lambda^{\mathsf{b}}_t)$ then $g_t = \theta^{\mathsf{b}}_{a-} < d_t = \theta^{\mathsf{b}}_a$ and $\Delta A_a = 0$. We next use the following: for all $\varepsilon \in (0, \infty)$,

$$(180) \qquad \mathbf{P}\text{-a.s.} \quad \sum_{a\in[0,\infty)} \mathbf{1}_{\{\Delta A_a = 0\,;\, \Delta\theta^{\mathsf{b}}_a > \varepsilon\,;\, Y_a = J_a\}} = 0.$$

Before proving (180), let us complete the proof of (ii): first note that (180) and the previous arguments entail that a.s. for all $t \in (0, \infty)$, if $X_t > I_t$, then $Y(\Lambda^{\mathsf{b}}_t) > J(\Lambda^{\mathsf{b}}_t)$. Next observe that if $X_t = I_t$, then (i) implies that $I_t = J(\Lambda^{\mathsf{b}}_t) \leq Y(\Lambda^{\mathsf{b}}_t) \leq X_t$. This shows that if $X_t = I_t$, then $J(\Lambda^{\mathsf{b}}_t) = Y(\Lambda^{\mathsf{b}}_t)$, which completes the proof of (ii), provided that (180) holds true.

Proof of (180). Suppose that $\Delta\theta^{\mathsf{b}}_a > \varepsilon$ and that $\Delta A_a = 0$. Then $\theta^{\mathsf{b}}_{a-} = a + \gamma^{\mathsf{r}}(A_a-)$; since by definition $\theta^{\mathsf{b}}_a = a + \gamma^{\mathsf{r}}(A_a)$, we get $\Delta\theta^{\mathsf{b}}_a = (\Delta\gamma^{\mathsf{r}})(A_a)$. For all $x \in [0, \infty)$, we set $\lambda_x = \inf\{a \in [0, \infty) : A_a > x\}$. By Lemma 2.5, a.s. A is strictly increasing, which implies that $x \mapsto \lambda_x$ is continuous: we get $a = \lambda(A_a)$ and (180) is clearly a consequence of the following

$$(181) \qquad \mathbf{P}\text{-a.s.} \quad Q(\varepsilon) = \sum_{x\in[0,\infty)} \mathbf{1}_{\{\Delta A(\lambda_x) = 0\,;\, \Delta\gamma^{\mathsf{r}}(x) > \varepsilon\,;\, Y(\lambda_x) = J(\lambda_x)\}} = 0.$$

Let us prove (181): recall from (164) that $\mathbf{N}(\zeta \in dr)$ is the Lévy measure of the subordinator $\gamma^{\mathtt{r}}$; since (Y, A) and $X^{\mathtt{r}}$ are independent, we get

$$\mathbf{E}\big[Q(\varepsilon)\big|(Y,A)\big] = \mathbf{N}(\zeta > \varepsilon)\int_0^\infty dx\, \mathbf{1}_{\{\Delta A(\lambda_x)=0\,;\,Y(\lambda_x)=J(\lambda_x)\}} = \mathbf{N}(\zeta > \varepsilon)\int_0^\infty dA_a \mathbf{1}_{\{\Delta A_a=0\,;\,Y_a=J_a\}},$$

by an easy change of variable. Observe that $dA_a = \kappa\beta a\, da + \sum_{a'\in[0,\infty)} \Delta A_{a'}\delta_{a'}(da)$. Thus,

$$\mathbf{E}\big[Q(\varepsilon)\big|(Y,A)\big] = \mathbf{N}(\zeta > \varepsilon)\int_0^\infty \kappa\beta a\, da\, \mathbf{1}_{\{Y_a=J_a\}} = \mathbf{N}(\zeta > \varepsilon)\int_0^\infty \kappa\beta a\, da\, \mathbf{1}_{\{X(\theta^{\mathtt{b}}_a)=I(\theta^{\mathtt{b}}_a)\}},$$

since $X(\theta^{\mathtt{b}}_a) = Y_a$ (by Lemma 5.4 (i)) and since $I(\theta^{\mathtt{b}}_a) = J_a$ by (i). The change of variable $t = \theta^{\mathtt{b}}_a$, entails that $\int_0^\infty a\, da\, \mathbf{1}_{\{X(\theta^{\mathtt{b}}_a)=I(\theta^{\mathtt{b}}_a)\}} = \int_0^\infty \Lambda^{\mathtt{b}}_t\, d\Lambda^{\mathtt{b}}_t \mathbf{1}_{\{X_t=I_t\}}$. Since $\Lambda^{\mathtt{b}}$ is 1-Lipschitz, and since $\int_0^\infty dt\, \mathbf{1}_{\{X_t=I_t\}} = 0$ a.s., the previous arguments imply $\mathbf{E}\big[Q(\varepsilon)\big|(Y,A)\big] = 0$ and (181) and (180). This completes the proof of (ii).

Let us prove (iii): by standard results, $\mathscr{E}' := \{t \in [0,\infty) : X_t > I_t\}$ is open and if (g, d) is a connected component of $\mathscr{E}'$, then $X_g = X_d = I_g = I_d$ and for all $t \in (g, d)$, $X_{t-} \wedge X_t > I_g$. Recall that $\mathscr{E} = \big\{a \in [0,\infty) : Y_a > J_a\big\}$ and let $a \in \mathscr{E}$. Since $X(\theta^{\mathtt{b}}_a) = Y_a$ and since $I(\theta^{\mathtt{b}}_a) = J_a$ (by (i)), we get $\theta^{\mathtt{b}}_a \in \mathscr{E}'$; denote by (g, d) the connected component of $\mathscr{E}'$ such that $\theta^{\mathtt{b}}_a \in (g, d)$. By (i), $X_d = X_g = Y(\Lambda^{\mathtt{b}}_g) = Y(\Lambda^{\mathtt{b}}_d) = J(\Lambda^{\mathtt{b}}_g) = J(\Lambda^{\mathtt{b}}_d)$. We then set $l = \Lambda^{\mathtt{b}}_g$ and $r = \Lambda^{\mathtt{b}}_d$. This proves that $Y_l = Y_r = J_r = J_l$ and that for all $a \in [l, r]$, $J_a = J_l$. By (ii), $\mathscr{E}' := \{t \in [0,\infty) : Y(\Lambda^{\mathtt{b}}_t) > J(\Lambda^{\mathtt{b}}_t)\}$; since (g, d) is connected component of $\mathscr{E}'$, for all $t \in (g, d)$, we get $Y(\Lambda^{\mathtt{b}}_t) > J(\Lambda^{\mathtt{b}}_t) = J_l$ and thus $l < \Lambda^{\mathtt{b}}_t < r$. Namely, $\Lambda^{\mathtt{b}}((g, d)) = (l, r) \subset \mathscr{E}$; since neither l nor r are in $\mathscr{E}$, (l, r) is a connected component of $\mathscr{E}$. This easily entails (iii).

Let us prove (iv). First recall from Section 5.1.3 the notation: $\mathscr{Z} = \big\{t \in [0,\infty) : X_t = I_t\big\}$ and recall that the continuous process $t \mapsto -I_t$ is a local-time for $\mathscr{Z}$: in particular, recall from (161) that $\mathscr{Z} \cap (s, t) \neq \emptyset$ iff $I_t < I_s$. By (ii), $\mathscr{Z} = \big\{t \in [0,\infty) : Y(\Lambda^{\mathtt{b}}_t) = J(\Lambda^{\mathtt{b}}_t)\big\}$; it easily implies the following: $\mathscr{Z}^{\mathtt{b}} \cap (a, z) \neq \emptyset$ iff $\mathscr{Z} \cap (\theta^{\mathtt{b}}_a, \theta^{\mathtt{b}}_z) \neq \emptyset$ which is equivalent to $I(\theta^{\mathtt{b}}_z) = J_z < J_a = I(\theta^{\mathtt{b}}_a)$ (by (i)), which completes the proof of (iv).

Let us prove (v). Since H is continuous, $\mathcal{H}$ is càdlàg and $\mathcal{H}_{a-} = H(\theta^{\mathtt{b}}_{a-})$. If $\Delta\theta^{\mathtt{b}}_a = 0$, then $\mathcal{H}_{a-} = \mathcal{H}_a$. We next assume that $\Delta\theta^{\mathtt{b}}_a > 0$: by Lemma 5.4 (iii), for all $t \in (\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$, we get $X_t > X((\theta^{\mathtt{b}}_{a-})-) = X(\theta^{\mathtt{b}}_a)$; we then apply Lemma 5.1 to $t_0 = \theta^{\mathtt{b}}_{a-}$ and $t_1 = \theta^{\mathtt{b}}_a$: in particular we get $H_{t_0} = H_{t_1}$, namely: $\mathcal{H}_{a-} = \mathcal{H}_a$. This proves that a.s. $\mathcal{H}$ is continuous.

Recall from above that $\mathscr{E}' = \big\{t \in [0,\infty) : X_t > I_t\big\}$ and recall from (160) and (ii) that $\mathscr{E}' = \big\{t \in [0,\infty) : H_t > 0\big\} = \big\{t \in [0,\infty) : Y(\Lambda^{\mathtt{b}}_t) > J(\Lambda^{\mathtt{b}}_t)\big\}$. Then, observe that $a \in \mathscr{E}$ iff $X(\theta^{\mathtt{b}}_a) = Y_a > J_a = I(\theta^{\mathtt{b}}_a)$, which also equivalent to $\mathcal{H}_a = H(\theta^{\mathtt{b}}_a) > 0$. This implies (v) and it completes the proof of the lemma. ■

Proposition 5.7 *Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell^{\downarrow}_3$ satisfy (152). Then, a.s. for all $t \in [0,\infty)$, $H_t \geq \mathcal{H}(\Lambda^{\mathtt{b}}_t)$. Thus,*

$$\text{(182)} \qquad \mathbf{P}\text{-a.s. for all } a \geq b \geq 0, \quad \inf_{r\in[a,b]} \mathcal{H}_r = \inf_{s\in[\theta^{\mathtt{b}}_a, \theta^{\mathtt{b}}_a]} H_s \,.$$

Proof. Recall that $\theta^{\mathtt{b}}$ is the right-continuous inverse of $\Lambda^{\mathtt{b}}$ and that $H \circ \theta^{\mathtt{b}} = \mathcal{H}$. Let $t \in [0,\infty)$. Then, set $a = \Lambda^{\mathtt{b}}_t$, which entails that $\theta^{\mathtt{b}}_{a-} \leq t \leq \theta^{\mathtt{b}}_a$. If $\Delta\theta^{\mathtt{b}}_a = 0$, then $\theta^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t) = t$ and $H_t = \mathcal{H}(\Lambda^{\mathtt{b}}_t)$. Suppose next that $\Delta\theta^{\mathtt{b}}_a > 0$; by Lemma 5.4 (iii), $X_s > X((\theta^{\mathtt{b}}_{a-})-) = Y_a = X(\theta^{\mathtt{b}}_a)$, for all $s \in (\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$; thus, we can apply Lemma 5.1 with $t_0 = \theta^{\mathtt{b}}_{a-}$ and $t_1 = \theta^{\mathtt{b}}_a$, to get $H_t \geq H(\theta^{\mathtt{b}}_{a-}) = H(\theta^{\mathtt{b}}_a) = \mathcal{H}_a = \mathcal{H}(\Lambda^{\mathtt{b}}_t)$. We thus have proved the first point of the proposition.

Let us prove (182): since $H \circ \theta^{\mathtt{b}} = \mathcal{H}$, $\inf_{[a,b]} \mathcal{H} = \inf_{[a,b]} H \circ \theta^{\mathtt{b}} \geq \inf_{[\theta^{\mathtt{b}}_a, \theta^{\mathtt{b}}_a]} H$. But $H \geq \mathcal{H} \circ \Lambda^{\mathtt{b}}$ implies $\inf_{[\theta^{\mathtt{b}}_a, \theta^{\mathtt{b}}_a]} H \geq \inf_{[\theta^{\mathtt{b}}_a, \theta^{\mathtt{b}}_a]} \mathcal{H} \circ \Lambda^{\mathtt{b}}$. Since $\Lambda^{\mathtt{b}}$ is non-decreasing and continuous, and since $\Lambda^{\mathtt{b}}(\theta^{\mathtt{b}}_t) = t$, we finally get $\inf_{[\theta^{\mathtt{b}}_a, \theta^{\mathtt{b}}_a]} \mathcal{H} \circ \Lambda^{\mathtt{b}} = \inf_{[a,b]} \mathcal{H}$, which entails (182). ■

We next recall the following result due to Aldous & Limic [4] (Proposition 14, p. 20).

Proposition 5.8 (Proposition 14 [4]) *Let $\alpha, \beta \in [0, \infty)$, $\kappa \in (0, \infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$ satisfy (38). Then the following holds true.*

(*i*) *For all $a \in [0, \infty)$, $\mathbf{P}(Y_a = J_a) = 0$.*

(*ii*) *$\mathbf{P}$-a.s. the set $\{a \in [0, \infty) : Y_a = J_a\}$ contains no isolated points.*

(*iii*) *Set $M_a = \max\{r - l;\ r \geq l \geq a : (l, r)$ is an excursion interval of $Y - J$ above $0\}$. Then, $M_a \to 0$ in probability as $a \to \infty$.*

Proof. The process $(Y_{s/\kappa})_{s\in[0,\infty)}$ is the the process $W^{\kappa',-\tau,\mathbf{c}}$ in [4], where $\kappa' = \beta/\kappa$ and $\tau = \alpha/\kappa$ (note that the letter κ plays another role in [4]). Then (i) (resp. (ii) and (iii)) is Proposition 14 [4] (b) (resp. resp. (d) and (c)). ∎

Thanks to Proposition 5.8 (iii), the excursion intervals of $Y - J$ above 0 can be listed as follows

$$\{a \in [0, \infty) : Y_a > J_a\} = \bigcup_{k\geq 1} (l_k, r_k) \ . \tag{183}$$

where $\zeta_k = r_k - l_k$, $k \geq 1$ is non-decreasing. Then, as a consequence of Theorem 2 in Aldous & Limic [4], p. 4, we recall the following.

Proposition 5.9 (Theorem 2 [4]) *Let $\alpha, \beta \in [0, \infty)$, $\kappa \in (0, \infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$ satisfy (38). Then, $(\zeta_k)_{k\geq 1}$, the ordered sequence of lengths of the excursions of $Y - J$ above 0, is distributed as the $(\beta/\kappa, \alpha/\kappa, \mathbf{c})$-multiplicative coalescent (as defined in [4]) taken at at time 0. In particular, we get a.s. $\sum_{k\geq 1} \zeta_k^2 < \infty$.*

5.2.5 Embedding into a Lévy tree. Proof of Proposition 2.8.

We now explain how continuous multiplicative graphs are embedded in Lévy trees. First, let us index properly the excursions of H above 0. To that end, recall that Lemma 5.6 (v) asserts that the excursion intervals of $Y - J$ above 0 are exactly the excursion intervals of $\mathcal{H}$ above 0. Similarly, (160) asserts that the excursion intervals of $X - I$ above 0 are exactly the excursion intervals of H above 0. By Lemma 5.6 (ii), (l, r) is an excursion interval of $\mathcal{H}$ above 0 iff $l = \Lambda^{\mathsf{b}}(\mathbf{l})$ and $r = \Lambda^{\mathsf{b}}(\mathbf{r})$, where $(\mathbf{l}, \mathbf{r})$ is an excursion interval of H above 0. Consequently, the excursion interval of H above zero can be ordered according to (183):

$$\bigcup_{k\geq 1} (\mathbf{l}_k, \mathbf{r}_k) = \big\{t \in [0, \infty) : H_t > 0\big\} \quad \text{where} \quad l_k = \Lambda^{\mathsf{b}}(\mathbf{l}_k) \text{ and } r_k = \Lambda^{\mathsf{b}}(\mathbf{r}_k) \text{ satisfy} \tag{184}$$
$$\bigcup_{k\geq 1} (l_k, r_k) = \big\{a \in [0, \infty) : \mathcal{H}_a > 0\big\} \text{ with } \zeta_k = r_k - l_k,\ k \geq 1, \text{ decreasing.}$$

Next, we set for all $k \geq 1$,

$$\forall s \in [0, \infty), \quad \mathbf{H}_k(s) = H_{(\mathbf{l}_k + s)\wedge \mathbf{r}_k} \quad \text{and} \quad \mathtt{H}_k(s) = \mathcal{H}_{(l_k+s)\wedge r_k}. \tag{185}$$

We also set $\boldsymbol{\zeta}_k = \mathbf{r}_k - \mathbf{l}_k$. Recall from (49) the definition of the pseudometric d_h coded by a function h. As a consequence of (182) in Lemma 5.7, we get for all $k \geq 1$,

$$\forall a, b \in [0, \zeta_k], \quad d_{\mathtt{H}_k}(a, b) = d_{\mathbf{H}_k}(\theta_a^{\mathsf{b}}, \theta_b^{\mathsf{b}}) \ . \tag{186}$$

Recall from (50) in Section 2.2.2 that (T_h, d_h, ρ_h, m_h) stands for the rooted compact measured real tree coded by h and recall that $p_h : [0, \zeta_h) \to T_h$ is the canonical projection. To simplify notation, we set

$$\big(\mathbf{T}_k, \delta_k, \rho_k, \mathrm{m}_k^*\big) := \big(T_{\mathbf{H}_k}, d_{\mathbf{H}_k}, \rho_{\mathbf{H}_k}, \mathbf{m}_{\mathbf{H}_k}\big).$$

Then (186) implies the following: set $\mathcal{T}_k = p_{\mathbf{H}_k}(\theta^{\mathsf{b}}([0,\zeta_k]))$, then

(187) $$\big(\mathcal{T}_k, \delta_{k|\mathcal{T}_k\times\mathcal{T}_k}, \rho_k, \mathrm{m}^*_k(\cdot\cap\mathcal{T}_k)\big) \text{ is isometric to } \big(T_{\mathtt{H}_k}, d_{\mathtt{H}_k}, \rho_{\mathtt{H}_k}, m_{\mathtt{H}_k}\big).$$

Namely, we view the tree coded by $\mathtt{H}_k$ as a compact subtree (namely, a compact connected subset) of the Lévy tree coded by $\mathbf{H}_k$. Next, recall from (60) that $\mathbf{\Pi}_k = ((s^k_p, t^k_p))_{1\le p\le \mathbf{p}_k}$ is the set of pinching times of the excursion $\mathtt{H}_k$ and recall from (61) that $(\mathbf{G}_k, d_k, \rho_k, \mathbf{m}_k)$ is the compact metric space coded by $\mathtt{H}_k$ and the pinching setup $(\mathbf{\Pi}_k, 0)$ as defined in (53). We then set $\mathbf{\Pi}^*_k = \big(p_{\mathbf{H}_k}(\theta^{\mathsf{b}}_{s^k_p}), p_{\mathbf{H}_k}(\theta^{\mathsf{b}}_{t^k_p})\big)_{1\le p\le \mathbf{p}_k}$ and thanks to (186), we see that:

(188) $(\mathbf{G}_k, d_k, \rho_k, \mathbf{m}_k)$ is isometric to the $(\mathbf{\Pi}^*_k, 0)$-metric space
associated to $\big(\mathcal{T}_k, \delta_{k\,|\mathcal{T}_k\times\mathcal{T}_k}, \rho_k, \mathbf{m}^*_k(\cdot\cap\mathcal{T}_k)\big)$.

To summarise, up to the identifications given by (187) and (188), *the k-th largest component $\mathbf{G}_k$ of the multiplicative continuous random graph is obtained as a finitely pinched metric space associated with the real tree $\mathcal{T}_k$ coded by $\mathtt{H}_k$ that is a subtree of the real tree $\mathbf{T}_k$ coded by $\mathbf{H}_k$*. This allows to prove Proposition 2.8 as follows.

Proof of Proposition 2.8. Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} \in \ell^{\downarrow}_3$. Recall from (37) that for all $\lambda \in [0,\infty)$, $\psi(\lambda) = \alpha\lambda + \frac{1}{2}\beta\lambda^2 + \sum_{j\ge1}\kappa c_j\big(e^{-\lambda c_j} - 1 + \lambda c_j\big)$, that is assumed to satisfy (152). We then introduce the following exponents:

$$\gamma = \sup\{r\in[0,\infty) : \lim_{\lambda\to\infty}\psi(\lambda)\lambda^{-r} = \infty\} \quad\text{and}\quad \eta = \inf\{r\in[0,\infty) : \lim_{\lambda\to\infty}\psi(\lambda)\lambda^{-r} = 0\}\;.$$

Recall from (162) that $\mathbf{N}$ stands for the excursion measure of the ψ-height process H above 0 and denote by (T_H, d_H, ρ_H, m_H) the rooted compact measured real tree coded by H. Theorem 5.5 in Le Gall & D. [22], p. 590, asserts that if $\gamma > 1$, then $\mathbf{N}(dH)$-a.e. $\dim_{\mathrm{H}}(T_H) = \eta/(\eta-1)$ and $\dim_{\mathrm{p}}(T_H) = \gamma/(\gamma-1)$ (this statement is a specific case of Theorem 5.5 in [22] where $E = [0,\infty)$). Moreover, in the proof of the Theorem 5.5 [22], two estimates for the local upper- and lower-densities of the mass measure m_H are given at (45) and (46) in [22], p. 593: namely, for all $0 < u < \eta/(\eta-1)$ and $0 < v < \gamma/(\gamma-1)$, $\mathbf{N}(dH)$-a.e. for m_H-almost all $\sigma \in T_H$, $\limsup_{r\to0} r^{-u} m_H(B(\sigma,r)) < \infty$ and $\liminf_{r\to0} r^{-v} m_H(B(\sigma,r)) < \infty$ (actually, within the notations of [22], if $E = [0,\infty)$, then $d(E) = 1$ and $\kappa(d\sigma) = m_H(d\sigma)$). Since $\mathbf{T}_k$ is the tree coded by $\mathbf{H}_k$, then, the previous estimates and Thm 5.5 in [22] show that for all $0 < u < \eta/(\eta-1)$ and $0 < v < \gamma/(\gamma-1)$

(189) $$\mathbf{P}\text{-a.s. for all } k\ge1,\quad \dim_{\mathrm{H}}(\mathbf{T}_k) = \frac{\eta}{\eta-1} \quad\text{and}\quad \dim_{\mathrm{p}}(\mathbf{T}_k) = \frac{\gamma}{\gamma-1}$$
$$\text{and for } \mathrm{m}^*_k\text{-almost all } \sigma\in\mathbf{T}_k \quad \limsup_{r\to0}\frac{\mathrm{m}^*_k(B(\sigma,r))}{r^u} < \infty \quad\text{and}\quad \liminf_{r\to0}\frac{\mathrm{m}^*_k(B(\sigma,r))}{r^v} < \infty.$$

We now apply Lemma C.1 in Appendix Section C to $E_0 = \mathbf{T}_k$, $E = \mathcal{T}_k$ and $E' = \mathbf{G}_k$ to derive from (189) that $\mathbf{P}$-a.s. for all $k\ge1$, $\dim_{\mathrm{H}}(\mathbf{G}_k) = \eta/(\eta-1)$ and $\dim_{\mathrm{p}}(\mathbf{G}_k) = \gamma/(\gamma-1)$.

Thus, to complete the proof of Proposition 2.8, it remains to prove that the exponents γ and η are given by (62) when $\beta = 0$: set $\pi(dr) = \sum_{j\ge1}\kappa c_j\delta_{c_j}$, the Lévy measure of X. By an immediate calculation, we get $\psi'(\lambda) - \alpha = \int_{(0,\infty)}(1 - e^{-\lambda r})r\pi(dr)$. We next introduce

$$J(x) = x^{-1}\!\int_0^x\! du\int_{(u,\infty)} r\pi(dr) = \int_{(0,\infty)} r(1\wedge(r/x))\pi(dr) = \sum_{j\ge1}\kappa c_j^2(1\wedge(c_j/x))$$

as in Proposition 2.8 (ii). As explained in Bertoin's book [6] Chapter III, general arguments on the Laplace exponents of subordinators entail that there exist two universal constants $k_1, k_2 \in (0,\infty)$ such that

$$k_1 J(1/\lambda) \le \psi'(\lambda) - \alpha \le k_2 J(1/\lambda)\;.$$

Since $\psi(\lambda) \leq \lambda\psi'(\lambda) \leq 4\psi(\lambda)$, by convexity, the previous inequality entails: $\gamma = 1 + \sup\{r \in (0,\infty) : \lim_{0+} x^r J(x) = \infty\}$ and $\eta = 1 + \inf\{r \in (0,\infty) : \lim_{0+} x^r J(x) = 0\}$, which complete the proof of Proposition 2.8. ■

6 Limit theorems.

6.1 Proof of Theorem 2.14.

In this section we fix $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$, $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$. We assume that

$$\text{(190)} \qquad \int^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty \quad \text{where} \quad \psi(\lambda) = \alpha\lambda + \tfrac{1}{2}\beta\lambda^2 + \sum_{j\leq 1} \kappa c_j \big(e^{-\lambda c_j} - 1 + \lambda c_j\big),\ \lambda \in [0,\infty).$$

We fix the following *independent* processes: a standard linear Brownian motion $B = (B_t)_{t\in[0,\infty)}$ and for all $j \geq 1$, $(N_j(t))_{t\in[0,\infty)}$, an homogeneous Poisson process with jump-rate κc_j. All processes have initial value 0. Recall from (36) that

$$\text{(191)} \qquad \forall t \in [0,\infty), \quad X_t^{\mathtt{b}} = \alpha t + \sqrt{\beta} B_t + \sum_{j\geq 1}^{\perp} c_j \big(N_j(t) - c_j \kappa t\big).$$

(we sum with respect to the L^2-norm of the supremum). Then, $X^{\mathtt{b}}$ is a spectrally positive Lévy process with Laplace exponent ψ and initial value 0. Recall that $\mathbf{E}[c_j(N_j(t)-1)_+] = c_j\big(e^{-\kappa c_j t} - 1 + \kappa c_j t\big)? \leq \frac{1}{2}(\kappa t)^2 c_j^3$, then, recall from (39) that it makes sense to set

$$\text{(192)} \qquad \forall t \in [0,\infty), \quad A_t = \tfrac{1}{2}\kappa\beta t^2 + \sum_{j\geq 1} c_j \big(N_j(t) - 1\big)_+ \quad \text{and} \quad Y_t = X_t^{\mathtt{b}} - A_t.$$

Let $(X_t^{\mathtt{r}})_{t\in[0,\infty)}$ be an independent copy of $X^{\mathtt{b}}$ and recall from (41) the following:

$$\text{(193)} \qquad \forall x, t \in [0,\infty), \quad \gamma_x^{\mathtt{r}} = \inf\{s \in [0,\infty) : X_s^{\mathtt{r}} < -x\} \quad \text{and} \quad \theta_t^{\mathtt{b}} = t + \gamma_{A_t}^{\mathtt{r}}.$$

Let $a_n, b_n \in (0,\infty)$, and $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, satisfy (66), $(\mathbf{C1})$–$(\mathbf{C3})$: namely $a_n \to \infty$, $b_n/a_n \to \infty$, $b_n^2/a_n \to \beta_0 \in [0,\beta]$, $a_n b_n/\sigma_1(\mathtt{w}_n) \to \kappa \in (0,\infty)$ and

$$\text{(194)} \qquad (\mathbf{C1}): \ \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big) \xrightarrow[n\to\infty]{} \alpha \qquad (\mathbf{C2}): \ \frac{b_n}{a_n^2} \cdot \frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)} \xrightarrow[n\to\infty]{} \beta + \kappa\sigma_3(\mathbf{c}),$$

$$\text{(195)} \qquad (\mathbf{C3}): \ \forall j \in \mathbb{N}^*, \quad \frac{w_j^{(n)}}{a_n} \xrightarrow[n\to\infty]{} c_j.$$

In this section we admit Proposition 2.11 and Proposition 2.12 that are proved in Section 6.3.2.

Recall that $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ is the set of positive integers. For all $n \in \mathbb{N}^*$, let $(N_j^{\mathtt{w}_n}(\cdot))_{j\geq 1}$ be independent homogeneous Poisson processes, the jumps rate of $N_j^{\mathtt{w}_n}$ being $w_j^{(n)}/\sigma_1(\mathtt{w}_n)$. Recall from (28) and (29) that for all $t \in [0,\infty)$,

$$\text{(196)} \qquad X_t^{\mathtt{b},\mathtt{w}_n} = -t + \sum_{j\geq 1} w_j^{(n)} N_j^{\mathtt{w}_n}(t), \quad A_t^{\mathtt{w}_n} = \sum_{j\geq 1} w_j^{(n)} (N_j^{\mathtt{w}_n}(t) - 1)_+ \ \text{and} \ Y_t^{\mathtt{w}_n} = X_t^{\mathtt{b},\mathtt{w}_n} - A_t^{\mathtt{w}_n}.$$

Let $X^{\mathtt{r},\mathtt{w}_n}$ be an independent copy of $X^{\mathtt{b},\mathtt{w}_n}$. Recall from (28) and (30) in Lemma 2.3 the following definition for all $x, t \in [0,\infty)$:

$$\text{(197)} \qquad \gamma_x^{\mathtt{r},\mathtt{w}_n} = \inf\{s \in [0,\infty) : X_s^{\mathtt{r},\mathtt{w}_n} < -x\} \quad \text{and} \quad \theta_t^{\mathtt{b},\mathtt{w}_n} = t + \gamma_{A_t^{\mathtt{w}_n}}^{\mathtt{r},\mathtt{w}_n}.$$

We shall use several time the following result from Ethier & Kurtz [25].

Lemma 6.1 (Lemma 8.2 [25]) *For all $n \in \mathbb{N}$, let $0 = \mathbf{s}_0^n < \mathbf{s}_1^n < \mathbf{s}_2^n < \ldots$ be a sequence of r.v. such that a.s. $\lim_{k\to\infty} \mathbf{s}_k^n = \infty$. Fix $z \in (0,\infty)$ and set $\mathbf{k}_n = \max\{k \in \mathbb{N} : \mathbf{s}_k^n < z\}$. Then*

$$\lim_{\eta\to 0+} \sup_{n\in\mathbb{N}} \mathbf{P}\Big(\min_{1\leq k\leq \mathbf{k}_n} \mathbf{s}_k^n - \mathbf{s}_{k-1}^n < \eta \Big) = 0 \quad \Longleftrightarrow \quad \lim_{\eta\to 0+} \sup_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} \mathbf{P}\big(\mathbf{s}_k^n < z\,;\, \mathbf{s}_{k+1}^n - \mathbf{s}_k^n < \eta\big) = 0.$$

Proof: see Lemma 8.2 in Ethier & Kurtz [25] (Chapter 3, p. 134). ■

Recall from (144) the càdlàg modulus of continuity of $y \in \mathbf{D}([0,\infty),\mathbb{R})$: for all $z, \eta \in (0,\infty)$:

$$w_z(y,\eta) = \inf\big\{ \max_{1\leq i\leq r} \mathtt{osc}(y, [t_{i-1}, t_i)\,)\,;\ 0 = t_0 < \ldots < t_r = z\ :\ \min_{1\leq i\leq r-1} (t_i - t_{i-1}) \geq \eta\ \big\}, \tag{198}$$

where for all interval I, $\mathtt{ocs}(y, I) = \sup\{|y(s) - y(t)|;\ s, t \in I\}$. We shall use a tightness result for *increasing processes* that is a consequence of Proposition 8.3 in Ethier & Kurtz [25]. To recall this statement, we need to introduce the following notation: let $y \in \mathbf{D}([0,\infty),\mathbb{R})$ be nonnegative, increasing and such that $y(t) \to \infty$ as $t \to \infty$; then for all $\varepsilon \in (0,\infty)$, we recursively define times $(\tau_k^\varepsilon(y))_{k\in\mathbb{N}}$ by setting

$$\tau_0(y) = 0 \quad \text{and} \quad \tau_{k+1}^\varepsilon(y) = \inf\big\{t > \tau_k^\varepsilon(y) \colon\ y(t) - y\big(\tau_k^\varepsilon(y)\big) > \varepsilon\big\}. \tag{199}$$

Observe that if $w_z(y,\eta) > \varepsilon$, then there exists $k \in \mathbb{N}$ such that $\tau_k^\varepsilon(y) \leq z$ and $\tau_{k+1}^\varepsilon(y) - \tau_k^\varepsilon(y) < \eta$. This combined with Lemma 8.2 [25] (recalled above as Lemma 6.1) immediately entails the following.

Lemma 6.2 *For all $n \in \mathbb{N}$, let $(R_t^n)_{t\in[0,\infty)}$ be a càdlàg nondecreasing process such that a.s. $\lim_{t\to\infty} R_n(t) = \infty$. Then, the laws of the R^n are tight in $\mathbf{D}([0,\infty),\mathbb{R})$ if for any t, the laws of the $R_n(t)$, $n \in \mathbb{N}$ are tight on $\mathbb{R}$ and if*

$$\forall z, \varepsilon \in (0,\infty), \qquad \lim_{\eta\to 0+} \limsup_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} \mathbf{P}\big(\tau_k^\varepsilon(R^n) < z\,;\ \tau_{k+1}^\varepsilon(R^n) - \tau_k^\varepsilon(R^n) < \eta\big) = 0. \tag{200}$$

Proof: see Lemma 8.1 and Proposition 8.3 in Ethier & Kurtz [25] (Chapter 3, pp. 134-135). ■

We immediately apply Lemma 6.2 in combination to the estimates in Lemma 4.2 with to prove tightness of a rescaled version of $A^{\mathtt{w}_n}$.

Lemma 6.3 *Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} \in \ell_3^\downarrow$ satisfy (190). Let $a_n, b_n \in (0,\infty)$ and $bw_n \in \ell_f^\downarrow$, $n \in \mathbb{N}$, satisfy (66) and $(\mathbf{C1})$–$(\mathbf{C3})$. The the laws of $(\frac{1}{a_n} A_{b_n t}^{\mathtt{w}_n})_{t\in[0,\infty)}$ are tight on $\mathbf{D}([0,\infty),\mathbb{R})$.*

Proof: we repeatedly use the following estimate on Poisson r.v. N with mean $r \in (0,\infty)$:

$$\mathbf{E}\big[(N-1)_+\big] = e^{-r} - 1 + r \quad \text{and} \quad \mathbf{var}\big((N-1)_+\big) = r^2 - (e^{-r} - 1 + r)(e^{-r} + r) \leq r^2. \tag{201}$$

By the definition(196), we get $\mathbf{E}[A_t^{\mathtt{w}_n}] = \sum_{j\geq 1} w_j^{(n)} (e^{-tw_j^{(n)}/\sigma_1(\mathtt{w}_n)} - 1 + \frac{tw_j^{(n)}}{\sigma_1(\mathtt{w}_n)}) \leq \frac{t^2\sigma_3(\mathtt{w}_n)}{2\sigma_1(\mathtt{w}_n)^2}$. Thus, by $(\mathbf{C1})$–$(\mathbf{C3})$ and the Markov inequality, we get

$$\limsup_{n\to\infty} \mathbf{P}\Big(\tfrac{1}{a_n} A_{b_n t}^{\mathtt{w}_n} \geq x\Big) \leq \tfrac{1}{2} x^{-1} t^2 \kappa \big(\kappa\sigma_3(\mathbf{c}) + \beta\big) \xrightarrow[x\to\infty]{} 0.$$

This shows that for any $t \in [0,\infty)$, the laws of the $\frac{1}{a_n} A_{b_n t}^{\mathtt{w}_n}$ are tight on $\mathbb{R}$.

We next prove (200) with $R_t^n = \frac{1}{a_n} A_{b_n t}^{\mathtt{w}_n}$, $t \in [0,\infty)$. To that end we fix $z, \varepsilon \in (0,\infty)$ and $k \in \mathbb{N}$, and we set $T_n := \tau_k^\varepsilon(R^n)$. Then, (141) in Lemma 4.2 with $a = a_n\varepsilon$, $T = b_n T_n$, $t = b_n\eta$ and $t_0 = b_n z$ implies the following:

$$\begin{aligned}
\mathbf{P}\big(\tau_k^\varepsilon(R^n) < z\,;\ \tau_{k+1}^\varepsilon(R^n) - \tau_k^\varepsilon(R^n) < \eta\big) &= \mathbf{P}\big(b_n T_n < b_n z\,;\ A_{b_n T_n + b_n\eta}^{\mathtt{w}_n} - A_{b_n T_n}^{\mathtt{w}_n} > a_n\varepsilon\big) \\
&\leq (a_n\varepsilon)^{-1} b_n\eta\big(b_n z + \tfrac{1}{2} b_n\eta\big) \frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)^2} \\
&\leq \varepsilon^{-1}\eta(z+\eta) \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \frac{b_n\sigma_3(\mathtt{w}_n)}{a_n^2\sigma_1(\mathtt{w}_n)}.
\end{aligned}$$

Then $(\mathbf{C1})$–$(\mathbf{C3})$ entails (200) and Lemma 6.2 completes the proof. ■

Lemma 6.4 *Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} \in \ell_3^{\downarrow}$ satisfy (190). Let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, satisfy (66),* $(\mathbf{C1})$–$(\mathbf{C3})$. *Then*

$$\Big(\big(\tfrac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n t}\big)_{t\in[0,\infty)}, \big(N_j^{\mathtt{w}_n}(b_n t)\big)_{t\in[0,\infty)};\, j\geq 1 \Big) \xrightarrow[n\to\infty]{} \big(X^{\mathtt{b}}, N_j;\, j\geq 1\big) \tag{202}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R})^{\mathbb{N}}$ equipped with the product topology.

Proof. Let $u \in \mathbb{R}$. Note that

$$\mathbf{E}\big[\exp(iuN_j^{\mathtt{w}_n}(b_n t))\big] = \exp(-tb_n w_j^{(n)}(1-e^{iu})/\sigma_1(\mathtt{w}_n)) \longrightarrow \exp(-t\kappa c_j(1-e^{iu}))$$

by (66) and $(\mathbf{C3})$. Thus, for all $t\in[0,\infty)$, $N_j^{\mathtt{w}_n}(b_n t) \to N_j(t)$ in law. Next fix $k\geq 1$ and set:

$$\forall t\in[0,\infty), \quad Q_t^n = \frac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n t} - \sum_{1\leq j\leq k} a_n^{-1} w_j^{(n)} N_j^{\mathtt{w}_n}(b_n t) \quad \text{and} \quad Q_t = X^{\mathtt{b}}_t - \sum_{1\leq j\leq k} c_j N_j(t)\;.$$

Since we assume that Proposition 2.11 holds true, $\frac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n t} \to X^{\mathtt{b}}_t$ weakly on $\mathbb{R}$. Since Q_t^n (resp. Q_t) is independent of $(N_j^{\mathtt{w}_n})_{1\leq j\leq k}$ (resp. independent of $(N_j)_{1\leq j\leq k}$), we easily check

$$\mathbf{E}\big[e^{iuQ_t^n}\big] = \mathbf{E}\big[e^{iuX^{\mathtt{b},\mathtt{w}_n}_{b_n t}/a_n}\big] \Big/ \prod_{1\leq j\leq k} \mathbf{E}\big[e^{-iu\frac{w_j^{(n)}}{a_n} N_j^{\mathtt{w}_n}(b_n t)}\big] \xrightarrow[n\to\infty]{} \mathbf{E}\big[e^{iuX^{\mathtt{b}}_t}\big] \Big/ \prod_{1\leq j\leq k} \mathbf{E}\big[e^{-iuc_j N_j(t)}\big] = \mathbf{E}\big[e^{iuQ_t}\big].$$

Thus, $Q_t^n \to Q_t$ weakly on $\mathbb{R}$. Since Lévy processes weakly converge in $\mathbf{D}([0,\infty),\mathbb{R})$ iff unidimensional marginals weakly converge on $\mathbb{R}$ (see Lemma B.8 in Appendix Section B, with precise references), we get $Q^n \to Q$ and for all $j\geq 1$, $N_j^{\mathtt{w}_n}(b_n\cdot) \to N_j$, weakly on $\mathbf{D}([0,\infty),\mathbb{R})$.

Since $Q^n, N_1^{\mathtt{w}_n}, \ldots, N_k^{\mathtt{w}_n}$ are independent Lévy processes, they have a.s. no common jump-times and Lemma B.2 (in Appendix, Section B) asserts that

$$(Q_t^n, N_1^{\mathtt{w}_n}(b_n t), \ldots, N_k^{\mathtt{w}_n}(b_n t))_{t\in[0,\infty)} \longrightarrow (Q, N_1, \ldots, N_k) \text{ weakly on } \mathbf{D}([0,\infty),\mathbb{R}^{k+1}).$$

Since $X^{\mathtt{b},\mathtt{w}_n}$ is a linear combination of Q^n and the $(N_j^{\mathtt{w}_n})_{1\leq j\leq k}$, we get:

$$\big((\tfrac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n t}, N_1^{\mathtt{w}_n}(b_n t), \ldots, N_k^{\mathtt{w}_n}(b_n t)\big)_{t\in[0,\infty)} \longrightarrow (X^{\mathtt{b}}, N_1, \ldots, N_k) \text{ weakly on } \mathbf{D}([0,\infty),\mathbb{R}^{k+1}),$$

which implies the weaker statement: $(\frac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, N_1^{\mathtt{w}_n}(b_n\cdot), \ldots, N_k^{\mathtt{w}_n}(b_n\cdot)) \longrightarrow (X^{\mathtt{b}}, N_1, \ldots, N_k)$, weakly on $(\mathbf{D}([0,\infty),\mathbb{R}))^{k+1}$ equipped with the product topology. Since it holds true for all k, an elementary result (see Lemma B.7 in Appendix, Section B) entails (202). ■

Lemma 6.5 *Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} \in \ell_3^{\downarrow}$ satisfy (190). Let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, satisfy (66),* $(\mathbf{C1})$–$(\mathbf{C3})$. *Then*

$$\Big(\big(\tfrac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n t}\big)_{t\in[0,\infty)}, \big(\tfrac{1}{a_n} A^{\mathtt{w}_n}_{b_n t}\big)_{t\in[0,\infty)} \Big) \xrightarrow[n\to\infty]{} \big(X^{\mathtt{b}}, A\big) \quad \textit{weakly on } \mathbf{D}([0,\infty),\mathbb{R})^2. \tag{203}$$

Proof: Lemma 6.3 and Lemma 6.4 imply that the laws of $(\frac{1}{a_n} A^{\mathtt{w}_n}_{b_n\cdot}, \frac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, N_j^{\mathtt{w}_n}(b_n\cdot); j\geq 1)$ are tight on $\mathbf{D}([0,\infty),\mathbb{R})^{\mathbb{N}}$ equipped with the product topology. We want to prove that there is a unique limiting law: let $(n(p))_{p\in\mathbb{N}}$ be an increasing sequence of integers such that

$$\Big(\frac{1}{a_{n(p)}} A^{\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}, \frac{1}{a_{n(p)}} X^{\mathtt{b},\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}, N_j^{\mathtt{w}_{n(p)}}(b_{n(p)}\cdot); j\geq 1\Big) \xrightarrow[p\to\infty]{} \big(A', X^{\mathtt{b}}, N_j; j\geq 1\big), \tag{204}$$

holds weakly on $\mathbf{D}([0,\infty),\mathbb{R})^{\mathbb{N}}$. Since $\mathbf{D}([0,\infty),\mathbb{R})^{\mathbb{N}}$ equipped with the product topology is a Polish space, Skorokod's representation theorem applies and without loss of generality (but with a slight abuse of notation), we can assume that (204) holds true $\mathbf{P}$-almost surely on $\mathbf{D}([0,\infty),\mathbb{R})^{\mathbb{N}}$.

Set $A_t = \frac{1}{2}\kappa\beta t^2 + \sum_{j\geq 1} c_j \big(N_j(t)-1\big)_+$, $t\in[0,\infty)$. Then, to prove (203), we claim that it is sufficient to prove that for all $t\in[0,\infty)$,

$$\text{(205)} \qquad \frac{1}{a_{n(p)}} A^{\mathtt{w}_{n(p)}}_{b_{n(p)}t} \longrightarrow A_t \quad \text{in probability}$$

because it implies a.s. that $A = A'$. *Indeed*, let t such that $\Delta A'_t = \Delta A_t = 0$ and let q, q' be rational numbers such that $q < t < q'$; thus, $A^{\mathtt{w}_{n(p)}}_{b(n(p))q} \leq A^{\mathtt{w}_{n(p)}}_{b(n(p))t} \leq A^{\mathtt{w}_{n(p)}}_{b(n(p))q'}$; since $\Delta A'_t = 0$, we get a.s. $A^{\mathtt{w}_{n(p)}}_{b_{n(p)}t}/a_{n(p)} \to A'_t$; the convergence in probability entails that $A_q \leq A'_t \leq A_{q'}$; since it holds true for all rational numbers q, q' such that $q<t<q'$, we get $A_{t-} \leq A'_t \leq A_t$ which implies $A_t = A'_t$ since $\Delta A_t = 0$. Thus, a.s. A and A' coincide on the dense subset $\{t\in[0,\infty) : \Delta A'_t = \Delta A_t = 0\}$: it entails that a.s. $A = A'$ and the law of $(A, X^{\mathtt{b}}, N_j; j\geq 1)$ is the unique weak limit of the the laws of $(\frac{1}{a_n}A^{\mathtt{w}_n}_{b_n\cdot}, \frac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, N^{\mathtt{w}_n}_j(b_n\cdot); j\geq 1)$.

So, we only need to prove (205). To simplify notation we define $\mathtt{v}_n \in \ell^{\downarrow}_f$ by

$$\text{(206)} \qquad \forall j?\in\mathbb{N}^*, \quad v^{(n)}_j = w^{(n)}_j/a_n$$

By $(\mathbf{C3})$, $v^{(n)}_j \to c_j$; by (66) and $(\mathbf{C2})$,$b_n\sigma_1(\mathtt{v}_n)\to\kappa$ and $\sigma_3(\mathtt{v}_n)\to\sigma_3(\mathbf{c}) + \beta/\kappa$. We next claim that there exists $j_n \to \infty$ such that

$$\text{(207)} \qquad \lim_{n\to\infty} v^{(n)}_{j_n} = 0 \quad \lim_{n\to\infty} \sum_{1\leq j\leq j_n} (v^{(n)}_j)^3 = \sigma_3(\mathbf{c}) \quad \text{and} \quad \lim_{n\to\infty} \sum_{j>j_n} (v^{(n)}_j)^3 = \beta/\kappa.$$

Indedd, suppose first that $\sup\{j\geq 1 : c_j > 0\} = \infty$ and set $j_n = \sup\{j \geq 1 : \sum_{1\leq i\leq j} (v^{(n)}_i)^3 \leq \sigma_3(\mathbf{c})\}$, with the convention that $\sup\emptyset = 0$. Here $j_n\to\infty$, and it is easy to check that it satisfies (207).

Next suppose that $j_* = \sup\{j\geq 1 : c_j > 0\} < \infty$. Clearly $\sum_{1\leq j\leq j_*} (v^{(n)}_j)^3 \to \sigma_3(\mathbf{c})$. Thus, $\sum_{j>j_*} (v^{(n)}_j)^3 \to \beta/\kappa$, that is strictly positive as implied by (190). Thus, it is possible to find a sequence (j_n) that tends to ∞ sufficiently slowly to get $\sum_{j_*<j\leq j_n} (v^{(n)}_j)^3 \to 0$, which implies (207). □

We fix $t\in[0,\infty)$. Let $k\in\mathbb{N}$ to be specified further; since $j_n\to\infty$, we can assume p such that $k < j_{n(p)}$. To simplify, we set $\xi^p_j = v^{(n(p))}_j \big(N^{\mathtt{w}_{n(p)}}_j(b_{n(p)}t) - 1\big)_+$ and $\xi_j = c_j\big(N_j(t)-1\big)_+$ and

$$D^{k,p}_t = \sum_{1\leq j\leq k} \xi^p_j - \xi_j, \quad R^{k,p}_t = \sum_{k<j\leq j_{n(p)}} \xi^p_j - \sum_{j>k} \xi_j, \quad C^p_t = \sum_{j>j_{n(p)}} \xi^p_j - \mathbf{E}[\xi^p_j] \quad \text{and} \quad d_p(t) = \tfrac{1}{2}\kappa\beta t^2 - \sum_{j>j_{n(p)}} \mathbf{E}[\xi^p_j].$$

Thus, $A^{\mathtt{w}_{n(p)}}(b_{n(p)}t)/a_{n(p)} - A_t = D^{k,p}_t + R^{k,p}_t + C^p_t - d_p(t)$ and we prove that each term in the right-hand side goes to 0 in probability.

We first show that $d_p(t)\to 0$. Since $N^{\mathtt{w}_{n(p)}}_j(b_{n(p)}t)$ is a Poisson r.v. with mean $r_{p,j}$ that is equal to $v^{(n(p))}_j b_{n(p)}t/\sigma_1(\mathtt{v}_{n(p)})$, by (201) we get $\mathbf{E}\big[\xi^p_j\big] = v^{(n(p))}_j \big(e^{-r_{p,j}} - 1 + r_{p,j}\big)$. We next use the following elementary inequality:

$$\text{(208)} \qquad \forall y\in[0,\infty), \quad 0 \leq \tfrac{1}{2}y^2 - \big(e^{-y} - 1 + y\big) \leq \tfrac{1}{2}y^2(1-e^{-y}) \leq \tfrac{1}{2}\, y^2 \wedge y^3 \,.$$

that holds true since $y^{-2}(e^{-y} - 1 + y) = \int_0^1 dv \int_0^v dw\, e^{-wy}$. Thus

$$0 \leq \sum_{j>j_{n(p)}} \tfrac{1}{2} v^{(n(p))}_j r^2_{p,j} - \mathbf{E}\big[\xi^p_j\big] \leq \sum_{j>j_{n(p)}} \tfrac{1}{2} v^{(n(p))}_j r^3_{p,j} \leq \tfrac{1}{2} v^{(n(p))}_{j_{n(p)}} \frac{(b_{n(p)}t)^3}{\sigma_1(\mathtt{v}_{n(p)})^3} \sum_{j>j_{n(p)}} (v^{(n(p))}_j)^3 \longrightarrow 0,$$

by (207). Next, note that $\sum_{j>j_{n(p)}} v^{(n(p))}_j r^2_{p,j} = (b_{n(p)}t/\sigma_1(\mathtt{v}_{n(p)}))^2 \sum_{j>j_{n(p)}} (v^{(n(p))}_j)^3 \longrightarrow \kappa\beta t^2$, which implies that $d_p(t)\to 0$ as $p\to\infty$.

We next consider C^p_t: by (201), $\mathbf{var}(\xi^p_j) \le (v_j^{(n(p))})^2 r^2_{p,j}$. Since the ξ^p_j are independent, we get

$$\mathbf{E}\big[(C^p_t)^2\big] = \sum_{j>j_{n(p)}} \mathbf{var}(\xi^p_j) \le v_{j_{n(p)}}^{(n(p))} \frac{(b_{n(p)}t)^2}{\sigma_1(\mathtt{v}_{n(p)})^2} \sum_{j>j_{n(p)}} (v_j^{(n(p))})^3 \longrightarrow 0$$

by (207), which proves that $C^p_t \to 0$ in probability when $p\to\infty$.

We next deal with $R^{k,p}_t$. By (201), (207) and (208), we first get:

$$(209)\quad 0 \le \sum_{k<j\le j_{n(p)}} \mathbf{E}\big[\xi^p_j\big] \le \sum_{k<j\le j_{n(p)}} \tfrac{1}{2} v_j^{(n(p))} r^2_{p,j} = \frac{1}{2}\frac{(b_{n(p)}t)^2}{\sigma_1(\mathtt{v}_{n(p)})^2} \sum_{k<j\le j_{n(p)}} (v_j^{(n(p))})^3 \xrightarrow[p\to\infty]{} \tfrac{1}{2}(\kappa t)^2 \sum_{j>k} c_j^3$$

Similarly, observe that $\mathbf{E}[\xi_j] = c_j\big(e^{-\kappa t c_j} - 1 + \kappa t c_j\big) \le \frac{1}{2}(\kappa t)^2 c_j^3$. This inequality combined with (209) entail:

$$(210)\qquad \limsup_{p\to\infty} \mathbf{E}\big[|R^{k,p}_t|\big] \le (\kappa t)^2 \sum_{j>k} c_j^3 \xrightarrow[k\to\infty]{} 0.$$

Finally, we consider $D^{k,p}$. Since a.s. t is not a jump-time of N_j, a.s. $v_j^{n(p)}(N_j^{\mathtt{w}_{n(p)}}(b_{n(p)}t) - 1)_+ \to c_j(N_j(t)-1)_+$. Thus, for all $k\in\mathbb{N}$, a.s. $D^{k,p}_t \to 0$. These limits combined with (209) (and with the convergence to 0 in probability of C^p_t and $d_p(t)$) easily imply (205), which complete the proof of the lemma. ■

Lemma 6.6 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^\downarrow$ satisfy (190). Let $a_n, b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^\downarrow$, $n\in\mathbb{N}$, satisfy (66),* (**C1**)–(**C3**). *Then,*

$$(211)\quad \Big(\big(\tfrac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n t}, \tfrac{1}{a_n}A^{\mathtt{w}_n}_{b_n t}, \tfrac{1}{a_n}Y^{\mathtt{w}_n}_{b_n t}\big)\Big)_{t\in[0,\infty)} \xrightarrow[n\to\infty]{\textit{weakly}} \big((X^{\mathtt{b}}_t, A_t, Y_t)\big)_{t\in[0,\infty)} \ \textit{in}\ \mathbf{D}([0,\infty),\mathbb{R}^3).$$

Proof: without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that the convergence in (203) holds true $\mathbf{P}$-almost surely. We first prove that $((\frac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, \frac{1}{a_n}A^{\mathtt{w}_n}_{b_n\cdot})) \to ((X^{\mathtt{b}}, A))$ a.s. in $\mathbf{D}([0,\infty),\mathbb{R}^2)$ thanks to Lemma B.1 (iii) (a standard result recalled in Appendix, Section B). To that end, first recall that by definition, the jumps of A (resp. of $A^{\mathtt{w}_n}$) are jumps of $X^{\mathtt{b}}$ (resp. of $X^{\mathtt{b},\mathtt{w}_n}$): namely if $\Delta A_t > 0$, then $\Delta X^{\mathtt{b}}_t = \Delta A_t$ and if $\Delta X^{\mathtt{b}}_t = 0$, then $\Delta A_t = 0$. The same holds true with $A^{\mathtt{w}_n}$ and $X^{\mathtt{b},\mathtt{w}_n}$.

Let $t\in(0,\infty)$. First suppose that $\Delta A_t > 0$. Thus, $\Delta X^{\mathtt{b}}_t = \Delta A_t$. By Lemma B.1 (i), there exists a sequence of times $t_n \to t$ such that $\frac{1}{a_n}\Delta A^{\mathtt{w}_n}_{b_n t_n} \to \Delta A_t$. Thus, for all sufficiently large n, $\frac{1}{a_n}\Delta A^{\mathtt{w}_n}_{b_n t_n} > 0$, which entails $\frac{1}{a_n}\Delta A^{\mathtt{w}_n}_{b_n t_n} = \frac{1}{a_n}\Delta X^{\mathtt{b},\mathtt{w}_n}_{b_n t_n}$ and we get $\frac{1}{a_n}\Delta X^{\mathtt{b},\mathtt{w}_n}_{b_n t_n} \to \Delta A_t = \Delta X^{\mathtt{b}}_t$. Suppose next that $\Delta A_t = 0$; by Lemma B.1 (i), there exists a sequence of times $t_n \to t$ such that $\frac{1}{a_n}\Delta X^{\mathtt{b},\mathtt{w}_n}_{b_n t_n} \to \Delta X^{\mathtt{b}}_t$. Since $\Delta A_t = 0$, Lemma B.1 (ii) entails that $\frac{1}{a_n}\Delta A^{\mathtt{w}_n}_{b_n t_n} \to \Delta A_t = 0$. In both cases, we have proved that for all $t\in(0,\infty)$, there exists a sequence of times $t_n \to t$ such that $\frac{1}{a_n}\Delta X^{\mathtt{b},\mathtt{w}_n}_{b_n t_n} \to \Delta X^{\mathtt{b}}_t$ and $\frac{1}{a_n}\Delta A^{\mathtt{w}_n}_{b_n t_n} \to \Delta A_t$: by Lemma B.1 (iii), it implies that $((\frac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, \frac{1}{a_n}A^{\mathtt{w}_n}_{b_n\cdot})) \to ((X^{\mathtt{b}}, A))$ a.s. in $\mathbf{D}([0,\infty),\mathbb{R}^2)$. This entails (211), since the function $(x,a)\in\mathbb{R}^2 \mapsto (x,a,x-a)\in\mathbb{R}^3$ is Lipschitz and since $X^{\mathtt{b},\mathtt{w}_n} - A^{\mathtt{w}_n} = Y^{\mathtt{w}_n}$ and $X^{\mathtt{b}} - A = Y$. ■

Recall that $X^{\mathtt{r},\mathtt{w}_n}$ (resp. $X^{\mathtt{r}}$) is an independent copy of $X^{\mathtt{b},\mathtt{w}_n}$ (resp. of $X^{\mathtt{b}}$). Recall from (197) (resp. from (193)) the definition of $\gamma^{\mathtt{r},\mathtt{w}_n}$ (resp. of $\gamma^{\mathtt{r}}$).

Lemma 6.7 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^\downarrow$ satisfy (190). Let $a_n, b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^\downarrow$, $n\in\mathbb{N}$, satisfy (66),* (**C1**)–(**C3**). *Then,*

$$(212)\quad \Big(\big(\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n x}\big)_{x\in[0,\infty)}, \big(\tfrac{1}{a_n}X^{\mathtt{r},\mathtt{w}_n}_{b_n t}\big)_{t\in[0,\infty)}\Big) \xrightarrow[n\to\infty]{} (\gamma^{\mathtt{r}}, X^{\mathtt{r}}) \quad \textit{weakly on}\ \mathbf{D}([0,\infty),\mathbb{R})^2$$

Proof: recall that Proposition 2.11 asserts that $\frac{1}{a_n}X^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot} \to X^{\mathtt{r}}$ weakly on $\mathbf{D}([0,\infty),\mathbb{R})$. Let $0 \leq x_1 < \ldots < x_k$. Since $\gamma^{\mathtt{r}}$ is subordinator, it has no fixed time-discontinuity; thus, a.s. $\Delta\gamma^{\mathtt{r}}_{x_j} = \ldots = \Delta\gamma^{\mathtt{r}}_{x_k} = 0$; a standard result recalled in Lemma B.3 in Appendix Section B, implies the joint convergence:

$$\big(\tfrac{1}{a_n}X^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot}\, ;\, \tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n x_1}, \ldots, \tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n x_k}\big) \longrightarrow \big(X^{\mathtt{r}}; \gamma^{\mathtt{r}}_{x_1}, \ldots, \gamma^{\mathtt{r}}_{x_k}\big)$$

Since the $\gamma^{\mathtt{r},\mathtt{w}_n}$ are Lévy processes, Theorem B.8 (in Appendix Section B) entails that $\frac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n\cdot} \to \gamma^{\mathtt{r}}$, which easily entails (212). ■

Lemma 6.8 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^{\downarrow}$ satisfy (190). Let $a_n, b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^{\downarrow}$, $n\in\mathbb{N}$, satisfy (66), $(\mathbf{C1})$–$(\mathbf{C3})$. Recall from (197) the definition of $\theta^{\mathtt{b},\mathtt{w}_n}$. Then, the laws of the processes $(\frac{1}{b_n}\theta^{\mathtt{b},\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)}$ are tight on $\mathbf{D}([0,\infty),\mathbb{R})$.*

Proof: to simplify notation we set $R^n_t = \frac{1}{b_n}\theta^{\mathtt{b},\mathtt{w}_n}_{b_n t} - t = \frac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}(A^{\mathtt{w}_n}_{b_n t})$; we only need to prove that the R^n are tight on $\mathbf{D}([0,\infty),\mathbb{R})$. We use Lemma 6.2. To that end, first observe that for all $K, z\in(0,\infty)$,

$$\mathbf{P}(R^n_t > K) = \mathbf{P}\big(\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}(A^{\mathtt{w}_n}(b_n t)) > K\big) \leq \mathbf{P}\big(\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n z} > K\big) + \mathbf{P}\big(\tfrac{1}{a_n}A^{\mathtt{w}_n}_{b_n t} > z\big)\, .$$

This easily implies that the laws of the R^n_t are tight on $[0,\infty)$ since it is the case of the laws of $\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n z}/b_n$ and of $A^{\mathtt{w}_n}_{b_n t}/a_n$.

Next, denote by $\mathscr{F}_t$ the σ-field generated by the r.v. $N^{\mathtt{w}_n}_j(s)$ and $\gamma^{\mathtt{w}_n\mathtt{r}}(A^{\mathtt{w}_n}_s)$ with $s\in[0,t]$ and $j\geq 1$; note that $N^{\mathtt{w}_n}_j(t+\cdot)-N^{\mathtt{w}_n}_j(t)$ are independent of $\mathscr{F}_t$. Fix $\varepsilon\in(0,\infty)$ and recall from (199) the definition of the times $\tau^{\varepsilon}_k(R^n)$: clearly $b_n\tau^{\varepsilon}_k(R^n)$ is a $(\mathscr{F}_t)$-stopping times. Next, fix $k\in\mathbb{N}$ and set

$$\forall x\in[0,\infty), \quad \mathbf{g}(x) = \tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}\big(a_n(x + \tfrac{1}{a_n}A^{\mathtt{w}_n}(b_n\tau^{\varepsilon}_k(R^n))\big) - \tfrac{1}{b_n}\gamma^{\mathtt{b},\mathtt{w}_n}\big(A^{\mathtt{w}_n}(b_n\tau^{\varepsilon}_k(R^n)\big)$$

Clearly, $\mathbf{g}$ has the same law as $\frac{1}{b_n}\gamma^{\mathtt{b},\mathtt{w}_n}_{a_n\cdot}$; we also set $\mathbf{u}_\varepsilon = \inf\{x\in[0,\infty) : \mathbf{g}(x) > \varepsilon\}$. Then, the definition of $\tau^{\varepsilon}_{k+1}(R^n)$ in (199) implies:

$$\tau^{\varepsilon}_{k+1}(R^n) = \inf\big\{t > \tau^{\varepsilon}_k(R^n) : \tfrac{1}{a_n}A^{\mathtt{w}_n}(b_n t) - \tfrac{1}{a_n}A^{\mathtt{w}_n}(b_n\tau^{\varepsilon}_k(R^n)) > \mathbf{u}_\varepsilon\big\}$$

Fix $z,\eta\in(0,\infty)$ and set $q_{n,k}(\eta) = \mathbf{P}\big(\tau^{\varepsilon}_k(R^n) < z\, ;\, \tau^{\varepsilon}_{k+1}(R^n) - \tau^{\varepsilon}_k(R^n) \leq \eta\big)$. Thus, we get:

$$\begin{aligned}
q_{n,k}(\eta) &\leq \mathbf{P}\big(b_n\tau^{\varepsilon}_k(R^n) < b_n z\, ;\, A^{\mathtt{w}_n}(b_n\eta + b_n\tau^{\varepsilon}_k(R^n)) - A^{\mathtt{w}_n}(b_n\tau^{\varepsilon}_k(R^n)) > a_n\mathbf{u}_\varepsilon\big)\\
&\leq \mathbf{P}\big(b_n\tau^{\varepsilon}_k(R^n) < b_n z\, ;\, A^{\mathtt{w}_n}(b_n\eta + b_n\tau^{\varepsilon}_k(R^n)) - A^{\mathtt{w}_n}(b_n\tau^{\varepsilon}_k(R^n)) > a_n y\big) + \mathbf{P}(\mathbf{u}_\varepsilon \leq y)\\
&\leq y^{-1}\eta(z + \tfrac{1}{2}\eta)\frac{a_n b_n}{\sigma_1(\mathtt{w}_n)}\frac{b_n\sigma_3(\mathtt{w}_n)}{a_n^2\sigma_1(\mathtt{w}_n)} + \mathbf{P}(\mathbf{u}_\varepsilon \leq y)\\
&\leq y^{-1}\eta(z + \tfrac{1}{2}\eta)\frac{a_n b_n}{\sigma_1(\mathtt{w}_n)}\frac{b_n\sigma_3(\mathtt{w}_n)}{a_n^2\sigma_1(\mathtt{w}_n)} + \mathbf{P}\big(\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n y} > \varepsilon\big)
\end{aligned}$$

by (141) in Lemma 4.2 applied to the $(\mathscr{F}_t)$-stopping time $T = b_n\tau^{\varepsilon}_k(R^n)$ to $t_0 = b_n z$, $t = b_n\eta$ and $a = a_n y$. Thus,

$$\limsup_{n\to\infty}\sup_{k\in\mathbb{N}} q_{n,k}(\eta) \leq y^{-1}\eta(z+\eta)\kappa(\beta + \kappa\sigma_3(\mathbf{c})) + \mathbf{P}\big(\gamma^{\mathtt{r}}_y > \varepsilon\big) \xrightarrow[\eta\to 0+]{} \mathbf{P}\big(\gamma^{\mathtt{r}}_y > \varepsilon\big) \xrightarrow[y\to 0+]{} 0,$$

which completes the proof by Lemma 6.2. ■

Lemma 6.9 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^{\downarrow}$ satisfy (190). Let $a_n, b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^{\downarrow}$, $n\in\mathbb{N}$, satisfy (66), $(\mathbf{C1})$–$(\mathbf{C3})$. Recall from (193) the definition of $\theta^{\mathtt{b}}$. Then*

$$\Big(\big(\tfrac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, \tfrac{1}{a_n}A^{\mathtt{w}_n}_{b_n\cdot}, \tfrac{1}{a_n}Y^{\mathtt{w}_n}_{b_n\cdot}\big), \tfrac{1}{b_n}\theta^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}, \tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n\cdot}, \tfrac{1}{a_n}X^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot}\Big) \xrightarrow[n\to\infty]{} \big((X^{\mathtt{b}}, A, Y), \theta^{\mathtt{b}}, \gamma^{\mathtt{r}}, X^{\mathtt{r}}\big) \tag{213}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^3)\times\mathbf{D}([0,\infty),\mathbb{R})^3$ equipped with the product-topology.

Proof: by Lemmas 6.6, 6.7 and 6.8, the laws of the processes on the left hand side of (213) are tight on $\mathbf{D}([0,\infty),\mathbb{R}^3)\times\mathbf{D}([0,\infty),\mathbb{R})^3$; we only need to prove that the joint law of the processes on the right hand side of (213) is the unique limiting law: to that end, let $(n(p))_{p\in\mathbb{N}}$ be an increasing sequence of integers such that

$$(214)\qquad \Big(\big(\tfrac{1}{a_{n(p)}}X^{\mathtt{b},\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot},\tfrac{1}{a_{n(p)}}A^{\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot},\tfrac{1}{a_{n(p)}}Y^{\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}\big),\tfrac{1}{b_{n(p)}}\theta^{\mathtt{b},\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot},\tfrac{1}{b_{n(p)}}\gamma^{\mathtt{r},\mathtt{w}_{n(p)}}_{a_{n(p)}\cdot},\tfrac{1}{a_{n(p)}}X^{\mathtt{r},\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}\Big) \xrightarrow[p\to\infty]{} \big((X^{\mathtt{b}},A,Y),\theta',\gamma^{\mathtt{r}},X^{\mathtt{r}}\big).$$

Actually, we only have to prove that $\theta'=\theta^{\mathtt{b}}$. Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that (214) holds true $\mathbf{P}$-almost surely. Since A has no fixed time of discontinuity, a.s. for all $q\in\mathbb{Q}\cap[0,\infty)$, $\Delta A_q=0$, and thus $A^{\mathtt{w}_{n(p)}}_{b_{n(p)}q}/a_{n(p)}\to A_q$. Since $\gamma^{\mathtt{r}}$ has no fixed discontinuity and since it is independent of A, a.s. for all $q\in\mathbb{Q}\cap[0,\infty)$, $\Delta\gamma^{\mathtt{r}}(A_q)=0$, which easily entails that $\gamma^{\mathtt{r},\mathtt{w}_{n(p)}}(A^{\mathtt{w}_{n(p)}}(b_{n(p)}q))/b_{n(p)}\to\gamma^{\mathtt{r}}(A_q)$; thus, $\theta^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}q)/b_{n(p)}\to\theta^{\mathtt{b}}_q$ for all $q\in\mathbb{Q}\cap[0,\infty)$. Therefore, $\theta'=\theta^{\mathtt{b}}$, which completes the proof. ■

Lemma 6.10 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^{\downarrow}$ satisfy (190). Let $a_n,b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^{\downarrow}$, $n\in\mathbb{N}$, satisfy (66), (**C1**)–(**C3**). Recall from (193) the definition of $\theta^{\mathtt{b}}$. Then*

$$(215)\qquad \mathscr{Q}_n(1)=\Big(\big(\tfrac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot},\tfrac{1}{a_n}A^{\mathtt{w}_n}_{b_n\cdot},\tfrac{1}{a_n}Y^{\mathtt{w}_n}_{b_n\cdot},\tfrac{1}{b_n}\theta^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}\big),\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}_{a_n\cdot},\tfrac{1}{a_n}X^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot}\Big) \xrightarrow[n\to\infty]{} \big((X^{\mathtt{b}},A,Y,\theta^{\mathtt{b}}),\gamma^{\mathtt{r}},X^{\mathtt{r}}\big)$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^4)\times\mathbf{D}([0,\infty),\mathbb{R})^2$ equipped with the product-topology.

Proof: without loss of generality (but with a slight abuse of notation), Skorokod's representation theorem allows to assume that (213) holds $\mathbf{P}$-a.s. and to simplify notation, we set $R^n=\frac{1}{a_n}(X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot},A^{\mathtt{w}_n}_{b_n\cdot},Y^{\mathtt{w}_n}_{b_n\cdot})$ and $R=(X^{\mathtt{b}},A,Y)$. Let $a\in(0,\infty)$.

Suppose first that $\Delta R_a\neq 0$. By Lemma B.1 (i), there is $s_n\to a$ such that $R^n_{s_n-}\to R_{a-}$, $R^n_{s_n}\to R_a$ and thus $\Delta R^n_{s_n}\to\Delta R_a$. If $\Delta Y_a>0$, then, by definition, $\Delta X^{\mathtt{b}}_a=\Delta Y_a$ and $\Delta A_a=0$, and Lemma 5.4 (iii) asserts that $\Delta\theta^{\mathtt{b}}_a=0$ and Lemma B.1 (ii) asserts that $\frac{1}{b_n}\Delta\theta^{\mathtt{b},\mathtt{w}_n}(b_ns_n)\to\Delta\theta^{\mathtt{b}}_a=0$.

We next suppose that $\Delta R_a\neq 0$ but $\Delta Y_a=0$; then, by definition, we get $\Delta X^{\mathtt{b}}_a=\Delta A_a>0$. Since $\gamma^{\mathtt{r}}$ is independent from R, it a.s. has no jump at the times A_{a-} and A_a; therefore:

$$\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}(A^{\mathtt{w}_n}_{b_ns_n-})\to\gamma^{\mathtt{r}}(A_{a-})\quad\text{and}\quad\tfrac{1}{b_n}\gamma^{\mathtt{r},\mathtt{w}_n}(A^{\mathtt{w}_n}_{b_ns_n})\to\gamma^{\mathtt{r}}(A_a)\;.$$

This implies that $\frac{1}{b_n}\Delta\theta^{\mathtt{b},\mathtt{w}_n}(b_ns_n)\to\Delta\theta^{\mathtt{b}}_a=\gamma^{\mathtt{r}}(A_a)-\gamma^{\mathtt{r}}(A_{a-})$.

We finally suppose that $\Delta R_a=0$; by Lemma B.1 (i), there exists a sequence $s'_n\to a$ such that $\frac{1}{b_n}\Delta\theta^{\mathtt{b},\mathtt{w}_n}(b_ns'_n)\to\Delta\theta^{\mathtt{b}}_a$. Since, $\Delta R_a=0$, Lemma B.1 (ii) entails that $\Delta R^n_{s'_n}\to\Delta R_a$.

Then, we have proved that for all $a\in(0,\infty)$, there exists a sequence $s''_n\to a$ such that $\frac{1}{b_n}\Delta\theta^{\mathtt{b},\mathtt{w}_n}(b_ns''_n)\to\Delta\theta^{\mathtt{b}}_a$ and $\Delta R^n_{s''_n}\to\Delta R_a$. By Lemma B.1 (iii), $(R^n,\frac{1}{b_n}\theta^{\mathtt{b},\mathtt{w}_n}(b_n\cdot))\to(R,\theta^{\mathtt{b}})$ a.s. on $\mathbf{D}([0,\infty),\mathbb{R}^4)$, which completes the proof. ■

Recall next that for all $t\in[0,\infty)$ and all $n\in\mathbb{N}$,

$$(216)\qquad \Lambda^{\mathtt{b},\mathtt{w}_n}_t=\inf\big\{s\in[0,\infty):\theta^{\mathtt{b},\mathtt{w}_n}_s>t\big\},\quad \Lambda^{\mathtt{b}}_t=\inf\big\{s\in[0,\infty):\theta^{\mathtt{b}}_s>t\big\},$$

that $\Lambda^{\mathtt{r},\mathtt{w}_n}_t=t-\Lambda^{\mathtt{b},\mathtt{w}_n}_t$ and that $\Lambda^{\mathtt{r}}_t=t-\Lambda^{\mathtt{b}}_t$. We next prove that under the assumptions of Lemma 6.10 and with the notation $\mathscr{Q}_n(1)$ in (215), the following convergence holds true

$$(217)\qquad \mathscr{Q}_n(2)=\big(\mathscr{Q}_n(1),\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot},\tfrac{1}{b_n}\Lambda^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot}\big)\xrightarrow[n\to\infty]{}\big((X^{\mathtt{b}},A,Y,\theta^{\mathtt{b}}),\gamma^{\mathtt{r}},X^{\mathtt{r}},\Lambda^{\mathtt{b}},\Lambda^{\mathtt{r}}\big)$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^4)\times\mathbf{D}([0,\infty),\mathbb{R})^2\times\mathbf{C}([0,\infty),\mathbb{R})^2$ equipped with the product-topology.

Indeed, without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that the convergence in (215) holds $\mathbf{P}$-a.s.; we fix $t\in(0,\infty)$; since $\theta^{\mathtt{b}}$ is strictly increasing, standard arguments entail $\Lambda^{\mathtt{b},\mathtt{w}_n}(b_nt)/b_n\to\Lambda^{\mathtt{b}}_t$. Since $\Lambda^{\mathtt{b}}$ is non-decreasing and continuous, a theorem due to Dini implies that $\frac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}\to\Lambda^{\mathtt{b}}$ uniformly on all compact subsets; it entails a similar convergence for $\Lambda^{\mathtt{r}}$, which completes the proof of (217). □

Lemma 6.11 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^{\downarrow}$ satisfy (190). Let $a_n,b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^{\downarrow}$, $n\in\mathbb{N}$, satisfy (66), (**C1**)–(**C3**). Then, the laws of the processes $(\frac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nt}))_{t\in[0,\infty)}$ and $(\frac{1}{a_n}X^{\mathtt{r},\mathtt{w}_n}(\Lambda^{\mathtt{r},\mathtt{w}_n}_{b_nt}))_{t\in[0,\infty)}$ are tight on $\mathbf{D}([0,\infty),\mathbb{R})$.*

Proof. Fix $t\in[0,\infty)$; then for all $t_0,K\in(0,\infty)$, note that:

$$\mathbf{P}\Big(\sup_{s\in[0,t]}\tfrac{1}{a_n}|X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_ns})|>K\Big)\leq\mathbf{P}\Big(\sup_{s\in[0,t_0]}\tfrac{1}{a_n}|X^{\mathtt{b},\mathtt{w}_n}_{b_ns}|>K\Big)+\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nt}>t_0\big).$$

Then, deduce from (217) that

$$\lim_{K\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big(\sup_{s\in[0,t]}\tfrac{1}{a_n}|X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_ns})|>K\Big)\leq\limsup_{n\to\infty}\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nt}>t_0\big)\xrightarrow[t_0\to\infty]{}0\,.$$

By a similar argument $\lim_{K\to\infty}\limsup_{n\to\infty}\mathbf{P}\big(\sup_{s\in[0,t]}|X^{\mathtt{r},\mathtt{w}_n}(\Lambda^{\mathtt{r},\mathtt{w}_n}_{b_ns})|>a_nK\Big)=0$.

Next, recall from (24) that a.s. for all $n\in\mathbb{N}$ and for all $t\in[0,\infty)$

$$X^{\mathtt{w}_n}_t=X^{\mathtt{b},\mathtt{w}_n}_{\Lambda^{\mathtt{b},\mathtt{w}_n}_t}+X^{\mathtt{r},\mathtt{w}_n}_{\Lambda^{\mathtt{r},\mathtt{w}_n}_t}\,. \tag{218}$$

Recall from (144) that for all $y\in\mathbf{D}([0,\infty),\mathbb{R})$, $w_z(y,\eta)$ stands for the η-càdlàg modulus of continuity of y on $[0,z]$. Fix $z_1,z,z_0,\eta,\varepsilon\in(0,\infty)$ and recall Lemma 4.3: by (145), we easily get:

$$\begin{aligned}\mathbf{P}\big(w_{z_1}\big(\tfrac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}),\,\eta\big)>2\varepsilon\big)\;&\leq\mathbf{P}\big(w_{z+\eta}\big(\tfrac{1}{a_n}X^{\mathtt{w}_n}_{b_n\cdot},\eta\big)>\varepsilon\big)+\mathbf{P}\big(w_{z_0}\big(\tfrac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot},\eta\big)>\varepsilon\big)\\&\quad+\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nz_1}>z_0\big)+\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nz}\leq z_0\big)\end{aligned}$$

By Lemma 2.2, $X^{\mathtt{w}_n}$ has the same law as $X^{\mathtt{b},\mathtt{w}_n}$ and $X^{\mathtt{r},\mathtt{w}_n}$: by Proposition 2.11, the laws of the processes $\frac{1}{a_n}X^{\mathtt{w}_n}_{b_n\cdot}$ (or equivalently of $\frac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}$) are tight on $\mathbf{D}([0,\infty),\mathbb{R})$. Consequently,

$$\begin{aligned}&\lim_{\eta\to0}\limsup_{n\to\infty}\mathbf{P}\big(w_{z_1}\big(\tfrac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}),\,\eta\big)>2\varepsilon\big)\\&\qquad\leq\limsup_{n\to\infty}\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nz_1}>z_0\big)+\limsup_{n\to\infty}\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nz}\leq z_0\big)\\&\qquad\xrightarrow[z\to\infty]{}\;\mathbf{P}\big(\tfrac{1}{b_n}\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nz_1}>z_0\big)\xrightarrow[z_0\to\infty]{}0,\end{aligned}$$

since the laws of the processes $\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}/b_n$ are tight by (217). This proves that the laws of the processes $(\frac{1}{a_n}X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_nt}))_{t\in[0,\infty)}$ are tight on $\mathbf{D}([0,\infty),\mathbb{R})$. We derive a similar result for the red processes by a quite similar argument based on (146) in Lemma 4.3. ■

Recall (218) and recall from (43) that $X_t=X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t)+X^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t)$ for all $t\in[0,\infty)$.

Proposition 6.12 *Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}\in\ell_3^{\downarrow}$ satisfy (190). Let $a_n,b_n\in(0,\infty)$ and $\mathtt{w}_n\in\ell_f^{\downarrow}$, $n\in\mathbb{N}$, satisfy (66), (**C1**)–(**C3**). Recall from (217) the notation $\mathscr{Q}_n(2)$. Then*

$$\begin{aligned}\mathscr{Q}_n(3)=\big(\mathscr{Q}_n(2),\,\tfrac{1}{a_n}\big(X^{\mathtt{b},\mathtt{w}_n}_{\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}},\,X^{\mathtt{r},\mathtt{w}_n}_{\Lambda^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot}},\,X^{\mathtt{w}_n}_{b_n\cdot}\big)\big)&\\\xrightarrow[n\to\infty]{}\big((X^{\mathtt{b}},A,Y,\theta^{\mathtt{b}}),\gamma^{\mathtt{r}},X^{\mathtt{r}},\Lambda^{\mathtt{b}},\Lambda^{\mathtt{r}},(X^{\mathtt{b}}_{\Lambda^{\mathtt{b}}},X^{\mathtt{r}}_{\Lambda^{\mathtt{r}}},X)\big),&\end{aligned}\tag{219}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^4)\times\mathbf{D}([0,\infty),\mathbb{R})^2\times\mathbf{C}([0,\infty),\mathbb{R})^2\times\mathbf{D}([0,\infty),\mathbb{R}^3)$ equipped with the product-topology.

Proof: we first prove the following

$$\mathscr{Q}'_n(3) = \big(\mathscr{Q}_n(2), \tfrac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n\cdot}}, \tfrac{1}{a_n} X^{\mathtt{r},\mathtt{w}_n}_{\Lambda^{\mathtt{r},\mathtt{w}_n}_{b_n\cdot}}\big) \xrightarrow[n\to\infty]{} \big((X^{\mathtt{b}}, A, Y, \theta^{\mathtt{b}}), \gamma^{\mathtt{r}}, X^{\mathtt{r}}, \Lambda^{\mathtt{b}}, \Lambda^{\mathtt{r}}, X^{\mathtt{b}}_{\Lambda^{\mathtt{b}}}, X^{\mathtt{r}}_{\Lambda^{\mathtt{r}}}\big), \tag{220}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^4) \times \mathbf{D}([0,\infty),\mathbb{R})^2 \times \mathbf{C}([0,\infty),\mathbb{R})^2 \times \mathbf{D}([0,\infty),\mathbb{R})^2$ equipped with the product-topology. Note that the laws of $\mathscr{Q}'_n(3)$ are tight thanks to (217) and Lemma 6.11. We only need to prove that the joint law of the processes on the right hand side of (220) is the unique limiting law: to that end, let $(n(p))_{p\in\mathbb{N}}$ be an increasing sequence of integers such that

$$\mathscr{Q}'_{n(p)}(3) \xrightarrow[p\to\infty]{} \big((X^{\mathtt{b}}, A, Y, \theta^{\mathtt{b}}), \gamma^{\mathtt{r}}, X^{\mathtt{r}}, \Lambda^{\mathtt{b}}, \Lambda^{\mathtt{r}}, Q^{\mathtt{b}}, Q^{\mathtt{r}}\big) \tag{221}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^4) \times \mathbf{D}([0,\infty),\mathbb{R})^2 \times \mathbf{C}([0,\infty),\mathbb{R})^2 \times \mathbf{D}([0,\infty),\mathbb{R})^2$ equipped with the product topology. Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that the convergence in (221) holds $\mathbf{P}$-a.s. and we only need to prove that $Q^{\mathtt{b}} = X^{\mathtt{b}} \circ \Lambda^{\mathtt{b}}$ and $Q^{\mathtt{r}} = X^{\mathtt{r}} \circ \Lambda^{\mathtt{r}}$.

To that end, recall first from Lemma 5.4 (iii) that if $\Delta\theta^{\mathtt{b}}_a > 0$, then $\Delta Y_a = 0$; thus, if $\Delta Y_a > 0$, then $\Delta\theta^{\mathtt{b}}_a = 0$ and by Lemma 5.4 (ii), there exists a unique time $t \in [0,\infty)$ such that $\Lambda^{\mathtt{b}}_t = a$. This implies that the set of time $S_1 = \big\{t \in [0,\infty) : \Delta Y(\Lambda^{\mathtt{b}}_t) > 0\big\}$ is countable. We next set $S_2 = \{\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a; a \in [0,\infty) : \Delta\theta^{\mathtt{b}}_a > 0\}$ and $S = S_1 \cup S_2$; then S is countable. We next fix $t \in (0,\infty)\backslash S$. We first assume that $(\Delta X^{\mathtt{b}})(\Lambda^{\mathtt{b}}_t) = 0$, then by Lemma B.1 (ii), $\Delta X^{\mathtt{b},\mathtt{w}_{n(p)}}(\Lambda^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}t))/a_{n(p)} \to (\Delta X^{\mathtt{b}})(\Lambda^{\mathtt{b}}_t) = 0$, since $\Lambda^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}t)/b_{n(p)} \to \Lambda^{\mathtt{b}}_t$.

We next assume that $\Delta X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t) > 0$. Since $t \notin S_1$, $\Delta Y(\Lambda^{\mathtt{b}}_t) = 0$, $\Delta X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t) = \Delta A(\Lambda^{\mathtt{b}}_t) > 0$, by definition: we then set $a = \Lambda^{\mathtt{b}}_t$ and we necessarily get $\Delta\theta^{\mathtt{b}}_a > 0$ and $t \in [\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a]$; since $t \notin S_2$, we then get $t \in (\theta^{\mathtt{b}}_{a-}, \theta^{\mathtt{b}}_a)$. To simplify notation we set

$$R^p = \big(\tfrac{1}{a_{n(p)}} X^{\mathtt{b},\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}, \tfrac{1}{a_{n(p)}} A^{\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}, \tfrac{1}{a_{n(p)}} Y^{\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}, \tfrac{1}{b_{n(p)}} \theta^{\mathtt{b},\mathtt{w}_{n(p)}}_{b_{n(p)}\cdot}\big) \quad \text{and} \quad R = (X^{\mathtt{b}}, A, Y, \theta^{\mathtt{b}}) \; .$$

By (221), $R^p \to R$ a.s. on $\mathbf{D}([0,\infty),\mathbb{R}^4)$. Since $\Delta\theta^{\mathtt{b}}_a > 0$, a is a jump-time of R. By Lemma B.1 (i), there is a sequence $s_p \to a$ such that $(R^p_{s_p-}, R^p_{s_p}) \to (R_{a-}, R_a)$: in particular, we get $X^{\mathtt{b},\mathtt{w}_{n(p)}}(s_p)/a_{n(p)} \to X^{\mathtt{b}}_a = X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t)$. It also implies that $\theta^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}s_p-)/b_{n(p)} \to \theta^{\mathtt{b}}_{a-}$ and $\theta^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}s_p)/b_{n(p)} \to \theta^{\mathtt{b}}_a$; thus, for all sufficiently large p, we get

$$\tfrac{1}{b_{n(p)}} \theta^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}s_p-) < t < \tfrac{1}{b_{n(p)}} \theta^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}s_p) \quad \text{and thus} \quad \Lambda^{\mathtt{b},\mathtt{w}_{n(p)}}_{b_{n(p)}t} = s_p,$$

which implies that $X^{\mathtt{b},\mathtt{w}_{n(p)}}\big(\Lambda^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}t)\big)/a_{n(p)} \to X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t)$.

Thus, we have proved a.s. for all $t \in [0,\infty)\backslash S$ that $X^{\mathtt{b},\mathtt{w}_{n(p)}}\big(\Lambda^{\mathtt{b},\mathtt{w}_{n(p)}}(b_{n(p)}t)\big)/a_{n(p)} \to X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t)$. Since S is countable, it easily implies that $Q^{\mathtt{b}} = X^{\mathtt{b}} \circ \Lambda^{\mathtt{b}}$.

We next prove that $Q^{\mathtt{r}} = X^{\mathtt{r}} \circ \Lambda^{\mathtt{r}}$: to that end, set $S_3 = \{t \in [0,\infty) : (\Delta X^{\mathtt{r}})(\Lambda^{\mathtt{r}}_t) > 0\}$. Lemma 5.5 (ii) entails that a.s. S_3 is countable and by Lemma B.1 (ii), a.s. for all $t \in [0,\infty)\backslash S_3$, we get $X^{\mathtt{r},\mathtt{w}_{n(p)}}\big(\Lambda^{\mathtt{r},\mathtt{w}_{n(p)}}(b_{n(p)}t)\big)/a_{n(p)} \to X^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t)$; this easily entails that a.s. $Q^{\mathtt{r}} = X^{\mathtt{r}} \circ \Lambda^{\mathtt{r}}$, which completes the proof of (220).

We now prove (219): without loss of generality (but with a slight abuse of notation), Skorokod's representation theorem allow to assume that (220) holds $\mathbf{P}$-a.s. By Lemma 5.5 (iii), a.s. for all $t \in [0,\infty)$, $\Delta Q^{\mathtt{b}}_t \Delta Q^{\mathtt{r}}_t = 0$, and Lemma B.1 entails:

$$\Big(\big(\tfrac{1}{a_n} X^{\mathtt{b},\mathtt{w}_n}_{\Lambda^{\mathtt{b},\mathtt{w}_n}_{b_n t}}, \tfrac{1}{a_n} X^{\mathtt{r},\mathtt{w}_n}_{\Lambda^{\mathtt{r},\mathtt{w}_n}_{b_n t}}\big)\Big)_{t\in[0,\infty)} \xrightarrow[n\to\infty]{} \big((Q^{\mathtt{b}}_t, Q^{\mathtt{r}}_t)\big)_{t\in[0,\infty)} \quad \text{a.s. on } \mathbf{D}([0,\infty),\mathbb{R}^2).$$

which implies (219) since $X^{\mathtt{w}_n}_t = X^{\mathtt{b},\mathtt{w}_n}(\Lambda^{\mathtt{b},\mathtt{w}_n}_t) + X^{\mathtt{r},\mathtt{w}_n}(\Lambda^{\mathtt{r},\mathtt{w}_n}_t)$ and $X_t = X^{\mathtt{b}}(\Lambda^{\mathtt{b}}_t) + X^{\mathtt{r}}(\Lambda^{\mathtt{r}}_t)$. ■

Recall from (32) the definition of the height process $H^{\mathtt{w}_n}$ associated with $X^{\mathtt{w}_n}$ and recall from (18) the definition of $\mathcal{H}^{\mathtt{w}_n}$ the definition of the height process associated with $Y^{\mathtt{w}_n}$. Recall from (34) in Lemma 2.4 that $\mathcal{H}^{\mathtt{w}_n} = H^{\mathtt{w}_n} \circ \theta^{\mathtt{b},\mathtt{w}_n}$. Let $\alpha, \beta, \kappa, \mathbf{c}$ satisfy (44) and recall from (45) the definition of $(H_t)_{t\in[0,\infty)}$, the height process associated with X: H is a continuous process and note that (45) implies that H is a measurable functional of X. Recall next from Proposition 2.7 that $\mathcal{H} = H \circ \theta^{\mathtt{b}}$ and that $\mathcal{H}$ is continuous too: $\mathcal{H}$ is the height process associated with Y. Then, recall from (33) the definition of the offspring distribution $\mu_{\mathtt{w}_n}$ and denote by $(Z_k^{\mathtt{w}_n})_{k\in\mathbb{N}}$ a Galton-Watson Markov chain with initial state $Z_0^{\mathtt{w}_n} = \lfloor a_n \rfloor$ and offspring distribution $\mu_{\mathtt{w}_n}$; recall from (71) Assumption **(C4)**:

$$(222) \qquad \textbf{(C4)}: \quad \exists \delta \in (0,\infty), \qquad \liminf_{n\to\infty} \mathbf{P}\big(Z^{\mathtt{w}_n}_{\lfloor b_n\delta/a_n\rfloor} = 0\big) > 0 \, .$$

Proposition 6.13 *Let $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} \in \ell_3^{\downarrow}$ satisfy (190). Let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, satisfy (66),* **(C1)**–**(C4)**. *Recall from (219) the notation $\mathscr{Q}_n(3)$. Then,*

$$(223) \quad \mathscr{Q}_n(4) = \big(\mathscr{Q}_n(3), \tfrac{a_n}{b_n} H^{\mathtt{w}_n}_{b_n \cdot}, \tfrac{a_n}{b_n} \mathcal{H}^{\mathtt{w}_n}_{b_n \cdot}\big) \xrightarrow[n\to\infty]{} \big((X^{\mathtt{b}}, A, Y, \theta^{\mathtt{b}}), \gamma^{\mathtt{r}}, X^{\mathtt{r}}, \Lambda^{\mathtt{b}}, \Lambda^{\mathtt{r}}, (X^{\mathtt{b}}_{\Lambda^{\mathtt{b}}}, X^{\mathtt{r}}_{\Lambda^{\mathtt{r}}}, X), H, \mathcal{H}\big),$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^4) \times \mathbf{D}([0,\infty),\mathbb{R})^2 \times \mathbf{C}([0,\infty),\mathbb{R})^2 \times \mathbf{D}([0,\infty),\mathbb{R}^3) \times \mathbf{C}([0,\infty),\mathbb{R})^2$ equipped with the product-topology.

Proof: we first prove that

$$(224) \quad \mathscr{Q}'_n(4) = \big(\mathscr{Q}_n(3), \tfrac{a_n}{b_n} H^{\mathtt{w}_n}_{b_n \cdot}\big) \xrightarrow[n\to\infty]{} \mathscr{Q}'(4) = \big((X^{\mathtt{b}}, A, Y, \theta^{\mathtt{b}}), \gamma^{\mathtt{r}}, X^{\mathtt{r}}, \Lambda^{\mathtt{b}}, \Lambda^{\mathtt{r}}, (X^{\mathtt{b}}_{\Lambda^{\mathtt{b}}}, X^{\mathtt{r}}_{\Lambda^{\mathtt{r}}}, X), H\big),$$

weakly on the appropriate product-space. By Proposition 2.12, the laws of the processes $\frac{b_n}{a_n} H^{\mathtt{w}_n}_{b_n \cdot}$ are tight on $\mathbf{C}([0,\infty),\mathbb{R})$. Then, the laws of $\mathscr{Q}'_n(4)$ are tight thanks to (219). We only need to prove that the law of $\mathscr{Q}'(4)$ is the unique limiting law, which is an easy consequence of (219), of the joint convergence (72) in Proposition 2.12 and of the fact that H is a measurable deterministic functional of X.

To complete the proof of the lemma, we use a general (deterministic) result on Skorokhod's convergence for the composition of functions that is recalled in Theorem B.5, in Appendix Section B.1. Without loss of generality (but with a slight abuse of notation), Skorokod's representation theorem allows to assume that (224) holds $\mathbf{P}$-a.s.: since $\frac{a_n}{b_n} H^{\mathtt{w}_n}_{b_n \cdot} \to H$ a.s. on $\mathbf{C}([0,\infty),\mathbb{R})$, since $\frac{1}{b_n}\theta^{\mathtt{b},\mathtt{w}_n}_{b_n \cdot} \to \theta^{\mathtt{b}}$ a.s. on $\mathbf{D}([0,\infty),\mathbb{R})$ and finally we since $\mathcal{H} = H \circ \theta^{\mathtt{b}}$ is a.s. continuous, Theorem B.5 (i) applies and asserts that $\frac{a_n}{b_n} \mathcal{H}^{\mathtt{w}_n}_{b_n \cdot} \to \mathcal{H}$, which completes the proof of the proposition. ■

Proof of Theorem 2.14. We only have to deal with the convergence of the sequences of pairs of pinching times $\mathbf{\Pi}_{\mathtt{w}_n}$. To that end, we denote by $\mathscr{Q}(4)$ the right member of (223) and thanks to Skorokod's representation theorem (but with a slight abuse of notation) we can assume without loss of generality that (223) holds almost surely: namely, a.s. $\mathscr{Q}_n(4) \to \mathscr{Q}(4)$; next, we couple the $\mathbf{\Pi}_{\mathtt{w}_n}$ and $\mathbf{\Pi}_{\mathtt{w}}$ as follows.

- Let $\mathcal{R} = \sum_{i\in\mathtt{I}} \delta_{(t_i,r_i,u_i)}$ a Poisson point measure on $[0,\infty)^3$ with intensity the Lebesgue measure $dtdrdv$ on $[0,\infty)^3$. We assume that $\mathcal{R}$ is independent of $\mathscr{Q}(4)$ and of $(\mathscr{Q}_n(4))_{n\in\mathbb{N}}$.
- We set $\kappa_n = a_n b_n / \sigma_1(\mathtt{w}_n)$ and for all $t \in [0,\infty)$ we set $\mathtt{Z}^n_t = \frac{1}{a_n}(Y^{\mathtt{w}_n}_{b_n t} - J^{\mathtt{w}_n}_{b_n t})$, where we recall that $J^{\mathtt{w}_n}_{b_n t} = \inf_{s\in[0,b_n t]} Y^{\mathtt{w}_n}_s$. We then set $S_n = \{(t,r,v) \in [0,\infty)^3 : 0 < r < \mathtt{Z}^n_t \text{ and } 0 \le v \le \kappa_n\}$ and we define
$$\mathcal{P}_n = \sum_{i\in\mathtt{I}} \mathbf{1}_{\{(t_i,r_i,u_i)\in S_n\}} \delta_{(t_i,r_i,u_i)} = \sum_{1\le p<\mathbf{p}_n} \delta_{(t^n_p, r^n_p, v^n_p)},$$
where the indexation is such that the finite sequence $(t^n_p)_{1\le p<\mathbf{p}_n}$ increases. (Note that since $\mathtt{Z}^n$ is eventually null, $\mathcal{P}_n$ is a finite point process.)

– For all $t\in[0,\infty)$, for all $r\in\mathbb{R}$ and for all $z\in\mathbf{D}([0,\infty),\mathbb{R})$, we set

$$\tau(z,t,r)=\inf\big\{s\in[0,t]:\inf_{u\in[s,t]}z(u)>r\big\} \text{ with the the convention that } \inf\emptyset=\infty. \tag{225}$$

Then, we set

$$\tfrac{1}{b_n}\boldsymbol{\Pi}_{\mathtt{w}_n}=\big((s^n_p,t^n_p)\big)_{1\leq p<\mathbf{p}_n} \quad\text{where}\quad s^p_n=\tau(\mathtt{Z}^n,t^n_p,r^n_p),\ 1\leq p<\mathbf{p}_n. \tag{226}$$

Thanks to (15) and (16), we see that conditionally given $Y^{\mathtt{w}_n}$, $\frac{1}{b_n}\boldsymbol{\Pi}_{\mathtt{w}_n}$ has the right law. By convenience, we set $(s^n_p,t^n_p)=(-1,-1)$, for all $p\geq\mathbf{p}_n$.

Similarly, we set $\mathtt{Z}^\infty_t=Y_t-J_t$, where $J_t=\inf_{s\in[0,t]}Y_s$ and we also set $S=\{(t,r,v)\in[0,\infty)^3: 0<r<\mathtt{Z}^\infty_t \text{ and } 0\leq v\leq\kappa\}$; then we define:

$$\mathcal{P}=\sum_{i\in\mathtt{I}}\mathbf{1}_{\{(t_i,r_i,u_i)\in S\}}\delta_{(t_i,r_i,u_i)}=\sum_{p\geq1}\delta_{(t_p,r'_p,v_p)},$$

where the indexation is such that $(t_p)_{p\geq1}$ is increases. Then, set

$$\boldsymbol{\Pi}=\big((s_p,t_p)\big)_{p\geq1} \quad\text{where}\quad s_p=\tau(\mathtt{Z}^\infty,t_p,r'_p),\ p\geq1, \tag{227}$$

It is easy to check that $\boldsymbol{\Pi}$ has the right law conditionally given Y.

First observe that $\kappa_n\to\kappa>0$, by the last point of (66). Next, we prove that $\mathtt{Z}^n\to\mathtt{Z}^\infty$ a.s. in $\mathbf{D}([0,\infty),\mathbb{R})$: *indeed*, since Y has no negative jumps, J is continuous and by Lemma B.3 (ii), $(\frac{1}{a_n}J^{\mathtt{w}_n}_{b_nt})_{t\in[0,\infty)}\to(J_t)_{t\in[0,\infty)}$ a.s. in $\mathbf{C}([0,\infty),\mathbb{R})$. Since J is continuous, Y and J do not share any jump-times and by Lemma B.1 (iii), $(\frac{1}{a_n}(Y^{\mathtt{w}_n}_{b_nt},J^{\mathtt{w}_n}_{b_nt}))_{t\in[0,\infty)}\to((Y_t,J_t))_{t\in[0,\infty)}$ a.s. in $\mathbf{D}([0,\infty),\mathbb{R}^2)$, which entails that $\mathtt{Z}^n\to\mathtt{Z}^\infty$ a.s. in $\mathbf{D}([0,\infty),\mathbb{R})$.

Let us fix $a,b,c\in(0,\infty)$ such that

$$b>2\sup_{n\in\mathbb{N}\cup\{\infty\}}\ \sup_{s\in[0,a]}\mathtt{Z}^n_s \quad\text{and}\quad c>2\sup_{n\in\mathbb{N}\cup\{\infty\}}\kappa_n\,.$$

We introduce $\sum_{1\leq l\leq N}\delta_{(t^*_l,r^*_l,u^*_l)}:=\sum_{i\in\mathtt{I}}\mathbf{1}_{\{t_i<a\,;\,r_i<b\,;\,u_i<c\}}\delta_{(t_i,r_i,u_i)}$, where $(t^*_l)_{1\leq l\leq N}$ increases; here, N is Poisson r.v. with mean abc; note that conditionally given N, the law of the r.v. (t^*_l,r^*_l,u^*_l) is absolutely continuous with respect to Lebesgue measure. Therefore, a.s. for all $l\in\{1,\ldots,N\}$ (if any), $\Delta\mathtt{Z}^\infty_{t^*_l}=0$, $u^*_l\neq\kappa_\infty$, and $r^*_l\neq\mathtt{Z}^\infty_{t^*_l}$, and if $r^*_l<\mathtt{Z}^\infty_{t^*_l}$, then we get $\tau(\mathtt{Z}^\infty,t^*_l,r^*_l-)=\tau(\mathtt{Z}^\infty,t^*_l,r^*_l)$ since, by Lemma B.3 (iv), $r\mapsto\tau(\mathtt{Z}^\infty,t^*_l,r)$ is right-continuous and has therefore a countable number of times of discontinuities. Since $\Delta\mathtt{Z}^\infty_{t^*_l}=0$, Lemma B.1 (ii) entails that $\mathtt{Z}^n_{t^*_l}\to\mathtt{Z}^\infty_{t^*_l}$, and for all sufficiently large n, $u^*_l\neq\kappa_n$, $u^*_l\neq\kappa_n$ and $r^*_l\neq\mathtt{Z}^n_{t^*_l}$, and when $r^*_l<\mathtt{Z}^n_{t^*_l}$, by Lemma B.3 (iv), $\tau(\mathtt{Z}^n,t^*_l,r^*_l)\to\tau(\mathtt{Z}^\infty,t^*_l,r^*_l)$. This proves that if $t_p<a$, then $(s^n_p,t^n_p)\to(s_p,t_p)$. Since a can be arbitrarily large, we get $\frac{1}{b_n}\boldsymbol{\Pi}_{\mathtt{w}_n}\to\boldsymbol{\Pi}$ a.s. in $(\mathbb{R}^2)^{\mathbb{N}^*}$equipped with the product topology. This, combined with the a.s. convergence $\mathscr{Q}_n(4)\to\mathscr{Q}(4)$, entails Theorem 2.14. ■

6.2 Proof of Theorem 2.15 and proof of Theorem 2.16

Let $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c}=(c_j)_{j\geq1}\in\ell^{\downarrow}_3$. Recall that ψ satisfies (190). Recall from (39) the definition of Y; recall from Proposition 2.7, the definition of $\mathcal{H}$, the height process associated with Y; recall the notation, $J_t=\inf_{s\in[0,t]}Y_s$, $t\in[0,\infty)$. Lemma 5.6 (v) in Section 5.2 asserts that the excursions of $\mathcal{H}$ above 0 and those of $Y-J$ above 0 are the same. As recalled in Proposition 5.8, Proposition 14 in Aldous & Limic [4] asserts that these excursions can be indexed in the decreasing order of their length. Namely,

$$\{t\in[0,\infty):\mathcal{H}_t>0\}=\{t\in[0,\infty):Y_t>J_t\}=\bigcup_{k\geq1}(l_k,r_k)\,, \tag{228}$$

where the sequence $\zeta_k = l_k - r_k$, $k \geq 1$, decreases. Moreover, the sequence $(\zeta_k)_{k\geq 1}$ appears as the law of a version of the multiplicative coalescent at a fixed time: see Theorem 2 in Aldous & Limic [4] (recalled in Proposition 5.9). In particular, it implies that a.s. $\sum_{k\geq 1} \zeta_k^2 < \infty$. We then recall from (59) in Section 2.2.3 the definition of *excursion processes* of $\mathcal{H}$ and $Y - J$ above 0:

$$\forall k \geq 1, \ \forall t \in [0, \infty), \qquad \mathtt{H}_k(t) = \mathcal{H}_{(l_k+t)\wedge r_k} \quad \text{and} \quad \mathtt{Y}_k(t) = Y_{(l_k+t)\wedge r_k} - J_{l_k}. \tag{229}$$

Recall from (46) and (47) the definition of $\mathbf{\Pi} = \big((s_p, t_p)\big)_{p\geq 1}$: namely, conditionally given Y, let

$$\mathcal{P} = \sum_{p\geq 1} \delta_{(t_p, y_p)} \text{ be a Poisson pt. meas. on } [0, \infty)^2 \text{ with intensity } \kappa \mathbf{1}_{\{0<y<Y_t - J_t\}}\, dt\, dy \tag{230}$$

and set

$$\mathbf{\Pi} = \big((s_p, t_p)\big)_{p\geq 1} \quad \text{where} \quad s_p = \inf\big\{s \in [0, t_p] : \inf_{u\in[s,t_p]} Y_u - J_u > y_p\big\}, \ \ p \geq 1. \tag{231}$$

Let $a_n, b_n \in (0, \infty)$, and $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, satisfy (66) and (**C1**)–(**C4**). Recall from (13) the definition of $Y^{\mathtt{w}_n}$; recall from (18) the definition of $\mathcal{H}^{\mathtt{w}_n}$, the height process associated to $Y^{\mathtt{w}_n}$. Recall from (15) and (16) the definition of $\mathbf{\Pi}_{\mathtt{w}_n}$. For all $t \in [0, \infty)$, to simplify notations, we introduce the following:

$$Y_t^{(n)} := \tfrac{1}{a_n} Y^{\mathtt{w}_n}_{b_n t}, \quad J_t^{(n)} := \inf_{s\in[0,t]} Y_s^{(n)}, \quad \mathcal{H}_t^{(n)} := \tfrac{a_n}{b_n} \mathcal{H}^{\mathtt{w}_n}_{b_n t} \quad \text{and} \quad \mathbf{\Pi}^{(n)} := \tfrac{1}{b_n} \mathbf{\Pi}_{\mathtt{w}_n} =: \big((s_p^n, t_p^n)\big)_{1\leq p<\mathbf{p}_n}. \tag{232}$$

Recall from Section 2.2.2 that the excursion intervals of $Y^{(n)} - J^{(n)}$ above 0 are the same as the excursions intervals of $\mathcal{H}^{(n)}$ above 0; let $\mathbf{q}_{\mathtt{w}_n}$ stands for the number of such intervals. Namely,

$$\big\{t \in [0, \infty) : \mathcal{H}_t^{(n)} > 0\big\} = \big\{t \in [0, \infty) : Y_t^{(n)} > J_t^{(n)}\big\} = \bigcup_{1\leq k\leq \mathbf{q}_{\mathtt{w}_n}} (l_k^n, r_k^n) \tag{233}$$

where the indexation is such that the $\zeta_k^n := r_k^n - l_k^n$ are nonincreasing and such that $l_k^n < l_{k+1}^n$ if $\zeta_k^n = \zeta_{k+1}^n$ (within the notation of Section 2.2.2, $l_k^n = l_k^{\mathtt{w}_n}/b_n$, $r_k^n = r_k^{\mathtt{w}_n}/b_n$ and $\zeta_k^n = \zeta_k^{\mathtt{w}_n}/b_n$).

6.2.1 Proof of Theorem 2.15.

We keep the previous notation. By Theorem 2.14, we get

$$\big(Y^{(n)}, \mathcal{H}^{(n)}, \mathbf{\Pi}^{(n)}\big) \xrightarrow[n\to\infty]{} \big(Y, \mathcal{H}, \mathbf{\Pi}\big) \tag{234}$$

weakly on $\mathbf{D}([0, \infty), \mathbb{R}) \times \mathbf{C}([0, \infty), \mathbb{R}) \times (\mathbb{R}^2)^{\mathbb{N}^*}$, equipped with product topology (recall that here we use the following convention: the finite sequence $\mathbf{\Pi}^{(n)} = \big((s_p^n, t_p^n)\big)_{1\leq p<\mathbf{p}_n}$ is extended by setting $(s_p^n, t_p^n) = (-1, -1)$ for all $p > \mathbf{p}_n$). Thanks to Skorokod's representation theorem (but with a slight abuse of notation) *we can assume without loss of generality that (234) holds* $\mathbf{P}$*-a.s.* We first prove the following lemma.

Lemma 6.14 *We keep the previous notations and we assume that (234) holds a.s. Then, for all $k, n \geq 1$, there exists a sequence $j(n, k) \in \{1, \ldots, \mathbf{q}_n\}$ such that*

$$\mathbf{P}\text{-a.s. for all } k \geq 1, \qquad \big(l^n_{j(n,k)}, r^n_{j(n,k)}\big) \xrightarrow[n\to\infty]{} \big(l_k, r_k\big). \tag{235}$$

Proof. Fix $k\geq 1$ and let $t_0\in(l_k,r_k)$; note that $l_k=\sup\{t\in[0,t_0]:\mathcal{H}_t=0\}$ and $r_k=\inf\{t\in[t_0,\infty):\mathcal{H}_t=0\}$. For all $n\geq 1$, set $\gamma(n)=\sup\{t\in[0,t_0):\mathcal{H}^{(n)}_t=0\}$ and $\delta(n)=\inf\{t\in[t_0,\infty):\mathcal{H}^{(n)}_t=0\}$. Let q and r be such that $l_k<q<t_0<r<r_k$. Since $\inf_{t\in[q,r]}\mathcal{H}_t>0$, for all sufficiently large n, we get $\inf_{t\in[q,r]}\mathcal{H}^{(n)}_t>0$, which implies that $\gamma(n)\leq q$ and $r\leq\delta(n)$. This easily implies that $\limsup_{n\to\infty}\gamma(n)\leq l_k$ and $r_k\leq\liminf_{n\to\infty}\delta(n)$.

Let q and r be such that $q<l_k$ and $r_k<r$. Since $\mathcal{H}_{l_k}=\mathcal{H}_{r_k}=0$, (179) in Lemma 5.6 (iv) implies that $J_q>J_{t_0}>J_r$. Since J is continuous, Lemma B.1 (iii) entails that $J^{(n)}\to J$ a.s. in $\mathbf{C}([0,\infty),\mathbb{R})$. Thus, for all sufficiently large n, $J^{(n)}_q>J^{(n)}_{t_0}>J^{(n)}_r$; by definition, it implies that $Y^{(n)}-J^{(n)}$ (and thus $\mathcal{H}^{(n)}$) take the value 0 between the times q and t_0 and between the times t_0 and r: namely, for all sufficiently large n, $\gamma(n)\geq q$ and $\delta(n)\leq r$. This easily entails $\liminf_{n\to\infty}\gamma(n)\geq l_k$ and $r_k\geq\limsup_{n\to\infty}\delta(n)$, and we have proved that $\lim_{n\to\infty}\gamma(n)=l_k$ and $\lim_{n\to\infty}\delta(n)=r_k$.

Let $n_0\geq 1$ be such that for all $n\geq n_0$, $\mathcal{H}^{(n)}_{t_0}>0$. Then, for all $n\geq n_0$, there exists $j(n,k)\in\{1,\dots,\mathbf{q}_{\mathtt{w}_n}\}$ such that $\gamma(n)=l^n_{j(n,k)}$ and $\delta(n)=r^n_{j(n,k)}$; for all $n\leq n_1$, we take for instance $j(n,k)=1$. Then, (235) holds true which completes the proof. ■

We next recall from Proposition 2.17, Section 2.3.4 (this result an immediate consequence of Proposition 7 in Aldous & Limic [4]) that $\sum_{1\leq k\leq\mathbf{q}_{\mathtt{w}_n}}(\zeta^n_k)^2\to\sum_{k\geq 1}(\zeta_k)^2$ weakly on $[0,\infty)$ as $n\to\infty$. We use this result to prove the following joint convergence.

Lemma 6.15 *We keep the previous notations. Then*

$$\mathscr{Q}_n(5):=\Big(Y^{(n)},\mathcal{H}^{(n)},\mathbf{\Pi}^{(n)},\sum_{1\leq k\leq\mathbf{q}_{\mathtt{w}_n}}(\zeta^n_k)^2\Big)\xrightarrow[n\to\infty]{}\mathscr{Q}(5):=\Big(Y,\mathcal{H},\mathbf{\Pi},\sum_{k\geq 1}(\zeta_k)^2\Big)\tag{236}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R})?\times\mathbf{C}([0,\infty),\mathbb{R})?\times?(\mathbb{R}^2)^{\mathbb{N}^*}\times[0,\infty)$, *equipped with product topology.*

Proof. The laws of the $\mathscr{Q}_n(5)$ are tight by (234) and the weak convergence $\sum_{1\leq k\leq\mathbf{q}_{\mathtt{w}_n}}(\zeta^n_k)^2\to\sum_{k\geq 1}(\zeta_k)^2$. We only need to prove that the law of $\mathscr{Q}(5)$ is the unique limiting law: to that end, let $(n(p))_{p\in\mathbb{N}}$ be an increasing sequence of integers such that $\mathscr{Q}_{n(p)}(5)\to(Y,\mathcal{H},\mathbf{\Pi},Z)$ weakly. Actually, we only have to prove that $Z=\sum_{k\geq 1}(\zeta_k)^2$. Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that $\mathscr{Q}_{n(p)}(5)\to(Y,\mathcal{H},\mathbf{\Pi},Z)$ holds true $\mathbf{P}$-a.s. Then, by Lemma 6.14, observe that for all $l\geq 1$,

$$Z\xleftarrow[n\to\infty]{}\sum_{1\leq k\leq\mathbf{q}_{\mathtt{w}_n}}(\zeta^n_k)^2\geq\sum_{1\leq k\leq l}(\zeta^n_{j(n,k)})^2\xrightarrow[n\to\infty]{}\sum_{1\leq k\leq l}(\zeta_k)^2.\tag{237}$$

Set $Z'=\sum_{k\geq 1}(\zeta_k)^2$; by letting l go to ∞ in (237), we get $Z\geq Z'$, which implies $Z=Z'$ a.s. since Z and Z' have the same law. This completes the proof of the lemma. ■

Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that (236) holds true a.s. on $\mathbf{D}([0,\infty),\mathbb{R})?\times\mathbf{C}([0,\infty),\mathbb{R})\times?(\mathbb{R}^2)^{\mathbb{N}^*}?\times[0,\infty)$, equipped with product topology. We next prove the following.

Lemma 6.16 *Assume that (236) holds true almost surely. We keep the previous notations. Then,*

$$\mathbf{P}\text{-a.s. for all } k\geq 1,\qquad (l^n_k,r^n_k)\xrightarrow[n\to\infty]{}(l_k,r_k).\tag{238}$$

Proof. Let $\varepsilon\in(0,\infty)$ and k_ε be such that $\zeta_k>\varepsilon$ for all $k\in\{1,\dots,k_\varepsilon\}$ and $\zeta_k<\varepsilon$ for all $k>k_\varepsilon$. Let $k'_\varepsilon\geq k_\varepsilon$ be such that $\sum_{k>k'_\varepsilon}(\zeta_k)^2<\varepsilon^2/3$. Let $n_0\geq 1$ be such that for all $n\geq n_0$,

$$\Big|\sum_{1\leq k\leq\mathbf{q}_{\mathtt{w}_n}}(\zeta^n_k)^2-\sum_{k\geq 1}(\zeta_k)^2\Big|<\varepsilon^2/3,\qquad\sum_{1\leq k\leq k'_\varepsilon}\big|(\zeta^n_{j(n,k)})^2-(\zeta_k)^2\big|<\varepsilon^2/3$$
$$\text{and}\quad\max_{1\leq k\leq k'_\varepsilon}\big|\zeta_k-\zeta^n_{j(n,k)}\big|<\min_{1\leq k\leq k'_\varepsilon}|\varepsilon-\zeta_k|.\tag{239}$$

Set $S_n = \{1, \ldots, \mathbf{q}_{\mathtt{W}_n}\} \backslash \{j(n,1), \ldots, j(n,k'_\varepsilon)\}$; then the previous inequality imply for all $n \geq n_0$, that $\sum_{k \in S_n} (\zeta_k^n)^2 < \varepsilon^2$. Thus, for all $n \geq n_0$, if $k \in S_n$, then $\zeta_k^n < \varepsilon$; the last inequality of (239) implies for all $k \in \{k_\varepsilon + 1, \ldots, k'_\varepsilon\}$, $\zeta^n_{j(n,k)} < \varepsilon$ and that for all $k \in \{1, \ldots, k_\varepsilon\}$, $\zeta^n_{j(n,k)} > \varepsilon$. This implies that for all sufficiently large n, $j(n,k) = k$, for all $k \in \{1, \ldots, k_\varepsilon\}$ and (235) in Lemma 6.14 implies $(l_k^n, r_k^n) \to (l_k, r_k)$ a.s. for all $k \in \{1, \ldots, k_\varepsilon\}$, which entails (241) since ε can be choosen arbitrarily small. ■

Recall from (229) the definition of the excursions $\mathtt{H}_k$ and $\mathtt{Y}_k$ of resp. $\mathcal{H}$ and $Y - J$ above 0. We define the (rescaled) excursion of $Y^{(n)} - J^{(n)}$ and of $\mathcal{H}^{(n)}$ above 0 as follows:

$$\text{(240)} \qquad \forall k \geq 1,\ \forall t \in [0, \infty), \qquad \mathtt{H}_k^{(n)}(t) = \mathcal{H}^{(n)}_{(l_k^n + t) \wedge r_k^n} \quad \text{and} \quad \mathtt{Y}_k^{(n)}(t) = Y^{(n)}_{(l_k^n + t) \wedge r_k^n} - J^{(n)}_{l_k^n}.$$

As an immediate consequence of (236), Lemma 6.16 and Lemma B.4 (iii) in Appendix Section B, we get the following result.

Lemma 6.17 *Assume that (236) holds true almost surely. We keep the previous notations. Then,*

$$\text{(241)} \qquad \mathbf{P}\text{-a.s. for all } k \geq 1, \qquad \big(\mathtt{Y}_k^{(n)}, \mathtt{H}_k^{(n)}, l_k^n, r_k^n\big) \xrightarrow[n \to \infty]{} \big(\mathtt{Y}_k, \mathtt{H}_k, l_k, r_k\big).$$

in $\mathbf{D}([0,\infty), \mathbb{R}) \times \mathbf{C}([0,\infty), \mathbb{R}) \times [0,\infty)^2$.

Recall from (230) and (231) the definition of $\mathbf{\Pi} = \big((s_p, t_p)\big)_{p \geq 1}$ and recall from (232) the notation $\mathbf{\Pi}^{(n)} = \big((s_p^n, t_p^n)\big)_{1 \leq p \leq \mathbf{p}_n}$. We next prove the following.

Lemma 6.18 *Assume that (236) holds almost surely. We keep the previous notations. Then,* $\mathbf{P}$-*a.s. for all* $p \geq 1$, *there exists* $k \geq 1$ *such that* $l_k < s_p \leq t_p < r_k$ *and for all sufficiently large* n, $l_k^n < s_p^n < t_p^n < r_k^n$ *and* $(l_k^n, s_p^n, t_p^n, r_k^n) \to (l_k, s_p, t_p, r_k)$.

Proof. By Proposition 5.8 (i), $\mathbf{P}$-a.s. for all $p \geq 1$, $Y_{t_p} > J_{t_p}$ and there exists $k \geq 1$ such that $t_p \in (l_k, r_k)$. By Lemma 5.6 (iii), we get $Y_{l_k} - J_{l_k} = 0$; recall that $y_p \in (0, Y_{t_p} - J_{t_p})$ and recall from (231) that $s_p = \inf \big\{ s \in [0, t_p] : \inf_{u \in [s, t_p]} Y_u - J_u > y_p \big\}$; thus, we get $l_k < s_p \leq t_p < r_k$ and the proof is completed by (236) that asserts that $(s_p^n, t_p^n) \to (s_p, t_p)$ and by Lemma 6.16 that asserts that $(l_k^n, r_k^n) \to (l_k, r_k)$. ■

Recall from (60) that for all $k \geq 1$,

$$\mathbf{\Pi}_k = \big((s_p^k, t_p^k)\big)_{1 \leq p \leq \mathbf{p}_k} \text{ where } (t_p^k)_{1 \leq p \leq \mathbf{p}_k} \text{ increases and where}$$
$$\text{the } (l_k + s_p^k, l_k + t_p^k) \text{ are exactly the terms } (s_{p'}, t_{p'}) \text{ of } \mathbf{\Pi} \text{ such that } t_{p'} \in [l_k, r_k],$$

and similarly recall from (56) the definition of the sequence of pinching times $(\mathbf{\Pi}_k^{\mathtt{W}})_{1 \leq k \leq \mathbf{q}_{\mathtt{W}_n}}$: namely, in their rescaled version,

$$\tfrac{1}{b_n} \mathbf{\Pi}_k^{\mathtt{W}_n} = \big((s_p^{n,k}, t_p^{n,k})\big)_{1 \leq p \leq \mathbf{p}_k^n} \text{ where } (t_p^{n,k})_{1 \leq p \leq \mathbf{p}_k^n} \text{ increases and where}$$
$$\text{the } (l_k^n + s_p^{n,k}, l_k^n + t_p^{n,k}) \text{ are exactly the terms } (s_{p'}^n, t_{p'}^n) \text{ of } \mathbf{\Pi}_{\mathtt{W}}^{(n)} \text{ such that } t_{p'}^n \in [l_k^n, r_k^n].$$

Thus, Lemma 6.18 immediately entails that

$$\mathbf{P}\text{-a.s. for all } k \geq 1, \quad \tfrac{1}{b_n} \mathbf{\Pi}_k^{\mathtt{W}_n} \xrightarrow[n \to \infty]{} \mathbf{\Pi}_k.$$

This convergence combined with Lemma 6.17 implies Theorem 2.15. ■

6.2.2 Proof of Theorem 2.16.

Keep the previous notation and recall that $\big(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}, d_k^{\mathtt{w}_n}, \varrho_k^{\mathtt{w}_n}, \mathbf{m}_k^{\mathtt{w}_n}\big)$, $1 \le k \le \mathbf{q}_{\mathtt{w}_n}$, stand for the connected components of the $\mathtt{w}_n$-multiplicative random graph $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$. Here, $d_k^{\mathtt{w}_n}$ stands for the graph-metric on $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$, $\mathbf{m}_k^{\mathtt{w}_n}$ is the restriction to $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ of the measure $\mathbf{m}_{\mathtt{w}_n} = \sum_{j\ge 1} w_j^{(n)} \delta_j$, $\varrho_k^{\mathtt{w}_n}$ is the first vertex of $\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}$ that is visited during the exploration of $\boldsymbol{\mathcal{G}}_{\mathtt{w}_n}$, and the indexation is such that $\mathbf{m}_1^{\mathtt{w}_n}\big(\boldsymbol{\mathcal{G}}_1^{\mathtt{w}_n}\big) \ge \ldots \ge \mathbf{m}_{\mathbf{q}_{\mathtt{w}_n}}^{\mathtt{w}_n}\big(\boldsymbol{\mathcal{G}}_{\mathbf{q}_{\mathtt{w}_n}}^{\mathtt{w}_n}\big)$.

Next, recall from (240) that $\mathtt{H}_k^{(n)}(\cdot)$ stands for k-th longest excursion of $\mathcal{H}^{\mathtt{w}_n}$ that is rescaled in time by a factor $1/b_n$ and in space by a factor a_n/b_n; recall that $\frac{1}{b_n}\boldsymbol{\Pi}_k^{\mathtt{w}_n} = \big((s_p^{n,k}, t_p^{n,k}); 1 \le p \le \mathbf{p}_k^n\big)$ is the ($1/b_n$-rescaled) finite sequence of pinching times of $\mathtt{H}_k^{(n)}$. Then, for all $k \in \{1, \ldots, \mathbf{q}_{\mathtt{w}_n}\}$ the compact measured metric space

$$\mathbf{G}_k^{(n)} := \big(\boldsymbol{\mathcal{G}}_k^{\mathtt{w}_n}, \tfrac{a_n}{b_n} d_k^{\mathtt{w}_n}, \varrho_k^{\mathtt{w}_n}, \tfrac{1}{b_n}\mathbf{m}_k^{\mathtt{w}_n}\big)$$

is isometric to $G\big(\mathtt{H}_k^{(n)}, \frac{1}{b_n}\boldsymbol{\Pi}_k^{\mathtt{w}_n}, \frac{a_n}{b_n}\big)$, the compact measured metric space coded by $\mathtt{H}_k^{(n)}$ and the pinching setup $(\frac{1}{b_n}\boldsymbol{\Pi}_k^{\mathtt{w}_n}, \frac{a_n}{b_n})$ as defined in (53).

Then recall from (229) that $\mathtt{H}_k(\cdot)$ stands for k-th longest excursion of $\mathcal{H}$ and recall from (60) that $\boldsymbol{\Pi}_k = \big((s_p^k, t_p^k); 1 \le p \le \mathbf{p}_k\big)$ is the finite sequence of pinching times of $\mathtt{H}_k$. Then, for all $k \ge 1$, the compact measured metric space

$$\mathbf{G}_k := \big(\mathbf{G}_k, \mathrm{d}_k, \varrho_k, \mathbf{m}_k\big)$$

is isometric to $G(\mathtt{H}_k, \boldsymbol{\Pi}_k, 0)$ that is the compact measured metric space coded by $\mathtt{H}_k$ and the pinching setup $(\boldsymbol{\Pi}_k, 0)$ as defined in (53).

Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that the convergence in Theorem 2.15 holds almost surely. Namely a.s. for all $k \ge 1$, $\big(\mathtt{H}_k^{(n)}, \zeta_k^n, \frac{1}{b_n}\boldsymbol{\Pi}_k^{\mathtt{w}_n}\big) \to \big(\mathtt{H}_k, \zeta_k, \boldsymbol{\Pi}_k\big)$ on $\mathbf{C}([0,\infty), \mathbb{R}) \times [0,\infty) \times (\mathbb{R}^2)^{\mathbb{N}^*}$. We next fix $k \ge 1$; then for all sufficiently large n, $\frac{1}{b_n}\boldsymbol{\Pi}_k^{\mathtt{w}_n}$ and $\boldsymbol{\Pi}_k$ have the same number of points: namely, $\mathbf{p}_k^n = \mathbf{p}_k$ and

$$\forall 1 \le p \le \mathbf{p}_k^n = \mathbf{p}_k, \quad (s_p^{n,k}, t_p^{n,k}) \xrightarrow[n\to\infty]{} (s_p^k, t_p^k) \; . \tag{242}$$

Recall from (63) the definition of the Gromov-Hausdorff-Prohorov distance $\boldsymbol{\delta}_{\mathrm{GHP}}$. We next apply Lemma 2.10 with $(h, h') = (\mathtt{H}_k, \mathtt{H}_k^{(n)})$, $(\Pi, \Pi') = (\boldsymbol{\Pi}_k, \frac{1}{b_n}\boldsymbol{\Pi}_k^{\mathtt{w}_n})$, $(\varepsilon, \varepsilon') = (0, a_n/b_n)$ and $\delta = \delta_n = \max_{1 \le p \le \mathbf{p}_k} |s_p^k - s_p^{n,k}| \vee |t_p^k - t_p^{n,k}|$. Then, by (65),

$$\boldsymbol{\delta}_{\mathrm{GHP}}(\mathbf{G}_k, \mathbf{G}_k^{(n)}) \le 6(\mathbf{p}_k + 1)\big(\|\mathtt{H}_k - \mathtt{H}_k^{(n)}\|_\infty + \omega_{\delta_n}(\mathtt{H}_k)\big) + 3a_n\mathbf{p}_k/b_n + |\zeta_k - \zeta_k^n|, \tag{243}$$

where $\omega_{\delta_n}(\mathtt{H}_k) = \max\{|\mathtt{H}_k(t) - \mathtt{H}_k(s)|; s, t \in [0,\infty) : |s - t| \le \delta_n\}$. By (242), $\delta_n \to 0$; since $\mathtt{H}_k$ is continuous and since it is null on $[\zeta_k, \infty)$, it is uniformly continuous and $\omega_{\delta_n}(\mathtt{H}_k) \to 0$; recall that $a_n/b_n \to 0$. Thus, the right member of (243) goes to 0 as $n \to 0$. Thus, we have proved that a.s. for all $k \ge 1$, $\boldsymbol{\delta}_{\mathrm{GHP}}(\mathbf{G}_k, \mathbf{G}_k^{(n)}) \to 0$, which implies Theorem 2.16. ■

6.3 Proof of the limit theorems for the Markovian processes.

6.3.1 Convergence of the Markovian queueing system: the general case.

We say that $\mathbb{R}$-valued spectrally positive Lévy processes $(R_t)_{t\in[0,\infty)}$ with initial value $R_0 = 0$ is *integrable* if for at least one $t \in (0,\infty)$ we have $\mathbf{E}[|R_t|] < \infty$. It implies that $\mathbf{E}[|R_t|] < \infty$ for all $t \in (0,\infty)$. We recall from Section B.2.1 in Appendix that there is a one-to-one correspondence between the laws of $\mathbb{R}$-valued spectrally positive Lévy processes $(R_t)_{t\in[0,\infty)}$ with initial value $R_0 = 0$ that are integrable and triplets (α, β, π), where $\alpha \in \mathbb{R}$, $\beta \in [0,\infty)$ and π is a Borel-measure on

$(0,\infty)$ such that $\int_{(0,\infty)} \pi(dr)\,(r\wedge r^2) < \infty$. More precisely, the correspondence is given by the Laplace exponent of spectrally positive Lévy processes: namely, for all $t, \lambda \in [0,\infty)$,

$$\mathbf{E}\big[e^{-\lambda R_t}\big] = e^{t\psi_{\alpha,\beta,\pi}(\lambda)}, \text{ where } \psi_{\alpha,\beta,\pi}(\lambda) = \alpha\lambda + \tfrac{1}{2}\beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\,\pi(dr). \tag{244}$$

Concerning the convergence of branching processes, we rely on a result due to Grimvall [27], that is recalled in Theorem B.11: this result states the convergence of rescaled Galton-Watson processes to Continuous State Branching Processes (CSBP for short). Namely, recall that $(Z_t)_{t\in[0,\infty)}$ is a conservative CSBP if it is a $[0,\infty)$-valued Feller Markov process obtained from spectrally positive Lévy processes via Lamperti's time-change, the law of the CSBP being completely characterised by the spectrally Lévy process and thus by its Laplace exponent that is usually called the *branching mechanism* of the CSBP, which is necessarily of the form (244): see Section B.2.2 for a brief account on CSBP.

Let $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n\in\mathbb{N}$. Define the law $\nu_{\mathtt{w}_n}$ and $\mu_{\mathtt{w}_n}$ by setting:

$$\nu_{\mathtt{w}_n} = \frac{1}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} w_j^{(n)} \delta_j \quad \text{and} \quad \forall k\in\mathbb{N}, \quad \mu_{\mathtt{w}_n}(k) = \sum_{j\geq 1} \frac{(w_j^{(n)})^{k+1}}{\sigma_1(\mathtt{w}_n)\, k!}\, e^{-w_j^{(n)}}. \tag{245}$$

Recall from Section 4.1.2, the definition of the Markovian LIFO-queueing system associated with the set of weights $\mathtt{w}_n$: clients arrive at unit rate; each client has a *type* that is a positive integer; the amount of service required by a client of type j is $w_j^{(n)}$; the types are i.i.d. with law $\nu_{\mathtt{w}_n}$. If one denotes by τ_k^n the time of arrival of the k-th client in the queue and by $\mathtt{J}_k^n$ his type, then the queueing system is entirely characterised by:

$$\mathscr{X}_{\mathtt{w}_n} = \sum_{k\geq 1} \delta_{(\tau_k^n, \mathtt{J}_k^n)}, \tag{246}$$

that is a Poisson point measure on $[0,\infty)\times\mathbb{N}^*$ whith intensity $\ell\otimes\nu_{\mathtt{w}_n}$, where ℓ stands for the Lebesgue measure on $[0,\infty)$. Next, for all $j\in\mathbb{N}^*$ and all $t\in[0,\infty)$, we introduce the following:

$$N_j^{\mathtt{w}_n}(t) = \sum_{k\geq 1} \mathbf{1}_{\{\tau_k^n \leq t\,;\, \mathtt{J}_k^n = j\}} \quad \text{and} \quad X_t^{\mathtt{w}_n} = -t + \sum_{k\geq 1} w^{(n)}_{\mathtt{J}_k^n} \mathbf{1}_{[0,t]}(\tau_k^n) = -t + \sum_{j\geq 1} w_j^{(n)} N_j^{\mathtt{w}_n}(t). \tag{247}$$

Observe that $(N_j^{\mathtt{w}_n})_{j\geq 1}$ are independent homogeneous Poisson processes with rates $w_j^{(n)}/\sigma_1(\mathtt{w}_n)$ and $X^{\mathtt{w}_n}$ is a càdlàg spectrally positive Lévy process.

Let $a_n, b_n \in (0,\infty)$, $n\in\mathbb{N}$ be two sequence that satisfy the following conditions.

$$a_n \text{ and } \frac{b_n}{a_n} \xrightarrow[n\to\infty]{} \infty, \quad \frac{b_n}{a_n^2} \xrightarrow[n\to\infty]{} \beta_0 \in [0,\infty), \quad \text{and} \quad \sup_{n\in\mathbb{N}} \frac{w_1^{(n)}}{a_n} < \infty. \tag{248}$$

It is important to note that these assumptions are weaker than (66): namely, we temporarily **do not** assume that $\frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \to \kappa \in (0,\infty)$, which explains why the possible limits in the theorem below are more general.

Theorem 6.19 *Let $\mathtt{w}_n \in \ell_f^{\downarrow}$ and $a_n, b_n \in (0,\infty)$, $n\in\mathbb{N}$, satisfy (248). Recall from (247) the definition of $X_t^{\mathtt{w}_n}$, recall from (245) the definition of $\mu_{\mathtt{w}_n}$ and let $(Z_k^{(n)})_{k\in?\mathbb{N}}$ be a Galton-Watson process with offspring distribution $\mu_{\mathtt{w}_n}$ and initial state $Z_0^{(n)} = \lfloor a_n \rfloor$. Then, the following convergences are equivalent.*

- (I) $\big(\frac{1}{a_n} Z^{(n)}_{\lfloor b_n t/a_n \rfloor}\big)_{t\in[0,\infty)} \longrightarrow (Z_t)_{t\in[0,\infty)}$ *weakly on* $\mathbf{D}([0,\infty),\mathbb{R})$.
- (II) $\big(\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n t}\big)_{t\in[0,\infty)} \longrightarrow (X_t)_{t\in[0,\infty)}$ *weakly on* $\mathbf{D}([0,\infty),\mathbb{R})$.

If (I) *or* (II) *hold true, then* Z *is necessarily a CSBP and* X *is an integrable* (α, β, π)*-spectrally positive Lévy process (as defined at the beginning of Section 6.3.1) whose Laplace exponent is the same as the branching mechanism of* Z*. Here* (α, β, π) *necessarily satisfies:*

$$\beta \geq \beta_0 \quad \text{and} \quad \exists r_0 \in (0,\infty) \text{ such that } \pi((r_0,\infty)) = 0 \, , \tag{249}$$

which implies $\int_{(0,\infty)} r^2 \, \pi(dr) < \infty$*. Moreover,* (I) $\Leftrightarrow$ (II) $\Leftrightarrow$ (IIIabc) $\Leftrightarrow$ ((IIIa)&(IV)) *where:*

- (IIIa) $\frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big) \longrightarrow \alpha.$
- (IIIb) $\frac{b_n}{(a_n)^2} \frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)} \longrightarrow \beta + \int_{(0,\infty)} r^2 \, \pi(dr).$
- (IIIc) $\frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} \frac{w_j^{(n)}}{a_n} f\big(w_j^{(n)}/a_n\big) \longrightarrow \int_{(0,\infty)} f(r) \, \pi(dr)$*, for all continuous bounded* $f : [0,\infty) \to \mathbb{R}$ *vanishing in a neighbourhood of* 0.
- (IV) $\frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} \frac{w_j^{(n)}}{a_n} \big(e^{-\lambda w_j^{(n)}/a_n} - 1 + \lambda \, w_j^{(n)}/a_n\big) \longrightarrow \psi_{\alpha,\beta,\pi}(\lambda) - \alpha\lambda,$ *for all* $\lambda \in (0,\infty)$*, where* $\psi_{\alpha,\beta,\pi}$ *is defined by (244).*

Proof. We easily check that $(X^{\mathtt{w}_n}_{b_n t}/a_n)$ is a $(\alpha_n, \beta_n, \pi_n)$-spectrally positive Lévy process where

$$\alpha_n = \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big), \quad \beta_n = 0 \quad \text{and} \quad \pi_n = \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} \frac{w_j^{(n)}}{a_n} \, \delta_{w_j^{(n)}/a_n}.$$

We immediately see that $\beta_n + \int r^2 \pi_n(dr) = b_n \sigma_3(\mathtt{w}_n)/a_n^2 \sigma_1(\mathtt{w}_n)$. Then, Theorem B.9 implies that (II) $\Leftrightarrow$ (IIIabc). We then apply Lemma A.3 to $\Delta^n_k = (X^{\mathtt{w}_n}_k - X^{\mathtt{w}_n}_{k-1})/a_n$ and $q_n = \lfloor b_n \rfloor$: it shows that the weak limit $X^{\mathtt{w}_n}_1 \to X_1$ is equivalent to the convergence of the Laplace exponents $\psi_{\alpha_n,\beta_n,\pi_n}(\lambda) \to \psi_{\alpha,\beta,\pi}(\lambda)$, for all $\lambda \in [0,\infty)$. Then note that the left member in (IV) is $\psi_{\alpha_n,\beta,\pi_n}(\lambda) - \alpha_n\lambda$. This shows that (II) $\Leftrightarrow$ ((IIIa)&(IV)).

It remains to prove that $\beta \geq \beta_0$ and that (I) $\Leftrightarrow$ (IIIabc). Let $(\zeta^n_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with law $\mu_{\mathtt{w}_n}$ as defined in (245). By Theorem B.11, (I) is equivalent to the weak convergence on $\mathbb{R}$ of the r.v. $R_n := \frac{1}{a_n} \sum_{1\leq k\leq \lfloor b_n \rfloor} \big(\zeta^n_k - 1\big)$. We next apply Lemma A.3 to $\Delta^n_k := a_n^{-1}(\zeta^n_k - 1)$: it asserts that (I) is equivalent to

$$\exists \psi \in \mathbf{C}([0,\infty),\mathbb{R}) : \quad \psi(0) = 0 \quad \text{and} \quad \forall \lambda \in [0,\infty), \; L_n(\lambda) := \mathbf{E}\big[e^{-\lambda R_n}\big] \xrightarrow[n\to\infty]{} e^{\psi(\lambda)}. \tag{250}$$

Let $(W^n_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence of r.v. with the same law as $w^{(n)}_{\mathtt{J}^n_1}$, where $\mathtt{J}^n_1$ has law $\nu_{\mathtt{w}_n}$. Namely, for all all measurable function $f : [0,\infty) \to [0,\infty)$,

$$\mathbf{E}\big[f(W^n_k)\big] = \frac{1}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} w_j^{(n)} f\big(w_j^{(n)}\big) \, .$$

Note that for all $k \in \mathbb{N}$, $\mu_{\mathtt{w}_n}(k) = \mathbf{E}[\,(W^n_1)^k e^{-W^n_1}/k!\,]$, which implies:

$$L_n(\lambda) = e^{\lambda \lfloor b_n \rfloor/a_n} \big(\mathbf{E}\big[e^{-\lambda \zeta^n_1/a_n}\big]\big)^{\lfloor b_n \rfloor} = e^{\lambda \lfloor b_n \rfloor/a_n} \big(\mathbf{E}\big[\exp\big(-W^n_1(1 - e^{-\lambda/a_n})\big)\big]\big)^{\lfloor b_n \rfloor}. \tag{251}$$

We next set $S^n_1 = \frac{1}{a_n} \sum_{1\leq k\leq \lfloor b_n \rfloor} \big(W^n_k - 1\big)$ and $\mathcal{L}_n(\lambda) = \mathbf{E}[\exp(-\lambda S^n_1)]$. By (251), we get:

$$\forall \lambda \in [0,\infty), \quad \mathcal{L}_n\big(a_n(1 - e^{-\lambda/a_n})\big) = L_n(\lambda) \, \exp\big(\lfloor b_n \rfloor (1 - e^{-\lambda/a_n}) - \lambda \lfloor b_n \rfloor/a_n\big)$$

Since $\lfloor b_n \rfloor (1 - e^{-\lambda/a_n}) - \lambda \lfloor b_n \rfloor/a_n + \frac{1}{2} b_n a_n^{-2} \lambda^2 = \mathcal{O}(b_n a_n^{-3}) \to 0$, then (250) is equivalent to

$$\exists \psi_0 \in \mathbf{C}([0,\infty),\mathbb{R}) : \quad \psi_0(0) = 0 \quad \text{and} \quad \forall \lambda \in [0,\infty), \; \lim_{n\to\infty} \mathcal{L}_n(\lambda) = e^{\psi_0(\lambda)}, \tag{252}$$

and if (250) or (252) holds true, then $\psi(\lambda)=\psi_0(\lambda)+\frac{1}{2}\beta_0\lambda^2$, for all $\lambda\in[0,\infty)$.

Next, by Lemma A.3 applied to $\Delta_k^n := a_n^{-1}(W_k^n - 1)$, we see that (252) is equivalent to the weak convergence $S_1^n \to S_1$ in $\mathbb{R}$ and Theorem B.10 asserts that is equivalent to the conditions (*Rw3abc*) with $\xi_1^n = W_1^n - 1$: namely, there exists a triplet $(\alpha^*, \beta^*, \pi^*)$ such that $\beta^*, \alpha^* \in [0,\infty)$, such that there exists $r_0 \in (0,\infty)$ satisfying $\pi^*([r_0,\infty))=0$ and such that the following holds true

$$\frac{b_n}{a_n}\mathbf{E}[\xi_1^n]=\frac{b_n}{a_n}\Big(\frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}-1\Big)\to -\alpha^*,\;\frac{b_n}{a_n^2}\mathbf{var}(\xi_1^n)=\frac{b_n}{a_n^2}\frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}-\frac{b_n}{a_n^2}\Big(\frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big)^2\to\beta^*+\int_{(0,\infty)}r^2\pi^*(dr)$$

$$\text{and}\quad b_n\mathbf{E}\big[f\big(\xi_1^n/a_n\big)\big]=\frac{a_nb_n}{\sigma_1(w_n)}\sum_{j\geq 1}\frac{w_j^{(n)}}{a_n}f\big(w_j^{(n)}/a_n\big)\to\int_{(0,\infty)}f(r)\,\pi^*(dr),$$

for all continuous bounded $f:[0,\infty)\to\mathbb{R}$ vanishing in a neighbourhood of 0. It is easy to see that these conditions are equivalent to (IIIabc) with $\alpha=\alpha^*$, $\beta=\beta_0+\beta^*$ and $\pi=\pi^*$. This completes the proof of the theorem. ■

We next recall from Section 4.1.2 that the Markovian $\mathtt{w}_n$-LIFO queueing system governed by $\mathscr{X}_{\mathtt{w}_n}$ induces a Galton-Watson forest $\mathbf{T}_{\mathtt{w}_n}$ with offspring distribution $\mu_{\mathtt{w}_n}$: informally, the clients are the vertices of $\mathbf{T}_{\mathtt{w}_n}$ and the server is the root (or the ancestor); the j-th client to enter the queue is a child of the i-th one if the j-th client enters when the i-th client is served; among siblings, the clients are ordered according to their time of arrival. We denote by $H_t^{\mathtt{w}_n}$ the number of clients waiting in the line right after time t; recall from (32) how $H^{\mathtt{w}_n}$ is derived from $X_n^{\mathtt{w}}$: namely, for all $s\leq t$, if one sets $I_t^{\mathtt{w}_n,s}=\inf_{r\in[s,t]}X_r^{\mathtt{w}_n}$, then:

$$H_t^{\mathtt{w}_n}=\#\big\{s\in[0,t]\;:\;I_t^{\mathtt{w}_n,s-}<I_t^{\mathtt{w}_n,s}\big\}. \tag{253}$$

We recall from Section 4.1.2 that $X^{\mathtt{w}_n}$ and $H^{\mathtt{w}_n}$ are close to respectively the Lukasiewicz path and the contour process of $\mathbf{T}_{\mathtt{w}_n}$. Therefore, the convergence results for Lukasiewicz paths and contours processes of Galton-Watson trees in Le Gall & D. [21] (and recalled in Appendix Theorem B.12, Section B.2.3) allow to prove the following theorem.

Theorem 6.20 *Let X be an integrable (α,β,π)-spectrally positive Lévy process, as defined at the beginning of Section 6.3.1. Assume that (249), that $\alpha\geq 0$ and that $\int^\infty dz/\psi_{\alpha,\beta,\pi}(z)<\infty$, where $\psi_{\alpha,\beta,\pi}$ is given by (244). Let $(H_t)_{t\in[0,\infty)}$ be the continuous height process derived from X as defined by (45).*

Let $\mathtt{w}_n\in\ell_f^{\downarrow}$ and $a_n,b_n\in(0,\infty)$, $n\in\mathbb{N}$, satisfy (248). Let $(Z_k^{(n)})_{k\in?\mathbb{N}}$ be a Galton-Watson process with offspring distribution $\mu_{\mathtt{w}_n}$ (defined by (245)), and initial state $Z_0^{(n)}=\lfloor a_n\rfloor$. Assume that the three conditions (IIIabc) *in Theorem 6.19 hold true and assume the following:*

$$\exists\,\delta\in(0,\infty),\qquad\liminf_{n\to\infty}\mathbf{P}\big(Z^{(n)}_{\lfloor b_n\delta/a_n\rfloor}=0\big)>0\,. \tag{254}$$

Then, the following joint convergence holds true:

$$\Big(\big(\tfrac{1}{a_n}X^{\mathtt{w}_n}_{b_nt}\big)_{t\in[0,\infty)},\big(\tfrac{a_n}{b_n}H^{\mathtt{w}_n}_{b_nt}\big)_{t\in[0,\infty)}\Big)\xrightarrow[n\to\infty]{}(X,H) \tag{255}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R})\times\mathbf{C}([0,\infty),\mathbb{R})$, equipped with the product topology. We also get:

$$\forall t\in[0,\infty),\quad\lim_{n\to\infty}\mathbf{P}\big(Z^{(n)}_{\lfloor b_nt/a_n\rfloor}=0\big)=e^{-v(t)}\quad\textit{where}\quad\int_{v(t)}^{\infty}\frac{dz}{\psi_{\alpha,\beta,\pi}(z)}=t. \tag{256}$$

Proof. Recall from Section 4.1.2 the definition of the Lukasiewicz path, the height and the contour process of $\mathbf{T}_{\mathtt{w}_n}$, that are respectively denoted by $(V_k^{\mathbf{T}_{\mathtt{w}_n}})_{k\in\mathbb{N}}$, $(\mathtt{Hght}_k^{\mathbf{T}_{\mathtt{w}_n}})_{k\in\mathbb{N}}$ and $(C_t^{\mathbf{T}_{\mathtt{w}_n}})_{t\in[0,\infty)}$. We first assume that (IIIabc) in Theorem 6.19 and that (254) hold true. Then, Theorem B.12 applies with $\mu_n := \mu_{\mathtt{w}_n}$: namely, the joint convergence (302) holds true and we get (256). Now recall from (119) the notation $N^{\mathtt{w}_n}(t) = \sum_{k\geq 1} \mathbf{1}_{[0,t]}(\tau_k^n)$ that is a homogeneous Poisson process with unit rate. Then, by Lemma B.6 (see Section B.1 in Appendix) the joint convergence (302) entails the following.

$$\mathscr{Q}_n(6) = \left(\tfrac{1}{a_n} V^{\mathbf{T}_{\mathtt{w}_n}}(N^{\mathtt{w}_n}_{b_n \cdot}),\ \tfrac{a_n}{b_n} \mathtt{Hght}^{\mathbf{T}_{\mathtt{w}_n}}(N^{\mathtt{w}_n}_{b_n \cdot}),\ \tfrac{a_n}{b_n} C^{\mathbf{T}_{\mathtt{w}_n}}_{b_n \cdot}\right) \xrightarrow[n\to\infty]{} \left(X, H, (H_{t/2})_{t\in[0,\infty)}\right)$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}) \times \mathbf{C}([0,\infty),\mathbb{R})^2$ equipped with the product topology. Here X is an integrable (α,β,π)-spectrally positive Lévy process (as defined at the beginning of Section 6.3.1) and H is the height process derived from X by (45). By Theorem 6.19, the laws of the processes $\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n \cdot}$ are tight in $\mathbf{D}([0,\infty),\mathbb{R})$. Thus, if one sets $\mathscr{Q}_n(7) = (\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n \cdot}, \mathscr{Q}_n(6))$, then the laws of the $\mathscr{Q}_n(7)$ are tight on $\mathbf{D}([0,\infty),\mathbb{R})^2 \times \mathbf{C}([0,\infty),\mathbb{R})^2$. Thus, to prove the weak convergence $\mathscr{Q}_n(7) \to (X, X, H, H_{\cdot/2}) := \mathscr{Q}(7)$, we only need to prove that the law of $\mathscr{Q}(7)$ is the unique limiting law: to that end, let $(n(p))_{p\in\mathbb{N}}$ be an increasing sequence of integers such that

$$\mathscr{Q}_{n(p)}(7) \xrightarrow[p\to\infty]{} \left(X', X, H, H_{\cdot/2}\right). \tag{257}$$

Actually, we only have to prove that $X' = X$. Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that (257) holds $\mathbf{P}$-almost surely. We next use (121) in Lemma 4.1: fix $t, \varepsilon, y \in (0,\infty)$, set $I^{\mathtt{w}_n}_t = \inf_{s\in[0,t]} X^{\mathtt{w}_n}_s$; by applying (121) at time $b_n t$, with $a = a_n \varepsilon$ and $x = a_n y$ we get

$$\mathbf{P}\left(\left|\tfrac{1}{a_n} V^{\mathbf{T}_{\mathtt{w}_n}}_{N^{\mathtt{w}_n}(b_n t)} - \tfrac{1}{a_n} X^{\mathtt{w}_n}_{b_n t}\right| > 2\varepsilon\right) \leq 1 \wedge \frac{4y}{\varepsilon^2 a_n} + \mathbf{P}\left(-\tfrac{1}{a_n} I^{\mathtt{w}_n}_{b_n t} > y\right) + \mathbf{E}\left[1 \wedge \frac{\frac{1}{a_n}(X^{\mathtt{w}_n}_{b_n t} - I^{\mathtt{w}_n}_{b_n t})}{\varepsilon^2 a_n}\right].$$

By Lemma B.3 (ii), $\frac{1}{a_{n(p)}}(X^{\mathtt{w}_{n(p)}}_{b_{n(p)} t} - I^{\mathtt{w}_{n(p)}}_{b_{n(p)} t}) \to X'_t - I'_t$ and $\frac{1}{a_{n(p)}} I^{\mathtt{w}_{n(p)}}_{b_{n(p)} t} \to I'_t$ almost surely, where we have set $I'_t = \inf_{s\in[0,t]} X'_s$. Thus,

$$\limsup_{p\to\infty} \mathbf{P}\left(\left|\tfrac{1}{a_{n(p)}} V^{\mathbf{T}_{\mathtt{w}_{n(p)}}}_{N^{\mathtt{w}_{n(p)}}(b_{n(p)} t)} - \tfrac{1}{a_{n(p)}} X^{\mathtt{w}_{n(p)}}_{b_{n(p)} t}\right| > 2\varepsilon\right) \leq \mathbf{P}\left(-I'_t > y/2\right) \xrightarrow[y\to\infty]{} 0$$

and (257) entails that for all $t \in [0,\infty)$ a.s. $X'_t = X_t$ and thus, a.s. $X' = X$.

We have proved that $\mathscr{Q}_n(7) \to (X, X, H, H_{\cdot/2}) := \mathscr{Q}(7)$, weakly on $\mathbf{D}([0,\infty),\mathbb{R})^2 \times \mathbf{C}([0,\infty),\mathbb{R})^2$. Without loss of generality (but with a slight abuse of notation), by Skorokod's representation theorem we can assume that the convergence holds true $\mathbf{P}$-almost surely. We next recall from (125) and (126) that:

$$M^{\mathtt{w}_n}(t) = 2N^{\mathtt{w}_n}(t) - H^{\mathtt{w}_n}_t, \quad C^{\mathbf{T}_{\mathtt{w}_n}}_{M^{\mathtt{w}_n}(t)} = H^{\mathtt{w}_n}_t \quad \text{and} \quad \sup_{s\in[0,t]} H^{\mathtt{w}_n}_s \leq 1 + \sup_{s\in[0,t]} \mathtt{Hght}^{\mathbf{T}_{\mathtt{w}_n}}_{N^{\mathtt{w}_n}(s)}.$$

Then, we fix $t, \varepsilon \in (0,\infty)$, and we apply (127) at time $b_n t$, with $a = b_n \varepsilon$ to get

$$\mathbf{P}\left(\sup_{s\in[0,t]} \left|\tfrac{1}{b_n} M^{\mathtt{w}_n}_{b_n s} - 2s\right| > 2\varepsilon\right) \leq 1 \wedge \frac{16t}{\varepsilon^2 b_n} + \mathbf{P}\left(1 + \sup_{s\in[0,t]} \tfrac{a_n}{b_n} \mathtt{Hght}^{\mathbf{T}_{\mathtt{w}}}_{N^{\mathtt{w}}(b_n s)} > \varepsilon a_n\right).$$

Since $\frac{a_n}{b_n} \mathtt{Hght}^{\mathbf{T}_{\mathtt{w}}}(N^{\mathtt{w}}(b_n \cdot)) \to H$ a.s. in $\mathbf{C}([0,\infty),\mathbb{R})$, it easily entails that $\frac{1}{2b_n} M^{\mathtt{w}_n}_{b_n \cdot}$ tends in probability to the identity map on $[0,\infty)$ in $\mathbf{C}([0,\infty),\mathbb{R})$. Since $H^{\mathtt{w}_n}_t = C^{\mathbf{T}_{\mathtt{w}_n}}(M^{\mathtt{w}_n}(t))$, and since $C^{\mathbf{T}_{\mathtt{w}_n}}(b_n \cdot) \to H(\cdot/2)$ a.s. in $\mathbf{C}([0,\infty),\mathbb{R})$, Lemma B.6 easily entails the joint convergence (255) weakly in $\mathbf{D}([0,\infty),\mathbb{R}) \times \mathbf{C}([0,\infty),\mathbb{R})$ equipped with the product topology. This completes the proof of the theorem. ■

As explained right after Theorem 2.3.1 in Le Gall & D. [21] (see Chapter 2, pp. 54-55) Assumption (254) is actually a necessary condition for the height process to converge. The following proposition provides a practical criterion implying (254).

Proposition 6.21 *Let X be a an integrable (α, β, π)-spectrally positive Lévy process, as defined at the beginning of Section 6.3.1. Assume that (α, β, π) satisfies (249), that $\alpha \geq 0$ and that $\int^{\infty} dz/\psi_{\alpha,\beta,\pi}(z) < \infty$, where $\psi_{\alpha,\beta,\pi}$ is given by (244). Let H be the continuous height process derived from X by (45). Let $\mathtt{w}_n \in \ell_f^{\downarrow}$ and $a_n, b_n \in (0, \infty)$, $n \in \mathbb{N}$, satisfy (248). Recall from (247) the definition of $X^{\mathtt{w}_n}$ and denote by ψ_n the Laplace exponent of $(\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n t})_{t\in[0,\infty)}$: namely, for all $\lambda \in [0, \infty)$,*

$$\psi_n(\lambda) = \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big)\lambda + \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} \frac{w_j^{(n)}}{a_n}\big(e^{-\lambda w_j^{(n)}/a_n} - 1 + \lambda\, w_j^{(n)}/a_n\big)\ . \tag{258}$$

We assume that the three conditions (IIIabc) *in Theorem 6.19 hold true. Then, (254) in Theorem 6.20 holds true if the following holds true,*

$$\lim_{y\to\infty} \limsup_{n\to\infty} \int_y^{a_n} \frac{d\lambda}{\psi_n(\lambda)} = 0\ . \tag{259}$$

To prove Proposition 6.21 we first prove a lemma that compares the total height of Galton-Watson trees with i.i.d. exponentially distributed edge-lengths and the total height of their discrete skeleton. More precisely, let $\rho \in (0, \infty)$ and let μ be a (sub)-critical offspring distribution whose generating function is denoted by $g_\mu(r) = \sum_{l\in\mathbb{N}} \mu(l) r^l$. Note that $g_\mu([0, 1]) \subset [0, 1]$; let $g_\mu^{\circ k}$ be the k-th iterate of g_μ, with the convention that $g_\mu^{\circ 0}(r) = r$, $r \in [0, 1]$. Let $\tau : \Omega \to \mathbb{T}$ be a random tree whose distribution is characterised as follows.

- The number of children of the ancestor (namely the r.v. $k_\varnothing(\tau)$) is a Poisson r.v. with mean ρ;
- For all $l \geq 1$, under $\mathbf{P}(\,\cdot\,|\, k_\varnothing(\tau) = l)$, the l subtrees $\theta_{[1]}\tau, \ldots, \theta_{[l]}\tau$ stemming from the ancestor $\varnothing$ are independent Galton-Watson trees with offspring distribution μ.

We next denote by Z_k the number of vertices of τ that are situated at height $k + 1$: namely, $Z_k = \#\{u \in \tau : |u| = k + 1\}$ (see Section 4.1.1 for the notation on trees). Then, $(Z_k)_{k\in\mathbb{N}}$ is a Galton-Watson process whose initial value Z_0 is distributed as a Poisson r.v. with mean ρ. We denote by $\Gamma(\tau)$ the total height of τ: namely, $\Gamma(\tau) = \max_{u\in\tau} |u|$ is the maximal graph-distance from the root $\varnothing$ and we get $(\Gamma(\tau) - 1)_+ = \max\{k \in \mathbb{N} : Z_k \neq 0\}$, with the convention that $\max \emptyset = 0$. Thus,

$$\mathbf{P}\big(\Gamma(\tau) < k + 1\big) = \mathbf{P}(Z_k = 0) = \exp\big(-\rho\big(1 - g_\mu^{\circ k}(0)\big)\big)\ . \tag{260}$$

We next equip each individual u of the family tree τ with an independent lifetime $e(u)$ that is distributed as follows.

- The lifetime $e(\varnothing)$ of $\varnothing$ is 0.
- Conditionnaly given τ, the r.v. $e(u)$, $u \in \tau\backslash\{\varnothing\}$ are independent and exponentially distributed r.v. with parameter $q \in (0, \infty)$.

Within our notation the *genealogical order* on τ is defined as follows: a vertex $v \in \tau$ is an ancestor of $u \in \tau$, which is denoted $v \preceq u$, if there exists $v' \in \mathbb{U}$ such that $u = v * v'$; $\preceq$ is a partial order on τ: it is the genealogical order. For all $u \in \tau$, we denote by $\zeta(u) = \sum_{\varnothing\preceq v\preceq u} e(v)$, the date of death of u; then, $\zeta(\overleftarrow{u})$ is the date of birth of u (recall here that $\overleftarrow{u}$ stands for the direct parent of u). For all $t \in [0, \infty)$, we next set $\mathtt{Z}_t = \sum_{u\in\tau\backslash\{\varnothing\}} \mathbf{1}_{[\zeta(\overleftarrow{u}),\zeta(u))}(t)$. Then $(\mathtt{Z}_t)_{t\in[0,\infty)}$ is a continuous-time Galton-Watson process (or a Harris process) with offspring distribution μ, with time parameter q and with Poisson(ρ)-initial distribution. We denote by $\Gamma = \max_{u\in\tau} \zeta(u)$ the extinction time of the population; then $\Gamma = \max\{t \in [0, \infty) : \mathtt{Z}_t \neq 0\}$. Standard results on continuous-time GW-processes imply the following. For all $t \in (0, \infty)$,

$$\mathbf{P}\big(\Gamma < t\big) = \mathbf{P}(\mathtt{Z}_t = 0) = e^{-\rho r(t)}, \quad \text{where} \quad \int_{r(t)}^1 \frac{dr}{g_\mu(1-r) - 1 + r} = qt\ . \tag{261}$$

For a formal proof, see for instance Athreya & Ney [5], Chapter III, Section 3, Equation (7) p. 106 and Section 4, Equation (1) p. 107.

We next compare $\Gamma(\tau)$ and Γ. To that end, we introduce $(\mathrm{e}_n)_{n\geq 1}$, a sequence of i.i.d. exponentially distributed r.v. with mean 1, and we set:

$$\forall\, \varepsilon \in (0,1), \quad \delta(\varepsilon) \;=\; \sup_{n\geq 1} \mathbf{P}\big(n^{-1}(\mathrm{e}_1+\ldots+\mathrm{e}_n) \notin (\varepsilon, \varepsilon^{-1})\big). \tag{262}$$

The law of large numbers easily implies that $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$. Note that $\mathtt{Z}_0=Z_0$. We first assume that $Z_0\neq 0$. Let $u^*\in\tau\backslash\{\varnothing\}$ be the first vertex in the lexicographical such that $|u^*|=\Gamma(\tau)$; since $\zeta(u^*)\leq\Gamma$ and since conditionally given τ, $\zeta(u^*)$ is the sum of $|u^*|$ (conditionally) independent exponential r.v. with parameter q, we get for all $t\in(0,\infty)$,

$$\mathbf{P}\big(\Gamma<t\,;\,\mathtt{Z}_0\neq 0\big)\leq\sum_{n\geq 1}\mathbf{P}\big(\,\Gamma(\tau)=n;\; Z_0\neq 0\,\big)\,\mathbf{P}\big(\mathrm{e}_1+\ldots+\mathrm{e}_n\leq qt\big)\;.$$

Then, let $\varepsilon\in(0,1)$ and observe that $\mathbf{P}\big(\mathrm{e}_1+\ldots+\mathrm{e}_n\leq qt\big)\leq\delta(\varepsilon)+\mathbf{1}_{\{n\leq qt/\varepsilon\}}$. Consequently,

$$\mathbf{P}\big(\Gamma<t\,;\,\mathtt{Z}_0\neq 0\big)\leq\delta(\varepsilon)+\mathbf{P}\big(\Gamma(\tau)\leq\lfloor qt/\varepsilon\rfloor\,;\,Z_0\neq 0\big).$$

If $\mathtt{Z}_0=Z_0=0$, $\Gamma=\Gamma(\tau)=0$, which implies that

$$\mathbf{P}\big(\Gamma<t\big)\leq\delta(\varepsilon)+\mathbf{P}\big(\Gamma(\tau)\leq\lfloor qt/\varepsilon\rfloor\big).$$

Thus by (261) and (260), we have proved the following lemma.

Lemma 6.22 *Let $\rho,q\in(0,\infty)$ and let μ be a (sub)-critical offspring distribution, whose generating function is denoted by g_μ; denote by $g_\mu^{\circ k}$ the k-th iterate of g_μ with the convention $g_\mu^{\circ k}(r)=r$, $r\in[0,1]$. For all $t\in(0,\infty)$, let $r(t)$ be such that*

$$\int_{r(t)}^{1}\frac{dr}{g_\mu(1-r)-1+r}=qt\;. \tag{263}$$

For all $\varepsilon\in(0,1)$, recall from (262) the definition of $\delta(\varepsilon)$. Then, the following holds true.

$$\forall\, t\in(0,\infty), \quad e^{-\rho r(t)}-\delta(\varepsilon)\;\leq\;\exp\big(-\rho\big(1-g_\mu^{\circ\lfloor tq/\varepsilon\rfloor}(0)\,\big)\big). \tag{264}$$

We are now ready to **prove Proposition 6.21.** Recall from (245) the definition of the offspring distribution $\mu_{\mathtt{w}_n}$. We apply Lemma 6.22 with $\mu=\mu_{\mathtt{w}_n}$, $\rho=a_n$, $q=b_n/a_n$ and we denote by $r_n(t)$ the solution of (263): the change of variable $\lambda=a_n r$ implies that $r_n(t)$ satisfies

$$\int_{a_n r_n(t)}^{a_n}\frac{d\lambda}{b_n\big(g_{\mu_{\mathtt{w}_n}}(1-\frac{\lambda}{a_n})-1+\frac{\lambda}{a_n}\big)}=t \tag{265}$$

Next, it is easy to check from (245) that $b_n\big(g_{\mu_{\mathtt{w}_n}}(1-\frac{\lambda}{a_n})-1+\frac{\lambda}{a_n}\big)=\psi_n(\lambda)$, where ψ_n is defined in (258). Then, Lemma 6.22 asserts for all $t\in(0,\infty)$ and for all $\varepsilon\in(0,1)$, that

$$e^{-a_n r_n(t)}-\delta(\varepsilon)\;\leq\;\exp\big(-a_n\big(1-g_{\mu_{\mathtt{w}_n}}^{\circ\lfloor tb_n/a_n\varepsilon\rfloor}(0)\,\big)\big)\quad\text{where}\quad\int_{a_n r_n(t)}^{a_n}\frac{d\lambda}{\psi_n(\lambda)}=t\;. \tag{266}$$

Next, fix $t\in(0,\infty)$ and set $C:=\limsup_{n\to\infty}a_n r_n(t)\in[0,\infty]$. Suppose that $C=\infty$. Then, there is an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}a_{n_k}r_{n_k}(t)=\infty$. Let $y\in(0,\infty)$; then, for all sufficiently large k, we have $a_{n_k}r_{n_k}(t)\geq y$, which entails

$$t=\int_{a_{n_k}r_{n_k}(t)}^{a_{n_k}}\frac{d\lambda}{\psi_{n_k}(\lambda)}\leq\int_{y}^{a_{n_k}}\frac{d\lambda}{\psi_{n_k}(\lambda)}$$

Thus, for all $y \in (0,\infty)$, $t \leq \limsup_{n\to\infty} \int_y^\infty d\lambda/\psi_n(\lambda)$, which contradicts Assumption (259). This proves that $C<\infty$. Since $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$, we can choose ε such that $\delta(\varepsilon)<\frac{1}{2}e^{-C}$; then, we set $\delta = t/\varepsilon$ and (266) implies that

$$\limsup_{n\to\infty} a_n\big(1-g_{\mu_{\mathtt{w}_n}}^{\circ\lfloor \delta b_n/a_n\rfloor}(0)\big) < \infty\ . \tag{267}$$

Recall that $(Z_k^{(n)})_{k\in\mathbb{N}}$ stands for a Galton-Watson branching process with offspring distribution $\mu_{\mathtt{w}_n}$ such that $Z_0^{(n)} = \lfloor a_n \rfloor$. Then,

$$\mathbf{P}\big(Z^{(n)}_{\lfloor \delta b_n/a_n\rfloor}=0\big) \;=\; \big(g_{\mu_{\mathtt{w}_n}}^{\circ\lfloor \delta b_n/a_n\rfloor}(0)\big)^{\lfloor a_n\rfloor}$$

and (267) easily implies that $\liminf_{n\to\infty}\mathbf{P}\big(Z^{(n)}_{\lfloor \delta b_n/a_n\rfloor}=0\big)>0$, which completes the proof of Proposition 6.21. ■

6.3.2 Proof of Propositions 2.11 and 2.12.

In this section we shall assume that the sequence $a_n, b_n \in (0,\infty)$ satisfy (248) **and** that $\frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \to \kappa \in (0,\infty)$. This dramatically restricts the possible limiting triplets (α,β,π). To see this point, we first prove the following lemma.

Lemma 6.23 *For all $n\in\mathbb{N}$, let $\mathrm{v}_n = (v_j^{(n)})_{j\geq 1} \in \ell_f^{\downarrow}$ and set $\phi_n(\lambda) = \sum_{j\geq 1} v_j^{(n)}\big(e^{-\lambda v_j^{(n)}} - 1 + \lambda v_j^{(n)}\big)$, for all $\lambda\in[0,\infty)$. Then, the following assertions are equivalent.*

- (L) *For all $\lambda\in[0,\infty)$, there exists $\phi(\lambda)\in[0,\infty)$ such that $\lim_{n\to\infty}\phi_n(\lambda)=\phi(\lambda)$.*
- (S) *There are $\mathbf{c}\in\ell_3^{\downarrow}$ and $\beta'\in[0,\infty)$ such that*

$$\forall j\in\mathbb{N}^*,\quad \lim_{n\to\infty} v_j^{(n)} = c_j \quad \textit{and} \quad \lim_{n\to\infty}\sigma_3(\mathrm{v}_n)-\sigma_3(\mathbf{c}) = \beta'.$$

Moreover, if (L) or (S) hold true, then β' is necessarily nonnegative and ϕ in (L) is given by

$$\forall\lambda\in[0,\infty),\quad \phi(\lambda) = \tfrac{1}{2}\beta'\lambda^2 + \sum_{j\geq 1} c_j\big(e^{-\lambda c_j} - 1 + \lambda c_j\big)\ . \tag{268}$$

Proof. We first prove $(S)\Rightarrow(L)$. We set $f(x)=e^{-x}-1+x$ for all $x\in[0,\infty)$. By elementary arguments,

$$\forall x\in[0,\infty)\quad 0\leq \tfrac{1}{2}x^2 - f(x) \leq \tfrac{1}{2}x^2(1-e^{-x}) \tag{269}$$

We set $\eta(x) = \sup_{y\in[0,x]} y^{-2}|\frac{1}{2}y^2 - f(y)|$; thus, $\eta(x)\leq\frac{1}{2}(1-e^{-x})\leq 1\wedge x$ and $\eta(x)\downarrow 0$ as $x\downarrow 0$.

Fix $\lambda\in[0,\infty)$ and define $\phi(\lambda)$ by (268). Fix $j_0\geq 2$ and observe the following.

$$\begin{aligned}\phi_n(\lambda)-\phi(\lambda) \;=\; & \sum_{1\leq j\leq j_0}\big(v_j^{(n)} f(\lambda v_j^{(n)}) - c_j f(\lambda c_j)\big) + \tfrac{1}{2}\lambda^2\Big(\sigma_3(\mathrm{v}_n)-\sigma_3(\mathbf{c})-\beta' + \sum_{1\leq j\leq j_0}\big(c_j^3 - (v_j^{(n)})^3\big)\Big)\\ & + \sum_{j>j_0}\big(v_j^{(n)} f(\lambda v_j^{(n)}) - \tfrac{1}{2}\lambda^2 (v_j^{(n)})^3\big) + \sum_{j>j_0}\big(\tfrac{1}{2}\lambda^2 c_j^3 - c_j f(\lambda c_j)\big).\end{aligned}$$

Then, note that:

$$\sum_{j>j_0}\big|v_j^{(n)} f(\lambda v_j^{(n)}) - \tfrac{1}{2}\lambda^2(v_j^{(n)})^3\big| \leq \lambda^2\eta\big(\lambda v_{j_0}^{(n)}\big)\sigma_3(\mathrm{v}_n)\ .$$

Similarly, $\sum_{j>j_0}\big|\frac{1}{2}\lambda^2 c_j^3 - c_j f(\lambda c_j)\big| \leq \lambda^2\eta\big(\lambda c_{j_0}\big)\sigma_3(\mathbf{c})$. Thus

$$\limsup_{n\to\infty}\big|\phi_n(\lambda)-\phi(\lambda)\big| \leq (\beta' + 2\sigma_3(\mathbf{c}))\lambda^2\eta\big(\lambda c_{j_0}\big) \xrightarrow[j_0\to\infty]{} 0\ ,$$

since $c_{j_0} \to 0$ as $j_0 \to \infty$. This proves (L) and (268).

Conversely, we assume (L). Note that $v_1^{(n)} f(v_1^{(n)}) \leq \phi_n(1)$. Thus, $x_0 = \sup_{n\in\mathbb{N}} v_1^{(n)} < \infty$. By (269), for all $y \in [0,x]$, $f(y) \geq \frac{1}{2}e^{-x}y^2$, which implies $\sigma_3(\mathtt{v}_n) \leq 2e^{x_0}\sup_{n\in\mathbb{N}}\phi_n(1) =: z_0$. Consequently, for all $n \in \mathbb{N}$, $(\sigma_3(\mathtt{v}_n), \mathtt{v}_n)$ belongs to the compact space $[0,z_0]\times[0,x_0]^{\mathbb{N}^*}$. Let $(q_n)_{n\in\mathbb{N}}$ be an increasing sequence of integers such that $\lim_{n\to\infty}\sigma_3(\mathtt{v}_{q_n}) = a$ and such that for all $j \geq 1$, $\lim_{n\to\infty} v_j^{(q_n)} = c'_j$. By Fatou, $\sigma_3(\mathbf{c}') \leq a$ and we then set $\beta' = a - \sigma_3(\mathbf{c}')$. By applying $(S) \Rightarrow (L)$ to $(\mathtt{v}_{q_n})_{n\in\mathbb{N}}$, we get $\phi(\lambda) = \frac{1}{2}\beta'\lambda^2 + \sum_{j\geq 1} c'_j\big(\exp(-\lambda c'_j) - 1 + \lambda c'_j\big)$, for all $\lambda \in [0,\infty)$. We easily show that it characterises β' and $\mathbf{c}'$. Thus, $((\sigma_3(\mathtt{v}_n), \mathtt{v}_n))_{n\in\mathbb{N}}$, has a unique limit point in $[0,z_0]\times[0,x_0]^{\mathbb{N}^*}$, which easily entails (S). ■

Lemma 6.24 *Let $\mathtt{w}_n \in \ell_f^\downarrow$ and $a_n, b_n \in (0,\infty)$, $n \in \mathbb{N}$, satisfy (66). Namely*

$$(270)\quad a_n \ \textit{and}\ \frac{b_n}{a_n} \xrightarrow[n\to\infty]{} \infty, \quad \frac{b_n}{a_n^2} \xrightarrow[n\to\infty]{} \beta_0 \in [0,\infty),$$
$$\sup_{n\in\mathbb{N}} \frac{w_1^{(n)}}{a_n} < \infty \quad \textit{and} \quad \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \xrightarrow[n\to\infty]{} \kappa \in (0,\infty).$$

Recall from (247) the definition of $X^{\mathtt{w}_n}$. Then the following assertions hold true.

(i) *Let us suppose that* (II) *in Theorem 6.19 holds true; namely, $\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n \cdot} \longrightarrow X$ weakly on $\mathbf{D}([0,\infty),\mathbb{R})$. Then, X is an integrable (α,β,π) spectrally Lévy process (as defined at the beginning of Section 6.3.1) and (α,β,π) necessarily satisfies:*

$$(271)\qquad \beta \geq \beta_0 \quad \textit{and} \quad \exists\, \mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^\downarrow : \ \pi = \sum_{j\geq 1} \kappa c_j \delta_{c_j}$$

and the following hold true:

$$(\mathbf{C1}): \ \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big) \xrightarrow[n\to\infty]{} \alpha \qquad (\mathbf{C2}): \ \frac{b_n}{a_n^2}\cdot\frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)} \xrightarrow[n\to\infty]{} \beta + \kappa\sigma_3(\mathbf{c}),$$
$$(\mathbf{C3}): \ \forall j \in \mathbb{N}^*, \quad \frac{w_j^{(n)}}{a_n} \xrightarrow[n\to\infty]{} c_j\,.$$

(ii) *Conversely,* $(\mathbf{C1})$–$(\mathbf{C3})$ *are equivalent to* (II), *and by Theorem 6.19, it is equivalent to* (I), *or to* (IIIabc) *or to* ((IIIa) & (IV)).

Proof. To simplify notation, we set $\kappa_n = a_n b_n/\sigma(\mathtt{w}_n)$. By the last point of (66) (that is recalled in (270)), $\kappa_n \to \kappa \in (0,\infty)$. We also set $v_j^{(n)} = w_j^{(n)}/a_n$ for all $j \geq 1$. We first prove (i), so we suppose Theorem 6.19 (II), which first implies that $\beta \geq \beta_0$; then recall that Theorem 6.19 (II) is equivalent to ($(\mathbf{C1})$ & (IV)) and Theorem 6.19 ((IV)) can be rewritten as follows: for all $\lambda \in [0,\infty)$,

$$\kappa_n \sum_{j\geq 1} v_j^{(n)}\big(e^{-\lambda v_j^{(n)}} - 1 + \lambda v_j^{(n)}\big) \xrightarrow[n\to\infty]{} \psi_{\alpha,\beta,\pi}(\lambda) - \alpha\lambda\,.$$

This entails Condition (L) in Lemma 6.23 with $\phi(\lambda) = (\psi_{\alpha,\beta,\pi}(\lambda) - \alpha\lambda)/\kappa$. Lemma 6.23 then implies that there are $\mathbf{c} \in \ell_3^\downarrow$ and $\beta' \in [0,\infty)$ such that for all $j \in \mathbb{N}^*$, $\lim_{n\to\infty} v_j^{(n)} = c_j$ and $\lim_{n\to\infty}\sigma_3(\mathtt{v}_n) - \sigma_3(\mathbf{c}) = \beta'$ and that

$$\tfrac{1}{2}\kappa^{-1}\beta\lambda^2 + \kappa^{-1}\int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\,\pi(dr) = \frac{\psi_{\alpha,\beta,\pi}(\lambda) - \alpha\lambda}{\kappa} = \phi(\lambda) = \tfrac{1}{2}\beta'\lambda^2 + \sum_{j\geq 1} c_j\big(e^{-\lambda c_j} - 1 + \lambda c_j\big)\,.$$

This easily entails that $\kappa\beta' = \beta$, $\pi = \sum_{j\geq 1}\kappa c_j\delta_{c_j}$ and we easily get $(\mathbf{C2})$ and $(\mathbf{C3})$.

We next prove (ii): we assume that $\alpha \in [0,\infty)$, that $\beta \geq \beta_0$ and that $\pi = \sum_{j\geq 1}\kappa c_j\delta_{c_j}$ where $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^\downarrow$. Then observe that $(\mathbf{C1})$ is (IIIa) in Theorem 6.19, that $(\mathbf{C2})$ is (IIIb) in Theorem

6.19; moreover, $(\mathbf{C3})$ easily entails (IIIc) in Theorem 6.19. Then Theorem 6.19 easily entails (ii). This completes the proof of the lemma. ■

Lemma 6.24 combined with Theorem 6.19 implies Proposition 2.11 (i), (ii) and (iii), and Lemma 6.24 combined with Theorem 6.20 implies Proposition 2.12.

It only remains to prove Proposition 2.11 (iv). Namely, fix $\alpha \in \mathbb{R}$, $\beta \in [0,\infty)$, $\kappa \in (0,\infty)$, and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$. We prove that there are sequences $a_n, b_n \in (0,\infty)$, $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, that satisfy (66) (recalled in (270)) with $\beta_0 \in [0,\beta]$ and $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$: first let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that $\lim_{n\to\infty} \rho_n = \infty$ and $\sum_{1\leq j\leq \rho_n} c_j + c_j^2 \leq n$, for all $n \geq c_1 + c_1^2$. We then define the following.

$$
q_j^{(n)} = \begin{cases} c_j & \text{if} \quad j \in \{1,\dots,\rho_n\}, \\ ((\beta-\beta_0)/\kappa)^{\frac{1}{3}} n^{-1} & \text{if} \quad j \in \{\rho_n+1,\dots,\rho_n+n^3\}, \\ u_n & \text{if} \quad j \in \{\rho_n+n^3+1,\dots,\rho_n+n^3+n^8\}, \\ \;\; 0 & \text{if} \quad j > \rho_n+n^3+n^8, \end{cases} \tag{272}
$$

where $u_n = n^{-3}$ if $\beta_0 = 0$ and $u_n = (\beta_0/\kappa)^{\frac{1}{3}} n^{-8/3}$ if $\beta_0 > 0$. We denote by $\mathtt{v}_n = (v_j^{(n)})_{j\geq 1}$ the non-increasing rearrangement of $\mathtt{q}_n = (q_j^{(n)})_{j\geq 1}$. Thus, we get $\sigma_p(\mathtt{v}_n) = \sigma_p(\mathtt{q}_n)$ for any $p \in (0,\infty)$ and we observe the following.

$$
\kappa\sigma_1(\mathtt{v}_n) \sim \begin{cases} \kappa n^5 & \text{if } \beta_0 = 0, \\ \kappa^{\frac{2}{3}} \beta_0^{\frac{1}{3}} n^{\frac{16}{3}} & \text{if } \beta_0 > 0, \end{cases} \tag{273}
$$

$$
\kappa\sigma_2(\mathtt{v}_n) \sim \begin{cases} \kappa n^2 & \text{if } \beta_0 = 0, \\ \kappa^{\frac{1}{3}} \beta_0^{\frac{2}{3}} n^{\frac{8}{3}} & \text{if } \beta_0 > 0, \end{cases} \quad \text{and} \quad \kappa\sigma_3(\mathtt{v}_n) \sim \kappa\sigma_3(\mathbf{c}) + \beta \,.
$$

We next set:

$$
b_n = \kappa\sigma_1(\mathtt{v}_n), \quad a_n = \frac{\kappa\sigma_1(\mathtt{v}_n)}{\kappa\sigma_2(\mathtt{v}_n) + \alpha} \quad \text{and} \quad w_j^{(n)} = a_n v_j^{(n)}, \; j \geq 1. \tag{274}
$$

We then see that $a_n b_n / \sigma_1(\mathtt{w}_n) = \kappa$, that $\sup_{n\in\mathbb{N}} w_1^{(n)}/a_n < \infty$. Moreover, we get

$$
\frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\Big) = \alpha, \quad \lim_{n\to\infty} \frac{b_n}{a_n^2} \cdot \frac{\sigma_3(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)} = \beta + \kappa\sigma_3(\mathbf{c}) \quad \text{and} \quad \lim_{n\to\infty} \frac{w_j^{(n)}}{a_n} = c_j
$$

which are the limits $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$. It is easy to derive from (273) and (274) that a_n and b_n/a_n tend to ∞ and that b_n/a_n^2 tends to β_0. This completes the proof of Proposition 2.11 (iv). ■

6.3.3 Proof of Proposition 2.13 (i).

Fix $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$. For all $\lambda \in [0,\infty)$, set $\psi(\lambda) = \alpha\lambda + \frac{1}{2}\beta\lambda^2 + \sum_{j\geq 1} \kappa c_j \big(e^{-\lambda c_j} - 1 + \lambda c_j\big)$ and we assume that $\int^{\infty} d\lambda/\psi(\lambda) < \infty$. Let $a_n, b_n \in (0,\infty)$ and $\mathtt{w}_n \in \ell_f^{\downarrow}$, $n \in \mathbb{N}$, satisfy (66) (recalled in (270)), $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$ (as recalled in Lemma 6.24). Recall from (247) the definition of $X^{\mathtt{w}_n}$; we denote by ψ_n the Laplace exponent of $\frac{1}{a_n} X^{\mathtt{w}_n}_{b_n \cdot}$. Namely,

$$
\forall \lambda \in [0,\infty), \quad \psi_n(\lambda) = \alpha_n \lambda + \frac{a_n b_n}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} \frac{w_j^{(n)}}{a_n} \big(e^{-\lambda w_j^{(n)}/a_n} - 1 + \lambda\, w_j^{(n)}/a_n\big) \,. \tag{275}
$$

where we have set $\alpha_n = \frac{b_n}{a_n}\big(1 - \frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)}\big)$. Proposition 6.21 and Lemma 6.24 prove that (75) Proposition 2.13 (i) entails $(\mathbf{C4})$.

It only remains to prove the last point of Proposition 2.13 (i): assume that $\beta_0 > 0$ in (66); let $V_n : \Omega \to [0,\infty)$ be a r.v. with law $\frac{1}{\sigma_1(\mathtt{w}_n)} \sum_{j\geq 1} w_j^{(n)} \delta_{w_j^{(n)}/a_n}$. First, observe the following.

$$\mathbf{E}[V_n] = \frac{\sigma_2(\mathtt{w}_n)}{a_n \sigma_1(\mathtt{w}_n)} = \frac{1}{a_n}\Big(1-\alpha_n \frac{a_n}{b_n}\Big) \quad \text{and} \quad \psi_n(\lambda) - \alpha_n \lambda = b_n \mathbf{E}\big[f\big(\lambda V_n\big)\big],$$

where we recall that $f(x) = e^{-x} - 1 + x$. Since f is convex and since $\alpha_n \geq 0$, by Jensen's inequality entails $\psi_n(\lambda) \geq b_n \mathbf{E}[f(\lambda V_n)] \geq b_n f(\lambda \mathbf{E}[V_n])$. Next, recall from (269) that $f(\lambda \mathbf{E}[V_n]) \geq \frac{1}{2}(\lambda \mathbf{E}[V_n])^2 \exp(-\lambda \mathbf{E}[V_n])$. Then, note that for all $\lambda \in [0, a_n]$, $\lambda \mathbf{E}[V_n] \leq 1$ and observe that $\mathbf{E}[V_n] \sim 1/a_n$ since $\alpha_n \to \alpha$ by $(\mathbf{C1})$. Therefore, there exists n_0 such that for all $n \geq n_0$ and for all $\lambda \in [0, a_n]$, $\psi_n(\lambda) \geq (8e)^{-1}(b_n/a_n^2)\lambda^2$. Since $\beta_0 > 0$, there exists $n_1 \geq n_0$ such that $b_n/a_n^2 \geq \beta_0/2$. Thus, we have proved that

$$\exists n_1 \in \mathbb{N} : \ \forall n \geq n_1, \ \forall \lambda \in [0, a_n], \quad \psi_n(\lambda) \geq \tfrac{1}{16e}\beta_0 \lambda^2 ,$$

which clearly implies (75). This completes the proof of Proposition 2.13 (i). ■

6.3.4 Proof of Proposition 2.13 (ii).

Let us mention that, here, we closely follows the counterexample given in Le Gall & D. [21], p. 55. Fix $\alpha, \beta \in [0,\infty)$, $\kappa \in (0,\infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$. For all $\lambda \in [0,\infty)$, set $\psi(\lambda) = \alpha\lambda + \frac{1}{2}\beta\lambda^2 + \sum_{j\geq 1} \kappa c_j \big(e^{-\lambda c_j} - 1 + \lambda c_j\big)$; assume that $\int^{\infty} d\lambda/\psi(\lambda) < \infty$. For all positive integers n, we next define $\mathbf{c}_n = (c_j^{(n)})_{j\geq 1}$ by setting

$$c_j^{(n)} = c_j \ \text{ if } j \leq n, \quad c_j^{(n)} = (\beta/(\kappa n))^{\frac{1}{3}} \ \text{ if } n < j \leq 2n \quad \text{and} \quad c_j^{(n)} = 0 \ \text{ if } j > n.$$

We also set $\psi_n(\lambda) = \alpha\lambda + \sum_{j\geq 1} \kappa c_j^{(n)} \big(\exp(-\lambda c_j^{(n)}) - 1 + \lambda c_j^{(n)}\big)$, $\lambda \in [0,\infty)$. Let $(U_t^n)_{t\in[0,\infty)}$, be a CSBP with branching mechanism ψ_n and with initial state $U_0^n = 1$. As $\lambda \to \infty$, observe that $\psi_n(\lambda) \sim (\alpha + \kappa\sigma_2(\mathbf{c}_n))\lambda$. Thus, $\int^{\infty} d\lambda/\psi_n(\lambda) = \infty$; by standard results on CSBP (recalled in Section B.2.2 in Appendix), we therefore get

$$\forall n \in \mathbb{N}, \ \forall t \in [0,\infty), \quad \mathbf{P}\big(U_t^n > 0\big) = 1 . \tag{276}$$

Let $Z = (Z_t)_{t\in[0,\infty)}$ stands for a CSBP with branching mechanism ψ and with initial state $Z_0 = 1$. Observe that for all $\lambda \in [0,\infty)$, $\lim_{n\to\infty} \psi_n(\lambda) = \psi(\lambda)$. By standard results on CSBP (see Helland [28], Theorem 6.1, p. 96), we get

$$U^n \xrightarrow[n\to\infty]{} Z \ \text{ weakly on } \mathbf{D}([0,\infty), \mathbb{R}). \tag{277}$$

Next, let us fix $n \in \mathbb{N}$. By Proposition 2.11 (iv) there exist sequences $\mathtt{w}_{n,p} = (w_j^{(n,p)})_{j\geq 1} \in \ell_f^{\downarrow}$ and $a_{n,p}, b_{n,p} \in (0,\infty)$, $p \in \mathbb{N}$, such that

$$\frac{a_{n,p} b_{n,p}}{\sigma_1(w_{n,p})} \to \kappa, \quad a_{n,p}, \frac{b_{n,p}}{a_{n,p}} \text{ and } \frac{a_{n,p}^2}{b_{n,p}} \xrightarrow[p\to\infty]{} \infty, \quad \frac{b_{n,p}}{a_{n,p}}\Big(1 - \frac{\sigma_2(\mathtt{w}_{n,p})}{\sigma_1(\mathtt{w}_{n,p})}\Big) \xrightarrow[p\to\infty]{} \alpha \tag{278}$$

$$\frac{b_{n,p}}{a_{n,p}^2} \cdot \frac{\sigma_3(\mathtt{w}_{n,p})}{\sigma_1(\mathtt{w}_{n,p})} \xrightarrow[p\to\infty]{} \kappa\sigma_3(\mathbf{c}_n) \quad \text{and} \quad \forall j \in \mathbb{N}^*, \quad \frac{w_j^{(n,p)}}{a_{n,p}} \xrightarrow[p\to\infty]{} c_j^{(n)}, \tag{279}$$

and the following weak limit holds true on $\mathbf{D}([0,\infty), \mathbb{R})$:

$$\Big(\tfrac{1}{a_{n,p}} Z^{(n,p)}_{\lfloor b_{n,p} t / a_{n,p} \rfloor}\Big)_{t\in[0,\infty)} \xrightarrow[p\to\infty]{} (U_t^n)_{t\in[0,\infty)} \tag{280}$$

where $(Z_k^{(n,p)})_{k\in\mathbb{N}}$ is a Galton-Watson process with initial state $Z_0^{(n,p)} = \lfloor a_{n,p} \rfloor$ and with offspring distribution $\mu_{\mathtt{w}_{n,p}}$ given by

$$\forall k\in\mathbb{N}, \quad \mu_{\mathtt{w}_{n,p}}(k) = \sum_{j\geq 1} \frac{(w_j^{(n,p)})^{k+1}}{\sigma_1(\mathtt{w}_{n,p})\, k!}\, e^{-w_j^{(n,p)}} . \tag{281}$$

By Portemanteau's theorem for all $t\in[0,\infty)$, $\liminf_{p\to\infty} \mathbf{P}\big(Z^{(n,p)}_{\lfloor b_{n,p}t/a_{n,p}\rfloor} > 0\big) \geq \mathbf{P}(U_t^n > 0) = 1$, by (276). Thus, there exists $p_n \in \mathbb{N}$, such that

$$\forall p\geq p_n, \qquad \mathbf{P}\big(Z^{(n,p)}_{\lfloor b_{n,p}n/a_{n,p}\rfloor} > 0\big) \geq 1 - 2^{-n} . \tag{282}$$

Without loss of generality we can furthermore assume the following:

$$a_{n,p_n}, \ \frac{b_{n,p_n}}{a_{n,p_n}} \text{ and } \frac{a^2_{n,p_n}}{b_{n,p_n}} \geq 2^n, \quad \Big|\frac{b_{n,p_n}}{a_{n,p_n}}\Big(1 - \frac{\sigma_2(\mathtt{w}_{n,p_n})}{\sigma_1(\mathtt{w}_{n,p_n})}\Big) - \alpha\Big| \leq 2^{-n}, \quad \Big|\frac{a_{n,p}b_{n,p}}{\sigma_1(w_{n,p})} - \kappa\Big| \leq 2^{-n}$$

$$\Big|\frac{b_{n,p_n}}{a^2_{n,p_n}} \cdot \frac{\sigma_3(\mathtt{w}_{n,p_n})}{\sigma_1(\mathtt{w}_{n,p_n})} - \kappa\sigma_3(\mathbf{c}_n)\Big| \leq 2^{-n} \quad \text{and} \quad \forall j\in\{1,\dots,n\}, \ \Big|\frac{w_j^{(n,p_n)}}{a_{n,p_n}} - c_j^{(n)}\Big| \leq 2^{-n}.$$

Set $a_n = a_{n,p_n}$, $b_n = b_{n,p_n}$ and $\mathtt{w}_n = \mathtt{w}_{n,p_n}$. Note that $\kappa\sigma_3(\mathbf{c}_n) \to \beta + \kappa\sigma_3(\mathbf{c})$ as $n\to\infty$. Thus, a_n, b_n and $\mathtt{w}_n$ satisfy satisfy (66) (recalled in (270)) with $\beta_0 = 0$, $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$. Set $Z_k^{(n)} = Z_k^{(n,p_n)}$. By (282), for all $\delta\in(0,\infty)$, and all integers $n\geq\delta$, we easily get $\mathbf{P}\big(Z^{(n)}_{\lfloor b_n\delta/a_n\rfloor} = 0\big) \leq \mathbf{P}\big(Z^{(n)}_{\lfloor b_n n/a_n\rfloor} = 0\big) \leq 2^{-n}$. Consequently, $\lim_{n\to\infty} \mathbf{P}\big(Z^{(n)}_{\lfloor b_n\delta/a_n\rfloor} = 0\big) = 0$, for all $\delta\in(0,\infty)$. Namely, $(\mathbf{C4})$ is not satisfied, which completes the proof of Proposition 2.13 (ii). ∎

6.3.5 Proof of Proposition 2.13 (iii).

Fix $\alpha,\beta\in[0,\infty)$, $\kappa\in(0,\infty)$ and $\mathbf{c} = (c_j)_{j\geq 1} \in \ell_3^{\downarrow}$. For all $\lambda\in[0,\infty)$, set $\psi(\lambda) = \alpha\lambda + \frac{1}{2}\beta\lambda^2 + \sum_{j\geq 1}\kappa c_j\big(e^{-\lambda c_j} - 1 + \lambda c_j\big)$; assume that $\int^{\infty} d\lambda/\psi(\lambda) < \infty$. We consider several cases.

• **Case 1**: we first assume that $\beta \geq \beta_0 > 0$. By Proposition 2.11 (iv) there exists $a_n, b_n, \mathtt{w}_n$ satisfying (66) (recalled in (270)) with $\beta_0 > 0$, $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$. But Proposition 2.13 (i) (proved in Section 6.3.3) asserts that $a_n, b_n, \mathtt{w}_n$ necessarily satisfy $(\mathbf{C4})$. This proves Proposition 2.13 (iii) in **Case 1**.

• **Case 2**: we next assume that $\beta > 0$ and $\beta_0 = 0$, and we set:

$$q_j^{(n)} = \begin{cases} c_j & \text{if } \ j\in\{1,\dots,n\}, \\ (\beta/\kappa)^{\frac{1}{3}} n^{-1} & \text{if } \ j\in\{n+1,\dots,n+n^3\}, \\ n^{-3} & \text{if } \ j\in\{n+n^3+1,\dots,n+n^3+n^8\}, \\ \ 0 & \text{if } \ j > n+n^3+n^8. \end{cases} \tag{283}$$

Denote by $\mathtt{v}_n = (v_j^{(n)})_{j\geq 1}$ the non-increasing rearrangement of $\mathtt{q}_n = (q_j^{(n)})_{j\geq 1}$. Thus, $\sigma_p(\mathtt{v}_n) = \sigma_p(\mathtt{q}_n)$ for any $p\in(0,\infty)$. Since $\sum_{1\leq j\leq n} c_j^p \leq c_1^p n$, we easily get $\kappa\sigma_1(\mathtt{v}_n) \sim \kappa n^5$, $\kappa\sigma_2(\mathtt{v}_n) \sim \kappa n^2$ $\kappa\sigma_3(\mathtt{v}_n) \to \beta + \kappa\sigma_3(\mathbf{c})$. We next set $b_n = \kappa\sigma_1(\mathtt{v}_n)$, $a_n = \kappa\sigma_1(\mathtt{v}_n)/(\kappa\sigma_2(\mathtt{v}_n) + \alpha)$ and for all $j\geq 1$, $w_j^{(n)} = a_n v_j^{(n)}$. Note that $a_n \sim n^3$. Then, it is easy to check that a_n, b_n and $\mathtt{w}_n$ satisfy (66) (recalled in (270)) with $\beta_0 = 0$, $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C3})$. Here observe that $\kappa = a_n b_n/\sigma_1(\mathtt{w}_n)$ and $b_n\big(1 - (\sigma_2(\mathtt{w}_n)/\sigma_1(\mathtt{w}_n))\big)/a_n = \alpha$.

We next prove that $(\mathbf{C4})$ holds true by proving that (75) in Proposition 2.13 (i) holds true. To that end, we introduce $f_\lambda(x) = x\big(e^{-\lambda x} - 1 + \lambda x\big)$, for all $x,\lambda\in[0,\infty)$, and we recall from (247) the definition of $X^{\mathtt{w}_n}$; we denote by ψ_n the Laplace exponent of $\frac{1}{a_n}X^{\mathtt{w}_n}_{b_n\cdot}$. We first observe that for all $\lambda\in[0,\infty)$

$$\psi_n(\lambda) = \alpha\lambda + \kappa\sum_{j\geq 1} f_\lambda(q_j^{(n)}) = \alpha\lambda + \kappa\sum_{1\leq j\leq n} f_\lambda(c_j) + \kappa n^3 f_\lambda((\beta/\kappa)^{\frac{1}{3}} n^{-1}) + \kappa n^8 f_\lambda(n^{-3}). \tag{284}$$

Set $s_0 = (\beta/\kappa)^{1/3}$ and recall from (269) that $f_\lambda(x) \geq \frac{1}{2}x^3\lambda^2 e^{-\lambda x}$. Thus, if $\lambda \in [1, 2n/s_0]$, then

$$\psi_n(\lambda) \geq \kappa n^3 f_\lambda((\beta/\kappa)^{\frac{1}{3}} n^{-1}) \geq \tfrac{1}{2} e^{-2}\beta\lambda^2 =: s_1\lambda^2.$$

Next observe that, $f_\lambda(x) \geq x(\lambda x - 1)$; thus, if $\lambda \in [2n/s_0, n^3]$, then

$$\psi_n(\lambda) \geq \kappa n^3 f_\lambda((\beta/\kappa)^{\frac{1}{3}} n^{-1}) \geq \kappa s_0 n^2 (s_0 n^{-1}\lambda - 1).$$

Thus, for all $y > 1$,

$$\int_y^{n^3} \frac{d\lambda}{\psi_n(\lambda)} \leq \int_y^{\frac{2n}{s_0}} \frac{d\lambda}{s_1\lambda^2} + \int_{\frac{2n}{s_0}}^{n^3} \frac{d\lambda}{\kappa s_0 n^2 \big(s_0 n^{-1}\lambda - 1\big)} \leq \frac{1}{s_1 y} + \frac{\log(s_0 n^2 - 1)}{\kappa s_0^2 n}.$$

Since $a_n \sim n^3$, it proves that ψ_n satisfies (75), and (**C4**) holds true. This proves Proposition 2.13 (iii) in **Case 2**.

• **Case 3**: We now assume that $\beta = \beta_0 = 0$. Let $\beta_n \in (0, \infty)$ be a sequence decreasing to 0. For all $n \in \mathbb{N}^*$, we set $\Psi_n(\lambda) = \psi(\lambda) + \frac{1}{2}\beta_n\lambda^2 = \alpha\lambda + \frac{1}{2}\beta_n\lambda^2 + \sum_{j\geq 1} \kappa c_j \big(e^{-\lambda c_j} - 1 + \lambda c_j\big)$. We now fix $n \in \mathbb{N}^*$; by **Case 2**, there exists $\mathtt{w}_{n,p} = (w_j^{(n,p)})_{j\geq 1} \in \ell_f^\downarrow$ and $a_{n,p}, b_{n,p} \in (0, \infty)$, $p \in \mathbb{N}$, that satisfy

$$\text{(285)} \qquad \frac{a_{n,p} b_{n,p}}{\sigma_1(w_{n,p})} = \kappa, \quad a_{n,p}, \frac{b_{n,p}}{a_{n,p}} \text{ and } \frac{a_{n,p}^2}{b_{n,p}} \xrightarrow[p\to\infty]{} \infty, \quad \frac{b_{n,p}}{a_{n,p}}\bigg(1 - \frac{\sigma_2(\mathtt{w}_{n,p})}{\sigma_1(\mathtt{w}_{n,p})}\bigg) = \alpha$$

$$\text{(286)} \qquad \frac{b_{n,p}}{a_{n,p}^2} \cdot \frac{\sigma_3(\mathtt{w}_{n,p})}{\sigma_1(\mathtt{w}_{n,p})} \xrightarrow[p\to\infty]{} \beta_n + \kappa\sigma_3(\mathbf{c}) \quad \text{and} \quad \forall j \in \mathbb{N}^*, \quad \frac{w_j^{(n,p)}}{a_{n,p}} \xrightarrow[p\to\infty]{} c_j.$$

and

$$\text{(287)} \quad \forall n \in \mathbb{N}^*, \ \forall t \in [0, \infty), \quad \lim_{n\to\infty} \mathbf{P}\big(Z^{(n,p)}_{\lfloor b_{n,p} t / a_{n,p} \rfloor} = 0\big) = e^{-v_n(t)} \quad \text{where} \quad \int_{v_n(t)}^{\infty} \frac{d\lambda}{\Psi_n(\lambda)} = t.$$

Here, $(Z_k^{(n,p)})_{k\in\mathbb{N}}$ is a Galton-Watson process with offspring distribution $\mu_{\mathtt{w}_{n,p}}$ given by (245) and where $Z_0^{(n,p)} = \lfloor a_{n,p} \rfloor$. Let $v : (0, \infty) \to (0, \infty)$ be such that $t = \int_{v(t)}^\infty d\lambda/\psi(\lambda)$ for all $t \in (0, \infty)$. Since $\Psi_n(\lambda) \geq \psi(\lambda)$, we get $\int_{v(t)}^\infty d\lambda/\psi(\lambda) = t \leq \int_{v_n(t)}^\infty d\lambda/\psi(\lambda)$; thus $v_n(t) \leq v(t)$. Thus, there exists $p_n \in \mathbb{N}$ such that for all $p \geq p_n$, $\mathbf{P}\big(Z^{(n,p)}_{\lfloor b_{n,p}/a_{n,p} \rfloor} = 0\big) \geq \frac{1}{2}\exp(-v_n(1)) \geq \frac{1}{2}\exp(-v(1))$. Without loss of generality, we can assume that $a_{n,p_n}, b_{n,p_n}/a_{n,p_n}$ and $a_{n,p_n}^2/b_{n,p_n} \geq 2^n$, that for all $1 \leq j \leq n$, $|w_j^{(n,p_n)}/a_{n,p_n} - c_j| \leq 2^{-n}$ and

$$\bigg|\frac{b_{n,p_n}}{a_{n,p_n}^2} \cdot \frac{\sigma_3(\mathtt{w}_{n,p_n})}{\sigma_1(\mathtt{w}_{n,p_n})} - \kappa\sigma_3(\mathbf{c})\bigg| \leq 2\beta_n \longrightarrow 0.$$

If one set $a_n = a_{n,p_n}$, b_{n,p_n} and $\mathtt{w}_n = \mathtt{w}_{n,p_n}$, then we have proved that $a_n, b_n, \mathtt{w}_n$ satisfy (66) (recalled in (270)) with $\beta = \beta_0 = 0$, and (**C1**)–(**C4**), which proves Proposition 2.13 (iii) in **Case 3**. This completes the proof of Proposition 2.13 (iii). ■

7 Proof of Lemma 2.18.

We first prove the following lemma.

Lemma 7.1 *Let $\ell : (0, 1] \to (0, \infty)$ be a measurable slowly varying function such that for all $x_0 \in (0, 1)$, $\sup_{x\in[x_0,1]} \ell(x) < \infty$. Then, for all $\delta \in (0, \infty)$, there exist $\eta_\delta \in (0, 1]$ and $c_\delta \in (1, \infty)$ such that*

$$\text{(288)} \qquad \forall y \in (0, \eta_\delta), \ \forall z \in (y, 1], \quad \frac{1}{c_\delta}\Big(\frac{z}{y}\Big)^{-\delta} \leq \frac{\ell(z)}{\ell(y)} \leq c_\delta \Big(\frac{z}{y}\Big)^{\delta}.$$

Proof. The measurable version of the representation theorem for slowly varying functions (see for instance [16]) implies that there exist two measurable functions $c: (0,1] \to \mathbb{R}$ and $\varepsilon: (0,1] \to [-1,1]$ such that $\lim_{x\to 0+} c(x) = \gamma \in \mathbb{R}$, such that $\lim_{x\to 0+} \varepsilon(x) = 0$, and such that $\ell(x) = \exp(c(x) + \int_x^1 ds\, \varepsilon(s)/s)$, for all $x \in (0,1]$. Since, $\sup_{x\in[x_0,1]} \ell(x) < \infty$, for all $x_0 \in (0,1)$, we can assume without loss of generality that c is bounded. Fix $\delta \in (0,\infty)$ and let $\eta_\delta \in (0,1]$ be such that $\sup_{(0,\eta_\delta]} |\varepsilon| \le \delta$. Fix $y \in (0,\eta_\delta)$ and $z \in (y,1]$; if $z \le \eta_\delta$, then note that $\int_y^z ds\, |\varepsilon(s)|/s \le \delta \log(z/y)$; if $\eta_\delta \le z$, then observe that $\int_y^z ds\, |\varepsilon(s)|/s \le \delta \log(\eta_\delta/y) + \int_{\eta_\delta}^1 ds\, |\varepsilon(s)|/s \le \delta \log(z/y) + \log(1/\eta_\delta)$. Thus

$$\eta_\delta e^{-2\|c\|_\infty} \Big(\frac{z}{y}\Big)^{-\delta} \le \frac{\ell(z)}{\ell(y)} = \exp\Big(c(z) - c(y) - \int_y^z ds\, \frac{\varepsilon(s)}{s}\Big) \le \eta_\delta^{-1} e^{2\|c\|_\infty} \Big(\frac{z}{y}\Big)^{\delta},$$

which implies the desired result. ■

Recall from (84) that $W: \Omega \to [0,\infty)$ is a r.v. such that $r := \mathbf{E}[W] = \mathbf{E}[W^2] < \infty$ and such that $\mathbf{P}(W \ge x) = x^{-\rho} L(x)$ where L is a slowly varying function at ∞ and $\rho \in (2,3)$. Recall from (85) that for all $y \in [0,\infty)$, we have set $G(y) = \sup\{x \in [0,\infty) : \mathbf{P}(W \ge x) \ge 1 \wedge y\}$. Note that G is non increasing and that it is null on $[1,\infty)$. Then, $G(y) = y^{-1/\rho} \ell(y)$, where ℓ is slowly varying at 0. Recall from (87) that $\kappa, q \in (0,\infty)$ and that $a_n \sim q^{-1} G(1/n)$, $w_j^{(n)} = G(j/n)$, $j \ge 1$, and $b_n \sim \kappa \sigma_1(\mathtt{w}_n)/a_n$.

Fix $a \in [1,2]$ and observe that $\sigma_a(\mathtt{w}_n) = \sum_{1\le j<n} \int_0^{G(1/n)} dz\, a z^{a-1} \mathbf{1}_{\{z \le G(j/n)\}}$. But observe that $z < G(y)$ implies $y \le P(W \ge z)$, which implies $z \le G(y)$. Thus,

$$\begin{aligned}
\sigma_a(\mathtt{w}_n) &= \sum_{1\le j<n} \int_0^{G(1/n)} dz\, a z^{a-1} \mathbf{1}_{\{j \le n\mathbf{P}(W\ge z)\}} = \int_0^{G(1/n)} dz\, a z^{a-1} \sum_{1\le j<n} \mathbf{1}_{\{j \le n\mathbf{P}(W\ge z)\}} \\
&= \int_0^{G(1/n)} dz\, a z^{a-1} \lfloor n\mathbf{P}(W \ge z) \rfloor = \int_0^{G(1/n)} dz\, a z^{a-1} n\mathbf{P}(W\ge z) - \int_0^{G(1/n)} dz\, a z^{a-1} \{n\mathbf{P}(W \ge z)\} \\
&= n \int_0^\infty dz\, a z^{a-1} \mathbf{P}(W \ge z) - \int_{G(1/n)}^\infty dz\, a z^{a-1} n\mathbf{P}(W\ge z) - \int_0^{G(1/n)} dz\, a z^{a-1} \{n\mathbf{P}(W \ge z)\}.
\end{aligned} \tag{289}$$

Note that $\int_0^\infty dz\, a z^{a-1} \mathbf{P}(W \ge z) = \mathbf{E}[W^a] < \infty$. Recall from (86) that $\mathbf{P}(W = G(1/n)) = 0$, which easily implies that $\mathbf{P}(W \ge G(1/n)) = 1/n$. Thus,

$$n\mathbf{P}(W \ge z) = \mathbf{P}(W \ge z)/\mathbf{P}(W \ge G(1/n)) = (z/G(1/n))^{-\rho} L(z)/L(G(1/n))$$

and by (289) and the change of variable $z \mapsto z/G(1/n)$, we get

$$\sigma_a(\mathtt{w}_n) = n\mathbf{E}[W^a] - G(\tfrac{1}{n})^a \int_1^\infty dz\, a z^{a-1-\rho} \frac{L(zG(\frac{1}{n}))}{L(G(\frac{1}{n}))} - G(\tfrac{1}{n})^a \int_0^1 dz\, a z^{a-1} \Big\{ z^{-\rho} \frac{L(zG(\frac{1}{n}))}{L(G(\frac{1}{n}))} \Big\}.$$

The measurable version of the representation theorem for slowly varying functions (see for instance [16]) implies that there exist two measurable functions $c: (0,\infty) \to \mathbb{R}$ and $\varepsilon: (0,\infty) \to [-1,1]$ such that $\lim_{x\to\infty} c(x) = \gamma \in \mathbb{R}$, such that $\lim_{x\to\infty} \varepsilon(x) = 0$, and such that $L(x) = \exp(c(x) + \int_1^x ds\, \varepsilon(s)/s)$, for all $x \in (0,\infty)$. We then set $u = (\rho - a)/2$ that is a strictly positive quantity since $a \le 2 < \rho$. Let n_0 be such that for all $n \ge n_0$, $\sup_{s\in[1,\infty)} |\varepsilon(sG(1/n))| \le u$. Thus, for all $z \in [1,\infty)$,

$$0 \le z^{a-1-\rho} \frac{L(zG(\frac{1}{n}))}{L(G(\frac{1}{n}))} = z^{a-1-\rho} \exp\Big(c(zG(\tfrac{1}{n})) - c(G(\tfrac{1}{n})) + \int_1^z ds \frac{\varepsilon(sG(\frac{1}{n}))}{s}\Big) \le e^{2\|c\|_\infty} z^{-1-u}$$

Since for all $z \in [1,\infty)$, $L(zG(1/n))/L(G(1/n)) \to 1$, dominated convergence entails:

$$\lim_{n\to\infty} \int_1^\infty dz\, a z^{a-1-\rho} \frac{L(zG(\frac{1}{n}))}{L(G(\frac{1}{n}))} = \frac{a}{\rho - a} \quad \text{and} \quad \lim_{n\to\infty} \int_0^1 dz\, a z^{a-1} \Big\{ z^{-\rho} \frac{L(zG(\frac{1}{n}))}{L(G(\frac{1}{n}))} \Big\} = \int_0^1 dz\, a z^{a-1} \{z^{-\rho}\}.$$

We then set $Q_a = a/(\rho - a) + \int_0^1 dz\, a z^{a-1}\{z^{-\rho}\}$ and since $a_n \sim q^{-1}G(1/n)$, we have proved that

$$\sigma_a(\mathtt{w}_n) = n\mathbf{E}[W^a] - q^a Q_a (a_n)^a + o((a_n)^a). \tag{290}$$

Recall that $r = \mathbf{E}[W] = \mathbf{E}[W^2]$ and take (290) with $a = 1$ to get $\sigma_1(\mathtt{w}_n) - rn \sim -Q_1 n^{1/\rho}\ell(1/n)$ since $a_n \sim q^{-1}n^{1/\rho}\ell(1/n)$; thus $b_n \sim \kappa q r n^{1-1/\rho}/\ell(1/n)$. It implies that a_n and b_n/a_n go to ∞ and that $b_n/a_n^2 \to 0$. Moreover for all $j \geq 1$, $w_j^{(n)}/a_n \to q j^{-1/\rho}$. This implies that a_n, b_n and $\mathtt{w}_n$ satisfy (66) with $\beta_0 = 0$ (and **(C3)**). Since $a_n b_n \sim \kappa \sigma_1(\mathtt{w}_n) \sim \kappa r n$, (290) with $a = 1$ and 2 implies

$$\frac{\sigma_2(\mathtt{w}_n)}{\sigma_1(\mathtt{w}_n)} = \frac{nr - q^2 Q_2 a_n^2 + o(a_n^2)}{nr - qQ_1 a_n + o(a_n)} = 1 - \kappa q^2 Q_2 \frac{a_n}{b_n} + o\Big(\frac{a_n}{b_n}\Big) = 1 - \alpha_0 \frac{a_n}{b_n} + o\Big(\frac{a_n}{b_n}\Big)\,, \tag{291}$$

where $\alpha_0 = \kappa q^2 Q_2$ as defined in (88).

Next recall that for all $\alpha \in [0, \infty)$, $w_j^{(n)}(\alpha) = (1 - \frac{a_n}{b_n}(\alpha - \alpha_0)) w_j^{(n)}$. Then, (291) entails that $\sigma_2(\mathtt{w}_n(\alpha))/\sigma_1(\mathtt{w}_n(\alpha)) = 1 - \alpha a_n/b_n + o(a_n/b_n)$. Namely, $\mathtt{w}_n(\alpha)$ satisfies **(C1)**. Since $w_j^{(n)}(\alpha) \sim_n w_j^{(n)}$, $\mathtt{w}_n(\alpha)$ also satisfies **(C3)** with $c_j = q j^{-1/\rho}$, $j \geq 1$.

Let us proves that $(\mathtt{w}_n(\alpha))$ satisfies **(C2)**. First observe that $\sigma_3(\mathtt{w}_n(\alpha)) \sim \sigma_3(\mathtt{w}_n)$. So we only need to prove that the $\mathtt{w}_n$ satisfy **(C2)**. To that end, for all n and $j \geq 1$, we set $f_n(j) = (G(j/n)/G(1/n))^3 = j^{-3/\rho}\ell^3(j/n)/\ell^3(1/n)$ and $\delta = \frac{1}{2}(\frac{3}{\rho} - 1)$ that is strictly positive. We apply Lemma 7.1 to ℓ^3: let $c_\delta \in (1, \infty)$ and $\eta_\delta \in (0, 1]$ such that (288) holds true; then, for all $n > 1/\eta_\delta$, $0 \leq f_n(j) \leq c_\delta j^{-1-\delta}$. Since for all $j \geq 1$, $\lim_{n\to\infty} f_n(j) = j^{-3/\rho}$, by dominated convergence we get:

$$G(1/n)^{-3}\sigma_3(\mathtt{w}_n) = \sum_{1\leq j\leq n} f_n(j) \xrightarrow[n\to\infty]{} \sum_{j\geq 1} j^{-3/\rho} = q^{-3}\sigma_3(\mathbf{c}),$$

which easily implies **(C2)**.

Let us prove that $\mathtt{w}_n(\alpha)$ satisfies **(C4)** thanks to (75) in Proposition 2.13. To that end, we fix $n \in \mathbb{N}^*$ and $\lambda \in [0, \infty)$ such that $\lambda \in [1, a_n]$. For all $x \in [0, \infty)$, recall that $f_\lambda(x) = x(e^{-\lambda x} - 1 + \lambda x)$ and for all $j \geq 1$, set

$$\phi_n(j) = f_\lambda\Big(\frac{w_j^{(n)}(\alpha)}{a_n}\Big) = f_\lambda\Big(q_n j^{-1/\rho}\frac{\ell(j/n)}{\ell(1/n)}\Big) \text{ where } \quad q_n = \Big(1 - \tfrac{a_n}{b_n}(\alpha - \alpha_0)\Big)\frac{G(1/n)}{a_n} \sim q\,.$$

To simplify, we also set $\kappa_n = a_n b_n/\sigma_1(\mathtt{w}_n(\alpha))$; note that $\kappa_n \sim \kappa$. Let $\delta \in (0, \infty)$ to be specified further; by Lemma 7.1 and the previous arguments, there exists $c_\delta \in (0, \infty)$ and n_δ such that for all $n \geq n_\delta$, $w_j^{(n)}(\alpha)/a_n \geq c_\delta j^{-\delta - 1/\rho}$ and $\kappa_n \geq \frac{1}{2}\kappa$, which entails $\kappa_n \phi_n(j) \geq \frac{1}{2}\kappa f_\lambda(c_\delta j^{-\delta - 1/\rho})$. We next set:

$$\alpha_n := \frac{b_n}{a_n}\Big(1 - \frac{\sigma_2(\mathtt{w}_n(\alpha))}{\sigma_1(\mathtt{w}_n(\alpha))}\Big) \sim \alpha$$

Recall from (74) that ψ_n stands for the the Laplace exponent of $(\frac{1}{a_n}X^{\mathtt{w}_n(\alpha)}_{b_n t})_{t\in[0,\infty)}$. The previous inequalities then imply that

$$\psi_n(\lambda) - \alpha_n\lambda = \sum_{1\leq j<n} \kappa_n\phi_n(j) \geq \tfrac{1}{2}\kappa \sum_{1\leq j<n} f_\lambda(c_\delta j^{-\delta - \frac{1}{\rho}}) \geq \tfrac{1}{2}\kappa \int_1^n dx\, f_\lambda(c_\delta x^{-\delta - \frac{1}{\rho}})\,.$$

We set $a = \rho/(1 + \rho\delta)$, namely $1/a = \delta + 1/\rho$ and we use the change of variables $y = \lambda x^{-1/a}$ in the last member of the inequality to get

$$\begin{aligned}
\forall n \geq n_\delta,\ \forall \lambda \in [1, a_n], \quad \psi_n(\lambda) - \alpha_n\lambda \;&\geq\; \tfrac{1}{2}\kappa a \lambda^{a-1}\int_{\lambda n^{-1/a}}^{\lambda} dy\, y^{-a-1} f_1(c_\delta y) \\
&\geq\; \tfrac{1}{2}\kappa a \lambda^{a-1}\int_{a_n n^{-1/a}}^{1} dy\, y^{-a-1} f_1(c_\delta y).
\end{aligned}$$

Now observe that $a_n n^{-1/a} \sim q^{-1} n^{-\delta}\ell(1/n) \to 0$. Thus, without loss of generality, we can assume that for all $n \geq n_\delta$, $a_n n^{-1/a} \leq 1/2$. Then, we set $K_\delta = \frac{1}{2}\kappa a \int_{1/2}^1 dy\, y^{a-1} f_1(c_\delta y) > 0$ and we have proved that for all $n \geq n_\delta$, and for all $\lambda \in [1, a_n]$, $\psi_n(\lambda) - \alpha_n\lambda \geq K_\delta \lambda^{a-1}$. Since $\rho > 2$ it is possible to choose a sufficiently small $\delta > 0$ such that $a - 1 = \rho/(1 + \rho\delta) - 1 > 1$. Then, we get (75) in Proposition 2.13 (i) which implies **(C4)**. This completes the proof of Lemma 2.18. ■

A Laplace exponents.

We state here a proposition on the Laplace transform of measures on $\mathbb{R}$. To that end, we briefly recall standard results on the Laplace transform of finite measures on $[0,\infty)$ and on $[0,\infty]$. Namely, let μ be a Borel-measure on the compact space $[0,\infty]$; its Laplace transform is given by $L_\mu(\lambda) = \int_{[0,\infty)} e^{-\lambda x}\,\mu(dx)$, for all $\lambda \in (0,\infty)$. In particular, we take $L_\mu(0) = L_\mu(0+) = \mu([0,\infty))$. Let μ, ν be finite Borel measures on $[0,\infty]$. Recall that if $\mu([0,\infty]) = \nu([0,\infty])$ and if $I = \{\lambda \in (0,\infty) : L_\mu(\lambda) = L_\nu(\lambda)\}$ has a limit point in $(0,\infty)$, then $\mu = \nu$. The *continuity theorem* for Laplace transform can stated as follows: let μ and μ_n, $n \in \mathbb{N}$, be finite Borel measures on $[0,\infty]$. Then, the following holds true.

$$\mu_n \xrightarrow[n\to\infty]{\text{weak}} \mu \iff \lim_{n\to\infty} \mu_n([0,\infty]) = \mu([0,\infty]) \text{ and } \lim_{n\to\infty} L_{\mu_n}(\lambda) = L_\mu(\lambda),\ \lambda \in [0,\infty). \tag{292}$$

We next easily deduce from (292) the following lemma.

Lemma A.1 *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $[0,\infty)$. Let $I \subset (0,\infty)$ having a limit point in $(0,\infty)$; let $L : I \to [0,\infty)$ be such that for all $\lambda \in I$, $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L(\lambda)$. Then, there exists a probability measure μ on $[0,\infty]$ such that $\mu_n \to \mu$ weakly on $[0,\infty]$. If furthermore the μ_n are tight on $[0,\infty)$, then $\mu(\{\infty\}) = 0$.*

Proof. Since $[0,\infty]$ is compact, $\{\mu_n; n \in \mathbb{N}\}$ is tight on $[0,\infty]$; by (292), the Laplace transform of two limiting probability measures coincide on I: there are therefore equal. ■

Let μ be a finite Borel-measure on $\mathbb{R}$; we extends its Laplace transform on $\mathbb{R}$ by simply setting for all $\lambda \in \mathbb{R}$, $L_\mu(\lambda) = \int_{\mathbb{R}} e^{-\lambda x}\mu(dx) \in [0,\infty]$. Let us mention that if in a right-neighbourhood of 0, L_μ and L_ν are finite and coincide, then $\mu = \nu$. We easily prove the following result.

Lemma A.2 *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $[0,\infty)$. Suppose that there exists $\lambda^* \in (0,\infty)$ such that for all $\lambda \in [0,\lambda^*]$, $\Lambda(\lambda) := \lim_{n\to\infty} L_{\mu_n}(-\lambda)$ exists and is finite. Then, $\mu_n \to \mu$ weakly on $[0,\infty)$, $\Lambda(\lambda) = L_\mu(-\lambda)$, $\lambda \in [0,\lambda^*)$, wich implies that $\lim_{\lambda\to 0+} \Lambda(\lambda) = 1$.*

Proof. For all $\lambda_0 \in (0,\lambda^*)$, set $\nu_{n,\lambda_0}(dx) = e^{\lambda_0 x}\mu_n(dx)/L_{\mu_n}(-\lambda_0)$ that is a well-defined probability measure. Note that for all $\lambda \in [\lambda_0 - \lambda^*, \lambda_0]$, $L_{\nu_{n,\lambda_0}}(\lambda) = L_{\mu_n}(\lambda - \lambda_0)/L_{\mu_n}(-\lambda_0) \to \Lambda(\lambda_0 - \lambda)/\Lambda(\lambda_0)$. This limit for $\lambda < 0$ entails that the ν_{n,λ_0} are tight on $[0,\infty)$; the same limit for $\lambda > 0$ combined with Lemma A.1 implies that there is a probability measure ν_{λ_0} on $[0,\infty)$ such that $\nu_{n,\lambda_0} \to \nu_{\lambda_0}$ weakly on $[0,\infty)$. Since $\mu_n(dx) = L_{\mu_n}(-\lambda_0)e^{-\lambda_0 x}\nu_{n,\lambda_0}(dx)$, we easily see that $\mu_n \to \mu := \Lambda(\lambda_0)e^{-\lambda_0 x}\nu_{\lambda_0}(dx)$ weakly on $[0,\infty)$ we easily check that $L_\mu(-\lambda) = \Lambda(-\lambda)$ for all $\lambda \in [0,\lambda^*)$. ■

We next recall a result essentially due to Grimvall [27] (Theorem 2.1, p. 1029).

Lemma A.3 *For all $n \in \mathbb{N}$, let $(\Delta^n_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence of real valued r.v. such that there exists $a \in (0,\infty)$ such that:*

$$\forall n, k \in \mathbb{N}, \quad \mathbf{P}(\Delta^n_k \geq -a) = 1\,. \tag{293}$$

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of integers that tends to ∞. We then set $Y_n = \sum_{0\leq k\leq q_n} \Delta^n_k$ and $L_n(\lambda) = \mathbf{E}\big[e^{-\lambda Y_n}\big]$ that is well-defined for all $\lambda \in [0,\infty)$ thanks to (293). Then, the following assertions are equivalent.

(a) The r.v. Y_n converge in law to a real-valued r.v. Y.

(b) There exists a function $L : [0,\infty) \to [0,\infty)$ that is right-continuous at 0, such that $L(0) = 1$ and such that $\lim_{n\to\infty} L_n(\lambda) = L(\lambda)$ for all $\lambda \in [0,\infty)$.

Moreover, if (a) or (b) is satisfied, then $L(\lambda) = \mathbf{E}[\exp(-\lambda Y)]$ and L is positive and continuous. Furthermore, the convergence $\lim_{n\to\infty} L_n = L$ holds true uniformly on every compact subset of $(0, \infty)$.

Proof. Grimvall's Theorem 2.1 [27] (p. 1029) asserts $(a) \Rightarrow (b)$. It also asserts that if (a) holds true, then $L(\lambda) = \mathbf{E}[\exp(-\lambda Y)]$ and $\lim_{n\to\infty} L_n = L$ uniformly on every compact subset of $(0, \infty)$.

It only remains to prove that $(b) \Rightarrow (a)$: first suppose that Y_{p_n} is a subsequence that converges in distribution to Y': by applying $(a) \Rightarrow (b)$, we get $L(\lambda) = \mathbf{E}[\exp(-\lambda Y')]$, $\lambda \in [0, \infty)$; as mentioned earlier, if the Laplace transform of a real-valued random variable is finite in a right-neighbourhood of 0, then it characterizes its law. Consequently, the laws of Y_n have at most one weak limit. Therefore, we only need to prove that the laws of the Y_n are tight on $\mathbb{R}$.

Since $[-\infty, \infty]$ is compact, the laws of the Y_n are tight on $[-\infty, \infty]$ and we only need to prove that for all increasing sequence of integers $(n_p)_{p\in\mathbb{N}}$ such that $Y_{n_p} \to Y$ in law on $[-\infty, \infty]$, we necessarily get $\mathbf{P}(|Y| = \infty) = 0$. To that end, first note that the convergence $Y_{n_p} \to Y$ in law on $[-\infty, \infty]$ implies that $(Y_{n_p})_{+/-} \to (Y)_{+/-}$ in law on $[0, \infty]$. By (292), we get

$$\lim_{p\to\infty} \mathbf{E}\big[\exp(-\lambda (Y_{n_p})_+)\big] = \mathbf{E}\big[\exp(-\lambda (Y)_+)\big]$$

for all $\lambda \in [0, \infty)$. Since $L_n(\lambda) = \mathbf{E}\big[\exp(\lambda (Y_n)_-)\big] + \mathbf{E}\big[\exp(-\lambda (Y_n)_+)\big] - 1$, we get

$$\lim_{p\to\infty} \mathbf{E}\big[\exp(\lambda (Y_{n_p})_-)\big] = L(\lambda) + 1 - \mathbf{E}\big[\exp(-\lambda (Y)_+)\big].$$

This easily entails that the laws of the $(Y_{n_p})_-$ are tight on $[0, \infty)$. Thus $\mathbf{P}(Y = -\infty) = 0$. We then apply Lemma A.2 to the laws of the r.v. $(Y_{n_p})_-$ and as $p \to \infty$ we get $\mathbf{E}\big[\exp(\lambda (Y)_-)\big] = L(\lambda) + 1 - \mathbf{E}\big[\exp(-\lambda (Y)_+)\big]$ and as $\lambda \to 0+$, since $\mathbf{E}\big[\exp(\lambda (Y)_-)\big]$ and $L(\lambda)$ tend to 1, we get $\mathbf{P}((Y)_+ < \infty) = \lim_{\lambda\to 0+} \mathbf{E}\big[\exp(-\lambda (Y)_+)\big] = 1$, which completes the proof of the lemma. ■

B Skorokod's topology.

B.1 General results.

In this section, we adapt and we recall from Jacod & Shiryaev's book [30] results on Skorokod's topology and weak convergence on $\mathbf{D}([0, \infty), \mathbb{R}^d)$ that are used in the proofs.

Lemma B.1 (Propositions 2.1 & 2.2 in [30]) *Let $x_n \to x$ in $\mathbf{D}([0, \infty), \mathbb{R}^d)$ and let $y_n \to y$ in $\mathbf{D}([0, \infty), \mathbb{R}^{d'})$. Then, the following holds true.*

(i) For all $t \in [0, \infty)$, there exists a sequence of times $t_n \to t$ such that $x_n(t_n-) \to x(t-)$, $x_n(t_n) \to x(t)$ and thus, $\Delta x_n(t_n) \to \Delta x(t)$.

(ii) For all $t \in [0, \infty)$ such that $\Delta x(t) = 0$ and for all sequences of times $s_n \to t$, we get $x_n(s_n-) \to x(t)$ and $x_n(s_n) \to x(t)$, and thus $\Delta x_n(s_n) \to 0$.

(iii) Assume for all $t \in (0, \infty)$ that there is a sequence of times $t_n \to t$ such that $\Delta x_n(t_n) \to \Delta x(t)$ and $\Delta y_n(t_n) \to \Delta y(t)$. Then $((x_n(t), y_n(t))_{t\in[0,\infty)} \longrightarrow ((x(t), y(t))_{t\in[0,\infty)}$ for the Skorokod topology on $\mathbf{D}\big([0, \infty), \mathbb{R}^{d+d'}\big)$. In particular, this joint convergence holds true whenever x and y have no common jump-time.

Proof. See Jacod & Shiryaev [30], Chapter VI, Section 2, pp. 337-338. More precisely, for (i) (resp. (ii)), see [30], Proposition 2.1 a) (resp. (b.5)); for (iii), see [30], Proposition 2.2 b). ■

As an immediate consequence of the Lemma B.1 (iii), we get the following lemma.

Lemma B.2 *Let $k \in \mathbb{N}^*$. For all $n \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$, let $R^n_j(\cdot)$ and $R_j(\cdot)$ be $\mathbb{R}^{d_j}$-valued càdlàg processes. Assume that $(R^n_1, \ldots, R^n_k) \to (R_1, \ldots, R_k)$ weakly on $\mathbf{D}([0, \infty), \mathbb{R}^{d_1}) \times \ldots \times$*

$\mathbf{D}([0,\infty),\mathbb{R}^{d_k})$ *equipped with the product topology. Assume that a.s. the processes* $R_1,\ldots,R_k$ *have no (pairwise) common jump-times. Then,*

$$((R_1^n(t),\ldots,R_k^n(t)))_{t\in[0,\infty)} \xrightarrow[n\to\infty]{} ((R_1(t),\ldots,R_k(t)))_{t\in[0,\infty)}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}^d)$*, where* $d=d_1+\ldots+d_k$*.*

Lemma B.3 *Let* $y_n\to y$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$*. Then the following holds true.*

(i) *Let* $s,t\in[0,\infty)$ *be such that* $s<t$ *and such that* $\Delta y(s)=\Delta y(t)=0$*. Then, for all* $(s_n,t_n)\to(s,t)$*, we get* $\inf_{[s_n,t_n]}y_n\to\inf_{[s,t]}y$*.*

(ii) *Suppose that* $t\in[0,\infty)\mapsto\inf_{s\in[0,t]}y(s)$ *is a continuous function. Then, the following convergence* $(\inf_{s\in[0,t]}y_n(s))_{t\in[0,\infty)}\to(\inf_{s\in[0,t]}y(s))_{t\in[0,\infty)}$ *holds uniformly on every compact subsets.*

Next, for all $r\in[0,\infty)$ *and all* $z\in\mathbf{D}([0,\infty),\mathbb{R})$*. We set* $\gamma_r(z)=\inf\{t\in[0,\infty):z(t)<-r\}$*, with the convention that* $\inf\emptyset=\infty$*. Note that* $r\mapsto\gamma_r(z)$ *is a non-decreasing* $[0,\infty]$*-valued càdlàg function. Then, we get the following.*

(iii) *Suppose that* $t\in[0,\infty)\mapsto\inf_{s\in[0,t]}y(s)$ *is continuous. Then, for all* $r\in[0,\infty)$ *such that* $\gamma_r(y)<\infty$ *and* $\Delta\gamma_r(y)=0$*, we get* $\gamma_r(y_n)\to\gamma_r(y)$*.*

For all $t\in[0,\infty)$*, all* $r\in\mathbb{R}$ *and all* $z\in\mathbf{D}([0,\infty),\mathbb{R})$ *we next set*

(294) $$\tau(z,t,r)=\inf\big\{s\in[0,t]:\inf_{u\in[s,t]}z(u)>r\big\}\ \textit{with the the convention that}\ \inf\emptyset=\infty.$$

Then, the following holds true.

(iv) *Suppose that* $y(t)>0=y(0)$*. Then,* $r\in[0,y(t))\mapsto\tau(y,t,r)$ *is right-continuous and non-decreasing. Furthermore, suppose that* $\Delta y(t)=0$ *and that* $r\in(0,y(t))$ *satisfies* $\tau(y,t,r-)=\tau(y,t,r)$*. Then, for all* $(t_n,r_n)\to(t,r)$*,* $\tau(y_n,t_n,r_n)\to\tau(y,t,r)$*.*

Proof: since $y_n\to y$ in $\mathbf{D}([0,\infty),\mathbb{R})$ there is a sequence of continuous increasing functions $\lambda_n:[0,\infty)\to[0,\infty)$, $n\in\mathbb{N}$, such that $\lambda_n(0)=0$, such that $\sup_{t\in[0,\infty)}\big|\lambda_n(t)-t\big|\to0$ and such that $\sup_{s\in[0,p]}|y_n-y(\lambda_n(s))|\to0$ as $n\to\infty$ for all $p\in\mathbb{N}$ (take the inverse of λ_n in Theorem 1.14 in Jacod & Shiryaev [30], Chapter VI, Section 1.b, p. 328). To simplify we set $s'_n=\lambda_n(s_n)$ and $t'_n=\lambda_n(t_n)$; note that $(s'_n,t'_n)\to(s,t)$ and that $\inf_{[s_n,t_n]}y_n-\inf_{[s'_n,t'_n]}y\to0$. Next observe that for all $\varepsilon>0$,

$$\inf_{[s-\varepsilon,t+\varepsilon]}y\le\liminf_{n\to\infty}\inf_{[s'_n,t'_n]}y\le\limsup_{n\to\infty}\inf_{[s'_n,t'_n]}y\le\inf_{[s+\varepsilon,t-\varepsilon]}y.$$

Since $\Delta y(s)=\Delta y(t)=0$, we get $\lim_{\varepsilon\to0}\inf_{[s-\varepsilon,t+\varepsilon]}y=\lim_{\varepsilon\to0}\inf_{[s+\varepsilon,t-\varepsilon]}y=\inf_{[s,t]}y$, which entails (i). The point (ii) is an immediate consequence of a well-known theorem due to Dini.

Under the assumption that $t\in[0,\infty)\mapsto\inf_{s\in[0,t]}y(s)$ is continuous, (iii) is a consequence of Proposition 2.11, Chapter VI, Section 2a p. 341 in Jacod & Shiryaev [30] applied to the functions $t\in[0,\infty)\mapsto\inf_{s\in[0,t]}y_n(s)$: to be specific, for all $r\in[0,\infty)$, set $S^n_r=\inf\{t\in[0,\infty):\inf_{s\in[0,t]}y_n(s)\le-r\}$ and $S_r=\inf\{t\in[0,\infty):\inf_{s\in[0,t]}y(s)\le-r\}$; then $r\mapsto S_r$ is left continuous with right-limits (see Lemma 2.10 (b) [30], p. 340) and Proposition 2.11 [30] p. 341 asserts the following: if $S_r=S_{r+}$, then $S^n_r\to S_r$. Now, observe that $S_{r+}=\gamma_r(y)$, $S^n_{r+}=\gamma_r(y_n)$, $S_r=\gamma_{r-}(y)$ and $S_r=\gamma_{r-}(y_n)$, which implies (iii).

Let us prove (iv): suppose $y(t)>0=y(0)$; it is easy to check that $r\in[0,y(t))\mapsto\tau(y,t,r)$ is right-continuous and nondecreasing. Suppose next that $\Delta y(t)=0$ and that $r\in(0,y(t))$ satisfies $\tau(y,t,r-)=\tau(y,t,r)$. Let $q\in(\tau(y,t,r),t)$ be such that $\Delta y(q)=0$; then $\inf_{[q,t]}y>r$; by (i), for all sufficiently large n, we get $\inf_{[q,t_n]}y_n>r_n$ and thus, $\tau(y_n,t_n,r_n)\le q<t_n$. This easily entails that $\limsup_{n\to\infty}\tau(y_n,t_n,r_n)\le\tau(y,t,r)$. Next, fix $q<\tau(y,t,r-)$ such that $\Delta y(q)=0$: then, there exists $r'\in(0,r)$ such that $q<\tau(y,t,r')$, which implies that $\inf_{[q,t]}y\le r'<r$; by (i), for all sufficiently large n, we get $\inf_{[q,t_n]}y_n<r_n$ and thus, $q\le\tau(y_n,t_n,r_n)$. This easily entails that $\liminf_{n\to\infty}\tau(y_n,t_n,r_n)\ge\tau(y,t,r-)$, which implies the desired result. ■

Lemma B.4 *Let* $r_n \to r$ *in* $[0,\infty)$ *and let* $y_n \to y$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$. *Assume that* $\Delta y(r)=0$. *Then the following holds true.*

(i) $(y_n(t\wedge r_n))_{t\in[0,\infty)} \to (y(t\wedge r))_{t\in[0,\infty)}$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$.

(ii) $(y_n(r_n+t))_{t\in[0,\infty)} \to (y(r+t))_{t\in[0,\infty)}$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$.

(iii) *Let* $l_n \in [0,r_n]$ *be such that* $l_n \to l$. *Assume that* $\Delta y(l)=0$. *Then* $(y_n((l_n+t)\wedge r_n))_{t\in[0,\infty)} \to (y((l+t)\wedge r))_{t\in[0,\infty)}$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$.

Proof. Denote by Λ the set of continuous increasing functions $\lambda : [0,\infty) \to [0,\infty)$, such that $\lambda(0)=0$. Recall from Jacod & Shiryaev [30] (Theorem 1.14, Chapter VI, Section 1.b, p. 328) that $y_n \to y$ in $\mathbf{D}([0,\infty),\mathbb{R})$ if and only if there exists $\lambda_n \in \Lambda$, $n\in\mathbb{N}$, such that $\sup_{t\in[0,\infty)} |\lambda_n(t)-t| \to 0$ and for all positive integers p, $\sup_{t\in[0,p]} |y_n(t)-y(\lambda_n(t))| \to 0$. Let $p \geq 1+\sup_{n\geq 1} r_n$; set $\varepsilon_n = \sup_{t\in[0,p]} |y_n(t)-y(\lambda_n(t))|$.

We first prove (i): first observe that for all $t\in[0,\infty)$, $|y_n(t\wedge r_n)-y(r\wedge\lambda_n(t))| \leq \varepsilon_n + \delta_n(t)$ where $\delta_n(t) = |y(r\wedge\lambda_n(t))-y(\lambda_n(r_n\wedge t))|$. Observe that $\delta_n(t)=0$ if $t \leq r_n \wedge \lambda_n^{-1}(r)$. Set $\eta_n = \sup\{|r-r\wedge\lambda_n(t)| + |r-\lambda_n(r_n\wedge t)|; t \geq r_n\wedge\lambda_n^{-1}(r)\}$. Then observe that $\eta_n \to 0$ and that $\delta_n(t) \leq \mathtt{osc}(y,[r-\eta_n, r+\eta_n])$ that is the oscillation of y on $[r-\eta_n, r+\eta_n]$ as defined in (142). Since $\Delta y(r)=0$, we get $\mathtt{osc}(y,[r-\eta_n, r+\eta_n]) \to 0$ (see for instance Jacod & Shiryaev [30], Proposition 2.1, Chapter VI, Section 2.a, p. 337). Thus, $\sup_{t\in[0,\infty)} |y_n(t\wedge r_n)-y(r\wedge\lambda_n(t))| \to 0$, which implies (i).

Let us prove (ii). Set $\phi_n(t)=\lambda_n(r_n+t)-r$ and $\rho_n = |r_n-r| + \sup_{t\in[0,\infty)} |\lambda_n(t)-t|$. Then $\sup_{t\in[0,\infty)} |\phi_n(t)-t| \leq \rho_n \to 0$; note that ϕ_n is continuous, increasing but $\phi_n(0)$ may be distinct from 0. We modify ϕ_n in the following way: for all $t\in[0,\rho_n]$, set $\varphi_n(t)=t$, for all $t\in[\rho_n, 3\rho_n]$, set $\varphi_n(t)=\rho_n+(2\rho_n)^{-1}(t-\rho_n)(\phi_n(3\rho_n)-\rho_n)$ and for all $t\geq 3\rho_n$, set $\varphi_n(t)=\phi_n(t)$. Clearly, $\varphi_n\in\Lambda$ and we check that $\sup_{t\in[0,\infty)} |\varphi_n(t)-t| \leq 3\rho_n \to 0$. Then observe that for all $t\in[0,\infty)$,

$$
\begin{aligned}
|y_n(r_n+t)-y(r+\varphi_n(t))| &\leq |y_n(r_n+t)-y(\lambda_n(r_n+t))| + |y(r+\phi_n(t))-y(r+\varphi_n(t))| \\
&\leq \varepsilon_n + \mathtt{osc}(y,[r-6\rho_n, r+6\rho_n])
\end{aligned}
$$

which implies (ii) since $\varepsilon_n + \mathtt{osc}(y,[r-6\rho_n, r+6\rho_n]) \to 0$. Then note that (iii) is an immediate consequence of (i) and (ii). This completes the proof. ■

Theorem B.5 (Theorem 3.1 in Whitt [42]) *Let* $h_n \to h$ *and* $\lambda_n \to \lambda$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$. *We assume that* $\lambda_n(0)=0$ *and that* λ_n *is nondecreasing. Then, the following holds true.*

(i) *If* $h_n \to h$ *in* $\mathbf{C}([0,\infty),\mathbb{R})$, *then* $h_n\circ\lambda_n \to h\circ\lambda$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$.

(ii) *If* $\lambda_n \to \lambda$ *in* $\mathbf{C}([0,\infty),\mathbb{R})$ *and if* λ *is strictly increasing, then* $h_n\circ\lambda_n \to h\circ\lambda$ *in* $\mathbf{D}([0,\infty),\mathbb{R})$.

Proof: See Whitt [42], Theorem 3.1, p. 75. ■

We use Theorem B.5 (ii) several times under the following form.

Lemma B.6 *Let* $(\beta_n)_{n\in\mathbb{N}}$ *be a sequence of nonnegative real numbers such that* $\beta_n \to \infty$. *For all* $n\in\mathbb{N}$, *let* $(\sigma_k^n)_{k\geq 1}$ *be an increasing sequence of random times such that* $\lim_{k\to\infty}\sigma_k^n = \infty$; *then, for all* $t\in[0,\infty)$, *we set* $M_t^n = \sum_{k\geq 1} \mathbf{1}_{[0,t]}(\sigma_k^n)$. *Let* $(R^n)_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}$*-valued càdlàg processes. We first assume that* $R^n \to R$ *weakly on* $\mathbf{D}([0,\infty),\mathbb{R})$. *We also assume that there is a deterministic strictly increasing* $\lambda\in\mathbf{C}([0,\infty),\mathbb{R})$ *such that* $\frac{1}{\beta_n} M^n_{\beta_n \cdot} \to \lambda$ *weakly on* $\mathbf{C}([0,\infty),\mathbb{R})$. *Then,*

$$
\left(R^n_{\beta_n^{-1} M^n_{\beta_n t}}\right)_{t\in[0,\infty)} \xrightarrow[n\to\infty]{} (R_{\lambda(t)})_{t\in[0,\infty)} \tag{295}
$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R})$. *In particular, this result applies if* M^n *are homogeneous Poisson processes with unit rate and* λ *is the identity map.*

Proof. We set $\lambda_n(t) = M^n(\beta_n t)/\beta_n$. Since λ is deterministic, Slutzky's argument implies that $(R^n, \lambda_n) \to (R, \lambda)$ weakly on $\mathbf{D}([0,\infty), \mathbb{R}) \times \mathbf{C}([0,\infty), \mathbb{R})$ and Theorem B.5 (ii) implies (295). To complete the proof of the Lemma, assume that M^n are homogeneous Poisson processes with unit rate. By Doob's L^2 inequality, $(\beta_n^{-1} M^n_{\beta_n t})_{t\in[0,\infty)} \to \mathrm{Id}$, weakly on $\mathbf{D}([0,\infty), \mathbb{R})$, where Id stands for the identity map on $[0,\infty)$. ∎

We next recall the following elementary lemma whose proof is left to the reader.

Lemma B.7 *Let E be a Polish space. For all $n, k \in \mathbb{N}$, let X_k and X_k^n be E-valued r.v. such that for all $k \in \mathbb{N}$, $(X_0^n, \ldots, X_k^n) \to (X_0, \ldots, X_k)$ weakly on E^{k+1} equipped with the product topology. Then $(X_k^n)_{k\in\mathbb{N}} \to (X_k)_{k\in\mathbb{N}}$ weakly on $E^{\mathbb{N}}$ equipped with the product topology.*

B.2 Weak limits of Lévy processes, of random walks and of branching processes.

B.2.1 Lévy processes and rescaled random walks.

We first recall from Jacod & Shiryaev [30] the following standard theorem on functional limits of Lévy processes that is used several times in the proofs.

Theorem B.8 *Let $(R_t^n)_{t\in[0,\infty)}$, $n \in \mathbb{N}$, be of $\mathbb{R}$-valued Lévy processes with initial value 0. Then, the following assertions are equivalent.*

(a) *There exists a time $t \in (0,\infty)$ such that the r.v. R_t^n converge weakly on $\mathbb{R}$.*

(b) *The processes R^n weakly converge on $\mathbf{D}([0,\infty), \mathbb{R})$.*

Moreover, if (a) or (b) holds true, then the limit of the process R^n is necessarily a Lévy process.

Proof. This is a consequence of Corollary 3.6 in Jacod & Shiryaev [30], Chapter VII, Section 3.a, p. 415. To understand the notation and the terminology, let us mention that in [30], a *PIIS* stands for a Lévy process and that the form of the *characteristics* of a PIIS is given in Corollary 4.19, Chapter II, Section 4.c, p. 107. ∎

Let us briefly recall some notation. Let $(R_t)_{t\in[0,\infty)}$ be a $\mathbb{R}$-valued Lévy process with initial value $R_0 = 0$. We assume it is *spectrally positive*, namely that R has no negative jump: a.s. for all $t \in [0,\infty)$, $\Delta R_t \geq 0$. We also assume that the process is *integrable*: namely, we assume that there exists a certain $t \in (0,\infty)$ such that $\mathbf{E}[|R_t|] < \infty$. Let us mention that if R is integrable, then $\mathbf{E}[|R_t|] < \infty$ *for all* $t \in [0,\infty)$. There is a one-to-one correspondence between the laws of integrable spectrally positive Lévy processes and triplets (α, β, π) where $\alpha \in \mathbb{R}$, $\beta \in [0,\infty)$ and π is a Borel-measure on $(0,\infty)$ such that $\int_{(0,\infty)} \pi(dr)\,(r \wedge r^2) < \infty$; the correspondence is given via the Laplace exponent of R (that is well-defined): namely, for all $t, \lambda \in [0,\infty)$,

$$\text{(296)} \qquad \mathbf{E}\big[e^{-\lambda R_t}\big] = e^{t\psi_{\alpha,\beta,\pi}(\lambda)}, \text{ where } \psi_{\alpha,\beta,\pi}(\lambda) = \alpha\lambda + \tfrac{1}{2}\beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\, \pi(dr).$$

We shall say that R is an integrable (α, β, π)-spectrally Lévy process to mean that its Laplace exponent is given by (296). We next recall the following specific version of a standard limit-theorem for Lévy processes.

Theorem B.9 *Let $(R^n)_{n\in\mathbb{N}}$ be a sequence of integrable $(\alpha_n, \beta_n, \pi_n)$-spectrally positive Lévy processes. Assume that there exists $r_0 \in (0,\infty)$ such that for all $n \in \mathbb{N}$, $\pi_n([r_0,\infty)) = 0$, which implies: $\int_{(0,\infty)} r^2\, \pi_n(dr) < \infty$. Let R be a $\mathbb{R}$-valued càdlàg process. Then, the following assertions are equivalent:*

- *(Lv1) : $R_1^n \longrightarrow R_1$ weakly on $\mathbb{R}$.*
- *(Lv2) : $R^n \longrightarrow R$ weakly on $\mathbf{D}([0,\infty, \mathbb{R})$.*

If $(Lv1)$ *or* $(Lv2)$ *hold true, then* R *is necessarily an integrable* (α, β, π)*-spectrally positive Lévy process such that* $\pi([r_0, \infty)) = 0$*, which entails* $\int_{(0,\infty)} r^2 \, \pi(dr) < \infty$*. Moreover,* $(Lv1)$ *or* $(Lv2)$ *are equivalent to the following conditions:*

- $(Lv3a)$: $\alpha_n \longrightarrow \alpha$.
- $(Lv3b)$: $\beta_n + \int_{(0,\infty)} r^2 \pi_n(dr) \longrightarrow \beta + \int_{(0,\infty)} r^2 \, \pi(dr)$.
- $(Lv3c)$: $\int_{(0,\infty)} f(r) \, \pi_n(dr) \longrightarrow \int_{(0,\infty)} f(r) \, \pi(dr)$*, for all bounded continuous* $f : \mathbb{R} \to \mathbb{R}$ *vanishing on a neighbourhood of* 0.

Proof. (Lv1) ⇔ (Lv2) is a specific case of Corollary 3.6 in Jacod & Shiryaev [30], Chapter VII, Section 3.a, p. 415 (already recalled in Theorem B.8). For the proof of (Lv1) ⇔ (Lv3abc), see Theorem 2.14 in Jacod & Shiryaev [30], Chapter VII, Section 2.a, p. 398. ■

Here is the random walk version of the previous theorem.

Theorem B.10 *Let* $a_n, b_n \in (0, \infty)$*,* $n \in \mathbb{N}$*, that both tend to* ∞*. For all* $n \in \mathbb{N}$*, let* $(\xi_k^n)_{k \in \mathbb{N}}$ *be an i.i.d. sequence of real-valued r.v. Assume that there exists* $r_0 \in (0, \infty)$ *such that for all* $n, k \in \mathbb{N}$*,* $\mathbf{P}(a_n r_0 \geq \xi_k^n \geq -r_0) = 1$*. For all* $t \in [0, \infty)$*, set* $R_t^n = a_n^{-1} \sum_{1 \leq k \leq \lfloor b_n t \rfloor} \xi_k^n$*. Let* R *be a* $\mathbb{R}$*-valued càdlàg process. Then, the following assertions are equivalent:*

- $(Rw1)$: $R_1^n \longrightarrow R_1$ *weakly on* $\mathbb{R}$.
- $(Rw2)$: $R^n \longrightarrow R$ *weakly on* $\mathbf{D}([0, \infty, \mathbb{R})$.

If $(Rw1)$ *or* $(Rw2)$ *hold true, then* R *is necessarily an integrable* (α, β, π)*-spectrally positive Lévy process such that* $\pi([r_0, \infty)) = 0$*, which entails* $\int_{(0,\infty)} r^2 \, \pi(dr) < \infty$*. Moreover,* $(Rw1)$ *or* $(Rw2)$ *are equivalent to the following conditions:*

- $(Rw3a)$: $b_n a_n^{-1} \mathbf{E}\big[\xi_1^n\big] \longrightarrow -\alpha$.
- $(Rw3b)$: $b_n a_n^{-2} \mathbf{var}(\xi_1^n) \longrightarrow \beta + \int_{(0,\infty)} r^2 \pi(dr)$.
- $(Rw3c)$: $b_n \mathbf{E}\big[f\big(\xi_1^n / a_n\big)\big] \longrightarrow \int_{(0,\infty)} f(r) \, \pi(dr)$*, for all bounded continuous* $f : \mathbb{R} \to \mathbb{R}$ *vanishing on a neighbourhood of* 0.

Proof. (Rw1) ⇔ (Rw3abc) is a specific case of Theorem 2.36 Jacod & Shiryaev [30], Chapter VII, Section 2.c p. 404. The equivalence (Rw1) ⇔ (Rw2) is standard: see for instance Theorem 3.2 p. 342 in Jacod [29]. ■

B.2.2 Continuous state branching processes and rescaled Galton-Watson processes.

We next recall converge theorems for rescaled Galton-Watson processes to integrable Continuous State Branching Processes (CSBP for short). Recall that $(Z_t)_{t \in [0,\infty)}$ is an integrable CSBP if it is a $[0, \infty)$-valued Feller Markovian process whose absorbing state is 0 and that satisfies $\mathbf{E}[Z_t] < \infty$ for all $t \in [0, \infty)$; transition probabilities are characterised by a function $\psi : [0, \infty) \to \mathbb{R}$ called the *branching mechanism*; ψ is necessarily the Laplace exponent of an integrable spectrally positive process: namely, it is the form $\psi = \psi_{\alpha,\beta,\pi}$ as in (296). The branching mechanism characterises the transition probabilities as follows: for all $t, s, \lambda \in [0, \infty)$,

$$\mathbf{E}\big[e^{-\lambda Z_{s+t}} \big| Z_s\big] = e^{-Z_s u_t(\lambda)}, \text{ where } \quad u_t(\lambda) = \lambda - \int_0^t \psi(u_s(\lambda)) \, ds. \tag{297}$$

Since $\psi = \psi_{\alpha,\beta,\pi}$ is as in (296), $\psi'(0+) = \alpha$ and the equation on the right-hand side has a unique solution. Since ψ is convex and since $\psi(0) = 0$, it has at most one positive root; denote by q the largest root of ψ; then, the equation on the right hand side of (297) is equivalent to the following.

$$\forall t \in [0, \infty), \ \forall \lambda \in (0, \infty) \backslash \{q\}, \qquad \int_{u_t(\lambda)}^{\lambda} \frac{dz}{\psi(z)} = t. \tag{298}$$

This easily implies the following conditions of non-absorption in 0:

$$\mathbf{P}\big(\exists t : Z_t = 0\big) = 0 \iff \int^{\infty} \frac{dz}{\psi(z)} = \infty. \tag{299}$$

We shall say that Z satisfies the *Grey condition* if it has a positive probability to be absorbed in 0, namely if $\int^{\infty} dz/\psi(z) < \infty$; in that case, one can show that $\mathbf{P}(\exists t : Z_t = 0) = \mathbf{P}(\lim_{t\to\infty} Z_t = 0)$ and if a.s. $Z_0 = x$, then we get:

$$\mathbf{P}(Z_t = 0) = e^{-xv(t)} \quad \text{where } v \text{ satisfies} \quad \int_{v(t)}^{\infty} \frac{dz}{\psi(z)} = t. \tag{300}$$

We refer to Bingham [15] for more details on CSBP. We next recall the following convergence result from Grimvall [27].

Theorem B.11 (Theorems 3.1 & 3.4 [27]) *Let $a_n, b_n \in (0,\infty)$, $n \in \mathbb{N}$, such that both a_n and b_n/a_n tend to ∞. For all $n \in \mathbb{N}$, let μ_n be a probability measure on $\mathbb{N}$, let $(Z_k^{(n)})_{k\in\mathbb{N}}$ be a Galton-Watson process with offspring distribution μ_n and initial state $Z_0^{(n)} = \lfloor a_n \rfloor$, and let $(\zeta_k^n)_{k\in\mathbb{N}}$ be an i.i.d. sequence of r.v. with law μ_n. Then, the following assertions are equivalent.*

(Br1): $\frac{1}{a_n}\sum_{1\leq k\leq \lfloor b_n \rfloor} \big(\zeta_k^n - 1\big) \longrightarrow R_1$ weakly on $\mathbb{R}$, and R_1 is integrable and it has a spectrally positive infinitely divisible law whose Laplace exponent ψ.

(Br2): $\big(\frac{1}{a_n} Z^{(n)}_{\lfloor b_n t/a_n \rfloor}\big)_{t\in[0,\infty)} \longrightarrow (Z_t)_{t\in[0,\infty)}$ weakly on $\mathbf{D}([0,\infty),\mathbb{R})$ and Z is an integrable CSBP with branching mechanism ψ.

Proof. See Theorem 3.1 p. 1030 and Theorem 3.4 p. 1040 in Grimvall [27]; in [27], $b_n/a_n = n$, however, the above extension is straightforward. ■

B.2.3 Height and contour processes of Galton-Watson trees.

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of (sub)critical offspring distributions. For all μ_n, we denote by $\mathbf{T}_n$ a Galton-Watson forest with offspring μ_n as defined in Section 4.1.1. Recall from this section the definition of the Lukasiewicz path, the height and the contour processes of $\mathbf{T}_n$ that are denoted respectively by $(V_k^{\mathbf{T}_n})_{k\in\mathbb{N}}$, $(\mathtt{Hght}_k^{\mathbf{T}_n})_{k\in\mathbb{N}}$ and $(C_t^{\mathbf{T}_n})_{t\in[0,\infty)}$. We shall use the following result from Le Gall & D. [21].

Theorem B.12 *Let X be an integrable (α,β,π)-spectrally positive Lévy process, as defined at the beginning of Section 6.3.1. Assume that $\alpha \geq 0$ and that $\int^{\infty} dz/\psi_{\alpha,\beta,\pi}(z) < \infty$, where $\psi_{\alpha,\beta,\pi}$ is given by (296). Let H be the continuous height process derived from X by (45). Let $a_n, b_n \in (0,\infty)$, $n \in \mathbb{N}$, be two sequences tending to ∞; for all $n \in \mathbb{N}$, let $\mathbf{T}_n$ be a GW(μ_n)-forest, where μ_n is a (sub)critical offspring distribution. Let $(Z_k^{(n)})_{k\in\mathbb{N}}$ be a Galton-Watson process with offspring distribution μ_n and initial state $Z_0^{(n)} = \lfloor a_n \rfloor$. We assume the following*

$$\frac{1}{a_n} V^{\mathbf{T}_n}_{\lfloor b_n \rfloor} \xrightarrow[n\to\infty]{\text{weakly on } \mathbb{R}} X_1 \quad \textit{and} \quad \exists\, \delta \in (0,\infty), \quad \liminf_{n\to\infty} \mathbf{P}\big(Z^{(n)}_{\lfloor b_n \delta / a_n \rfloor} = 0\big) > 0\,. \tag{301}$$

Then, the following joint convergence holds true:

$$\Big(\big(\tfrac{1}{a_n} V^{\mathbf{T}_n}_{\lfloor b_n t \rfloor}\big)_{t\in[0,\infty)}, \big(\tfrac{a_n}{b_n} \mathtt{Hght}^{\mathbf{T}_n}_{\lfloor b_n t \rfloor}\big)_{t\in[0,\infty)}, \big(\tfrac{a_n}{b_n} C^{\mathbf{T}_n}_{b_n t}\big)_{t\in[0,\infty)}\Big) \xrightarrow[n\to\infty]{} \big((X_t)_{t\in[0,\infty)}, (H_t)_{t\in[0,\infty)}, (H_{t/2})_{t\in[0,\infty)}\big) \tag{302}$$

weakly on $\mathbf{D}([0,\infty),\mathbb{R}) \times \mathbf{C}([0,\infty),\mathbb{R})^2$ equiped with the product topology. We also get

$$\forall t \in [0,\infty), \quad \lim_{n\to\infty} \mathbf{P}\big(Z^{(n)}_{\lfloor b_n t / a_n \rfloor} = 0\big) = e^{-v(t)} \quad \textit{where} \quad \int_{v(t)}^{\infty} \frac{dz}{\psi_{\alpha,\beta,\pi}(z)} = t. \tag{303}$$

Proof. Convergence (302) is a direct consequence of Corollary 2.5.1 in Le Gall & D. [21], Chapter 2, p. 69. Moreover, set $\gamma_n = \inf \big\{ k \in \mathbb{N} : V^{\mathbf{T}_n}(k) = -\lfloor a_n \rfloor \big\}$. Then, $\sup_{1\leq k\leq \gamma_n} \mathtt{Hght}^{\mathbf{T}_n}(k)$ is the total height of the $\lfloor a_n \rfloor$ first independent Galton-Watson trees $\theta_{[1]}\mathbf{T}_n, \ldots, \theta_{[\lfloor a_n \rfloor]}\mathbf{T}_n$. It is easy to deduce from the joint convergence (302) and Lemma B.3 (iii) that

$$\mathbf{P}\big(Z^{(n)}_{\lfloor b_n t/a_n \rfloor} = 0\big) = \mathbf{P}\Big(\sup_{1\leq k\leq \gamma_n} \mathtt{Hght}^{\mathbf{T}_n}_k < \lfloor b_n t/a_n \rfloor \Big) \xrightarrow[n\to\infty]{} \mathbf{P}\Big(\sup_{s\in[0,\gamma]} H_s \leq t \Big) = \mathbf{P}(Z_t = 0),$$

where $\gamma = \inf\{t \in [0,\infty) : X_t < -1\}$ and where Z is a CSBP with branching mechanism $\psi_{\alpha,\beta,\pi}$. Then, (300) implies (303). ■

C Metric spaces: pinching, coding and convergence.

C.1 Pinched metric spaces and their fractal dimensions.

Let (E,d) be a metric space. We briefly recall from section 2.2.2 the definition of pinched metrics: for all $i \in \{1,\ldots,p\}$, let $(x_i,y_i) \in E^2$; set $\mathbf{\Pi} = ((x_i,y_i))_{1\leq i\leq p}$; let $\varepsilon \in [0,\infty)$. Set $A_E = \{(x,y); x,y \in E\}$ and for all $e=(x,y)\in A_E$, we set $\underline{e} = x$ and $\overline{e} = y$. A path γ joining x to y is a sequence of $e_1,\ldots,e_q \in A_E$ such that $\underline{e}_1 = x$, $\overline{e}_q = y$ and $\overline{e}_i = \underline{e}_{i+1}$, for all $i\in\{1,\ldots,q-1\}$. Next, we set $A_{\mathbf{\Pi}} = \{(x_i,y_i),(y_i,x_i); 1\leq i\leq p\}$ and we define the length l_e of an edge e by setting $l_e = \varepsilon \wedge d(\underline{e},\overline{e})$ if $e\in A_{\mathbf{\Pi}}$ and $l_e = d(\underline{e},\overline{e})$ otherwise. The length of a path $\gamma=(e_1,\ldots,e_q)$ is given by $l(\gamma) = \sum_{1\leq i\leq q} l_{e_i}$. Then, recall from (52), that the $(\mathbf{\Pi},\varepsilon)$-pinched pseudo-distance between x and y in E is given by $d_{\mathbf{\Pi},\varepsilon}(x,y) = \inf\{l(\gamma); \gamma \text{ is a path joining } x \text{ to } y\}$. We easily check that

$$d_{\mathbf{\Pi},\varepsilon}(x,y) = d(x,y) \wedge \min \big\{ l(\gamma)\,;\; \gamma = (e_0, e'_0, \ldots, e_{r-1}, e'_{r-1}, e_r), \text{ a path joining } x \text{ to } y \text{ such that } e'_0, \ldots e'_{r-1} \in A_{\mathbf{\Pi}} \text{ and } r\leq p \big\}. \tag{304}$$

Clearly, $d_{\mathbf{\Pi},\varepsilon}$ is a pseudo-metric and we denote the equivalence relation $d_{\mathbf{\Pi},\varepsilon}(x,y)=0$ by $x \equiv_{\mathbf{\Pi},\varepsilon} y$; the quotient space $E/\!\equiv_{\mathbf{\Pi},\varepsilon}$ equipped with $d_{\mathbf{\Pi},\varepsilon}$ is the $(\mathbf{\Pi},\varepsilon)$-pinched metric space associated with (E,d). Recall that $\varpi_{\mathbf{\Pi},\varepsilon} : E \to E/\!\equiv_{\mathbf{\Pi},\varepsilon}$ stands for the canonical projection that is 1-Lipschitz. Note of course that if $\varepsilon>0$, then $d_{\mathbf{\Pi},\varepsilon}$ is a true metric on E, which is obviously identified with $E/\!\equiv_{\mathbf{\Pi},\varepsilon}$ and $\varpi_{\mathbf{\Pi},\varepsilon}$ is the identity map on E.

Next, set $S = \{x_i, y_i; 1\leq i\leq p\}$ and for all $x\in E$, set $d(x,S) = \min_{y\in S} d(x,y)$. Then, (304) immediately entails that

$$\forall x,y\in E, \quad d(x,y) \leq d(x,S) + d(y,S) \implies d(x,y) = d_{\mathbf{\Pi},\varepsilon}(x,y). \tag{305}$$

Then, for all $r\in(0,\infty)$, denote by $B_d(x,r)$ (resp. by $B_{d_{\mathbf{\Pi},\varepsilon}}(\varpi_{\mathbf{\Pi},\varepsilon}(x),r)$) the open ball in (E,d) (resp. in $(E/\equiv_{\mathbf{\Pi},\varepsilon}, d_{\mathbf{\Pi},\varepsilon})$) with center x (resp. $\varpi_{\mathbf{\Pi},\varepsilon}(x)$) and radius r. Then, (305) entails the following: if $x\in E\backslash S$ and if $0<r<\frac{1}{4}d(x,S))$, then

$$\varpi_{\mathbf{\Pi},\varepsilon} : B_d(x,r) \to B_{d_{\mathbf{\Pi},\varepsilon}}(\varpi_{\mathbf{\Pi},\varepsilon}(x),r) \text{ is a sujective isometry.} \tag{306}$$

Namely, outside the pinching points, the metric is locally unchanged.

We now prove a result on Hausdorff and packing dimensions that is used in the proof of Proposition 2.8. To that end, we suppose that there exists (E_0,d), a compact metric space such that $E\subset E_0$ and such that E is a compact subset of E_0. To simplify notation we set $(E',d',\varpi) = \big(E/\!\equiv_{\mathbf{\Pi},\varepsilon}, d_{\mathbf{\Pi},\varepsilon}, \varpi_{\mathbf{\Pi},\varepsilon}\big)$. We denote by $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{p}}$ resp. the Hausdorff and the packing dimensions.

Lemma C.1 *We keep the notations from above. We first assume that* $\dim_{\mathrm{H}}(E_0) \in (0,\infty)$ *and* $\dim_{\mathrm{p}}(E_0)\in(0,\infty)$. *Let* $a\in(0,\dim_{\mathrm{H}}(E_0))$ *and* $b\in(0,\dim_{\mathrm{p}}(E_0))$*; we assume that there is a finite measure* m_0 *on the Borel subsets of* E_0 *such that* $m_0(E)>0$ *and*

$$\text{for } m_0\text{-almost all } x\in E_0 \quad \limsup_{r\to 0} \frac{m_0(B_d(x,r))}{r^a} < \infty \quad \text{and} \quad \liminf_{r\to 0} \frac{m_0(B_d(x,r))}{r^b} < \infty. \tag{307}$$

Then, $a\leq \dim_{\mathrm{H}}(E') \leq \dim_{\mathrm{H}}(E_0)$ *and* $b\leq \dim_{\mathrm{p}}(E')\leq \dim_{\mathrm{p}}(E_0)$.

Proof. Since ϖ is Lipschitz, $\dim_{\mathrm{H}}(E') \leq \dim_{\mathrm{H}}(E) \leq \dim_{\mathrm{H}}(E_0)$, with the same inequality for packing dimensions. We set $m = m_0(\cdot \cap E)$ and $m' = m \circ \varpi^{-1}$ that is the pushforward measure of m via ϖ. Since $m(E) > 0$, (307) holds true with m_0 replace by m. Observe that (307) implies that m_0 has no atom. Thus, m has no atom and since there is a finite number of pinching points, (306) entails that (307) holds true for m' which entails $\dim_{\mathrm{H}}(E') \geq a$ and $\dim_{\mathrm{p}}(E') \geq b$ by standard comparison results on Hausdorff and packing measures due to Rogers & Taylor in [37] (Hausdorff case) and Taylor & Tricot in [38] (packing case) in Euclidian spaces and that have been extended in Edgar [23] (see Thm 4.15 and Proposition 4.24 for the Hausdorff case and see Theorem 5.9 for the packing case). ■

C.2 Proof of Lemma 2.10.

Let $h, h' : [0, \infty) \to [0, \infty)$ be two càdlàg processes. Recall from (48) the definition of ζ_h and $\zeta_{h'}$; we assume that ζ_h and $\zeta_{h'}$ are finite. Recall from Section 2.2.2 the definition of the rooted measured tree-like metric spaces coded by a function. We assume that h and h' are as in (a) or (b) in Remark 2.3, namely, either a pure-jump function with finitely many jumps, or a continuous function. Consequently, T_h and $T_{h'}$ are compact spaces. Let $\Pi = ((s_i, t_i))_{1 \leq i \leq p}$ and $\Pi' = ((s'_i, t'_i))_{1 \leq i \leq p}$ be two sequences such that $0 \leq s_i \leq t_i < \zeta_h$ and $0 \leq s'_i \leq t'_i < \zeta_{h'}$; let $\varepsilon, \varepsilon' \in [0, \infty)$. We assume that $\delta \in (0, \infty)$ is such that

$$\forall i \in \{1, \ldots, p\}, \quad |s_i - s'_i| \leq \delta \quad \text{and} \quad |t_i - t'_i| \leq \delta \; . \tag{308}$$

Recall from (53) the definition of the pinched (compact measured) metric spaces $G := G(h, \Pi, \varepsilon)$ and $G' := G(h', \Pi', \varepsilon')$. We want to prove that

$$\boldsymbol{\delta}_{\mathrm{GHP}}(G, G') \leq 6(p+1)\big(\|h - h'\|_\infty + \omega_\delta(h)\big) + 3p(\varepsilon \vee \varepsilon') + |\zeta_h - \zeta_{h'}| \; , \tag{309}$$

where $\boldsymbol{\delta}_{\mathrm{GHP}}$ stands for the pointed Gromov-Hausdorff-Prohorov distance (see (63) for a definition), where $\omega_\delta(h) = \max\big\{|h(s) - h(t)| \, ; \; s, t \in [0, \infty) : |s - t| \leq \delta\big\}$ and where $\|\cdot\|_\infty$ stands for the uniform norm on $[0, \infty)$. Note that h or h' are not necessarily continuous. Several key arguments of the proofs can be found in Le Gall & D. [22] (Lemma 2.3, p. 563), Addario-Berry, Goldschmidt & Broutin [2] (Lemma 21, p. 390) and Abraham, Delmas & Hoscheit [1] (Proposition 2.4); therefore our proof is brief.

We get (308) by bounding the *distorsion* of an explicit *correspondence* between G and G'. Namely, recall that a correspondence $\mathcal{R}$ between the two metric spaces (E, d) and (E', d') is a subset $\mathcal{R} \subset E \times E'$ such that for all $(x, x') \in E \times E'$, $\mathcal{R} \cap (\{x\} \times E')$ and $\mathcal{R} \cap (E \times \{x'\})$ are not empty; the *distorsion of* $\mathcal{R}$ is then given by $\mathrm{dis}(\mathcal{R}) = \sup\{|d(x, y) - d'(x', y')| ; (x, x') \in \mathcal{R}, (y, y') \in \mathcal{R}\}$. We first define a correspondence between T_h and $T_{h'}$. Recall that $p_h : [0, \zeta_h) \to T_h$ and $p_{h'} : [0, \zeta_{h'}) \to T_{h'}$ are the canonical projections and recall that the roots are defined by $p_h(0) = \rho_h$ and $p_{h'}(0) = \rho_{h'}$. We first set

$$\mathcal{R}_0 = \big\{(p_h(t), p_{h'}(t)) ; t \in [0, \infty)\big\} \cup \big\{(p_h(s_i), p_{h'}(s'_i)), (p_h(t_i), p_{h'}(t'_i)) ; 1 \leq i \leq p\big\},$$

where we have adopted the convention that $\rho_h = p_h(t)$ if $t \geq \zeta_h$ and $\rho_{h'} = p_{h'}(t)$ if $t \geq \zeta_{h'}$: indeed, recall that for all $t \geq \zeta_h$ (resp. $t \geq \zeta_{h'}$), $h(t) = 0$ (resp. $h'(t) = 0$), which implies $t \sim_h 0$ (resp. $t \sim_{h'} 0$). Then, $\mathcal{R}_0$ is clearly a correspondence between (T_h, d_h) and $(T_{h'}, d_{h'})$ and we easily check that $\mathrm{dis}(\mathcal{R}_0) \leq 4\big(\|h - h'\|_\infty + \omega_\delta(h)\big)$.

We next set $\boldsymbol{\Pi} = ((p_h(s_i), p_h(t_i)))_{1 \leq i \leq p}$ and $\boldsymbol{\Pi}' = ((p_{h'}(s'_i), p_{h'}(t'_i)))_{1 \leq i \leq p}$; recall that (G, d) (resp. (G', d')) stands for the $(\boldsymbol{\Pi}, \varepsilon)$-pinched metric space associated with (T_h, d_h) (resp. the $(\boldsymbol{\Pi}', \varepsilon')$-pinched metric space associated with $(T_{h'}, d_{h'})$); recall that $d = d_{\boldsymbol{\Pi}, \varepsilon}$ (resp. $d' = d_{\boldsymbol{\Pi}', \varepsilon'}$) is given by (304); we denote by $\varpi : T_h \to G$ and $\varpi' : T_{h'} \to G'$ the canonical projections and we set

$$\mathcal{R} = \big\{(\varpi(x), \varpi'(x')) ; (x, x') \in \mathcal{R}_0\big\} \; .$$

It is easy to check that $\mathcal{R}$ is a correspondence between (G,d) and (G',d'). Moreover, since the pinched metric can be expressed by finite sums as in (304) with at most $2p+1$ terms, we easily check that

$$\text{dis}(\mathcal{R}) \leq (p+1)\text{dis}(\mathcal{R}_0) + 2p(\varepsilon \vee \varepsilon') \leq 4(p+1)\big(\|h-h'\|_\infty + \omega_\delta(h)\big) + 2p(\varepsilon \vee \varepsilon') \; .$$

We next construct an ambient space into which G and G' are embedded: we first set $E = G \sqcup G'$ and we define $d_E : E^2 \to [0,\infty)$ as follows: first $d_{E|G\times G} = d$, $d_{E|G'\times G'} = d'$ and for all $x \in G$ and all $x' \in G'$,

$$d_E(x,x') = \inf \big\{ d(x,z) + \tfrac{1}{2}\text{dis}(\mathcal{R}) + d'(z',x') \, ; (z,z') \in \mathcal{R} \big\} \; .$$

Standard arguments easily imply that d_E is a distance on E. Note that the inclusion maps of resp. G and G' into E are isometries. Since G and G' are compact, so is (E, d_E). Moreover, we easily check that $d_E^{\text{Haus}}(G,G') \leq \frac{1}{2}\text{dis}(\mathcal{R})$. Recall that $\rho = \varpi(\rho_h)$, that $\rho' = \varpi'(\rho_{h'})$ and that $(\rho,\rho') \in \mathcal{R}$; thus, $d_E(\rho,\rho') \leq \frac{1}{2}\text{dis}(\mathcal{R})$.

Denote by $\mathcal{M}_f(E)$ the space of finite Borel measures; recall that for all $\mu, \nu \in \mathcal{M}_f(E)$, their Prohorov distance is $d_E^{\text{Pro}}(\mu,\nu) = \inf\{\eta \in (0,\infty) : \nu(K) \leq \mu(K^\eta) + \eta$ and $\mu(K) \leq \nu(K^\eta) + \eta,$ for all $K \subset E$ compact$\}$; here, $K^\eta = \{y \in E : d_E(y,K) \leq \eta\}$. Recall that m (resp. m') is the pushforward measure of the Lebesgue measure $\texttt{Leb}$ on $[0,\zeta_h)$ (resp. on $[0,\zeta_{h'})$) via the function $\varpi \circ p_h$ (resp. $\varpi' \circ p_{h'}$). Let $K \subset G$ be compact; set $C = (\varpi \circ p_h)^{-1}(K) \cap [0,\zeta_h]$: if h is a pure-jump function with finitely many jumps, C is a finite union of half-open half closed intervals; if h is continuous, so is $\varpi \circ p_h$ and C is also a compact of $[0,\zeta_h]$. We next set $C' = [0,\zeta_{h'}] \cap C$ and $K' = \varpi' \circ p_{h'}(C')$: if h' is continuous, then K' is a compact subset of G'; if h' is pure-jump function with finitely many jumps, then K' is a finite subset of G': it is also a compact subset. Note that $C' \subset (\varpi' \circ p_{h'})^{-1}(K')$. Thus, we get

$$m(K) = \texttt{Leb}(C) \leq \texttt{Leb}(C') + |\zeta_h - \zeta_{h'}| \leq \texttt{Leb}\big((\varpi' \circ p_{h'})^{-1}(K')\big) + |\zeta_h - \zeta_{h'}| = m'(K') + |\zeta_h - \zeta_{h'}|.$$

Then, observe that for all $x' \in K'$, there is $x \in K$ such that $(x,x') \in \mathcal{R}$, which implies $d_E(x,x') \leq \frac{1}{2}\text{dis}(\mathcal{R})$. It implies that $K' \subset K^\eta$, where $\eta = \frac{1}{2}\text{dis}(\mathcal{R})$. By exchanging the roles of m and m', we get $d_E^{\text{Pro}}(m,m') \leq \frac{1}{2}\text{dis}(\mathcal{R}) + |\zeta_h - \zeta_{h'}|$. Thus,

$$\boldsymbol{\delta}_{\text{GHP}}(G,G') \leq d_E^{\text{Haus}}(G,G') + d_E(\rho,\rho') + d_E^{\text{Pro}}(m,m') \leq \tfrac{3}{2}\text{dis}(\mathcal{R}) + |\zeta_h - \zeta_{h'}|$$

which entails (309). This completes the proof of Lemma 2.10. ■

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