The scaling limit of critical random graphs

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Plan

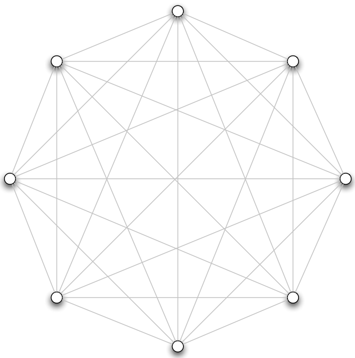
1. Random graphs
2. Exploration and branching processes
3. Large random trees
4. Comparing trees and Gromov–Hausdorff distance
5. Continuum random tree
6. Depth-first search
7. Cycle structure and distances

Erdős–Rényi random graphs

- n labelled vertices $\{1, 2, \ldots, n\}$
- $G_{n,p}$: each edge is present with probability $p \in [0, 1]$

Possible construction:

- each edge e gets an independent $[0, 1]$-uniform weight w_e
- $G = (V, E)$ with $V = \{1, \ldots, n\}$ and $E = \{e : w_e \leq p\}$.



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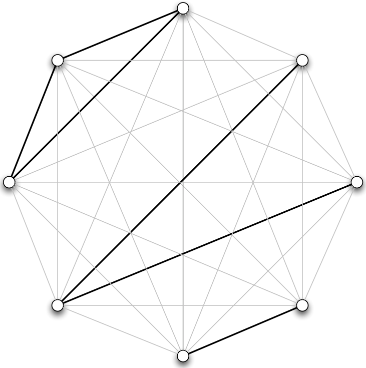
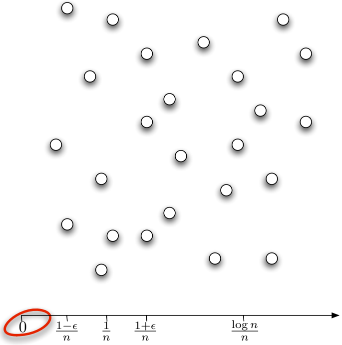


Evolution of random graphs

Different regimes:



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Different regimes:

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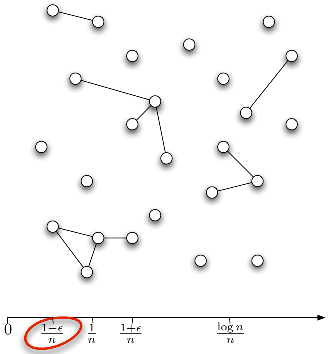

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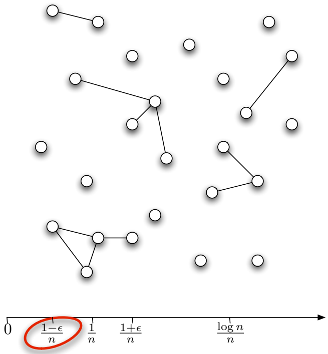

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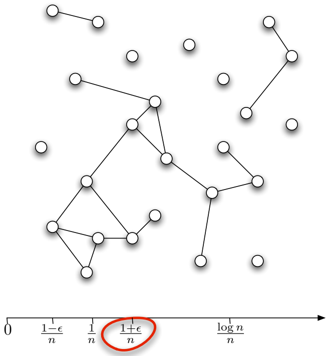

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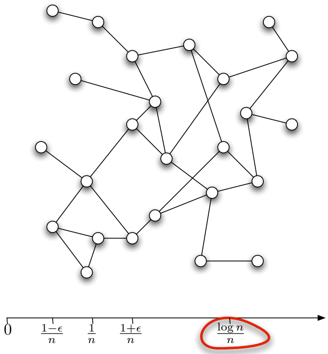
- $p = \frac{\log n}{n}$: the graph is connected



Random graphs and branching processes

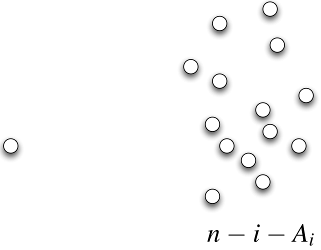
Exploration of the component structure:

- Active A, Explored E
- Start from one active vertex $E_0 = 0$, $A_0 = 1$, $U_0 = n - 1$
- $E_{i+1} = E_i + 1$ and $A_{i+1} = A_i - 1 + Bin(n - A_i - i, p)$

Explanation of the regimes for $p = c/n$

Number of new vertices $\mathbf{E}Bin(n - i, p) \approx c$:

- $c < 1$: subcritical
- $c = 1$: critical
- $c > 1$: supercritical



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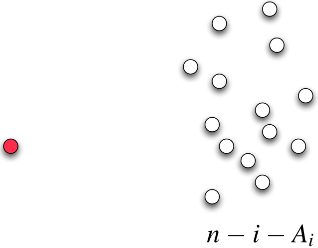

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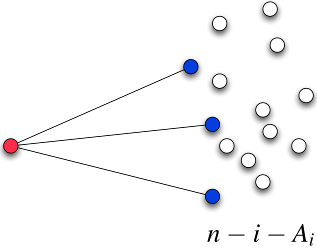

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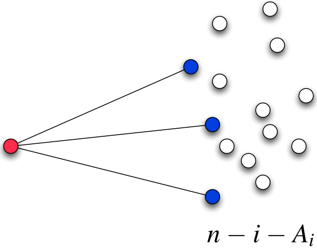

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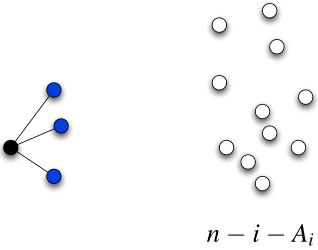

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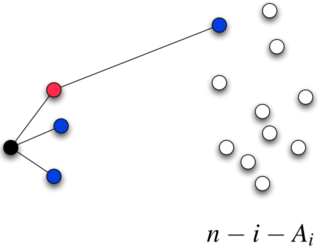

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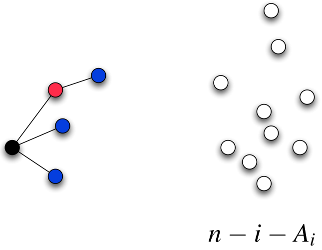

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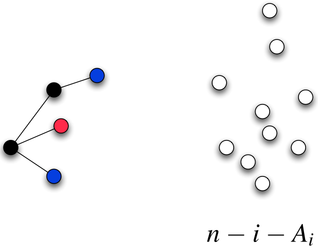

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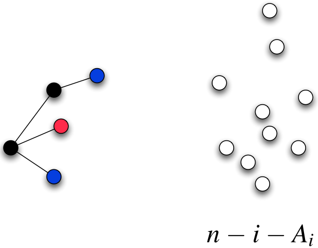

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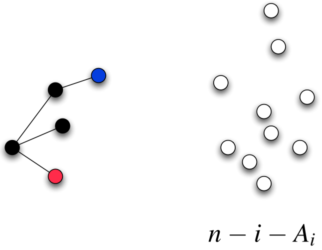

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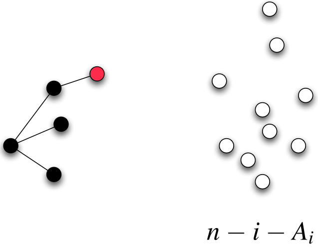

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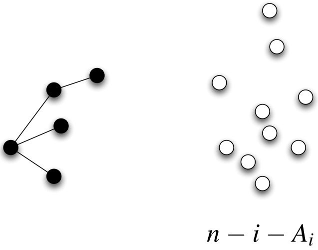

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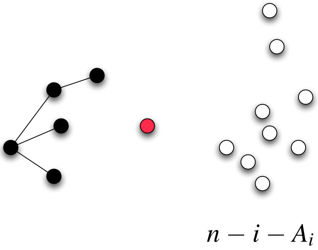

Inside the critical window

Bollobás, Aldous, Łuczak–Pittel–Wierman

Correct scaling

$p = 1/n + \lambda n^{-4/3}$ with $\lambda \in \mathbb{R}$

Analyze the walk (i, A_i), $i \geq 1$. The steps are independent and

$$A_{i+1} - A_i \approx Bin(n-i, p) - 1$$

So for $i \sim tn^{2/3}$,

$$\frac{\mathbf{E} A_{tn^{2/3}}}{n^{1/3}} = \lambda t - \frac{t^2}{2} + o(1)$$

And CLT for martingales:

$$\frac{A_{tn^{2/3}}}{n^{1/3}} \to \lambda t - \frac{t^2}{2} + W(t)$$

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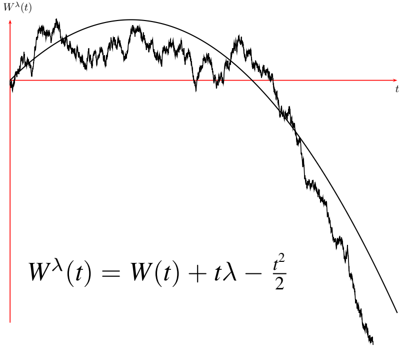
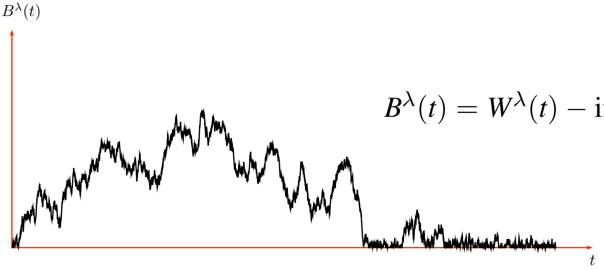


Inside the critical window: what happens

S_i^n size of the i-th largest component of $G_{n,p}$

Theorem (Aldous)

$n^{-2/3}\mathbf{S}^n = (n^{-2/3}S_1^n, n^{-2/3}S_2^n, \dots) \xrightarrow[n\to\infty]{d} S = (S_1, S_2, \dots)$ as a sequence in $\ell^2_{\searrow} = \{x = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i\geq 1} x_i^2 < \infty\}$.



$$B^\lambda(t) = W^\lambda(t) - \inf_{0<s<t} W^\lambda(s)$$

Strategy to describe random graphs

1. Decompose into components
2. Extract a tree from each component
3. Describe the trees
4. Describe how to put back surplus edges

Random Cayley trees

Rényi–Szekeres, Flajolet–Odlyzko

Cayley tree: tree on $\{1,\ldots,n\}$ picked uniformly among n^{n-2} labelled trees

- Iteration $T_{h+1}(z) = z\exp(T_h(z))$ and $T_0 = z$
- Galton–Watson with $\text{Poisson}(1)$ progeny conditioned on the size
- Aldous–Broder random walk construction

Distribution of distances $D_{i,j}$ between nodes i and j

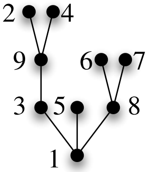
$$\mathbf{P}\left\{D_{1,2} \le x\sqrt{n}\right\} \to \int_0^x ye^{-y^2/2}dy \qquad \text{(Rayleigh)}$$

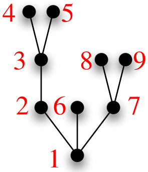
$$\mathbf{P}\left\{\max_i D_{1,i} \le x\sqrt{2n}\right\} \to 4x^{-3}\pi^{5/2}\sum_{k\ge 1} k^2 e^{-\pi^2k^2/x^2} \qquad \text{(theta)}$$

Representation of trees

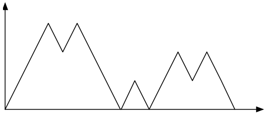
A canonical order of nodes:

- sort children by increasing label
- Depth-first order

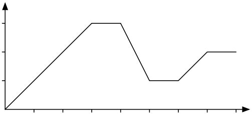




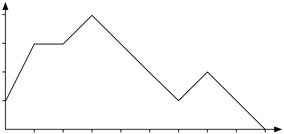
Three different encodings of trees as nonnegative paths:



Contour process C



Height process H



Depth-first walk X

Walks associated with large random trees

Let T_n be a Cayley tree of size n.

Theorem (Marckert–Mokkadem)

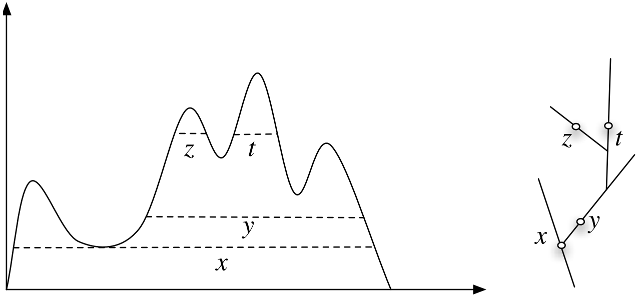
Let $e = (e(t), 0 \leq t \leq 1)$ be a standard Brownian excursion. Then,

$$\left(\frac{X(\lfloor n \cdot \rfloor)}{\sqrt{n}}; \frac{H(\lfloor n \cdot \rfloor)}{2\sqrt{n}}; \frac{C(\lfloor 2n \cdot \rfloor)}{2\sqrt{n}} \right) \xrightarrow[n \to \infty]{d} (e(\cdot); e(\cdot); e(\cdot))$$

Real trees

Aldous, Le Gall, Evans, ...

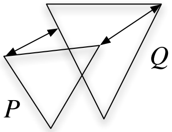
Continuous excursion f: $f(0) = f(\sigma) = 0, f(s) > 0,\ 0 < s < \sigma$.



Comparing metric spaces

Comparing subsets P and Q of a metric space: Hausdorff distance

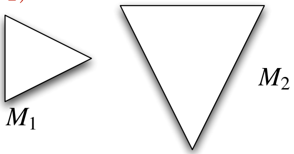
$$d_H(P,Q) = \inf\{\epsilon > 0 : P \subset \cup_{x\in Q} B(x,\epsilon) \text{ and } Q \subset \cup_{y\in P} B(y,\epsilon)\}.$$



Comparing metric spaces: Gromov–Hausdorff distance

$$d_{GH}((M_1,d_1);(M_2,d_2)) = \inf d_H(M_1,M_2),$$

where the infimum is over all metric spaces M containing both (M_1,d_1) and (M_2,d_2).



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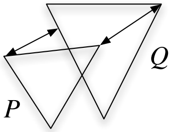

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The Brownian continuum random tree (CRT)

Let $\mathcal{T}(2e)$ be the real tree encoded by a standard Brownian excursion e

Theorem (Aldous)

Let T_n be a Cayley tree of size n, seen as a metric space with the graph distance. Then,

$$n^{-1/2} T_n \xrightarrow[n\to\infty]{d} \mathcal{T}(2e)$$

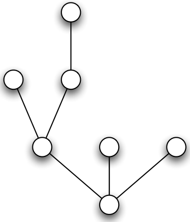
with the Gromov–Hausdorff distance.

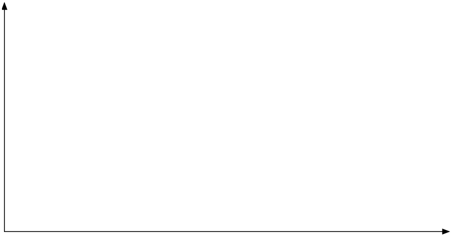
Some remarks:

- extends to all Galton–Watson trees with finite variance progeny
- the trees in random maps
- if infinite variance: Lévy trees, stable trees.

Understanding Depth-first search in graphs

$X(0) = 0$ and $X(i+1) - X(i) = \#$children of i minus 1





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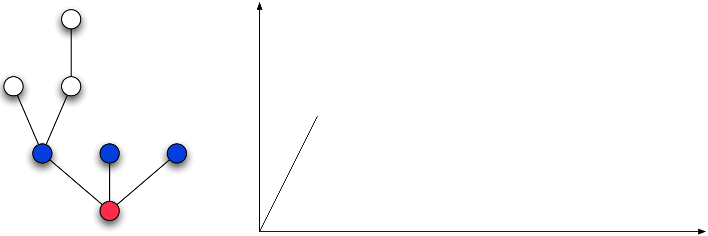

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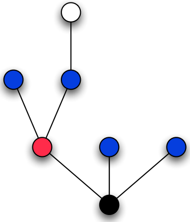

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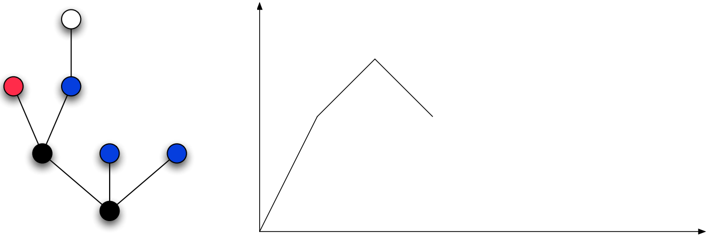

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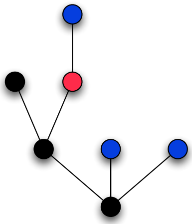

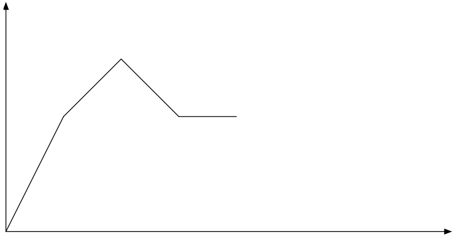

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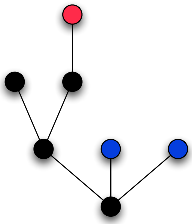

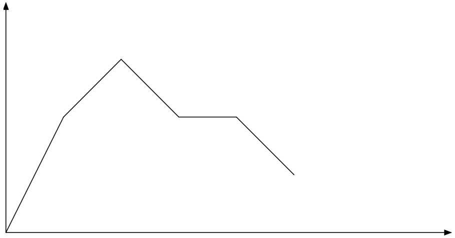

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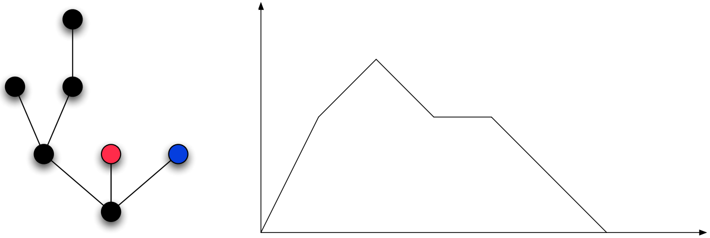

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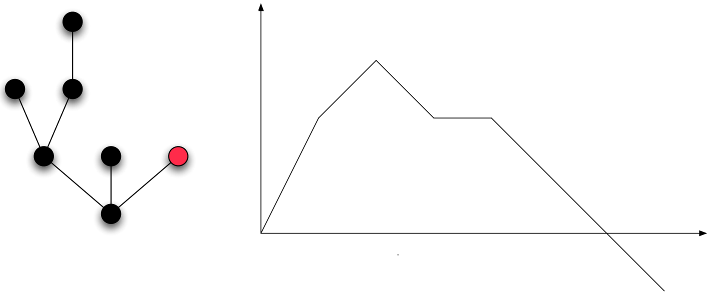

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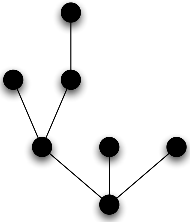

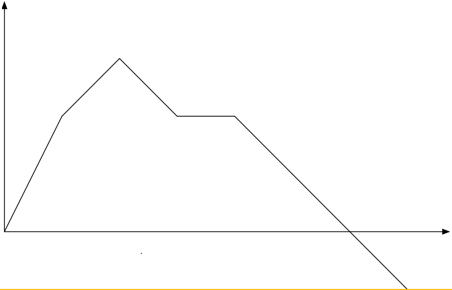

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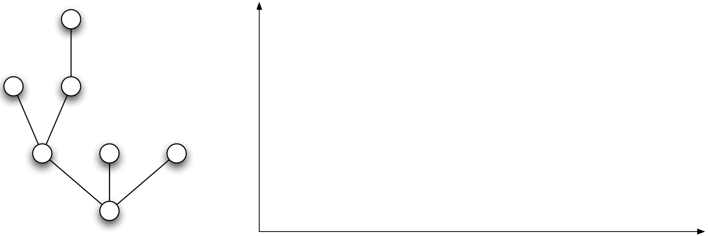

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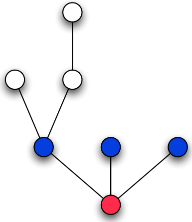

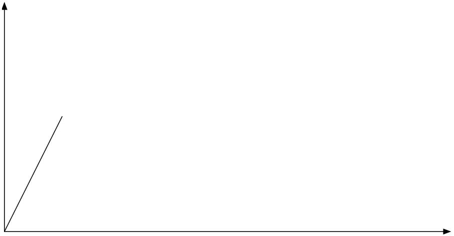

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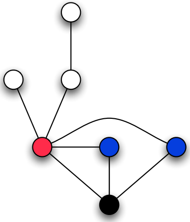

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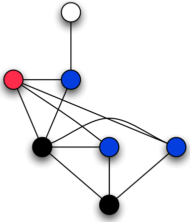

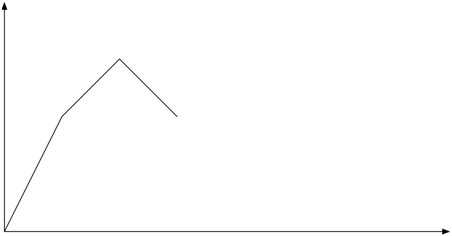

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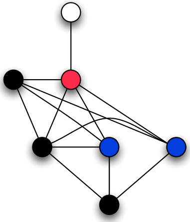

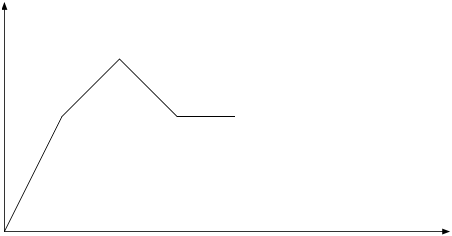

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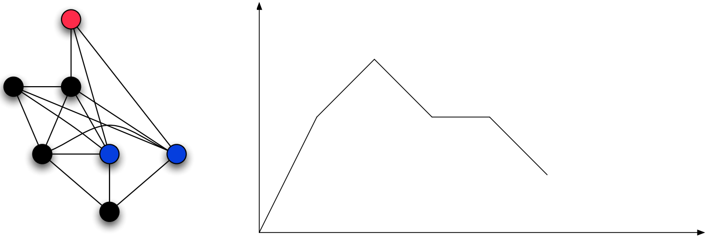

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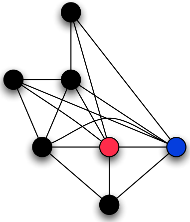

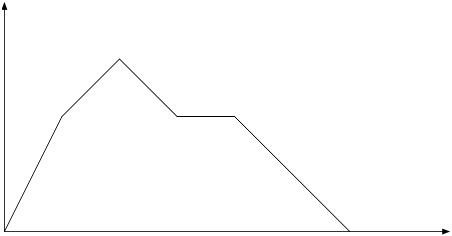

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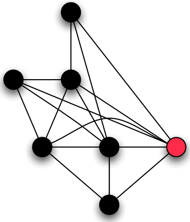

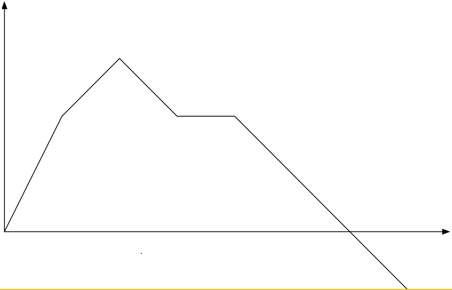

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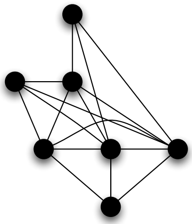

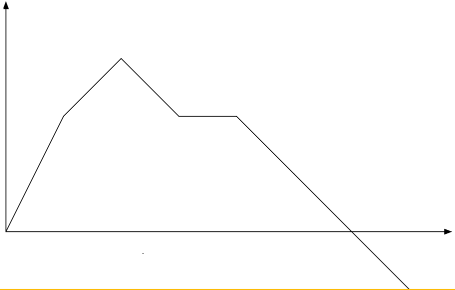

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Canonical tree and its area

Partition the graphs G according to their canonical tree $T(G)$

- Each graph $G \Rightarrow$ one canonical tree T
- Each canoninal tree $T \Rightarrow 2^{a(T)}$ graphs

Uniform connected graph with m vertices

- Pick $\tilde{T}_m = T$ with probability $\propto 2^{a(T)}$
- Add allowed edges each with probability $1/2$.

Uniform connected component of $G_{n,p}$ with m vertices

- Pick $\tilde{T}_m = T$ with probability $\propto (1-p)^{-a(T)}$
- Add each allowed edge with probability p.

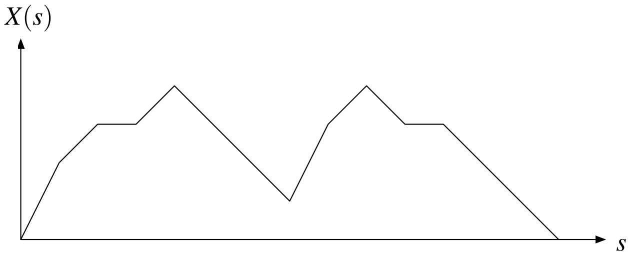
Uniform connected connected graph with m vertices and $m-1+s$ edges

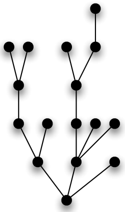
- Pick $\tilde{T}_m = T$ with probability $\propto \binom{a(T)}{s}$
- Add s random allowed edges.

Connected graphs as marked excursions

Bijection between:

- Excursion of length m with some points $(X, \mathcal{P})$.
- Connected graphs on m vertices $G^X(X, \mathcal{P})$





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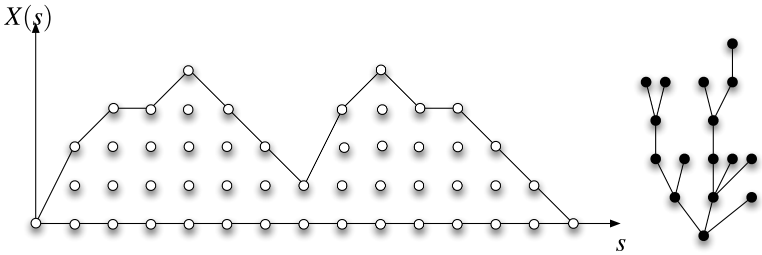

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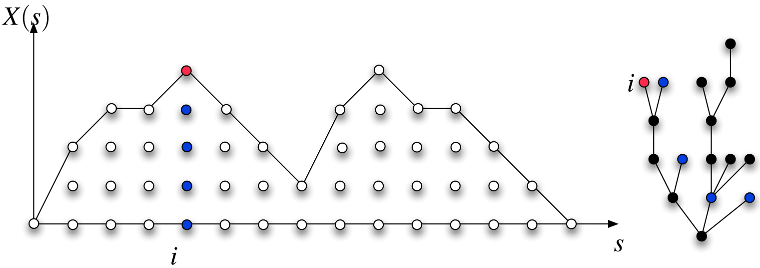

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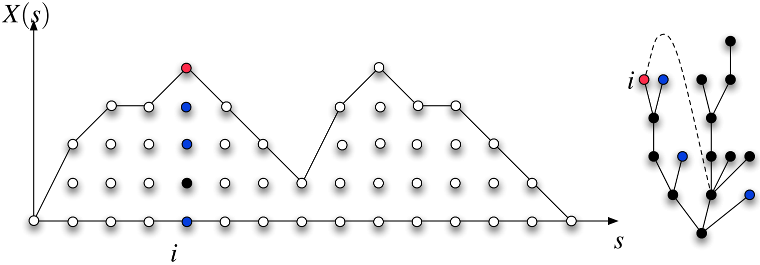

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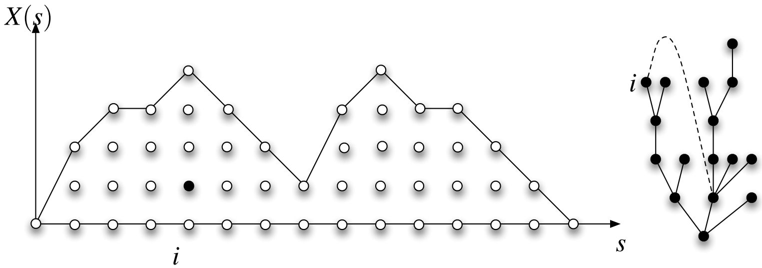


Limit of canonical trees

In the critical regime: $p \sim 1/n$ and $m \sim n^{2/3}$:

$$(1-p)^{-a(T_m)} \sim \exp(m^{-3/2}a(T_m)) \sim \exp(\int_0^1 e(s)ds)$$

Definition (Tilted excursion)

Let e be a standard Brownian excursion:

$$\mathbf{P}\left\{\tilde{e} \in \mathcal{B}\right\} = \frac{\mathbf{E}\left[\mathbf{1}[e \in \mathcal{B}] \exp(\int_0^1 e(s)ds)\right]}{\mathbf{E}\left[\exp(\int_0^1 e(s)ds)\right]}.$$

Let $\tilde{T}_m$ be picked such that $\mathbf{P}\left\{\tilde{T}_m = T\right\} \propto (1-m^{-3/2})^{-a(T)}$

Theorem

$$\left(\frac{\tilde{X}^n(\lfloor n\cdot \rfloor)}{\sqrt{n}};\frac{\tilde{C}^n(\lfloor 2n\cdot \rfloor)}{2\sqrt{n}};\frac{\tilde{H}^n(\lfloor n\cdot \rfloor)}{2\sqrt{n}}\right) \xrightarrow[n\to\infty]{d} (\tilde{e}(\cdot);\tilde{e}(\cdot);\tilde{e}(\cdot)).$$

The limit of tilted trees

Let $\mathcal{T}(\tilde{e}^{(\sigma)})$ the real tree encoded by a tilted excursion $\tilde{e}^{(\sigma)}$ of length σ.

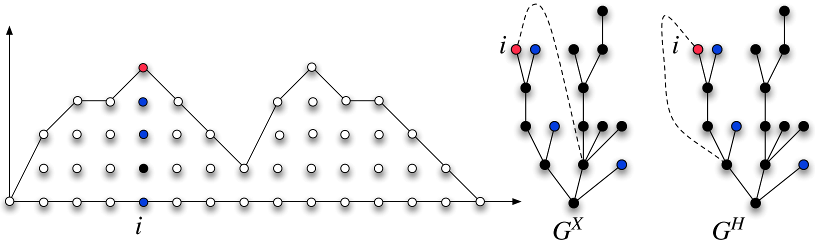
Theorem

Let $p \sim 1/n$ and $m \sim \sigma n^{2/3}$, $\sigma > 0$. Let G_m^p be a uniform connected component of $G_{n,p}$ on m vertices.

$$\frac{T(G_m^p)}{n^{1/3}} \xrightarrow[n\to\infty]{d} \mathcal{T}(2\tilde{e}^{(\sigma)}),$$

with the Gromov–Hausdorff distance.

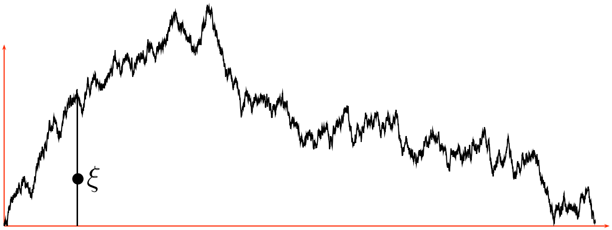
The limit of connected components: convergence



Limit using G^X

$m^{-1/2}\tilde{X}^m(\lfloor m\cdot \rfloor) \to \tilde{e}$ and marks converge to a Poisson point process $\mathcal{P}$

Limit of connected components: construction



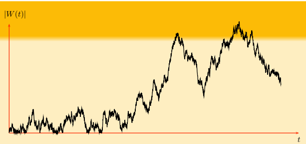
Characterization using G^X

$\mathcal{T}(\tilde{e})$ where each point (ξ_x, ξ_y) of $\mathcal{P}$ identifies the leaf ξ_x with the point at distance ξ_y from the root on the path $[[0, \xi_x]]$.

On reordering components

Representing a reflecting Brownian motion

- sequence of excursions above zero
- collection indexed by local time at zero
- Poisson point process in $(\mathbb{R}^+, \mathcal{E})$, Itô's measure $\mathbb{N}$



$$\mathcal{E} = \{f \in C(\mathbb{R}^+, \mathbb{R}^+) : f(0) = 0, \exists\, \sigma < \infty \text{ such that } f(x) > 0\, \forall\, x \in (0, \sigma), f(x) = 0, x > \sigma\}.$$

B^λ is inhomogeneous. Mesure $\mathbb{N}_t^\lambda$ for excursion starting at t.

$$\mathbb{N}_0^\lambda[\mathbf{1}[\mathcal{B}] | \sigma = x] = \frac{\mathbb{N}\left[\exp\left(\int_0^x W(s)ds\right) \mathbf{1}[\mathcal{B}] \middle| \sigma = x\right]}{\mathbb{N}\left[\exp\left(\int_0^x W(s)ds\right) \middle| \sigma = x\right]}.$$

The limit of critical random graphs

M_i^n the i-th largest connected component of $G_{n,p}$ and S_i^n its size.
$\mathbf{M}^n = (M_1^n, M_2^n, \dots)$ as a sequence of metric spaces, with distance

$$d(\mathbf{A}, \mathbf{B}) = \left(\sum_{i \geq 1} d_{GH}(A_i, B_i)^4 \right)^{1/4}$$

$n^{-2/3}\mathbf{S}^n = (n^{-2/3}S_1^n, \dots)$ in $\ell^2_{\searrow} = \{(x_1, x_x, \dots) : x_1 \geq \dots, \sum_{i \geq 1} x_i^2 < \infty\}$.

Theorem

$$(n^{-1/3}\mathbf{M}^n, n^{-2/3}\mathbf{S}^n) \xrightarrow[n\to\infty]{d} (\mathbf{M}, \mathbf{S}) \qquad \text{where}$$

- $\mathbf{S}$ is the ordered sequence of excursion lengths of B^λ
- Given $S = (S_1, S_2, \dots)$, $(M_1, M_2, \dots)$ are independent $g(\tilde{e}^{(S_i)}, \mathcal{P}_i)$.

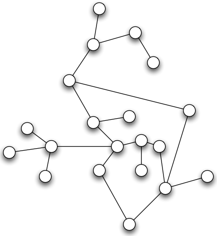
The diameter of critical random graphs

$$D^{(n)} = \max\left\{n^{-1/3}\mathrm{diam}(M_i^{(n)}) : M_i^{(n)} \in \mathbf{M}^{(n)}\right\}.$$

Theorem

- $\forall i$ there is $D_i \geq 0$ with $\mathbf{E}D_i < \infty$ such that $D_i^{(n)} \xrightarrow{d} D_i$
- there exists $D \geq 0$ with $\mathbf{E}D < \infty$ such that $D^{(n)} \xrightarrow{d} D$.

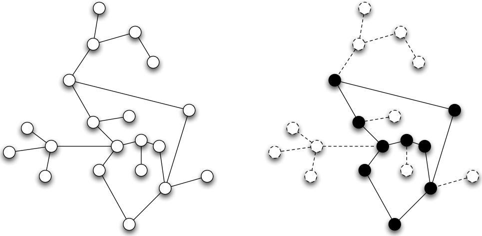
The cycle structure of connected components



A connected component C decomposes as

- one kernel (multigraph) K
- one rooted tree per vertex of K: vertex-trees
- one doubly rooted tree per edge of K: edge-trees

The cycle structure of connected components



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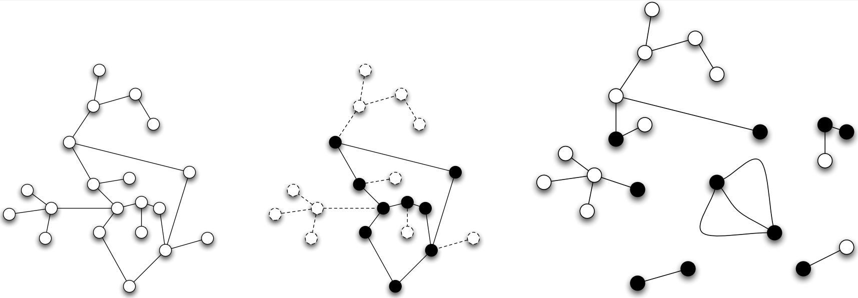

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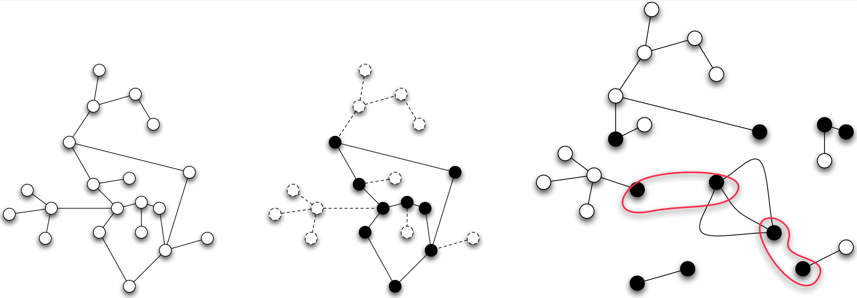

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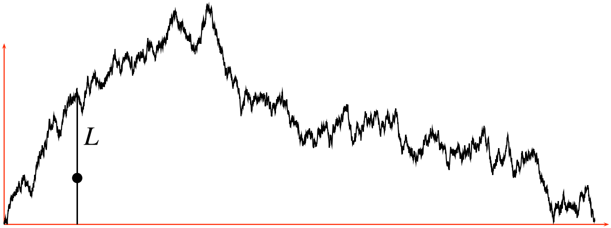
Unicyclic components

Connected component of $G_{n,p}$ with m vertices exactly 1 cycle
L_m the length of the unique cycle

Theorem

$$\mathbf{P}\{L_m \le x\sqrt{m}\} \to \mathbf{P}\{L \le x\} = \sqrt{\frac{2}{\pi}}\int_0^x e^{-y^2/2}dy$$



Limit distributions in random graphs

C a connected component of $G_{n,p}$ with m vertices and ≥ 2 cycles
$K(C)$ the kernel (unlabelled multigraph) with k edges
N_0: total number of vertices in vertex-trees
$N_1, N_2, \ldots, N_k$: number of vertices of the edge-trees $T_(e_1), T(e_2), \ldots, T(e_k)$

Theorem

- $\mathbf{P}\{K(C) \text{ is } 3-\text{regular}\} = 1 - O(\sqrt{m})$
- N_0 is bounded in probability
- Given any kernel K with k edges,
$$(\frac{N_1}{m}, \frac{N_2}{m}, \ldots, \frac{N_k}{m}) \xrightarrow[n\to\infty]{d} \text{Dirichlet}(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$$
- Given the sizes $(N_1, \ldots, N_k)$ the trees $T(e_1), T(e_2), \ldots, T(e_k)$ are independent Cayley trees.