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Systèmes de particules, EDP stochastiques et modèle d'Anderson continu

Spécialité MATHÉMATIQUES

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Spécialité : Mathématiques

présenté par

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Le mot de la fin - bien sûr - pour Laure et Yaelle qui savent si bien me ramener à ~~une autre réalité~~ la réalité. Et je ne saurais les en remercier assez.

[1]qu'on se rassure, cela ne concerne pas mes propres articles.
[2]mais je me suis bien gardé de le signaler aux rapporteurs.

Summary

This document presents the research activity that I carried out as a postdoc of Martin Hairer at Warwick University from September 2013 to August 2015, and since September 2015 as a maître de conférences (assistant professor) at Université Paris-Dauphine. The results obtained during my PhD, defended in October 2013, are not presented here.
I have chosen to split the material into three chapters:

- Chapter I - Particle systems.

  This chapter mainly focuses on a particle system called the simple exclusion process on the segment. I will first present a classification [Lab18] of the various scaling limits that the process displays according to the strength of the asymmetry imposed on its jump rates, and a convergence [Lab17] to the KPZ equation that presents specificities compared to previous results in the literature. Then I will expose a rather complete analysis of the asymptotic of the mixing times of the process, with a particular emphasis on the dependency of these mixing times on the strength of the asymmetry: these results [LL19a, LL19b] were obtained in collaboration with Hubert Lacoin (IMPA).
  The chapter also contains a few results on particle systems that share similarities with the simple exclusion process: a scaling limit [EL15] for reflected interfaces obtained in collaboration with Alison M. Etheridge (Univ. of Oxford), and a study [CLL19] of the mixing times of the adjacent walk on the continuous simplex carried out in collaboration with Pietro Caputo (Univ. Roma Tre) and Hubert Lacoin (IMPA).

- Chapter II - Singular SPDEs and regularity structures.

  This chapter is concerned with stochastic PDEs that are too singular for classical theories to apply, and which necessitate the use of elaborate renormalisation theories. Almost all my research activity on this topic relies on the theory of regularity structures. I will present a series of works in collaboration with Martin Hairer (Imperial College) on the construction of multiplicative stochastic heat equations on the full space [HL15, HL18], and on a generalisation [HL17] of the functional spaces of the theory of regularity structures. Finally, I will present a result [GL19] in collaboration with Paul Gassiat (Univ. Paris-Dauphine)

on the existence of densities for the Φ^4_3 equation on a torus, another example of SPDE that necessitates renormalisation.

- Chapter III - The continuous Anderson model.

  This chapter is devoted to the study of a random Schrödinger operator sometimes called the continuous Anderson Hamiltonian. First, I will present a result [Lab19] on the construction of this operator using the theory of regularity structures. Then, I will detail a thorough study [DL19b, DL19a] of the spectrum of this operator in dimension 1, in collaboration with Laure Dumaz (Univ. Paris-Dauphine). Finally, I will present another result [DL19c] in collaboration with Laure Dumaz on the bottom of the spectrum of the Stochastic Airy Operator at large temperature: an operator that arises in random matrices, and that shares similarities with the Anderson Hamiltonian.

Although some connections are mentioned between the chapters, each of them can be read independently of the others.

The results presented in this document are taken from research articles that are listed below. The articles that were written during my PhD are not presented here and do not appear on that list.

Publications and preprints

[CLL19] P. Caputo, C. Labbé, and H. Lacoin. Mixing time of the adjacent walk on the simplex. arXiv e-prints (2019). arXiv:1904.01088.

[DL19a] L. Dumaz and C. Labbé. Localization crossover for the continuous Anderson Hamiltonian in 1-d. in preparation (2019+).

[DL19b] L. Dumaz and C. Labbé. Localization of the continuous Anderson Hamiltonian in 1-D. Probability Theory and Related Fields (2019). doi:10.1007/s00440-019-00920-6.

[DL19c] L. Dumaz and C. Labbé. The stochastic Airy operator at large temperature. arXiv e-prints (2019). arXiv:1908.11273.

[EL15] A. M. Etheridge and C. Labbé. Scaling limits of weakly asymmetric interfaces. Comm. Math. Phys. 336, no. 1, (2015), 287–336. doi:10.1007/s00220-014-2243-2.

[GL19] P. Gassiat and C. Labbé. Existence of densities for the dynamic Φ^4_3 model. Annales de l'Institut Henri Poincaré , no. to appear(2019). arXiv:1711.08332.

[HL15] M. Hairer and C. Labbé. A simple construction of the continuum parabolic Anderson model on $\mathbf{R}^2$. Electron. Commun. Probab. 20, (2015), no. 43, 11. doi:10.1214/ECP.v20-4038.

[HL17] M. Hairer and C. Labbé. The reconstruction theorem in Besov spaces. J. Funct. Anal. 273, no. 8, (2017), 2578–2618. doi:10.1016/j.jfa.2017.07.002.

[HL18] M. Hairer and C. Labbé. Multiplicative stochastic heat equations on the whole space. J. Eur. Math. Soc. (JEMS) 20, no. 4, (2018), 1005–1054. doi:10.4171/JEMS/781.

[Lab17] C. Labbé. Weakly asymmetric bridges and the KPZ equation. Comm. Math. Phys. 353, no. 3, (2017), 1261–1298. doi:10.1007/s00220-017-2875-0.

[Lab18] C. Labbé. On the scaling limits of weakly asymmetric bridges. Probab. Surv. 15, (2018), 156–242. doi:10.1214/17-PS285.

[Lab19] C. Labbé. The continuous Anderson hamiltonian in $d \leq 3$. Journal of Functional Analysis 277, no. 9, (2019), 3187 – 3235. doi:https://doi.org/10.1016/j.jfa.2019.05.027.

[LL19a] C. Labbé and H. Lacoin. Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. Ann. Probab. 47, no. 3, (2019), 1541–1586. doi:10.1214/18-AOP1290.

[LL19b] C. Labbé and H. Lacoin. Mixing time and cutoff for the weakly asymmetric simple exclusion process. Annals of Applied Probability , no. to appear(2019+).

Contents

Chapter I

Particle systems

Most of this chapter is concerned with the so-called Simple Exclusion Process (SEP) on a segment. We present results on the scaling limits of this process in Sections I.1 and I.2, and on the asymptotic behaviour of its mixing times in Section I.3.
In Section I.4, we consider a variant of the SEP in which the associated height function (see below for a definition) is reflected on a hard wall and we present a result on the scaling limit of this process. Finally, Section I.5 is devoted to the adjacent walk on the continuous simplex (which is a continuous-space analogue of the SEP) and presents a result on the asymptotic behaviour of its mixing times.

We now recall the definition of the SEP[1] on a segment. Consider k particles on the lattice $[\![1, N]\!] := \{1, \ldots, N\}$, satisfying the so-called exclusion rule that prevents any two particles from sharing a same site, and evolving according to the following continuous-time dynamics: at rate p (resp. q) any particle jumps to its right (resp. to its left) unless the target site is already occupied. We also impose a zero-flux boundary condition to the particle system: a particle at site N (resp. at site 1) cannot jump to its right (resp. to its left). We refer to Figure I.1 for an illustration.
This system can also be seen as a collection of k (possibly biased) simple random walks in continuous-time. These random walks are essentially independent: the only interaction comes from the fact that they are reflected on each other.

This evolving particle system will be encoded by the processes $\eta_t(i)$ where, for any $i \in [\![1, N]\!]$, $\eta_t(i) = 1$ if the i-th site is occupied by a particle at time t, and $\eta_t(i) = 0$ otherwise. Although p and q do not need to sum up to 1 (recall that they are continuous-time jump rates), it is convenient

[1] I will never use the abbreviations SSEP, ASEP, WASEP in this document, and SEP will denote the generic situation where no assumption is made a priori on the jump rates.

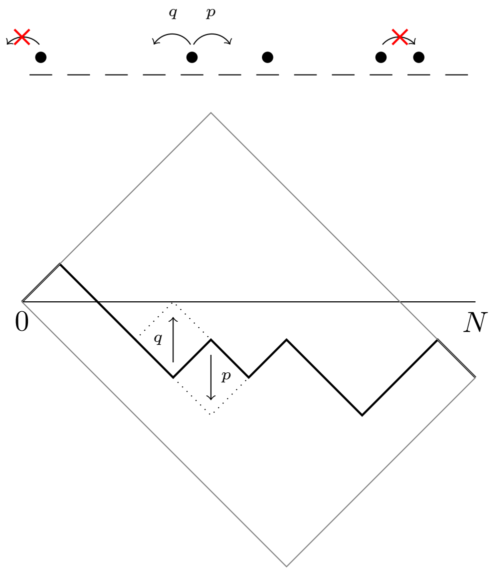

Figure I.1: A particle configuration and its corresponding height function. In light grey, the maximal and minimal height functions.

to impose

$$p + q = 1 \ , \quad p \geq 1/2 \ .$$

This implies that if the jump rates are biased, then the bias is to the right.

A convenient, equivalent representation of a particle configuration η is given by its associated so-called height function h. This is a lattice path that starts at 0:

$$h(0) = 0 \ ,$$

and makes ± 1 steps according to the presence/absence of a particle at the corresponding site:

$$h(x) - h(x-1) = 2\eta(x) - 1 \ , \quad x \in [\![1, N]\!] \ .$$

Note that the value at the endpoint is deterministic: $h(N) = 2k - N$. Note also that the set of all admissible height functions is endowed with a partial order for which the maximal[2] element $\wedge : x \mapsto \min(x, 2k - x)$ corresponds to the configuration with all particles to the left, and the minimal element $\vee : x \mapsto \max(-x, x - 2N + 2k)$ corresponds to the configuration with all particles to the right, see again Figure I.1.

The dynamics of the particles can easily be rephrased in terms of height functions: every upwards (resp. downwards) corner flips into its opposite

[2]The symbol $\wedge$ refers to the graph of the maximal height function which resembles a wedge: it is unfortunate that this symbol coincides with the usual notation for the minimum...

at rate p (resp. at rate q), see Figure I.1. In the case $p > 1/2$ where the jump rates are biased, this height function has therefore a tendency to go downwards. The height function at time t will be denoted h_t.

Since we are dealing with an irreducible Markov process on a finite state-space, it admits a unique invariant measure. Even when the jump rates are biased, the process is reversible w.r.t. its invariant measure: this is a consequence of our zero-flux boundary condition on the particle system! The detailed balance condition then immediately yields the following expression for the invariant measure:

$$\mu_N(h) = \frac{1}{Z_N}\Big(\frac{p}{1-p}\Big)^{-\frac{1}{2}A(h)} \quad \text{where } A(h) = \int_0^N (h - \vee)(x)dx \ .$$

Here Z_N is a normalisation constant, and $A(h)$ is the area between the height function h and the minimal height function $\vee$.
In the symmetric regime $p = q = 1/2$, this is nothing but the uniform measure on the set of admissible configurations, while in the asymmetric regime $p > q$ this measure favours height functions which are close to the minimal height function $\vee$.

I.1 A classification of the scaling limits of the SEP

Several questions can be asked on the SEP, among which: what does its invariant measure look like at large scale ? what is its hydrodynamic behaviour ? what are the fluctuations ? The literature contains already many answers to these questions in different settings (whole line, segment, torus, segment with reservoirs) and different regimes of asymmetry (symmetric/weakly asymmetric/asymmetric), see for instance [44].
The results presented in Sections I.1 and I.2 were driven by the desire to obtain a “complete” understanding of the SEP in a single setting, the segment, according to the strength of the asymmetry imposed to the jump rates. While some results were close to existing results in the literature, it turns out that some regimes of asymmetry display phenomena that were not observed before.

In Sections I.1 and I.2, we will always assume that[3]

$$p = \frac{1}{2} + \frac{1}{N^\beta}\ ,$$

for some $\beta \in (0,\infty]$. In other words, we will assume that the asymmetry is either weak $\beta \in (0,\infty)$, or null $\beta = +\infty$. The asymmetric regime, in which

[3]I opted for this parametrisation in [Lab18, Lab17] but the results still hold without imposing a polynomial (in N) decay of the bias $p-q$. In Section I.3 no a priori assumption will be made on the bias $p-q$ when it goes to 0.

p is larger than 1/2 and independent of N, will be considered later on.
To reduce the number of parameters, we also chose to work in a "centered" situation where the number of particles satisfies $k = N/2$ (implicitly, N will be even in the sequel). Graphically, the height functions are then bridges from $(0,0)$ to $(N,0)$ and the two extremal height functions $\wedge$ and $\vee$ are the opposite of one another. However, all the results can be extended mutatis mutandis to the case of a non-trivial density of particles, that is, $k/N \to \alpha$ for some $\alpha \in (0,1)$.

I.1.1 Scaling limit of the invariant measure

The main result obtained in [Lab18] on the scaling limit of the invariant measure is a Central Limit Theorem for the height function h under μ_N. Its statement can be informally presented as follows (see also Figure I.2):

- For $\beta > 1$, the mean of $h(xN)$ scales like $-N^{2-\beta}x(1-x)$, while the fluctuations around the mean are of order $\sqrt{N}$ and converge to a Brownian bridge on $[0,1]$. For $\beta > 3/2$, the mean is negligible compared to the fluctuations.
- For $\beta = 1$, the mean of $h(xN)$ scales like $-N\int_0^x \tanh(1-2y)dy$ and the fluctuations are given by a distorted Brownian bridge on $[0,1]$.
- For $\beta < 1$, the mean of $h(x)$ differs from the minimal height function $\vee(x)$ essentially only in a window of size $\mathcal{O}(N^\beta)$ around site $N/2$: in this window, the difference is of order N^β while the fluctuations scale like $N^{\beta/2}$ and converge to a Gaussian process on $\mathbb{R}$ which is a distorted Brownian bridge on $[0,1]$.

To provide the formal statement, we introduce

$$u_N(x) := \begin{cases} \frac{h(xN)-\Sigma_\beta^N(x)}{\sqrt{N}} , & x \in [0,1] , \quad \beta \geq 1 , \\ \frac{h(N/2+xN^\beta)-\Sigma_\beta^N(x)}{N^{\beta/2}} , & x \in \mathbb{R} , \quad \beta \in (0,1) , \end{cases}$$

where Σ_β^N asymptotically coincides with the mean of h under μ_N, and is defined as follows. For $\beta \geq 1$ we set

$$\Sigma_\beta^N\Big(\frac{\ell}{N}\Big) = -\sum_{i=1}^{\ell} \tanh\Big(\frac{N-2i+1}{N^\beta}\Big) , \quad \ell \in [\![0,N]\!] ,$$

and for $\beta \in (0,1)$

$$\Sigma_\beta^N\Big(\frac{\ell - N/2}{N^\beta}\Big) = -\sum_{i=1}^{\ell} \tanh\Big(\frac{N-2i+1}{N^\beta}\Big) , \quad \ell \in [\![0,N]\!] .$$

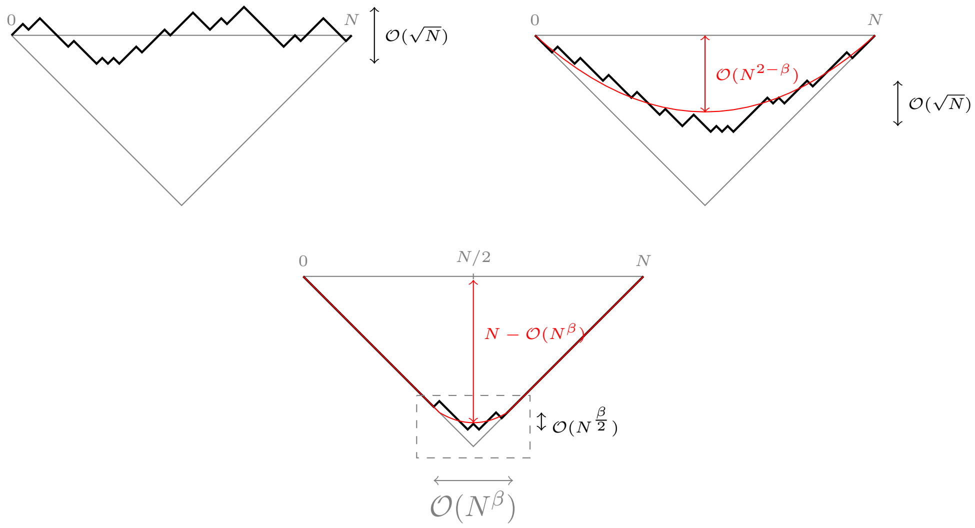

Figure I.2: Upper left $\beta > 3/2$, upper right $\beta \in [1, 3/2]$, bottom $\beta \in (0,1)$. The red curve is Σ_β^N: in the first case, it is negligible compared to the fluctuations so we have not drawn it.

Theorem I.1 ([Lab18]). Under the invariant measure μ_N, we have

$$u_N \overset{(d)}{\Longrightarrow} B_\beta \ , \quad N \to \infty \ .$$

When $\beta > 1$, the process B_β is a Brownian bridge on $[0,1]$. When $\beta = 1$, $(\sqrt{\operatorname{cotanh}(1)} B_\beta(r_1(x)), x \in [0,1])$ is a Brownian bridge where $r_1 : [0,1] \to [0,1]$ is defined by

$$r_1(x) = \frac{1}{2}(1 + \operatorname{artanh}((2x-1)\tanh(1))) \ .$$

When $\beta < 1$, $(B_\beta(r_\beta(x)), x \in [0,1])$ is a Brownian bridge where $r_\beta : [0,1] \to [-\infty, \infty]$ is defined by

$$r_\beta(x) = \frac{1}{2}\operatorname{artanh}(2x-1) \ .$$

To prove this result, we followed a strategy developed in a work [24] of Dobrushin and Hryniv where the fluctuations of a random walk conditioned to have a large area are derived. Let π_N be the measure induced by a simple random walk on $[\![1, N]\!]$ (under π_N, the walk does not necessarily come back to 0 after N steps). One notes that μ_N is obtained by tilting the measure π_N by $(p/q)^{-A(h)/2}$ and by conditioning this measure on the event $h(N) = 0$. This conditioning breaks the independence of the increments of the walk, making the measure μ_N not very convenient to deal with.
To circumvent this, one considers the auxiliary measure ν_N obtained by

tilting the measure π_N by $(p/q)^{-A(h)/2+\lambda h(N)}$ for some parameter λ chosen in such a way that, under ν_N, the expectation of $h(N)$ vanishes. In other words, we have replaced the "almost sure" conditioning by an "average" conditioning: the advantage being that the increments of the walk are still independent under ν_N. Then, one obtains a Local Limit Theorem under ν_N using arguments à la Gnedenko-Kolmogorov and one can then condition ν_N on the event $h(N) = 0$ whose probability can be evaluated thanks to the LLT.

Note that for $\beta = 0$ (asymmetric regime), the fluctuations of h under μ_N are of order 1 uniformly over N so that there is no analogue of Theorem I.1 in that regime.

Further results on the invariant measure will be collected in Section I.3.

I.1.2 Fluctuations at equilibrium

A natural subsequent question is to investigate the dynamics starting from μ_N. We look at the fluctuations of the height function at the same scale as in the CLT presented above. The scaling in time is taken diffusive. In the limit, we obtain a stochastic heat equation driven by a space-time white noise ξ. Here is the precise statement.

For $\beta \geq 1$, we set

$$u_N(t,x) := \frac{h(tN^2, xN) - \Sigma_\beta^N(x)}{\sqrt{N}} \;, \quad x \in [0,1] \;,\; t \geq 0 \;,$$

while for $\beta \in (0,1)$, we set

$$u_N(t,x) := \frac{h(tN^{2\beta}, N/2 + xN^\beta) - \Sigma_\beta^N(x)}{N^{\frac{\beta}{2}}} \;, \quad x \in \mathbb{R} \;,\; t \geq 0 \;.$$

Theorem I.2 ([Lab18]). Assume that the process starts from the invariant measure μ_N. Then, as $N \to \infty$, the process u_N converges in distribution to the process u where

1. For $\beta \in (1,\infty)$, u solves

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u + \xi \;, \quad x \in (0,1) \;, \\ u(t,0) = u(t,1) = 0 \;, \end{cases}$$

   started from an independent realisation of B_β,

2. For $\beta = 1$, u solves

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u + 2\partial_x \Sigma_1 \, \partial_x u + \sqrt{1-(\partial_x\Sigma_1)^2}\, \xi \;, \quad x \in (0,1) \;, \\ u(t,0) = u(t,1) = 0 \;, \end{cases}$$

started from an independent realisation of B_1, and where

$$\Sigma_1(x) = \lim_{N\to\infty} \Sigma_1^N(x)/N = -\int_0^x \tanh(1-2y)dy \ .$$

3. For $\beta \in (0,1)$, u solves

$$\partial_t u = \frac{1}{2}\partial_x^2 u + 2\partial_x\Sigma_\beta\, \partial_x u + \sqrt{1-(\partial_x\Sigma_\beta)^2}\,\xi \ , \quad x \in \mathbb{R} \ ,$$

started from an independent realisation of B_β, and where

$$\Sigma_\beta(x) = \lim_{N\to\infty} (\Sigma_\beta^N(x) + N/2)/N^\beta = -x - \int_{-x}^{\infty} \big(\tanh(2y) - 1\big)dy \ .$$

In all cases, convergence holds in the Skorohod space $\mathbb{D}([0,\infty),\mathcal{C}(I))$ with $I = [0,1]$ or $\mathbb{R}$.

Such a result is rather standard in the field of scaling limits of particle systems. At a technical level, the main ingredient is a Boltzmann-Gibbs principle, that allows to replace space-time averages of some non-linear functionals of the density field by linear functionals of the density field.

I.1.3 Hydrodynamic limit

We turn to the macroscopic behaviour of the evolving height function when it starts far from equilibrium. By macroscopic, we mean that we rescale the space interval $[0,N]$ onto $[0,1]$ and we rescale the height function by a prefactor $1/N$. On the other hand, the relevant time-scale depends on the strength of the asymmetry. We set:

$$m_N(t,x) := \frac{h(tN^{(1+\beta)\wedge 2}, xN)}{N} \ , \quad t \geq 0 \ , \quad x \in [0,1] \ .$$

Note that the scaling in time is N^2 for $\beta \geq 1$ and $N^{1+\beta}$ for $\beta < 1$.

Theorem I.3. [[Lab17, Lab18]] Assume[4] that the sequence of initial profiles $m_N(0,\cdot)$ converges to some deterministic function m_0. The process m_N converges in probability in the Skorohod space $\mathbb{D}([0,\infty),\mathcal{C}([0,1]))$ to the deterministic process m where:

1. If $\beta \in (1,\infty]$, m is the unique solution of the heat equation with Dirichlet b.c.

$$\begin{cases} \partial_t m = \frac{1}{2}\partial_x^2 m \ , \\ m(t,0) = m(t,1) = 0 \ , \quad m(0,\cdot) = m_0(\cdot) \ . \end{cases} \tag{I.1}$$

[4] This is a harmless assumption since the process m_N takes values in the space of 1-Lipschitz functions on $[0,1]$, which is compact for the topology of uniform convergence.

2. If $\beta = 1$, m is the solution of the following heat equation with non-linear drift with Dirichlet b.c.

$$\begin{cases} \partial_t m = \frac{1}{2}\partial_x^2 m + (\partial_x m)^2 - 1 \ , \\ m(t,0) = m(t,1) = 0 \ , \quad m(0,\cdot) = m_0(\cdot) \ . \end{cases} \tag{I.2}$$

3. If $\beta < 1$, m is the solution of the following Hamilton-Jacobi equation with Dirichlet b.c.

$$\begin{cases} \partial_t m = (\partial_x m)^2 - 1 \ , \\ m(t,0) = m(t,1) = 0 \ , \quad m(0,\cdot) = m_0(\cdot) \ . \end{cases} \tag{I.3}$$

We observe a competition between two terms: the Laplacian and a non-linearity. As the strength of the asymmetry increases, the non-linearity becomes predominant: this produces a crossover from parabolic PDEs ($\beta \geq 1$) to a hyperbolic PDE ($\beta < 1$). These equations have drastically different behaviours:

- the parabolic PDEs (I.1) and (I.2) take infinite time to reach their equilibrium profiles given respectively by the null function and by Σ_1,
- the hyperbolic PDE (I.3) reaches its equilibrium profile, given by the map $\vee : x \mapsto \max(-x, x-1)$, in finite time.

This observation will be of importance for the study of the mixing times of the SEP, see Section I.3.

The above two parabolic PDEs are well-posed. On the other hand, Hamilton-Jacobi equations do not admit strong ($\mathcal{C}^1$) solutions and generalised solutions (meaning: not $\mathcal{C}^1$) are in general not unique. One therefore needs to add further conditions to recover uniqueness: here the notion of solution is that of viscosity solution.

Our proof of the convergence for the two parabolic PDEs is relatively standard, and similar results were obtained in the literature [45]. On the other hand, the proof of convergence towards the Hamilton-Jacobi equation is more involved. Actually in this case, our proof is not performed directly at the level of the height function, but at the level of the density of particles. Indeed, we prove that the following measure-valued process

$$\rho_N(t, dx) = \frac{1}{N}\sum_{i=1}^{N} \eta_{tN^{1+\beta}}(i)\, \delta_{\frac{i}{N}}(dx) \ , \quad x \in (0,1) \ . \tag{I.4}$$

(which is nothing but an affine transformation[5] of the derivative of m_N) converges to the entropy solution of the following inviscid Burgers equation

[5] The reason for this affine transformation is simple: the occupation variables for the particles take values in $\{0,1\}$. Had they taken values in $\{-1,1\}$, we would not have had to apply any transformation.

with zero-flux b.c.:

$$\begin{cases} \partial_t \rho = -2\partial_x\big(\rho(1-\rho)\big)\ , \\ \rho(t,0)(1-\rho(t,0)) = \rho(t,1)(1-\rho(t,1)) = 0\ , \\ \rho(0,\cdot) = \rho_0(\cdot)\ . \end{cases} \tag{I.5}$$

Integrating the solution to this conservation law (and performing an affine transformation), one recovers the viscosity solution of (I.3).

Rezakhanlou [67] established a convergence result towards conservation laws that applies to a large class of asymmetric particle systems on the lattice $\mathbb{Z}^d$ (without boundary conditions). Our case falls into the scope of this result except that our asymmetry is weak (which is a minor issue) and that we have to deal with boundary conditions (this is more subtle). Bahadoran [7] extended the result of Rezakhanlou to the case where the particle system lives in a bounded domain in contact with reservoirs: the boundary conditions for the conservation laws are then of Dirichlet type. In (I.5), the boundary conditions do not carry over the value of the solution (Dirichlet) but on the flux (zero-flux) of the solution at the boundaries so that our case is still slightly different from [7]. Generally speaking, dealing with b.c. requires some care in the context of conservation laws: let us recall briefly a few facts on this topic.

Bardos, Le Roux and Nedelec [8] presented a solution theory for conservation laws endowed with Dirichlet b.c. They showed that the b.c. cannot in general be interpreted in the usual sense: the solution does not necessarily satisfy them at all times, but satisfies instead some inequalities involving the given b.c. These inequalities are usually referred to as the BLN conditions.

Remark I.4. The fact that Dirichlet b.c. are not necessarily satisfied is rather intuitive: one can impose a Dirichlet boundary condition where the flux is entering, but not where the flux is exiting. For concreteness, consider the linear conservation law

$$\partial_t \rho = a\partial_x \rho\ , \quad x \in [0,1]\ , \quad t \geq 0\ ,$$

for some fixed $a > 0$. The characteristics are given by $x(t) = -at + x(0)$. None of the characteristics hit the right boundary so that one can impose a boundary condition at $x = 1$. On the other hand, the characteristics hit the left boundary so that no boundary condition can be imposed at $x = 0$.

The case of zero-flux boundary condition is much neater and is due to Bürger, Frid and Karlsen [12]. They showed that the solution actually satisfies at almost all times t the zero-flux boundary condition imposed to the conservation law. In particular, the solution of (I.5) satisfies the zero-flux b.c.

It is interesting to note that the solution of (I.5) actually coincides with the solution of the same PDE with the following Dirichlet boundary conditions:

$$\rho(t,0) = 0 \;, \quad \rho(t,1) = 1 \;.$$

Of course, these b.c. must be interpreted in the BLN sense. The very simple proof of this connection between the two PDEs is presented in [Lab17].
Actually, this connection can easily be explained at the level of the particle system. In the regime of asymmetry $\beta < 1$, the SEP on the segment can roughly be obtained by considering a SEP on the whole lattice $\mathbb{Z}$ starting with the following initial condition: to the right of site N, we place only particles ; to the left of site 0 we place no particle ; on $[\![1, N]\!]$ we consider the same initial condition as for the SEP on the segment. Since the asymmetry goes to the right, the configuration outside $[\![1, N]\!]$ is essentially left unchanged by the dynamics. This configuration on $\mathbb{Z}$ emulates the above boundary conditions in the scaling limit: indeed there's a density equal to 1 of particles to the right of N and a density 0 to the left of 0. However, according to the chosen initial condition on $[\![1, N]\!]$, the system inside $[\![1, N]\!]$ may produce densities at the boundaries that are different from those coming from the system outside and this explains the BLN conditions.

Let us mention that we used the connection between the solutions with zero-flux b.c. and with Dirichlet b.c. in order to prove the convergence of (I.4) towards (I.5). Indeed, while the solution theory with zero-flux b.c. is simple, we did not find in the literature a characterisation of the solution that does not involve the trace of the solution at the boundaries (although we believe that such a formulation could be established): having to deal with the trace of the density of particles at the boundary is not very convenient technically. On the other hand, such a formulation exists in the case of Dirichlet b.c. and this was exploited by Bahadoran [7]. Therefore, we proved the convergence of the density of particles towards the solution with Dirichlet b.c. in order to avoid dealing with the trace of the solution at the boundaries.

Remark I.5. The third convergence result of Theorem I.3 also holds when the asymmetry $p - q$ is independent of N. In that case, instead of speeding up time by a factor $N^{1+\beta}$ we speed it up by $2N/(p-q)$. The result is then exactly the same. At a technical level, the proof of [Lab17] relies on a one block estimate which is not available anymore when the asymmetry is independent of N: the adaptation of the proof of this scaling limit to that setting is given in [LL19a].

I.2 Convergence to KPZ

We take $\beta < 1$ in this section. The solution of (I.3) is explicitly given by the Lax formula, adapted to the case with b.c., see for instance [52, Th 11.1].

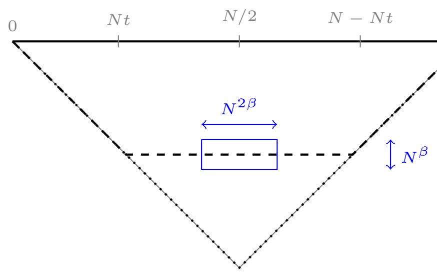

Figure I.3: A plot of (I.6): the bold black line is the initial condition, the dashed line is the solution at some time $0 < t < T = 1/2$, and the dotted line is the solution at the terminal time T. The blue box corresponds to the window where we see KPZ fluctuations.

The equilibrium profile is given by the map[6] $\vee : x \mapsto \max(-x, x-1)$. As explained previously, for any given initial condition the solution reaches this equilibrium profile in finite time. In the sequel, we restrict ourselves to the case of a flat initial profile: namely, $m_0(x) = 0$. The solution of (I.3) is then given by (see also Figure I.3)

$$m(t,x) = \max(-t, \vee(x)) = \max(-t, -x, x-1) \ . \tag{I.6}$$

The profile decreases at speed 1 uniformly in space, but is constrained to remain above the minimal (equilibrium) profile $\vee$. The hitting time of the equilibrium profile is $T = 1/2$.

This is a typical setting where one would expect the KPZ equation to arise in the fluctuations around the hydrodynamic limit. Let us recall the famous result of Bertini and Giacomin [11] on that topic. They considered a SEP on the infinite lattice $\mathbb{Z}$ with a bias $\epsilon^{1/2}$ to the right and starting from a flat initial profile. In this setting, the height function typically decays at constant speed, uniformly in space:

$$\epsilon^{1/2} h(t\epsilon^{-1}, x\epsilon^{-1/2}) \to -\frac{t}{2} \ , \quad \epsilon \downarrow 0 \ .$$

They showed that the fluctuation field

$$\epsilon^{1/2}\Big(h(t\epsilon^{-2}, x\epsilon^{-1}) + \frac{t}{2}\epsilon^{-3/2}\Big) \ ,$$

converges in law as $\epsilon \downarrow 0$ to the solution of the KPZ equation on the real line

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}(\partial_x u)^2 + \xi \ , \quad u(t=0,\cdot) = 0 \ ,$$

[6]Implicitly, we use the same symbol to denote the minimal height function and its macroscopic scaling limit.

where ξ is a space-time white noise.

This equation falls into a class of stochastic PDEs for which a (direct) notion of solution was missing until recently, we will come back to that point in Chapter II. However, if one applies formally the Hopf-Cole transform $u \mapsto z = e^u$ then one gets a multiplicative stochastic heat equation on the real line

$$\partial_t z = \frac{1}{2}\partial_x^2 z + z\xi \;, \quad z(t=0,\cdot) = 1 \;,$$

which admits a solution using Itô's integration.
The Hopf-Cole transform is rigorous if the noise is finite dimensional (projections of the white noise on finitely many Fourier modes for instance): it then boils down to applying Itô's formula. However, Itô's formula produces a second order term which involves the trace of the covariance of this finite dimensional noise: passing to the limit on the dimension of the noise, one gets an infinite term in the equation.
Nevertheless, if one starts from the solution z of the multiplicative SHE, then one can define $u = \log z$ (this is well-defined thanks to a result of Müller [61] that ensures that z is strictly positive at all times if the initial condition is non-negative and non-zero) and call u the solution to KPZ. This is the notion of solution used by Bertini and Giacomin: they showed that the Hopf-Cole transform of the fluctuation fields of the height function converges in law to z.

Given this convergence result, we are led to applying the same scaling as Bertini and Giacomin in our context:

$$u_N(t,x) := \frac{2}{N^\beta}\big(h(tN^{4\beta}, N/2 + xN^{2\beta}) + tN^{3\beta}\big) \;.$$

We have centered our field at the middle of the lattice for the sake of symmetry (see also Remark I.9).

Remark I.6. We have not sped up time by $(N^\beta/2)^4$ but by $N^{4\beta}$, nor rescaled space by $(N^\beta/2)^2$ but by $N^{2\beta}$, as this is more convenient for the discussion below. The only modification that it produces is that the noise in the KPZ equation will have a larger variance:

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}(\partial_x u)^2 + 2\xi \;, \quad u(t=0,\cdot) = 0 \;. \tag{I.7}$$

First, we note that the lattice size is N so that one cannot rescale space by more than N: this forces β to be in $(0, 1/2]$. At this point, one would expect that for $\beta = 1/2$ the fluctuation field should converge to a KPZ equation on $[-1/2, 1/2]$. However there is a further constraint on the time scaling that needs to be taken into account.

As we observed above, the hydrodynamic limit evolves on the time scale $N^{1+\beta}$ and reaches its equilibrium profile in finite time. Once the equilibrium profile is reached, we are no longer in a situation of interface growth, but rather in a regime of (reversible) equilibrium fluctuations. Consequently, the (irreversible) KPZ equation cannot arise after the hitting time of the macroscopic profile: this forces us to choose a time scaling for the fluctuations that does not evolve faster than $N^{1+\beta}$. Given that the Bertini-Giacomin scaling in time is $N^{4\beta}$, we need to take $\beta \le 1/3$.

Take $\beta \in (0, 1/3)$. The fluctuation field then evolves on a much slower scale than the hydrodynamic limit and we then obtain the KPZ equation on the real line (note that $N^{2\beta} \ll N$) and on the time interval $[0, \infty)$.

Theorem I.7 ([Lab17]). For $\beta \in (0, 1/3)$, the sequence u_N converges in law in the Skorohod space[7] $\mathbb{D}([0, \infty), \mathcal{C}(\mathbb{R}))$ to the solution u of the KPZ equation (I.7).

Take $\beta = 1/3$. Then the fluctuation field and the hydrodynamic limit evolve on the same time scale. From the discussion above, one could expect that the scaling limit of the fluctuation field is given by some KPZ-type equation on the real line that continuously vanishes as t gets close to the hitting time $T = 1/2$ of the equilibrium profile of the hydrodynamic limit. The following result shows that this is not the case.

Theorem I.8 ([Lab17]). For $\beta = 1/3$, the sequence u_N converges in law in the Skorohod space[7] $\mathbb{D}([0, T), \mathcal{C}(\mathbb{R}))$ to the solution u of the KPZ equation (I.7).

In other words, the scaling limit is given by the KPZ equation on the real line stopped at time T. This means that these fluctuations suddenly vanish at time T; let us give a simple explanation for this phenomenon.
At any time $t \in [0, T)$, the particle system is split into three zones: a high density zone $[\![1, Nt]\!]$, a low density zone $[\![N - Nt, N]\!]$ and, in between, the bulk where the density of particles is approximately $1/2$, we refer to Figure I.3. The KPZ equation arises in a window of order $N^{2\beta}$ around the middle point of the bulk: from the point of view of this window, the boundaries of the bulk are “at infinity” but move “at infinite speed”. Therefore, inside this window the system does not feel the effect of the boundary conditions until the very final time T where the boundaries of the bulk merge.

Remark I.9. One could also center the fluctuation field at some point rN with $r \in (0, 1)$. The analysis presented above carries through: the only difference lies in the value of the terminal time T which is $r \wedge (1 - r)$.

[7] The Skorohod space is endowed with the topology of uniform convergence on compact space-time sets.

Let us finally mention that convergence of discrete models to the KPZ equation has been established in other situations such as the SEP starting from the narrow wedge [5] or in a rarefaction fan [18], the SEP in contact with reservoirs [19, 33] or some variant of the SEP [20].

Future directions. In the regime $\beta = 1/3$, it would be interesting to investigate the fluctuations, in the Bertini-Giacomin scaling, in a moving frame centered at the position of the shock (roughly $x = Nt$). This would probably yield an equation which is spatially inhomogeneous, and whose behaviour towards $x \to +\infty$ is similar to KPZ, and towards $x \to -\infty$ is deterministic. Technically, this requires to deal with a moving frame and this is delicate in many respects.

I.3 Asymptotic of the mixing times of the SEP

In the present section, we will be interested in the following question: how much time does the SEP need to reach equilibrium when it starts from the "worst" initial condition ? The notion of distance to equilibrium that we will consider here is the total variation distance, whose definition is recalled below.
One may think that the hydrodynamic limit presented in Subsection I.1.3 already provides the answer to the above question. It turns out that our notion of distance to equilibrium requires microscopic information about the system, while the hydrodynamic limit only provides a macroscopic picture. We will see that, in some situations, the reasonable guess about the mixing times that the hydrodynamic provides is actually wrong.

For the moment, we only make the following assumptions

$$k \in [\![1, N/2]\!] , \quad p \in [1/2, 1] , \quad p + q = 1 .$$

Note that the assumption $k \leq N/2$ is not restrictive at all: by reversing the drift and exchanging the roles of particles and empty sites, one can recover the case $k \geq N/2$ from the case $k \leq N/2$.

Recall that the total variation distance between two probability measures ν and π on some measurable space $(\Omega, \mathcal{A})$ is given by

$$\|\nu - \pi\|_{TV} := \max_{A \in \mathcal{A}} \ \nu(A) - \pi(A) \in [0, 1] .$$

This distance can also be expressed as an infimum over all couplings $\mathbb{P}$ of two r.v. X and Y with laws ν and π respectively:

$$\|\nu - \pi\|_{TV} = \inf_{\mathbb{P}} \mathbb{P}(X \neq Y) .$$

Let us finally mention that in the case where Ω is countable and $\mathcal{A}$ is the collection of all subsets of Ω, we have

$$\|\nu - \pi\|_{TV} := \frac{1}{2}\sum_{x\in\Omega} |\nu(x) - \pi(x)| \ .$$

We denote by $P_t^h(\cdot)$ the law of the SEP at time t starting from the initial height function h, and we let $d_{N,k}(t)$ be the distance of the chain at time t to the invariant measure μ_N, starting from the "worst" initial condition:

$$d_{N,k}(t) := \max_h \|P_t^h - \mu_N\|_{TV} \ .$$

For any given threshold $\epsilon \in (0,1)$, we define the ϵ-Mixing Time

$$T_{N,k}(\epsilon) := \inf\{t \geq 0 : d_{N,k}(t) < \epsilon\} \ .$$

The distance to equilibrium is a non-increasing function of time. Typically, it decays exponentially fast to 0 with a rate given by the spectral gap of the generator of the Markov process at stake. However, when considering sequences of Markov processes (indexed by N say), there are situations where the graph of the distance to equilibrium has a completely different behaviour asymptotically in N: it remains close to 1 until some time t_N (which may go to infinity with N) and then falls abruptly to 0 right after t_N. In other words, the ϵ-Mixing Times are all equivalent to t_N when N goes to infinity. This is usually referred to as a cutoff phenomenon.
This phenomenon has been observed first in the context of card shuffling [23], and was then established in many other contexts [51]. We now recall the main results of the literature on the asymptotic behaviour of the mixing times of the SEP.

In 2004, Wilson [71] showed that in the symmetric case $p = q = 1/2$ and when there is a non-trivial density of particles, that is, $k/N \to \alpha$ as $N \to \infty$ with $\alpha \in (0, 1/2]$ then for all $\epsilon \in (0,1)$

$$\frac{1+o(1)}{\pi^2} N^2 \log N \leq T_{N,k}(\epsilon) \leq \frac{2(1+o(1))}{\pi^2} N^2 \log N \ ,$$

where $o(1)$ is a quantity that goes to 0 as $N \to \infty$. Such a situation is often referred to as a precutoff phenomenon: it shows some concentration phenomenon of the mixing times but does not necessarily imply an abrupt decay of the distance to equilibrium. However, it is expected that for non-pathological Markov chains a precutoff phenomenon implies a cutoff phenomenon.

In 2005, Benjamini, Berger, Hoffman and Mossel [10] showed that in the asymmetric case $p > 1/2$ (p independent of N), there exist two constants $0 < C < C'$ such that for any $\epsilon \in (0,1)$ and for all N large enough we have

$$CN < T_{N,k}(\epsilon) < C'N \ ,$$

thus establishing a precutoff phenomenon in this case as well.

In 2014, Lacoin [49] obtained the exact asymptotic of the mixing times of the SEP in the symmetric case $p = q = 1/2$:

$$T_{N,k}(\epsilon) \sim \frac{1}{\pi^2} N^2 \log k \ , \quad N \to \infty \ ,$$

under the only requirement that $k \to \infty$ as $N \to \infty$ (the density of particles may be equal to 0). This refined the result of Wilson, and established a cutoff phenomenon in the symmetric case.

In 2016, Levin and Peres [50] investigated the SEP in the entire weakly asymmetric regime:

$$p > 1/2 \ , \quad p \to 1/2 \quad \text{as } N \to \infty \ ,$$

and under the assumption that there is a non-trivial density of particles

$$k/N \to \alpha \in (0, 1/2] \ .$$

They identified three different regimes for the mixing times and established a precutoff phenomenon in each case: (the two constants $0 < C < C'$ are not the same in each case)

(A) If $p - q \lesssim 1/N$, then

$$CN^2 \log N \le T_{N,k}(\epsilon) \le C' N^2 \log N \ ,$$

(B) If $1/N \lesssim p - q \lesssim \log N/N$, then

$$C \frac{\log N}{(p-q)^2} \le T_{N,k}(\epsilon) \le C' \frac{\log N}{(p-q)^2} \ ,$$

(C) If $\log N/N \lesssim p - q$, then

$$C \frac{N}{p-q} \le T_{N,k}(\epsilon) \le C' \frac{N}{p-q} \ .$$

Despite the variety of situations covered, the article of Levin and Peres is remarkably short. Surprisingly, their proofs do not rely explicitly on the phenomenological transitions for the SEP that occur at the two thresholds $1/N$ and $\log N/N$. Let us briefly explain what happens at these two thresholds.

We have seen in Subsection I.1.3 that the hydrodynamic limit is a heat equation when the bias is negligible compared to $1/N$, a Hamilton-Jacobi equation when it is much larger than $1/N$ and a mixture of the two when the bias is of order $1/N$. As pointed out therein, the Hamilton-Jacobi equation

reaches equilibrium in finite time[8]: starting from the "worst" initial condition $\wedge$, the PDE takes a time $t_\alpha := (\sqrt{\alpha}+\sqrt{1-\alpha})^2/2$ to hit the equilibrium profile $\vee$. A natural guess would have been that the mixing time is exactly given by $2Nt_\alpha/(p-q)$ when $p-q \gg 1/N$: the result of Levin and Peres already shows that this is not true for $(p-q) \lesssim \log N/N$.
On the other hand, the heat equation with or without the non-linearity reaches equilibrium in infinite time so that the time scale of the hydrodynamic limit, that is N^2, has to be much smaller than the time scale of the mixing times whenever $p-q \lesssim 1/N$: this is in line with the result of Levin and Peres.

The transition occurring at $\log N/N$ is of a much more microscopic nature. From the shape of the macroscopic equilibrium profile $\vee$, we deduce that the density of particles vanishes on $[0, 1-\alpha]$: microscopically, it means that under μ_N, most particles lie at the right of site $N-k$. However, this does not provide a precise control on the leftmost particle under μ_N. In collaboration with Hubert Lacoin we showed, as a preliminary result in [LL19b], that the leftmost particle, under the invariant measure μ_N, is located:

- at a distance negligible compared to N from site $N-k$ if the bias is much larger than $\log N/N$,
- at a distance negligible compared to N from site 1 if the bias is much smaller than $\log N/N$.

When the bias is of order $\log N/N$, we observe a crossover between these two behaviours.
This knowledge on the location of the leftmost particle complements the information coming from the hydrodynamic limit and explains why, in regime (B) of Levin and Peres, the mixing time is much larger than the fixation time of the hydrodynamic limit. Indeed, in this regime the invariant measure has a complexity which is not reflected by the hydrodynamic limit.

In collaboration with Hubert Lacoin, we addressed essentially the same question as Levin and Peres. Our results, presented in two separate articles [LL19a, LL19b], identify two regimes for which we prove a cutoff phenomenon and a crossover regime in between, for which the existence of a cutoff phenomenon is left open. Before stating the results, we need to introduce the spectral gap of the process, that is, the distance between the first eigenvalue (which is zero) and the second eigenvalue of the generator of the SEP. Using the discrete Hopf-Cole transform, we showed in [LL19a, Sec 3.3] that

$$\mathrm{gap}_N = (\sqrt{p}-\sqrt{q})^2 + 4\sqrt{pq}\sin^2\Big(\frac{\pi}{2N}\Big)\ .$$

[8]In that section, the results were stated for $k = N/2$ and under the assumption that $p-q$ has a polynomial in N decay. These results can easily be generalised to the present case, see also Remark I.5.

Note that, whenever $p - q \to 0$, we have

$$\mathrm{gap}_N = \frac{(p-q)^2}{2} + \frac{\pi^2}{2N^2} + \mathcal{O}((p-q)^4) + \mathcal{O}(N^{-4}) \ .$$

We start with a regime that we called the small bias regime.

Theorem I.10 ([LL19b]). Under the assumption that

$$k \to \infty \quad \text{and} \quad p - q \ll \frac{\log k}{N} \ , \quad \text{as} \quad N \to \infty \ ,$$

we have for any given $\epsilon \in (0,1)$

$$T_{N,k}(\epsilon) \sim \frac{\log N}{2\,\mathrm{gap}_N} \ , \quad N \to \infty \ .$$

This result does not require the density of particles to be non-trivial (that is, to be in $(0,1)$): however, we ask the number of particles k to go to infinity. When the density is non-trivial, the small bias regime is the union of regimes (A) and (B) of Levin and Peres, except for the boundary case where the bias is exactly of order $\log N/N$ (note that this is of the same order as $\log k/N$ since the density is non-trivial). An asymptotic expansion of gap_N shows that our result is consistent with the results of Levin and Peres.

The proof consists of two steps. First the lower bound is obtained relatively easily using the method of Wilson: we show that at any time much smaller than the putative mixing time, the image of the evolving height function (starting from $\wedge$) through the eigenfunction associated with the spectral gap is far from equilibrium. Second the upper bound is obtained by controlling the area between the extremal height functions: this is the most involved step as it requires a very precise control on the evolution of this area.

We pass to what we called the large bias regime.

Theorem I.11 ([LL19a, LL19b]). Under the assumption that

$$\frac{k}{N} \to \alpha \in [0, 1/2] \quad \text{and} \quad \frac{\log k}{N} \ll p - q \ , \quad \text{as} \quad N \to \infty \ ,$$

we have for any given $\epsilon \in (0,1)$

$$T_{N,k}(\epsilon) \sim \frac{N}{p-q}(\sqrt{\alpha} + \sqrt{1-\alpha})^2 \ , \quad N \to \infty \ .$$

Our result shows that the mixing time in the large bias regime is given by the fixation time of the hydrodynamic limit. Let us emphasise that this fixation time is much smaller than the mixing time in the case where $1/N \ll p - q \ll \log k/N$, which belongs to the small bias regime.

When the density is non-trivial, the large bias regime coincides with regime (C) of Levin and Peres except for the boundary case where bias is exactly of order $\log N/N$. Here again our result is consistent with the result of Levin and Peres.

Let us say a few words about the proof. The lower bound is an immediate consequence of the hydrodynamic limit: the distance to equilibrium is close to 1 as long as the hydrodynamic limit has not reached its (macroscopic) equilibrium. The upper bound consists in proving that the evolving position of the leftmost particle in the system is close to what is suggested by the hydrodynamic limit, thus making the fixation time of the latter coincide with the mixing time.

We have not been able to establish a cutoff phenomenon in the crossover regime in between: this case is left open (and is probably quite hard).

Let us mention that the SEP is a projection of a larger object called the biased card shuffling. Consider a deck of N cards labeled from 1 to N and encoded by a permutation σ of $[\![1, N]\!]$: the label of the card in position i is given by $\sigma(i)$. For every pair $(i, i+1)$, one swaps the two adjacent cards at positions i and $i+1$ at rate p if the cards are in the decreasing order $(\sigma(i) > \sigma(i+1))$ and at rate q otherwise. Tracking the positions of the k cards of highest labels boils down to considering the SEP with k particles. In [LL19a] we showed that the mixing times of the biased card shuffling in the asymmetric regime (p is independent of N and larger than $1/2$) are equivalent to $2N/(p-q)$. This quantity is nothing but the maximum over k of the mixing times of the corresponding SEP. Our proof can certainly be extended to the entire large bias regime.

Although the asymptotic of the mixing time of the card shuffling in the symmetric regime was obtained in [49], our proof of Theorem I.11 does not yield the asymptotic of the mixing time of that process in the entire small bias regime. This is because we rely on a coupling for the SEP which cannot be extended into a coupling for the permutations. On the other hand, the symmetric regime is very special since the mixing time of the card shuffling is independent from the permutation one starts from: the proof in [49] then relied on censoring schemes that are available when one starts from the "maximal" configuration whose density w.r.t. the invariant measure in increasing.

Future directions. A natural subsequent question would be to determine the width of the cutoff window, that is, the time-scale at which the map $\epsilon \mapsto T_{N,k}(\epsilon)$ varies smoothly (of course, this scale has to be negligible compared to the magnitude of the mixing times since we have a cutoff phenomenon). On the circle and with symmetric rates, the width of the cutoff window was shown to be of order N^2 and the exact cutoff profile was obtained [48].

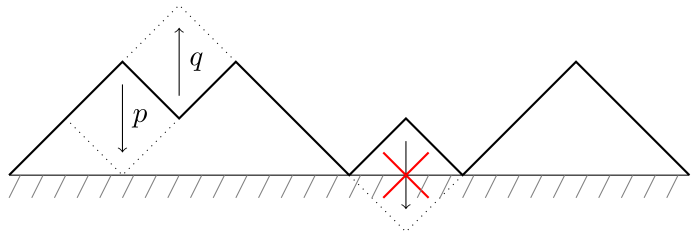

Figure I.4: An example of reflected SEP: upwards (resp. downwards) flips occur at rate q (resp. p), except for downwards flips that would make the interface negative which are not allowed.

I.4 Scaling limit of reflected interfaces

This section is concerned with a variant of the SEP that is easier to describe at the level of its height function. Consider the height function of the SEP with $k = N/2$ particles under the additional constraint that this height function remains non-negative. In other words, at height 0 there is a hard wall that prevents any upwards corner that sits on the wall to flip downwards. We refer to Figure I.4 for a picture.

The invariant measure is still given by the expression

$$\mu_N(h) = \frac{1}{Z_N}\Big(\frac{p}{1-p}\Big)^{-A(h)/2} ,$$

but the set of admissible height functions is smaller than before: it consists of all lattice paths h that satisfy

$$h(0) = h(N) = 0 , \quad h(x) - h(x-1) = \pm 1 , \quad h(x) \geq 0 , \forall x \in [\![1, N]\!] .$$

For $p = 1/2$, μ_N converges weakly to the law of the Brownian excursion on $[0,1]$. More generally, for any $\sigma \in \mathbb{R}$ if we take

$$p = \frac{1}{2} + \frac{\sigma}{N^{3/2}}(1 + o(1)) ,$$

then the measure μ_N converges to the law of the Brownian excursion X tilted by $e^{-2\sigma A(X)}$ where $A(X) = \int_0^1 X(t)dt$ is the area under the excursion.

In a work [EL15] in collaboration with Alison M. Etheridge, we considered the diffusively rescaled height function

$$u_N(t,x) = \frac{1}{\sqrt{N}} h(tN^2, xN) , \quad x \in [0,1] , t \geq 0 ,$$

and showed the following result.

Theorem I.12 ([EL15]). Assume that the process starts from its invariant measure μ_N, then u_N converges in law to the solution u of the following stochastic PDE

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u - 2\sigma + \xi + \eta \;, \quad x \in [0,1] \;, \quad t \geq 0 \;, \\ \int_{(0,\infty)\times(0,1)} u(t,x)\eta(dt,dx) = 0 \;, \\ u \geq 0 \;, \quad u(t,0) = u(t,1) = 0 \;. \end{cases} \tag{I.8}$$

starting from its invariant measure. Here ξ is a space-time white noise and η is a non-negative measure that prevents u from becoming negative.

This stochastic PDE is usually referred to as the reflected stochastic heat equation, and was introduced by Nualart and Pardoux [63]. It can be seen as an infinite dimensional version of the reflecting Brownian motion, the reflection measure η playing the role of the local time at 0.
Zambotti [73] showed that this stochastic PDE admits the law of the tilted (by its area, as described above) Brownian excursion as an invariant, reversible, measure.

Actually, our result is stronger than this. We identified a discrete reflection measure η_N, which is carried by the space-time points where the interface sits on the wall and is prevented from flipping downwards, and we showed that the pair (u_N, η_N) converges in law to the pair (u, η) (in an appropriate topology for the second coordinate).

Future directions. It is not difficult to come up with variants of the SEP whose invariant measures are discrete analogues of well-known measures on continuous paths. For instance, the law of the reflecting Brownian motion can be approximated by the law of a reflecting random walk, and this discrete law is left invariant by a very simple variant of the above model. Recent progress have been made on the construction of an SPDE that preserves the law of the reflecting Brownian motion, see [26], however the proof of the convergence of this discrete model is still open. Other laws can be considered: for instance, the law of the Brownian motion, starting from 1 and killed at its first hitting time of 0. This is a work in progress.

I.5 The adjacent walk on the continuous simplex

Consider $N-1$ points $X_1 < \ldots < X_{N-1}$ on the segment $[0,N]$ evolving according to the following continuous-time dynamics: for every $k \in [\![1, N-1]\!]$, at rate 1 one draws a uniform r.v. U on $[0,1]$ and one replaces the r.v. X_k by

$$UX_{k-1} + (1-U)X_{k+1} \;,$$

with the convention that $X_0 = 0$ and $X_N = N$.

This process admits the uniform measure on the simplex

$$S_N := \{(x_1, \ldots, x_{N-1}) : 0 < x_1 < \ldots < x_{N-1} < N\} ,$$

as an invariant measure. It is also natural to consider the spacings between the particles:

$$\eta_k := X_k - X_{k-1} , \quad k \in [\![1, N]\!] .$$

The invariant measure at the level of the particle spacings is given by a product measure of N exponential r.v. with the same parameter, conditioned on the event that their sum is equal to N. Note that the parameter of the exponential r.v. is arbitrary.

This dynamics shares some similarities with the SEP with symmetric jump rates. However the fact that the state-space is uncountable makes the analysis of its mixing times more involved. For instance, it is not clear a priori whether the generator of the process has pure point spectrum and the existence of a spectral gap is delicate.

In 2005, Randall and Winkler [66] showed that the mixing times $T_N(\epsilon)$, $\epsilon \in (0, 1)$, of this Markov process satisfy a precutoff phenomenon: there exist two constants $0 < C < C'$ such that for any $\epsilon \in (0, 1)$ and for all N large enough

$$CN^2 \log N \leq T_N(\epsilon) \leq C' N^2 \log N .$$

While their technique for proving the lower bound was essentially sharp, the upper bound was not. In a collaboration [CLL19] with Pietro Caputo and Hubert Lacoin, we aimed at establishing a cutoff phenomenon for this process. Actually, we considered a more general setting where the resampling law is not necessarily uniform over the segment formed by the nearest neighbours, but follows a $\beta(\alpha, \alpha)$ law for some $\alpha \geq 1$. Note that $\alpha = 1$ yields the uniform measure.

Remark I.13. One can remove the constraint that the particles live in $[0, N]$, and only impose them to live in $[0, \infty)$. Then, the $\beta(\alpha, \alpha)$ laws with $\alpha > 0$ are the only resampling laws that make the dynamics reversible w.r.t. to a product measure (at the level of the particle spacings). This can be checked by inspecting the constraints that the detailed balance condition imposes at a resampling event.

The beta resampling law with $\alpha \geq 1$ yields a reversible dynamics. The invariant measure, at the level of the particle spacings, is a product measure conditioned on the event $\eta_1 + \ldots + \eta_N = N$:

$$\mu_N = C_\alpha \mathbf{1}_{\{\eta_1 + \ldots + \eta_N = N\}} \prod_{k=1}^{N-1} \eta_k^{\alpha-1} d\eta_k .$$

Remark I.14. A very important technical tool in [CLL19] is the FKG inequality w.r.t. μ_N that holds for $\alpha \geq 1$ (by log-concavity of $x \mapsto x^{\alpha-1}$): this is the reason why we did not consider the case $\alpha \in (0,1)$, and restricted ourselves to the case $\alpha \geq 1$

Our first result in [CLL19] was the proof of existence of the spectral gap and its identification. Interestingly neither the gap, nor the associated eigenfunction depend on α.

Proposition I.15 ([CLL19]). For any $\alpha \geq 1$, the process admits a spectral gap given by

$$\mathrm{gap}_N := 1 - \cos(\frac{\pi}{N}) \;,$$

and the corresponding eigenfunction is

$$f(x) = \sum_{k=1}^{N-1} \sin(\frac{\pi k}{N})(x_k - k) \;, \quad x \in S_N \;.$$

While it was proved by Randall and Winkler [66] that $-\mathrm{gap}_N$ is an eigenvalue of the generator with eigenfunction f, the fact that this is actually the spectral gap of the generator was not established therein. The proof of this fact is not immediate, and relies on the FKG inequality w.r.t. μ_N. We also showed that the generator has pure point spectrum.

Our main result is the identification of the exact asymptotic of the mixing times, from which we deduce a cutoff phenomenon for the distance to equilibrium. Remarkably, it takes the same form as in the case of the SEP with symmetric jump rates.

Theorem I.16 ([CLL19]). For any $\alpha \geq 1$ and any $\epsilon \in (0,1)$ we have

$$T_N(\epsilon) \sim \frac{\log N}{2\,\mathrm{gap}_N} \;, \quad N \to \infty \;.$$

Let us make some comments about the proof of this theorem. The lower bound was (essentially) established in [66] using the method of Wilson (which consists in exhibiting an initial condition for which the expectation of $f(X_t)$ is much larger than the standard deviations of f under the invariant measure μ_N as long as t is smaller than the putative mixing time).
On the other hand, the upper bound of [66] was not sharp and refining their proof is our main contribution in [CLL19]. It required a fine analysis of the time evolution of the area between the two extremal initial conditions of the process: indeed, controlling the hitting time of zero of this area gives an upper bound on the mixing time.

Future directions. A bias in the resampling law can be introduced, and the model then resembles the SEP with asymmetric jump rates: the investigation of the asymptotic of the mixing times is then a natural research direction. This is one subject of the PhD thesis of Enguérand Petit, under the joint supervision of Cristina Toninelli (Univ. Paris-Dauphine) and myself.
With Pietro Caputo and Hubert Lacoin, we are now working on the mixing times of $\nabla\varphi$ interfaces in 1-d. Here again, these models share similarities with the height function of the SEP and the adjacent walk on the simplex, but to the best of our knowledge very little is known about their mixing times.

Chapter II

Singular SPDEs and regularity structures

This chapter is mostly concerned with the following stochastic PDE

$$\partial_t u = \partial_x^2 u + u \cdot \xi \;, \quad x \in \mathbb{R}^d \;, \quad t \geq 0 \;, \qquad u(0,\cdot) = u_0(\cdot) \;, \tag{II.1}$$

for the unknown $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. We have particular interest in the following two distinct settings:

- The parabolic Anderson model (PAM), where ξ is a space white noise and $d \leq 3$,
- The multiplicative stochastic heat equation (SHE), where ξ is a space-time white noise and $d = 1$.

We saw in Section I.2 that (SHE) is mapped onto the KPZ equation through the Hopf-Cole transform. The (PAM) can be seen informally as the density of a system of branching Brownian motions in the random environment ξ (roughly speaking: where $\xi(x)$ is negative, each Brownian motion is killed at rate $-\xi(x)$ and where $\xi(x)$ is positive each of them branches into two independent Brownian motions at rate $\xi(x)$). The rest of this introduction presents the difficulties that arise when constructing solutions to these equations, and explains briefly the content of the subsequent sections.

Generically, a solution of the PDE (II.1) is a function/distribution that satisfies

$$u = P * (u \cdot \xi) + P *_x u_0 \;, \tag{II.2}$$

where P is the heat kernel, and $*$, resp. $*_x$, denotes convolution in space-time, resp. in space only.

To prove existence and/or uniqueness of solution, one identifies a functional space in which the solution should live and sets up a fixed point argument in that space. To that end, one observes that the putative solution u should be a perturbation of the solution of the additive stochastic heat equation

$$\partial_t v = \Delta v + \xi \;, \quad t \geq 0 \;, \qquad v(0, \cdot) = 0 \;,$$

whose solution is explicitly given by

$$v = P * \xi \;.$$

The regularity of the latter object is well-understood. In parabolically scaled[1] Hölder spaces, v has space-time regularity $2 + \alpha$ where α is the Hölder regularity of ξ. The parameter α is itself given by $(-d/2)^-$ if ξ is a space white noise, and by $(-(2 + d)/2)^-$ if ξ is a space-time white noise. The notation x^- means $x - \epsilon$ for any arbitrary $\epsilon > 0$.

Therefore, if one believes that u is a perturbation of v, then it should have no better regularity than v. This suggests the Hölder space $\mathcal{C}^{2+\alpha}$ as a candidate for the fixed point argument. Coming back to the fixed point equation (II.2) that a solution must satisfy, we see that we need to make sense of the product $u \cdot \xi$ between an element of $\mathcal{C}^{2+\alpha}$ and an element of $\mathcal{C}^{\alpha}$. It is well-known that the product between elements of two such spaces is canonically[2] well-defined if and only if the sum of their regularity indices is positive; this result is often referred to as Young's theorem since the one dimensional case, in p-variation spaces, was first established by Young [72]. Applying this criterion to the spaces at stake, we obtain the condition $2 + 2\alpha > 0$. For (PAM), this is valid only if $d < 2$. For (SHE) this already fails if $d = 1$.

Such ill-defined products already arise when solvings SDEs (the sum of the Hölder regularities of the Brownian motion and its derivative is negative) and stochastic integrals (Itô, Stratonovich) provide alternative definitions of the corresponding products. However, stochastic integrals are not continuous w.r.t. the driving noise and this makes the solution theory of SDEs unstable: different approximations lead to different notions of solutions.
The Itô integral can be applied in the infinite dimensional setting of SPDE when the noise is white in time and the solution is adapted to the associated filtration: in particular, it provides a solution theory for (SHE), see [70]. Here again, this notion of solution is unstable: for instance, if one considers space-time regularisations of the white noise, the sequence of corresponding solutions does not converge to the Itô solution of (SHE). On the other hand, when the noise is constant in time as in (PAM) such stochastic integrals do

[1] This means that time regularity counts twice compared to space regularity: this setting is of course well-fitted to parabolic equations where one time derivative scales like two space derivatives.

[2] In the sense that it extends continuously the product of smooth functions.

not exist anymore and no notion of solution was available for $d \geq 2$ until recently.

Around 2013, two theories introduced novel frameworks in which one can make sense of these equations along with their stability under regularisation: the theory of paracontrolled distributions of Gubinelli, Imkeller and Perkowski [36] and the theory of regularity structures of Hairer [38]. We will present (briefly) the main ideas of these theories in Section II.1.

Originally, the two theories were able to construct solutions for SPDEs whose space variable lives in a torus. However, in terms of physical models, the equations would rather be considered on the full space $\mathbb{R}^d$. In Sections II.3 and II.4, we will present constructions [HL15, HL18] obtained in collaboration with Martin Hairer of (PAM) in dimensions $d = 2, 3$ and of (SHE) in dimension $d = 1$ on the full space $\mathbb{R}^d$. Let us mention that, although (SHE) makes sense via Itô's integration it is not stable under space-time regularisation procedures: the two theories actually provide the right frameworks in which one can connect such regularisations and the Itô solution.
In Section II.2, we will present a generalisation [HL17], carried out in collaboration with Martin Hairer, of the functional spaces of the theory of regularity structures to the general setting of Besov spaces, together with the associated Embedding Theorems that have some interesting consequences for the solution of (PAM). Finally, in Section II.5 we will present a result [GL19] in collaboration with Paul Gassiat on the existence of densities for the Φ^4_3 model, another SPDE whose construction relies on the two aforementioned theories.

II.1 A few words about the theory of regularity structures

The theory of paracontrolled distributions [36] and the theory of regularity structures [38] build on the following observation, due to Terry Lyons [53] and originally stated at the level of SDEs, but whose scope is more general: while it is not possible to construct a theory of integration that makes the solution map of an SDE continuous w.r.t. to the driving noise when this noise is rough (Brownian motion), one can recover continuity if one enhances the driving noise with iterated integrals of the driving noise against itself.

In terms of well-definiteness of the SDE, the actual definition of the iterated integrals is unimportant: one simply has to give them a meaning (via Itô or Stratonovich for instance). This fundamental observation lead to viewing the original solution map of SDE/SPDE as the composition of two maps: one, which is not continuous, that enhances the driving noise (and therefore makes some non-canonical choice) and another one, which is continuous, that associates to the enhanced driving noise a solution of the equation.

In the two aforementioned theories for SPDEs, this idea was fully exploited and the two maps alluded to above appear clearly. From now on, we will focus on the theory of regularity structures because all the results presented in the next sections rely on it. However, the situation is very similar in the theory of paracontrolled distributions.

The idea of enhancement of the driving noise gave rise to the notion of model in regularity structures. A model is a random object that consists of a driving noise and some additional non-linear functionals of this noise. The functionals that one needs to consider in the model are those that arise when solving the equation at stake. In the case of SDEs, only one functional is needed and it is $\int BdB$. In the case of (SHE) or (PAM), formal Picard iterations applied to (II.2) show that the functionals should be

$$\xi(P * \xi)\ , \xi(P * \xi(P * \xi))\ , \ldots$$

The first functional is not canonically well-defined as soon as $2 + 2\alpha \leq 0$, and the number of functionals that are not well-defined typically increases with the space dimension.

Remark II.1. At this point, we can distinguish two types of equations according as the number of ill-defined functionals is finite or infinite. When this number is finite, the equation is called subcritical by Hairer and his theory is restricted to this case. For (PAM), the equation is subcritical if $d < 4$ and for (SHE) it is subcritical for $d < 2$. Actually, when the equation is not subcritical it is no longer expected to be a perturbation of the linearised equation, and therefore the whole ansatz has to be reconsidered.

To construct a model, one then needs to give a meaning to these objects. In practice, this is done through a limiting procedure, which requires in some cases a renormalisation: one starts with a regularised version ξ_ϵ of the driving noise, for which all these functionals are well-defined, and then one tries to pass to the limit on the regularisation parameter ϵ. In the case of the first functional above, this is possible only if one subtracts a constant C_ϵ that diverges logarithmically for (PAM) in dimension 2, and polynomially for (PAM) in dimension 3 and for (SHE) in dimension 1.
Note that there is some degree of freedom in this procedure: one can shift C_ϵ by some finite constant without altering the convergence of the corresponding objects. This reflects the absence of canonical definition of these functionals.

The second map, that associates a solution to an enhanced driving noise, has also its counterpart in regularity structures. But it is itself again factorised into two maps: one that associates to an enhanced noise an abstract solution, and another one that associates to an abstract solution a genuine distribution/function. Let us explain briefly what abstract solution means.

Instead of viewing the solution as a function, one rather deals with a collection, indexed by the space-time points, of its generalised Taylor expansions. If there were no noise, or if the noise was smooth, these Taylor expansions would simply involve the values of the successive partial derivatives of the function at stake. In the presence of noise, one does not expect the solution to be a regular function (sometimes, it is only a distribution!) so that the order of the classical Taylor expansions would be very low. Building on the notion of controlled rough path introduced by Massimiliano Gubinelli [34], one considers Taylor expansions of the function/distribution not only on the basis of usual monomials as in the smooth case, but also on the basis of the functionals introduced in the construction of the model: hence the notion of generalised Taylor expansions.

To give a concrete example: the function $f(x) = 2x + 3B(x)$ is not differentiable and its classical Taylor expansion at any x_0 is only made of the value of the function at x_0: namely, $2x_0 + 3B(x_0)$.
However, one would expect that a notion of derivative of f could be defined if one were able to "remove" the irregularity coming from the Brownian motion. Actually, if one adds the functional B in the model, then the function f admits a generalised Taylor expansion at x_0 which is made of three coefficients:

- the value of the function $f_0(x_0) := 2x_0 + 3B(x_0)$,
- its derivative against the Brownian motion $f_{1/2^-}(x_0) := 3$,
- its "classical" derivative $f_1(x_0) := 2$.

This allows to replace f by $(f_\zeta(x_0), x_0 \in \mathbb{R}, \zeta \in \{0, 1/2^-, 1\})$ where $f_\zeta(x_0)$ is the coefficient of the ζ-th derivative of f at x_0. Hairer called these spaces of coefficients the spaces of modelled distributions. They are very similar to usual spaces of Hölder functions, and a calculus can be setup in this setting which allows to formulate a notion of fixed point for the original PDE at the abstract level of modelled distributions.
The map that associates to an abstract solution (a modelled distribution) a genuine distribution/function is what Hairer called the reconstruction operator: the construction of this operator is a cornerstone of the theory.

II.2 Besov spaces of modelled distributions

While the theory was originally defined in an L^∞ setting (the space of modelled distributions mimics the space of Hölder distributions), there are several motivations for considering a more general setting:

- The Dirac mass is a very natural initial condition: for (PAM) it corresponds to the case where the Brownian motions start from a deterministic position at time 0, and for (SHE) this is related to the so-called narrow wedge initial condition for KPZ, see [5]. The Dirac mass has regularity $-d$ in the Hölder scale, which makes it a very singular initial condition in the L^∞ setting. On the other hand, it is almost an L^1 function and it turns out that it belongs to the Besov space $\mathcal{B}^0_{1,\infty}$, whose definition is recalled below. Consequently, having at hand a solution theory for the (PAM) in an L^1 setting would probably allow one to start from a Dirac mass at time 0. This will be one of the results of Section II.4.

- The natural setup for Malliavin calculus is L^2: in particular, if one wants to look at derivative of the solution w.r.t. the noise in directions of its Cameron-Martin space (which is L^2 in the case of the white noise), then it is desirable to formulate the solution theory for the equation in an L^2 setting. We refer to Section II.5 for an application of Malliavin calculus in the setting of singular SPDEs.

- The theory of self-adjoint operators on a spatial domain D is usually defined in the Hilbert space $L^2(D)$. The construction of singular Schrödinger operators would therefore need to be carried out in an L^2 setting. This will be the content of Section III.1.

These considerations were a source of motivation for generalising the analytical setting of the theory of regularity structures to Besov-type spaces: this was the content of a work [HL17] in collaboration with Martin Hairer. The remainder of this section is devoted to presenting some aspects of this work[3].

Let us first recall the definition of the classical Besov space $\mathcal{B}^\gamma_{p,q}$, with $\gamma \in \mathbb{R}$, $p, q \in [1,\infty]$. The parameter γ controls the amplitude of the "oscillations" of the function/distribution: roughly speaking, at scale $\lambda \in (0,1)$, it oscillates at most like λ^γ. The parameters p and q control the integrability of the function/distribution respectively in space and in oscillations. Precise definitions can be found in [69] for instance, let us simply say that in the case where $\gamma \in (0,1)$, the function f belongs to $\mathcal{B}^\gamma_{p,q}$ if

$$\|f(x)\|_{L^p(dx)} < \infty\;,$$
$$\Big(\int_{h\in B(0,1)} \Big\|\frac{|f(x+h)-f(x)|}{|h|^\gamma}\Big\|^q_{L^p(dx)} \frac{dh}{|h|^d}\Big)^{\frac1q} < \infty\;.$$

Actually there are many equivalent definitions of these spaces: via the Fourier transform, using a wavelet analysis or dealing with the differences of

[3]The presentation of the articles is not chronological: we actually constructed (PAM) and (SHE) on the full space before we came up with a generalisation of the spaces of modelled distributions.

the function as above. Let us finally mention that when $p = q = \infty$, these are the Hölder spaces.

Our definition of Besov spaces of modelled distributions is an extension of the original definition of Hairer [38] in the L^∞ setting.

Remark II.2. The definition below is one of the very rare places in this document where specific notations from regularity structures are used. We chose not to introduce them in details since it would have been very long (and probably not so clear). For the reader not familiar with the theory, we suggest not to care about $\mathcal{T}_{<\gamma}$ and $\mathcal{A}_\gamma$, and to think of $|\mathfrak{s}|$ as being d, the dimension of the underlying space. On the other hand, the operator $\Gamma_{y,x}$ that appears in the definition is the operator that allows to re-expand at y a Taylor expansion at x. In the case of classical Taylor expansions, these are given by expressions like $(x-y)^{k-\ell}$ which arise when one re-expands the monomial $(X-x)^k$ in terms of $(X-y)^\ell$ for $0 \le \ell \le k$. In the case of generalised Taylor expansions, these are (complicated) random objects depending on the functionals added to the model.

Definition II.3. [[HL17]] For $\gamma \in \mathbb{R}$, let $\mathcal{D}^\gamma_{p,q}$ be the Banach space of all measurable maps $f : \mathbb{R}^d \to \mathcal{T}_{<\gamma}$ such that, for all $\zeta \in \mathcal{A}_\gamma$, we have:

1. Local bound:
$$\Big\| |f(x)|_\zeta \Big\|_{L^p(dx)} < \infty \;, \tag{II.3}$$

2. Translation bound:
$$\Big(\int_{h \in B(0,1)} \Big\| \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_\zeta}{\|h\|_{\mathfrak{s}}^{\gamma-\zeta}} \Big\|^q_{L^p(dx)} \frac{dh}{\|h\|_{\mathfrak{s}}^{|\mathfrak{s}|}} \Big)^{\frac{1}{q}} < \infty \;. \tag{II.4}$$

In [HL17], we presented the calculus associated to the spaces $\mathcal{D}^\gamma_{p,q}$ that is required to solve singular SPDEs with the theory of regularity structures: in particular, the reconstruction theorem (that allows to map modelled distributions to classical Besov spaces) and the convolution theorem (that allows to convolve modelled distributions with the heat kernel). The proofs of these theorems do not really require new ideas compared to their original versions in the Hölder settings, therefore we will not make any further comments on them. The main novelty in [HL17] lies in the Embedding Theorem that we proved at the level of modelled distributions.

Let us recall that classical Besov spaces enjoy the following Embedding Theorem (the second embedding is probably the most used, it increases integrability at the cost of decreasing regularity)

Theorem II.4 (Classical Embedding Theorem). For all $\gamma, \gamma' > 0$ and all $p, p', q, q' \in [1, \infty)$, the space $\mathcal{B}^\gamma_{p,q}$ is continuously embedded into $\mathcal{B}^{\gamma'}_{p',q'}$ if one of the following conditions holds:

1. $p' < p$, $q' = q$ and $\gamma' = \gamma$ [if the underlying space is bounded],
2. $p' > p$, $q' = q$ and $\gamma' \le \gamma - d(\frac{1}{p} - \frac{1}{p'})$,
3. $p' = p$, $q' > q$ and $\gamma' = \gamma$,
4. $p' = p$, $q' < q$ and $\gamma' < \gamma$.

The main result of [HL17] is the analogous result at the level of modelled distributions:

Theorem II.5 ([HL17]). For all $\gamma, \gamma' > 0$ and all $p, p', q, q' \in [1, \infty)$, the space $\mathcal{D}^{\gamma}_{p,q}$ is continuously embedded into $\mathcal{D}^{\gamma'}_{p',q'}$ if one of the following conditions holds:

1. $p' < p$, $q' = q$ and $\gamma' = \gamma$ [if the underlying space is bounded],
2. $p' > p$, $q' = q$ and $\gamma' < \gamma - |\mathfrak{s}|(\frac{1}{p} - \frac{1}{p'})$,
3. $p' = p$, $q' > q$ and $\gamma' = \gamma$,
4. $p' = p$, $q' < q$ and $\gamma' < \gamma$.

These four cases are the same as for classical Besov spaces, except the second one where an inequality becomes strict. This is a technical restriction, which could possibly be lifted.

Although the two statements are essentially the same, it was not clear a priori how to establish the result in the case of modelled distributions. In particular, the non-integer levels (corresponding to the noise) of the regularity structure give rise to complicated terms on which one does not have much control.

A key observation for us was the following: the difficulty of the proof of the Embedding Theorem for classical Besov spaces heavily depends on the norm one starts from. If one opts for a "continuous" norm (involving L^p and L^q norms), then it appears to be technical. On the other hand, if one starts from a "countable" norm, that is, a norm that relies on only countably many evaluations of the function/distribution (with a wavelet analysis for instance), then the proof of the Embedding Theorem boils down to continuous inclusions of ℓ^p-type spaces.

This key observation motivated the introduction of a space of modelled distributions with a countable norm. A very natural way to achieve this is by considering averages of a modelled distributions at all dyadic scales:

$$\bar{f}^n(x) := \frac{1}{|B(x, 2^{-n})|} \int_{y \in B(x,2^{-n})} \Gamma_{x,y} f(y) dy \;, \quad x \in 2^{-n}\mathbb{Z}^d \;.$$

At the level of these spaces of averages, the proof of the Embedding Theorem is simple as it relies on continuous embeddings of ℓ^p and ℓ^q spaces. The important point then is to check that these spaces of averages are isomorphic to the original $\mathcal{D}^{\gamma}_{p,q}$ spaces: this is the content of [HL17, Theorem 2.15].

Let us now mention a consequence of this Embedding Theorem. Suppose we have constructed the solution of (PAM) starting from a Dirac mass in some space $\mathcal{D}^{\gamma}_{1,\infty}$: as explained at the beginning of this section, working in L^1 is much better than in L^∞ in terms of regularity of the initial condition. The reconstruction theorem applied to this solution evaluated at any given time t yields an element of $\mathcal{B}^{\alpha}_{1,\infty}$ where α is the regularity of the roughest functional appearing in the description of the solution, namely $P * \xi$, that is $\alpha = 1/2^-$ in dimension 3.
The Classical Embedding Theorem ensures that a function in $\mathcal{B}^{\alpha}_{1,\infty}(\mathbb{R}^d)$ also lies in $\mathcal{B}^{\alpha'}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha' = \alpha - d$. In other words, the solution of (PAM) is (at least) a Hölder distribution of regularity $-5/2^-$.
On the other hand, applying the Embedding Theorem for modelled distributions, we see that the abstract solution lives in $\mathcal{D}^{\gamma'}_{\infty,\infty}$ where $\gamma' = \gamma - d$. Note that γ (that controls the depth of the generalised Taylor expansions) can be taken as large as desired. Applying the reconstruction theorem, one then gets an element of $\mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ so that the solution of (PAM) is a Hölder function of regularity $1/2^-$ as expected.
In other words, Embedding and Reconstruction do not commute. This is not so surprising since some information is lost upon reconstructing a modelled distribution.

II.3 PAM on $\mathbb{R}^2$

The construction of (PAM) on a 2-dimensional torus was one of the main achievements of paracontrolled distributions [36] and regularity structures [38]. The solution constructed therein is the limit in probability as $\epsilon \downarrow 0$ of the sequence of regularised equations

$$\partial_t u_\epsilon = \Delta u_\epsilon + u_\epsilon(\xi_\epsilon - C_\epsilon) \;, \tag{II.5}$$

where ξ_ϵ is a regularised version of ξ (obtained by convolving ξ with a smooth function that oscillates at scale ϵ and that approximates a Dirac mass), and where C_ϵ is a constant that diverges like $\log \epsilon^{-1}$ and is required to renormalise the functional $\xi(P * \xi)$, as explained in Section II.1.

In these constructions, one controls the Hölder regularities of the functions / distributions at stake. On a d-dimensional torus, the white noise lives almost surely in $\mathcal{C}^\alpha(\mathbb{T}^d)$ with $\alpha = -d/2^-$. On the other hand, on the full space $\mathbb{R}^d$ the white noise does not belong to any global Hölder space $\mathcal{C}^\alpha(\mathbb{R}^d)$: indeed, its Hölder semi-norms on, say, disjoint balls of radius 1 are unbounded IID r.v. so that their supremum is infinite a.s.
To get global bounds on the Hölder regularity of the white noise, one needs

to weigh the Hölder spaces by some function $w_\xi(x)$ that goes to ∞ sufficiently fast as $|x| \to \infty$. It turns out that $\sqrt{\log(1+|x|)}$ is optimal for the white noise, but for simplicity we take $w_\xi(x) = (1+|x|)^a$ for some small $a > 0$.

Since the space the noise lives in needs a weight, we deduce that the space the solution lives in needs a weight too: let us call this weight w. But then, the product $u \cdot \xi$ needs a priori the weight $w_\xi \cdot w$ and this seems to prevent us from closing the fixed point equation (II.2): the weight on the r.h.s. is a priori larger than the weight on the l.h.s.

To circumvent this difficulty, we relied on a trick that was already used in [41], and probably elsewhere in the PDE literature. For the solution, let us consider a weight that grows exponentially in time

$$w(t,x) = e^{t(1+|x|)} \;.$$

Since the heat kernel $P(t,x)$ integrates to 1 in space and is close to a Dirac mass at 0 for small t, the term

$$P * (u \cdot \xi)(t,x) = \int_0^t \int_{y \in \mathbb{R}^d} P(t-s, x-y)(u \cdot \xi)(s,y) ds \;,$$

needs a weight of order

$$\int_0^t w(s,x) w_\xi(x) = \int_0^t e^{s(1+|x|)}(1+|x|)^a ds \;,$$

which is smaller than $w(t,x) = e^{t(1+|x|)}$. In other words, the time-averaging induced by the convolution with the heat kernel combined with the exponential growth of the weight produces a weight which is not larger than the weight we started from.

In [HL15], we implemented this trick to construct the solution of (PAM) on $\mathbb{R}^2$. Moreover, although this SPDE requires renormalisation, we proposed a simple construction, in the spirit of the so-called Da Prato-Debussche trick [21], that spared us from using regularity structures or paracontrolled distributions. We now explain the main steps of this approach.

Consider the solution of the Poisson equation[4]:

$$\Delta Y_\epsilon = \xi_\epsilon \;,$$

and observe that if u_ϵ is the solution of (II.5) then $v_\epsilon(t,x) = u_\epsilon(t,x) e^{Y_\epsilon(x)}$ solves the following equation

$$\partial_t v_\epsilon = \Delta v_\epsilon + v_\epsilon Z_\epsilon - 2 \nabla v_\epsilon \cdot \nabla Y_\epsilon \;, \tag{II.6}$$

[4]In order to avoid divergences at infinity, we actually applied some cutoff on the Green function of the Laplacian so that Y_ϵ instead satisfies $\Delta Y_\epsilon = \xi_\epsilon + F * \xi_\epsilon$ for some compactly supported smooth function F. This produces some minor modifications in the subsequent equations.

where

$$Z_\epsilon := |\nabla Y_\epsilon|^2 - C_\epsilon \ .$$

A Wiener chaos analysis allows to prove that for

$$C_\epsilon := \mathbb{E}[|\nabla Y_\epsilon|^2] = \frac{1}{2\pi} \log \epsilon^{-1} + \mathcal{O}(1) \ ,$$

the sequence Z_ϵ converges in probability to a well-defined limit in some (weighted) space of Hölder distributions of regularity β for any $\beta < 0$. Note that this is nothing but the Wick renormalisation of $|\nabla Y|^2$.

It is then possible to check that, in terms of Hölder regularity, all the products in (II.6) are well-defined uniformly over ϵ. Considering exponential weights in the Hölder spaces of functions/distributions, we then showed the following result.

Theorem II.6 ([HL15]). Let u_0 be a Hölder distribution with regularity better than -1, and that grows at most exponentially fast at infinity. The sequence of processes v_ϵ converges uniformly on all compact sets of $(0, \infty) \times \mathbb{R}^2$, in probability as $\epsilon \to 0$, to a limit v which is the unique solution of

$$\partial_t v = \Delta v + vZ - 2\nabla v \cdot \nabla Y \ , \qquad v(0,x) = u_0(x) e^{Y(x)} \ .$$

As a consequence, u_ε converges in probability towards the process $u = ve^{-Y}$.

Remark II.7. This transformation is intimately related to the Hopf-Cole transform: the process $h = \log u$ formally satisfies

$$\partial_t h = \Delta h + |\nabla h|^2 + \xi \ ,$$

and Y is the stationary solution of the linearised equation where $|\nabla h|^2$ has been removed.

This construction only works in dimension 2: in dimension 3, the equation for v_ϵ involves ill-defined products and it is not possible to easily remove the divergences as in dimension 2.

II.4 Multiplicative SPDE on the full space

This section presents the construction of (PAM) on $\mathbb{R}^3$ and of (SHE) on $\mathbb{R}$ that we obtained with Martin Hairer in [HL18] in the framework of regularity structures. Let us stress again that the existence of solutions to (SHE) on $\mathbb{R}$ was already known using Itô's integration, but the understanding of the stability of solutions under space-time regularisation of the noise was missing.

Theorem II.8 ([HL18]). Consider either (PAM) where $d = 3$, or (SHE) where $d = 1$. Let u_0 be a Dirac mass at 0. There exists a divergent sequence of constants C_ϵ such that the sequence of solutions u_ϵ of

$$\partial_t u_\epsilon = \Delta u_\epsilon + u_\epsilon(\xi_\epsilon - C_\epsilon) \;, \tag{II.7}$$

converges uniformly on compact sets of $(0, \infty) \times \mathbb{R}^d$ to a limit u, in probability.

Provided that C_ϵ is suitably chosen, the limit is independent of the choice of mollifier ρ. In the case of (SHE), the limit can be chosen to coincide with the Itô solution to the multiplicative stochastic heat equation [70, 22].

To deal with the unboundedness of the underlying space, we incorporated exponential weights, as presented in the previous section, in the spaces of modelled distributions. Actually, one needs to consider weights that depend on the level in the regularity structure and this makes the definition of these spaces very heavy: we will not present any detail here. Let us simply mention that our exponential weights do not satisfy the very desirable property that $c \le w(t,x)/w(s,y) \le C$ for some constants $0 < c < C$, uniformly over all (t,x) and (s,y) at distance one from each other (however this property holds true if $t = s$). As a consequence, approximation arguments that would require evaluations slightly further in time would break the control on the weights and we thus needed extremely fine control on the time evolution of the objects at stake (in particular, for proving the reconstruction theorem).

To deal with a Dirac mass as an initial condition, following a suggestion of Khalil Chouk, we considered modelled distributions of L^1-type as presented in Section II.2. (Actually, the spaces of modelled distributions considered in [HL18] take a slightly different form from those in [HL17]: the latter article being posterior to the former).

The solution that we construct is global in time. This is a consequence of the linearity (in the solution) of the equation which allows to iterate the solution map on successive time-interval of the same length.

Our method can be applied to slightly more general situations: one can replace $u \cdot \xi$ by $g(u)\xi$ and construct solution on the whole space (but only locally in time) if g is of linear growth at infinity. If the non-linearity grows faster than linearly, then our method does not apply anymore since the exponential growth in time of the weight does not compensate the growth due to the non-linearity.

Let us mention that the construction of singular SPDE on unbounded spaces has been the subject of several recent works. Mourrat and Weber [60] constructed the solution of the stochastic quantization equation (II.8) on $\mathbb{R}^2$: their construction relies extensively on the negative sign in front of the non-linearity which allows to get a priori bounds on the solution. Gubinelli and Hofmanová [35] presented a construction of the same equation but in dimension 3 (and also of the elliptic version of the equation): their approach relies on a decomposition of the solution into a local irregular part, which is not weighted, and a global regular part which is weighted. This approach is probably the most robust of all.

II.5 Existence of densities for the Φ^4_3 model

In this section, we present a result [GL19] obtained in collaboration with Paul Gassiat on the stochastic quantization equation in dimension 3:

$$\partial_t u = \Delta u - u^3 + \xi \;, \quad x \in \mathbb{T}^3, \quad t \geq 0 \;. \tag{II.8}$$

This equation, when ξ is a space-time white noise, formally preserves the so-called Φ^4_3 measure. It arises in the Ising model near the critical temperature, see [58] for the case of dimension 2. However, this SPDE is ill-defined and falls into the class of equations that require renormalisation: the solution was constructed using regularity structures by Hairer [38] and using para-controlled distributions by Catellier and Chouk [15], and the renormalisation boils down to adding formally $+\infty \cdot u$ in the above equation.

In the article [GL19], we considered a noise ξ which is obtained by convolving space-time white noise with a kernel R satisfying some assumptions. Roughly speaking, these assumptions consist in asking that either the Cameron-Martin space of ξ is dense in $L^2(\mathbb{R}_+ \times \mathbb{T}^3)$, or that ξ is degenerate but "sufficiently rough", i.e. of Hölder regularity strictly less than -2 (recall that space-time white noise has Hölder regularity strictly less than $-5/2$). The existence of solutions to (II.8) in that setting is essentially granted by [16] and [38]. Note also that the case where $R = \delta_0$, that is, ξ is a white noise falls into this set of assumptions.
Our main result informally states that u admits a density with respect to the Lebesgue measure: since u is not a function but only a distribution, the precise statement carries over evaluations of u against test functions. To be more precise, let $\varphi_i, i = 1 \ldots n$ be $n \geq 1$ linearly independent functions in the (parabolically scaled) Besov space $\mathcal{B}^{1/2+\kappa}_{1,\infty}(\mathbb{R}_+ \times \mathbb{T}^3)$, for some $\kappa > 0$, and assume that they are all supported in $(0, T) \times \mathbb{T}^3$ for some $T > 0$.

Theorem II.9 ([GL19]). Assume the solution u of (II.8) starting from some $u_0 \in \mathcal{C}^{-2/3+}$ exists[5] up to time T almost surely. Then, the random variable

$$X = (\langle u, \varphi_1 \rangle, \ldots, \langle u, \varphi_n \rangle) \;,$$

admits a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

[5] Mourrat and Weber [59] showed that the explosion time of the solution is actually infinite when the driving noise is a space-time white noise. Their proof should carry through if we replaced the white noise by a more general driving noise: consequently, the assumption on the existence of the solution up to time T is probably not restrictive at all.

In the case of space-time white noise, the strong Feller property proved in [39] implies that for each $t > 0$, the law of $u(t, \cdot)$ is absolutely continuous w.r.t. the invariant measure for (II.8), which is a stronger statement than simply considering its finite-dimensional projections. Our result however can be applied to noises that are not white in time, where the Markovian theory is of course not accessible. In addition, we obtain densities for averages of our solution in space and time, and not just at a fixed time which is the case considered in virtually all of the literature. (Note that the existence of densities for space-time averages is in principle a strictly stronger statement than densities for a fixed time, as soon as the regularity required for the test functions allows for Dirac masses in t. For technical reasons this is however not the case in our theorem).

The proof relies on Malliavin's calculus and is implemented in the setting of regularity structures. Using the Bouleau-Hirsch criterion, one has to show that: 1) the random variable X of the statement is Malliavin differentiable, 2) the Malliavin derivative is almost surely non-degenerate. To carry out the first task, we had to consider the equation with a noise shifted by an element h in its Cameron-Martin space (a subset of L^2 under our assumptions):

$$\partial_t u = \Delta u - u^3 + \xi + h \;, \quad u(0, \cdot) = u_0 \tag{II.9}$$

and the associated tangent equation (formally obtained by differentiating u w.r.t. h):

$$\partial_t v = \Delta v - 3u^2 v + h \;, \quad v(0, \cdot) = 0. \tag{II.10}$$

To construct the solutions of these equations, one possibility is to enlarge the model and to view h as another noise. This strategy was followed in [14] for proving existence of densities for a generalised (PAM) equation, but it would require to build by hand many different stochastic objects in the case of Φ^4_3. Another possibility is to view $P * h$ as an element of an L^2-type Besov space of modelled distributions and to set up fixed point arguments in such spaces: this is the strategy we followed in [GL19]. Note that to make sense of the cubic non-linearity applied to elements in L^2 we relied on the Embedding Theorem for spaces of modelled distributions presented in Section II.2.

To carry out the second task, we worked with a backward representation of the Malliavin derivative, namely for a given test function $\varphi \in \mathcal{B}^{1/2+\kappa}_{1,\infty}$ supported in $(0, T) \times \mathbb{T}^3$, we considered w which is (formally) solution to

$$(-\partial_t - \Delta) w = -3u^2 w + \varphi, \quad w(T, \cdot) = 0 \;. \tag{II.11}$$

It is possible to show that the non-degeneracy of the Malliavin derivative of X is equivalent with proving the following implication

$$\Big(\langle w, h \rangle_{L^2([0,T]\times\mathbb{T}^3)} = 0 \quad \forall h \in \mathcal{H} \Big) \;\Rightarrow\; \varphi = 0,$$

where $\mathcal{H}$ is the Cameron-Martin space associated to the noise. Using the equation satisfied by w, a simple induction argument gives the implication

$$w = 0 \Rightarrow \varphi = 0,$$

so that when $\mathcal{H}$ is dense in L^2 the result follows immediately. When the noise is degenerate, we rely on our roughness assumption that allows to separate contributions coming from different levels in the regularity structures and show that $w = 0$ a.e., thus concluding the proof.

Chapter III

The continuous Anderson model

In 1958 the physicist P.W. Anderson [6] proposed the operator $-\Delta + V$ on $\mathbb{Z}^d$ as a simplified model for the Hamiltonian of a quantum particle evolving in a crystal: here the random disorder V models the impurities or defects in the crystal.
The spectrum of $-\Delta$ on $\mathbb{Z}^d$ is the interval $[0, 4]$. It contains no classical eigenfunctions: the point component of the spectral measure is null. The associated "eigenfunctions", which are sine functions, therefore do not belong to $\ell^2(\mathbb{Z}^d)$, and are delocalised in the sense that their "ℓ^2 mass" does not concentrate anywhere in space.
Anderson [6] argued that, for a strong enough disorder, parts of the spectrum of $-\Delta + V$ are made of classical eigenfunctions (the spectrum is pure point there) which are exponentially localised in space. This phenomenon is often referred to as Anderson localisation.
Physically, this produces a drastic change of behaviour for the wave function of the quantum particle: instead of diffusing in space, it remains in bounded spatial regions if initially the state of the particle lied in the subspace spanned by the point component.

Let us emphasise here the mechanism at play in this random operator: there is a competition between the Laplacian, that tends to spread the ℓ^2 mass of the eigenfunctions and, thus, to produce a continuous spectrum with delocalised eigenfunctions, and the disorder V, that tends to localise the eigenfunctions, and thus, to produce a point spectrum.

An important literature in the mathematics community is devoted to establishing rigorously this phenomenon. The case where $V = (V(k), k \in \mathbb{Z}^d)$ is a field of IID r.v. was extensively studied in the 80's and 90's, and the general results [1, 28, 29, 47] that emerged are essentially twofold:

- In dimension $d \geq 1$, the lower part of the spectrum is pure point and the eigenfunctions are localised,
- In dimension $d = 1$, the whole spectrum is pure point and the eigen-

functions are localised.

In the localised regime, the local statistics of the eigenvalues are often shown to be Poissonian, see [55].

Some continuous versions of this operator, namely $-\Delta + V$ on $\mathbb{R}^d$ with $-\Delta$ the continuous Laplacian, have been investigated. The case where V is a smooth gaussian field was considered in [27], and a result analogous to what is known in the discrete setting was established: in any dimension $d \geq 1$ the lower part of the spectrum is pure point.
In dimension $d = 1$, a very special setting was investigated by Gol'dscheid, Molchanov and Pastur [32] in the seventies: their disorder V is taken to be $F(B(x))$ where F is a positive smooth function and B is a stationary Brownian motion on a compact manifold. They showed that the spectrum is pure point and, later on, Molchanov [57] showed that the local statistics are Poissonian: we will come back to these works in Subsection III.2.4.

The central object of this chapter is the operator $-\Delta + \xi$ where Δ is the continuous Laplacian and the disorder is taken to be a white noise ξ on $\mathbb{R}^d$. Since the white noise is a universal object that arises as the scaling limit of fields of IID r.v. with finite variance, the corresponding operator appears formally as a scaling limit of the discrete operators mentioned at the beginning of this chapter. Actually, the intensity of the discrete disorders has to be tuned appropriately: a scaling argument shows that the operator $-N^2\Delta + N^{\frac{d}{2}}V$ restricted to $\mathbb{Z}^d \cap [0, NL]^d$ converges formally to $-\Delta + \xi$ on $(0, L)^d$. The intensity of the discrete disorder is therefore given by $N^{\frac{d}{2}-2}$ and goes to 0 as $N \to \infty$ whenever $d \leq 3$. Therefore, it is not clear at this point whether the operator $-\Delta + \xi$ on $(0, L)^d$ should have localised or delocalised eigenfunctions since we are in a situation which is not covered by the general results recalled above (the discrete disorder is vanishing).

Remark III.1. In dimension $d \geq 4$, the prefactor $N^{\frac{d}{2}-2}$ is constant or goes to infinity, which means that we are in situations which should already be very much localised. This is reflected in the fact that the corresponding continuous operator does not fall in the scope of renormalisation theories (it is not subcritical, see Remark II.1) as soon as $d \geq 4$.

The mere definition of the operator $-\Delta + \xi$, even in finite volume, is actually far from being obvious due to the irregularity of the white noise. In Section III.1, we will present results on the construction of the operator in any dimension $d \leq 3$. Section III.2 is devoted to the study of the spectrum when $d = 1$. Section III.3 presents results on the stochastic Airy operator, which shares similarities with the Anderson Hamiltonian in dimension 1, and which arises in the study of gaussian β-ensembles.

III.1 The continuous Anderson Hamiltonian

In this section, we set

$$\mathcal{H}_L f := -\Delta f + f \cdot \xi \ ,$$

where f is a function on $(0,L)^d$, Δ is the continuous Laplacian and ξ is a white noise on $(0,L)^d$. We will sometimes omit writing the subscript L in $\mathcal{H}_L$ when it plays no role in the discussion.

Let us present briefly the reasons why the construction of $\mathcal{H}$ is non-trivial. First of all, while $\mathcal{H}f$ is well-defined whenever f is a smooth (say $\mathcal{C}^2$) function, it never belongs to $L^2((0,L)^d)$. Indeed, the product $f \cdot \xi$ is not a function but only a distribution so that for $\mathcal{H}f$ to belong to L^2 one needs a subtle cancellation to happen between $-\Delta f$ and $f \cdot \xi$, and this requires $-\Delta f$ itself not to be a function. Consequently the putative domain of $\mathcal{H}$ does not contain any smooth function.

However, in dimension 1 Fukushima and Nakao [30] were able to construct the operator $\mathcal{H}$ under Dirichlet b.c. Let us recall the main steps of their construction (note that it can be adapted to cover other types of boundary conditions). One starts by proving that the bilinear form

$$\mathcal{E}(f,g) = \int \nabla f \nabla g + \int \xi f g \ ,$$

is closed in H_0^1. This ensures that H_0^1, endowed with the inner product $\mathcal{E}(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2}$ for some large enough $\lambda > 0$, is a Hilbert space: a classical representation theorem then allows to construct the resolvents. Due to the compactness of the injection of H_0^1 into L^2, the resolvents are compact, self-adjoint operators on L^2 so that they are associated with a self-adjoint operator $\mathcal{H}$ with pure point spectrum. This construction is completely deterministic and applies to any potential ξ which is the distributional derivative of a bounded function. Let us also point out that the construction does not yield much information on the domain of $\mathcal{H}$. However, one can guess that any element f in the domain of $\mathcal{H}$ should locally behave like $(-\Delta)^{-1}\xi$ so that the domain is made of random Hölder $3/2^-$ functions.

In dimension 2 and above, the term $\int \xi f g$ is no longer well-defined for $f, g \in H_0^1$, and it is possible to check that the bilinear form $\mathcal{E}$ is not closable. In fact, the domain of the form itself is random: one needs to consider the sum $\nabla f \nabla g + \xi f g$ as a whole and hope for a cancellation to happen for its integral to make sense.

Actually, an additional difficulty appears. Since any element in the domain of $\mathcal{H}$ should behave locally like $(-\Delta)^{-1}\xi$, the product $\xi \cdot (-\Delta)^{-1}\xi$ arises when applying the operator $\mathcal{H}$ to any element in its domain: while this distribution is well-defined in dimension 1 by Young's integration (recall that $(-\Delta)^{-1}$ improves regularity index by 2), it falls out of the scope of deterministic integration theories as soon as $d \geq 2$. This term needs to be renormalised by subtracting some infinite constant exactly like for (PAM), see Section II.1. Note that in dimension 3, there are other ill-defined products that need to be renormalised.

This suggests the following procedure. Given a regularised potential ξ_ϵ, the corresponding operator $-\Delta + \xi_\epsilon$ is well-defined and its domain is H^2 (up to the choice of boundary conditions). From the above discussion, this sequence of operators does not converge as $\epsilon \downarrow 0$. Instead, one considers a renormalised operator obtained by setting

$$\mathcal{H}_\epsilon := -\Delta + \xi_\epsilon + C_\epsilon \ .$$

for some appropriately chosen C_ϵ. One then expects $\mathcal{H}_\epsilon$ to converge, in some sense, to a limit that we call $\mathcal{H}$.

Such a result was proven by Allez and Chouk [2] in dimension 2, and under periodic boundary conditions. To give a meaning to the limiting operator, they adopted the theory of paracontrolled distributions [36].
Later on, Gubinelli, Ugurcan and Zachhuber [37] performed a construction, also based on paracontrolled distributions, of the operator in dimension 3 under periodic boundary conditions: they obtained several interesting functional inequalities and solved semi-linear PDEs involving this Hamiltonian. Roughly at the same time, we proposed a construction [Lab19] of the operator $\mathcal{H}$ in any dimension $d \leq 3$ and for both Dirichlet and periodic b.c., using the theory of regularity structures. Let us mention that Dirichlet b.c. is required for approximation arguments: for instance, one would expect that the behaviour of the operator on a large box is similar (in some sense) to the "joint behaviour" of the same operator, endowed with Dirichlet b.c., on disjoint sub-boxes.
Exactly like for (PAM) studied in Chapter II, in dimension $d \geq 4$, none of these theories apply anymore.

The main result of this section is the following. Let ρ be an even, smooth function integrating to 1 and supported in the unit ball of $\mathbb{R}^d$. Set $\rho_\epsilon(\cdot) := \epsilon^{-d}\rho(\cdot/\epsilon)$ for any $\epsilon > 0$, and consider the noise ξ_ϵ obtained by convolving white noise ξ with ρ_ϵ.

Theorem III.2 ([Lab19]). In any dimension $d \in \{1, 2, 3\}$ and under periodic or Dirichlet b.c., there exists a self-adjoint operator $\mathcal{H}$ on $L^2((0, L)^d)$ with pure point spectrum such that the following holds. For some suitably chosen sequence of constants C_ϵ, as $\epsilon \downarrow 0$ the eigenvalues/eigenfunctions $(\lambda_{\epsilon,n}, \varphi_{\epsilon,n})_{n\geq 1}$ of $\mathcal{H}_\epsilon$ converge in probability to the eigenvalues/eigenfunctions $(\lambda_n, \varphi_n)_{n\geq 1}$ of $\mathcal{H}$.

This construction relies on the theory of regularity structures, and most of the work consists in defining the resolvent operators through a fixed point problem. Once these resolvent operators are constructed, it is possible to show that they are continuous (in some sense) w.r.t. the driving noise and that they are compact and self-adjoint operators. The theorem then follows from classical arguments. Let us give some more details on the construction. The resolvent $G_a = (\mathcal{H} + a)^{-1}$ applied to some function $g \in L^2$ should be the fixed point of the map

$$f \mapsto (-\Delta + a)^{-1} g - (-\Delta + a)^{-1}(f \cdot \xi)\ . \qquad \text{(III.1)}$$

To deal with ill-defined products, this fixed point problem is lifted into an appropriate regularity structure (essentially the same as the one required to solve (PAM)). Since $g \in L^2$, the natural setting for solving (III.1) is an L^2-type space. Therefore, we rely on Besov-type spaces of modelled distributions introduced in [HL17], see Section II.2.

Let us explain the main difficulties coming from the boundary conditions. Most of the PDEs solved with the theory of regularity structures have been taken under periodic boundary conditions: this choice of b.c. does not induce any specific difficulty in our setting. On the other hand, if one opts for Dirichlet b.c. then the Green's function of the Laplacian is no longer translation invariant so that the construction of the model (in the sense of regularity structures, see Section II.1) and the identification of the renormalisation constants may become involved.

In a recent work [31] on parabolic SPDEs with b.c., Gerencsér and Hairer presented a nice trick to circumvent this difficulty. Using the reflection principle, one can write the Green's function under Dirichlet b.c. as the sum of the Green's function on the whole space plus a series of shifted versions of this same function. The singularities of these shifted Green's functions are localised at the boundary of the domain. Thus, one builds the model with the (translation invariant!) Green's function of the Laplacian on the whole space, and one deals "by hand" with the remaining kernels. This last part involves adding some weights near the boundary in the spaces of modelled distributions. An important difficulty in the present case comes from the interplay of these weights with the L^2 setting.

The second main result on this operator presented in [Lab19] is an estimate on the left tail of the distributions of the eigenvalues.

Theorem III.3 ([Lab19]). In the context of the previous theorem, for any $n \geq 1$ there exist two constants $a > b > 0$ such that for all $x > 0$ large enough we have

$$e^{-ax^{2-d/2}} \leq \mathbb{P}(\lambda_n < -x) \leq e^{-bx^{2-d/2}}\ . \qquad \text{(III.2)}$$

In dimension 1 and under periodic b.c., a much more precise result was established by Cambronero, Rider and Ramírez [13] on the first eigenvalue: they proved that for $L = 1$ the density of λ_1 is given by

$$\frac{4}{3\pi}|x|e^{-\frac{8}{3}|x|^{3/2}-\frac{1}{2}|x|^{1/2}}(1+o(1))\ , \quad x \to -\infty\ .$$

In dimension 2 and under periodic b.c., this result was established by Allez and Chouk [2] for the first eigenvalue - they also conjectured the present result in dimension 3. As a corollary of Theorem III.3, one deduces that the solution to (PAM)

$$\partial_t u = \Delta u - u \cdot \xi \;, \quad x \in (0, L)^d \;,$$

has no moments in dimension 3, and has finite moments in dimension 2 up to some finite time. Indeed, $u(t,x) = e^{-t\mathcal{H}} u_0(x)$ so that the moments of u are related to the exponential moments of the eigenvalues of $\mathcal{H}$.

The proof of Theorem III.3 essentially follows the strategy presented in [2]. For simplicity, let us consider the operator on $(0,1)^d$ since the size of the domain does not play any role here and since L will be used as a scaling parameter.
The key observation is the following: the n-th eigenvalue λ_n of the Anderson hamiltonian on $(0,1)^d$ coincides (up to a correction term due to renormalisation) with $L^2\,\widetilde{\lambda}_n$ where $\widetilde{\lambda}_n$ is the n-th eigenvalue of the Anderson hamiltonian on $(0,L)^d$ with potential $L^{d/2-2}\xi$. Taking $x \asymp L^2$, we deduce that to obtain (III.2) it suffices to bound from above and below the probability that $\widetilde{\lambda}_n < -c$ for some constant $c > 0$, uniformly over all large L.
The latter eigenvalue should be very close to 0 with large probability since the noise term vanishes as L goes to infinity: hence $\widetilde{\lambda}_n < -c$ should be a large deviation event. In particular, to prove the lower bound, we build some deterministic potential h whose n-th eigenvalue is less than $-c$ and then use the Cameron-Martin Theorem to estimate the probability that $\widetilde{\xi}$ is close to h: in our context, this part requires to adapt some arguments from [40, 42] on generalised convolutions encoded by labelled graphs.

Future directions. The construction of the operator opens the way to the investigation of its spectral properties. In a recent work [17], Chouk and Van Zuijlen determined the asymptotic behavior as $L \to \infty$ of the first eigenvalue of $\mathcal{H}_L$ in dimension 2. In collaboration with Giuseppe Cannizzaro (Univ. of Warwick) and Willem Van Zuijlen (TU Berlin) we have been working on the fluctuations of this first eigenvalue in dimensions 2 and 3. It would also be interesting to investigate the bulk of the spectrum of $\mathcal{H}_L$ in dimension 2 and 3, using techniques from the literature on Schrödinger operators.
Note that the behaviour in dimension 1 is now well-understood, see the next section.

III.2 Localisation of the Anderson Hamiltonian in 1-d

This section is devoted to the one dimensional setting and presents results obtained in collaboration with Laure Dumaz on the behaviour of the eigenvalues/eigenfunctions of $\mathcal{H}_L$ when $L \to \infty$. We will always assume that the operator is endowed with homogeneous Dirichlet boundary conditions, see Remark III.6 for other b.c. Before we present our results, we recall some recent results in random matrices on the localisation/delocalisation transition for the discrete Anderson Hamiltonian. Then, we provide a brief review of previous results on $\mathcal{H}_L$. Finally Subsections III.2.3 and III.2.4 present the results obtained with Laure Dumaz.

III.2.1 A motivating result coming from random matrices

Let us present some results contained in two works by Kritchevski, Valkó and Virág [46] and Rifkind and Virág [68]. They considered the following $N \times N$ matrix

$$M_N := \begin{pmatrix} 2+\xi(1) & -1 & & & \\ -1 & 2+\xi(2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2+\xi(N) \end{pmatrix}$$

where $\xi(k), k = 1, \ldots, N$ are IID r.v. This is nothing but the discrete Laplacian $-\Delta$ plus a diagonal disorder ξ. We will denote by λ_k, $k = 1, \ldots, N$ the eigenvalues of M_N.
If the law of ξ does not depend on N, then in the limit $N \to \infty$ the spectral properties of this model fall into the scope of the general results on Anderson localisation recalled at the very beginning of this chapter: the whole spectrum is made of localised eigenfunctions and the local statistics of the eigenvalues are Poissonian. In some sense, the disorder is too strong: in order to get delocalised eigenfunctions, one needs to scale down its amplitude. To that end, let us take

$$\xi(k) = \frac{\sigma}{N^\gamma} v(k) \ , \quad k = 1, \ldots, N \ ,$$

where $v(k), k = 1, \ldots, N$ are centered IID r.v. of variance 1 and $\sigma > 0$ is a tunable parameter. Since the disorder is vanishing, the density of states of M_N is that of the discrete Laplacian $-\Delta$, namely:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_k}(dE) = \rho(E)dE \ , \qquad \text{a.s.} \ ,$$

where

$$\rho(E) = \frac{1}{\pi\sqrt{E(4-E)}} \mathbb{1}_{\{E\in(0,4)\}} \ .$$

Kritchevski, Valkó and Virág [46] studied the local statistics of the eigenvalues in the bulk of the spectrum and showed that the transition between the localised and delocalised regime happens when $\gamma = 1/2$. More precisely, for any given energy $E \in (0,4)$ they showed that for $\gamma = 1/2$ the point process

$$(N\rho(E)(\lambda_k - E))_{k=1,\ldots,N} \ ,$$

converges in law to a random point process that they called Sch_τ, with $\tau = (\sigma\rho(E))^2$, and which is defined as follows

$$\mathrm{Sch}_\tau := \{\lambda \in \mathbb{R} : \varphi_{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z}\} \ ,$$

where φ_λ is a collection of coupled SDEs

$$d\varphi_\lambda(t) = \lambda dt + dB(t) + \Re(e^{-i\varphi_\lambda(t)} dW(t)) \ , \quad \varphi_\lambda(0) = 0 \ ,$$

driven by a real Brownian motion B and an independent complex Brownian motion W.

The point process Sch_τ interpolates between the localised and delocalised regimes. Indeed, when σ is sent to ∞, the parameter τ goes to ∞ and the point process Sch_τ converges in law to a Poisson point process (localised regime). On the other hand, when σ is sent to 0, the parameter τ goes to 0, and the point process Sch_τ converges to a deterministic point process that corresponds to the eigenvalues of the discrete Laplacian (delocalised regime). Actually, these two statements are not written explicitly in the literature, but they can be derived by adapting arguments in [3] for instance.

In [68], the scaling limit of the eigenfunctions of M_N, in the regime $\gamma = 1/2$ studied in [46], was obtained. They showed that, under an appropriate rescaling, these eigenfunctions converge in law to

$$\varphi(t) = \frac{1}{Z} \exp(-\frac{1}{\sqrt{2}} B_{t-U} - \frac{1}{4}|t-U|) \ , t \in [0,1] \ , \tag{III.3}$$

where U is a uniform r.v. on $[0,1]$, B is an independent two-sided Brownian motion and Z is a random constant. In other words, the eigenfunctions corresponding to the point process Sch_τ are delocalised.

As explained at the beginning of the chapter, the operator $-\Delta + \xi$ on $(0,1)$ is the formal scaling limit of the random matrix $N^2 M_N$ with $\gamma = 3/2$ and $\sigma = 1$. It can be checked that to obtain the operator on $(0,L)$ one needs to take $N^2 M_{NL}$ with $\gamma = 3/2$ and $\sigma = L^{3/2}$. Consequently, for fixed L the operator $-\Delta + \xi$ should be delocalised. However, by sending $L \to \infty$ on the continuous operator one can hope to recover a localised regime. Answers to these questions are provided in Subsections III.2.3 and III.2.4.

III.2.2 Existing results on the continuous Anderson Hamiltonian

Recall that in dimension 1, the operator $\mathcal{H}_L$ does not need renormalisation. Note also that the noise ξ can be seen as the distributional derivative of a Brownian motion B. By the result of Fukushima and Nakao [30] recalled in Section III.1, the operator has a pure point spectrum of multiplicity one bounded below: we will denote by $\lambda_1 < \lambda_2 < \ldots$ the successive eigenvalues, and by φ_k the corresponding eigenfunctions normalised in L^2.
Halperin [43] and later on Fukushima and Nakao [30] showed that the operator admits an integrated density of states:

$$N(E) := \lim_{L\to\infty} \frac{\#\{\text{ eigenvalues of } \mathcal{H}_L \le E\}}{L} \;,$$

where the convergence is almost sure. This quantity is given by the explicit formula

$$N(E) = \Big(\sqrt{2\pi}\int_0^\infty u^{-1/2} e^{-\frac{1}{6}u^3 - 2Eu} du\Big)^{-1} \;.$$

A meaningful quantity is the derivative of N, usually called the density of states: $n(E) = dN(E)/dE$. Roughly speaking, for large L, the typical spacing between two consecutive eigenvalues of $\mathcal{H}_L$ lying near E is of order $1/(Ln(E))$. Simple computations yield the following asymptotics:

$$n(E) \sim \frac{1}{2\pi\sqrt{E}} \;, \quad E \to +\infty \;,$$

and

$$n(E) \sim \frac{4|E|}{\pi} e^{-\frac{8}{3}|E|^{3/2}} \;, \quad E \to -\infty \;.$$

Note that the deterministic operator $-\Delta$ admits an integrated density of states given by $E \mapsto \sqrt{E}/\pi$ so that the two spectra behave similarly at $+\infty$, see Figure III.1. In some sense, the noise perturbs only the lower part of the spectrum: our results presented below provide a rigorous statement in that direction.

In a short article [54] written in 1994, McKean obtained the asymptotic behaviour of the smallest eigenvalue λ_1 of $\mathcal{H}_L$ when $L \to \infty$. He showed that there exists a deterministic function a_L, which is equivalent to $(\frac{3}{8}\ln L)^{2/3}$ when $L \to \infty$, such that the following convergence in distribution holds

$$-4\sqrt{a_L}(\lambda_1 + a_L) \Rightarrow \text{ Gumbel law} \;, \quad L \to \infty \;.$$

Note that λ_1 fluctuates in a tiny window of order $(\ln L)^{-1/3}$ around $-a_L$. Note also that this convergence is in line with the asymptotic behaviour of the integrated density of states: indeed $N(a_L)$ is roughly of order $1/L$.

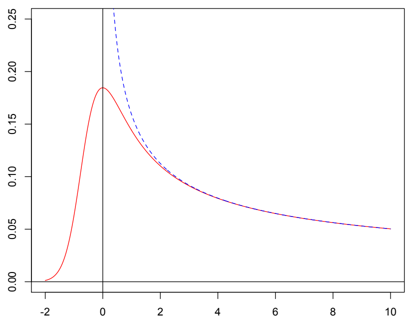

Figure III.1: Density of states of $\mathcal{H}_L$ (plain line) and of $-\Delta$ (dashed line).

III.2.3 Bottom of the spectrum

With Laure Dumaz, we first investigated the bottom of the spectrum of $\mathcal{H}_L$. To state the results, we need to introduce some notations. Let U_k be the point in $(0, L)$ where $|\varphi_k|$ reaches its maximum (if there are several points, then choose one arbitrarily). We also need to rescale the eigenfunctions onto $(0, 1)$ by setting $m_k(dx) = L\varphi_k^2(Lx)dx$, which is a probability measure on $(0, 1)$.

Theorem III.4 ([DL19b]). We have the following convergence[1] in law

$$\Big(4\sqrt{a_L}(\lambda_k + a_L), U_k/L, m_k\Big)_{k\geq 1} \Longrightarrow \Big(\lambda_k^\infty, U_k^\infty, \delta_{U_k^\infty}\Big)_{k\geq 1}\ , \quad L \to \infty\ ,$$

where $(\lambda_k^\infty, U_k^\infty)_{k\geq 1}$ are the atoms of a Poisson point process on $\mathbb{R} \times (0, 1)$ of intensity $e^x dx du$.

This result shows that the eigenvalues, in the fluctuation scale of McKean, converge to a Poisson point process: the intensity of this process being of course coherent with the Gumbel law obtained by McKean for the first eigenvalue. The result also shows that the eigenfunctions converge to Dirac masses, located at IID uniform points which are independent from the limiting eigenvalues.

A natural subsequent question is to determine the localisation length of the eigenfunctions. The following result shows that this length is of order $(\ln L)^{-1/3}$ and that the limiting shape of the eigenfunctions and of the driving Brownian motion, near the localisation centers, is deterministic.

[1] The topology that we consider on sequences is the product topology; as a consequence, this result only concerns the bottom of the spectrum of $\mathcal{H}_L$.

Theorem III.5 ([DL19b]). For any $k \geq 1$, the following pair of processes

$$h_k(x) := \frac{\sqrt{2}}{a_L^{1/4}} \Big|\varphi_k\Big(U_k + \frac{x}{\sqrt{a_L}}\Big)\Big| \;, \quad b_k(x) = \frac{1}{\sqrt{a_L}}\Big(B\Big(U_k + \frac{x}{\sqrt{a_L}}\Big) - B(U_k)\Big) \;, \quad x \in \mathbb{R} \;,$$

converges in probability, locally uniformly in space, to the following pair of deterministic processes

$$h(x) := \frac{1}{\cosh(x)} \;, \quad b(x) = -2\tanh(x) \;, \quad x \in \mathbb{R} \;.$$

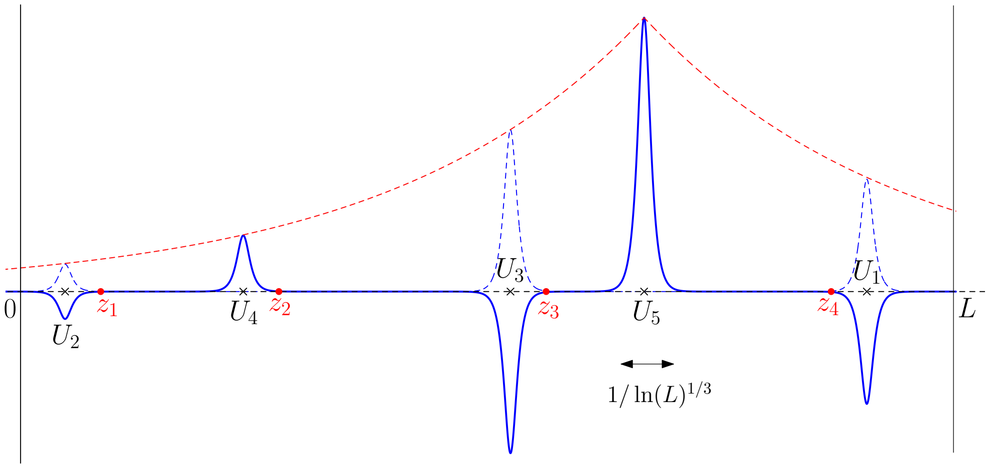

Figure III.2: A very schematic plot of the fifth eigenfunction φ_5 (blue plain line). The main peak lies at U_5, while near U_i, for every $i < 5$, the eigenfunction has peaks of smaller order. Observe that the height of the peak near U_i decays exponentially with the distance $|U_i - U_5|$. Red dots correspond to the zeros of the eigenfunction. The red dashed line illustrates the exponential decay of the eigenfunction.

We actually obtain more information on the eigenfunctions: let us describe informally the results (precise statements can be found in [DL19b]). For any $k \geq 1$, the k-th eigenfunction decays exponentially from its localisation center at rate $\sqrt{a_L}$: in other words, $|\varphi_k(x)|$ is roughly bounded by $Ce^{-\sqrt{a_L}|x-U_k|}$, see Figure III.2.
Moreover, the $k-1$ zeros of φ_k (excluding those at the boundaries) are very close to the localisation centers of the $k-1$ first eigenfunctions. In addition, $|\varphi_k|$ admits local maxima near the localisation centers of the $k-1$ first eigenfunctions with a deterministic shape given by the inverse of a hyperbolic cosine. This is illustrated on Figure III.2.

Remark III.6. The results are stated for homogeneous Dirichlet b.c., but also hold for homogeneous Neumann b.c. Actually, one can couple the two versions of the operators associated to these two b.c., and it turns out that the limiting r.v. are almost surely the same: namely, λ_k^∞ and U_k^∞ are a.s. the same for the two b.c. This can be generalised to any Robin b.c., that is, boundary conditions of the form $\varphi'(0)/\varphi(0) = -\varphi'(L)/\varphi(L) = \alpha \in (-\infty, \infty]$.

At a technical level, our proof relies extensively on the Riccati transform which is defined as follows. Suppose φ satisfies $\varphi(0) = 0$ and $-\varphi'' + \xi\varphi = \lambda\varphi$ on $(0, L)$. Then, it can be checked that the process $Y(t) = \varphi'(t)/\varphi(t)$ satisfies

$$dY(t) = (-\lambda - Y^2(t))dt + dB(t) \ , \quad Y(0) = +\infty \ .$$

For (λ, φ) to be an eigenvalue / eigenfunction pair of $\mathcal{H}_L$, it is necessary and sufficient that $\varphi(L) = 0$, which is equivalent with $Y(L) = -\infty$.
Consequently, by studying the coupled collection of SDEs

$$dX_a(t) = (a - X_a(t)^2)dt + dB(t) \ , \quad X_a(0) = +\infty \ ,$$

one identifies the set of eigenvalues as the set of values $-a$ such that $X_a(L) = -\infty$. Note that the lowest eigenvalues of $\mathcal{H}_L$ are typically negative and are expected to go to $-\infty$ as $L \to \infty$, so that, the meaningful parameters a in the study of the bottom of the spectrum of $\mathcal{H}_L$ are positive and very large.

Remark III.7. The Riccati transform is nothing but the Hopf-Cole transform! Indeed, Y is the logarithmic derivative of φ and the map $\varphi \mapsto Y$ sends a multiplicative equation onto an additive equation (w.r.t. the noise). In this context, the Hopf-Cole transform is applied in the converse direction than for the KPZ equation.

A typical realisation of X_a, for a large $a > 0$, comes down from $+\infty$ quickly and oscillates for a long time around the stable equilibrium point $x = \sqrt{a}$ of its potential $V_a(x) = x^3/3 - ax$. From time to time, an exceptional excursion away from $\sqrt{a}$ brings the diffusion to the unstable equilibrium point $x = -\sqrt{a}$: from that point, it either explodes quickly to $-\infty$ or comes back quickly to a neighbourhood of $x = \sqrt{a}$. If it explodes to $-\infty$, then it restarts immediately from $+\infty$.

To prove the above theorems, we performed a detailed analysis of this family of diffusions. A crucial observation is that the Riccati transforms of the eigenfunctions do not behave like typical solutions X_a of the above SDE. To illustrate this, let us consider the Riccati transform Y_1 of the first eigenfunction φ_1. We show that Y_1 behaves like a typical diffusion X_a up to a random time (of order L) at which it reaches the unstable equilibrium point $x = -\sqrt{a}$, and then, it oscillates around that point until time L. Since $-\sqrt{a}$ is unstable, these oscillations are very unlikely for the diffusion.
The reason for this atypical behaviour of Y_1 can be easily explained. If one

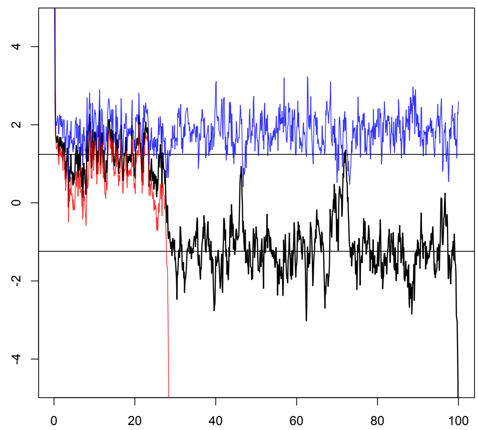

Figure III.3: Simulation of Y_1 (in black) and of two diffusions X_a (in red) and $X_{a'}$ (in blue) with $a < -\lambda_1 < a'$. The three processes are close to each other up to the random time at which X_a explodes: from that time on, monotonicity does not provide control on Y_1 and we then rely on the backward diffusions. Note that Y_1 is typical w.r.t. the forward diffusions up to the aforementioned random time, and is then typical w.r.t. the backward diffusions.

applies the Riccati transform backward in time, then one gets a process $\hat{Y}(t) = \varphi'(L-t)/\varphi(L-t)$ that satisfies

$$d\hat{Y}(t) = (\lambda + \hat{Y}^2(t))dt - dB(L-t)\ , \quad \hat{Y}(0) = -\infty\ .$$

The associated process $\hat{X}_a$ is equal in law to $-X_a$: its stable equilibrium point is at $x = -\sqrt{a}$ and $x = +\sqrt{a}$ is an unstable equilibrium point.
Our strategy of proof relies on the monotonicity of these collections of diffusions, and consists in controlling the Riccati transform forward in time with the forward diffusions X_a, see for instance Figure III.3, and backward in time with the diffusions $\hat{X}_a$: a (huge) number of technical estimates on typical realisations of these two sets of diffusions then provide controls on Y_1, which turn out to be atypical for both sets of diffusions.

III.2.4 From the bulk to the top of the spectrum

The subsequent results are taken from an article [DL19a] in collaboration with Laure Dumaz and which is still in preparation. They concern the behaviour as $L \to \infty$ of three distinct regions of the spectrum of $\mathcal{H}_L$. While the previous subsection focused on the bottom of the spectrum, here we investigate eigenvalues higher within the spectrum. Recall that $n(E)$ is the density of states, so that, when looking at the local statistics of the eigenvalues point process around E, the natural scaling is to zoom in at scale $Ln(E)$.

First, we look at the bulk of the spectrum, that is, at eigenvalues of order 1 uniformly over L.

Theorem III.8 (Bulk of the spectrum, [DL19a]). Fix some $E \in \mathbb{R}$. Then, the point process

$$\mathcal{Q}_E(dx) := \sum_{i\geq 1} \delta_{L\, n(E)(\lambda_i - E)}(dx) ,$$

converges in law to a Poisson point process on $\mathbb{R}$ of intensity dx as $L \to \infty$. Furthermore, the associated eigenfunctions decay exponentially with a rate of order 1.

This result probably implies Anderson localisation in the sense that the operator $-\Delta + \xi$ on $\mathbb{R}$ has pure point spectrum and its eigenfunctions are exponentially localised. However, the construction of the operator $-\Delta + \xi$ on the full space is not trivial and has not been performed[2] yet.
Technically, this result is inspired by a series [32, 56, 57] of works by Molchanov and co-authors on a relative of the present operator in which ξ is replaced by $F(B)$ for some smooth function F and some stationary Brownian motion B on a compact manifold. In particular, in [57] it is proven that the local statistics of the spectrum of the latter operator are Poissonian. A crucial formula in that work, that we adapted to our case, allows to express local statistics of the spectrum in terms of a concatenation of forward and backward diffusions.

Remark III.9. We were not able to understand the final argument of [57, Section 10 and 11] that shows that the local statistics of the spectrum are Poissonian. In [DL19a] we present a different approach based on the study of diffusions that allows to identify the law of these statistics.

Second, we consider the region of the spectrum where eigenvalues are of order L. Therein, the density of states $n(E)$ is of order $1/\sqrt{L}$. We show that in the scaling limit $L \to \infty$, we recover the point process Sch_τ introduced by Kritchevski, Valkó and Virág, whose definition was recalled in Subsection III.2.1. This result identifies the critical regime where the transition localisation/delocalisation occurs.

Theorem III.10 (Critical regime, [DL19a]). Assume that $E = E(L) \sim \alpha L$ for some $\alpha > 0$. Then, the point process

$$\mathcal{Q}_E(dx) := \sum_{i\geq 1} \delta_{\sqrt{L/\alpha}(\lambda_i - \alpha L) - \{2\sqrt{\alpha} L^{3/2}\}_{2\pi}} ,$$

[2] We were informed by Massimiliano Gubinelli and Baris Ugurcan that they have been working on the construction of this operator on $\mathbb{R}^d$ for $d \leq 3$.

converges to the point process $\mathrm{Sch}_{1/\alpha}$ introduced in [46]. Here $\{x\}_{2\pi}$ denotes the unique element $y \in [0, 2\pi)$ such that $x = y$ modulo 2π.

Actually, we also prove the joint convergence of these rescaled eigenvalues with the appropriately rescaled eigenfunctions towards a point process whose first marginal is $\mathrm{Sch}_{1/\alpha}$ and whose second marginal involves the law of the exponential of a Brownian motion plus drift as in (III.3).

Finally, we show that right above this critical region, that is, for eigenvalues much larger than L, the point process becomes, at first order, deterministic and coincides with the point process of eigenvalues of $-\Delta$.

Theorem III.11 (Top of the spectrum, [DL19a]). Assume that $E = E(L) \gg L$. Then, the random measure $\mathcal{Q}_E - \bar{\mathcal{Q}}_E$ converges in law to the null measure, where

$$\mathcal{Q}_E(dx) := \sum_{i \geq 1} \delta_{\frac{L}{\sqrt{E}}(\lambda_i - E)}(dx) \ , \quad \bar{\mathcal{Q}}_E(dx) := \sum_{i \geq 1} \delta_{\frac{L}{\sqrt{E}}(\bar{\lambda}_i - E)}(dx) \ ,$$

and $(\bar{\lambda}_i)_{i \geq 1}$ are the eigenvalues of the deterministic operator $-\Delta$ on $[0, L]$.

Future directions. The above results do not provide a complete picture of the spectrum of $\mathcal{H}_L$. We expect that the region in between the bulk and the critical regime behaves very much like the bulk, except that the rate of exponential decay of the eigenfunctions should be of order $1/E$: in particular, when we approach the critical regime, that is when $E \to L$, we recover delocalised eigenfunctions. Technically almost all our arguments for the bulk apply in that case, the only missing point is an estimate on the rate of convergence to equilibrium of a degenerate diffusion on the circle. We have some first results towards this convergence.
The region in between the bottom and the bulk should give rise to a Poisson point process for the eigenvalues, and we expect the eigenfunctions to decay exponentially at rate $\sqrt{|E|}$. Note that this is consistent with our result on the bottom of the spectrum. However, this regime is more involved technically and it is not clear that we will be able to treat that case.

We also intend to have a neater picture of the scaling behaviour of the spectrum of the matrix M_N from Subsection III.2.1. We have conjectured the localisation/delocalisation transition according to the strength of the disorder and the region in the spectrum where one is zooming in.

A natural research direction is now to investigate the asymptotic behaviour of the (PAM) in dimension 1 starting from a Dirac mass at time 0. The localisation phenomenon observed at the bottom of the Anderson Hamiltonian should imply an intermittency phenomenon for the (PAM): at any large time $t > 0$, the solution of the (PAM) is essentially carried on a few islands corresponding to the “supports” of the main eigenfunctions of the Anderson Hamiltonian restricted to a t-dependent ball.

III.3 The stochastic Airy operator at large temperature

This section is devoted to presenting a result [DL19c] obtained in collaboration with Laure Dumaz on the bottom of the spectrum of the Stochastic Airy Operator when the temperature $1/\beta$ is sent to ∞. We start by recalling the connection of this operator with the β-ensembles.

The (Gaussian) β-ensemble is the law of N interacting particles $\mu_1 > \ldots > \mu_N$ given by the density:

$$\frac{1}{Z_N^\beta}\prod_{i<j}|\mu_i-\mu_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N \mu_i^2}\;, \tag{III.4}$$

where $\beta > 0$ is an inverse temperature and Z_N^β is a partition function. In the special cases $\beta = 1$, 2 and 4, this measure coincides with the law of the eigenvalues of the Gaussian Orthogonal, Unitary and Symplectic ensembles. However, the connection with random matrices is not restricted to these three particular values of β: Dumitriu and Edelman [25] showed that for any $\beta > 0$, one can build a symmetric, tridiagonal random matrix whose eigenvalues distribution is given by (III.4).

The repulsion between particles increases with the parameter β: in particular, for fixed N and β goes to 0, the particles, multiplied by $\sqrt{\beta}$, converge in law to N IID Gaussian random variables. The behavior of these ensembles when N goes to infinity and the inverse temperature β is sent to zero has been the subject of recent works. In [9] the regime where N goes to infinity and β goes to 0 but $N\beta$ remains constant is considered: the local statistics in the bulk of the spectrum are shown to converge to a Poisson point process. In [62] an alternative proof of this convergence is presented and the intensity measure of the Poisson point process is given explicitly. Let us also cite the work [64] where it is shown that for $N\beta \to 0$ the bottom of the spectrum, properly rescaled, converges to a Poisson point process.

The result that we will present focuses on the case where N is sent to infinity first, and then β is sent to 0, so that, informally, we are in the regime $N\beta \to \infty$. The first step in that direction (sending N to infinity and keeping $\beta > 0$ fixed) is a result of Ramírez, Rider and Virág [65] that we now recall.

The scaling limit of the edge of the β-ensemble, in the regime where N goes to infinity and $\beta > 0$ is fixed, was obtained in [65]. They showed that for any $k \geq 1$, the k-dimensional vector $(N^{1/6}(2\sqrt{N}-\mu_i); i = 1\ldots k)$ converges in distribution to the k lowest eigenvalues of the following random operator called Stochastic Airy Operator (SAO)

$$\mathcal{A}_\beta f = -\partial_x^2 f + xf + \frac{2}{\sqrt{\beta}}\xi f\;,\quad x\in(0,\infty)\;, \tag{III.5}$$

endowed with homogeneous Dirichlet boundary condition at $f(0) = 0$. The potential ξ appearing in this operator is a white noise on $(0,\infty)$. It was shown in [65] that this operator is self-adjoint in $L^2(0,\infty)$ with pure point spectrum $\mu_1 < \mu_2 < \ldots$ of multiplicity one. The corresponding eigenfunctions $(\psi_k)_{k\ge1}$, normalized in $L^2(0,\infty)$, are Hölder functions of regularity index $3/2^-$.

In [4], the asymptotic behavior as $\beta \downarrow 0$ of the first eigenvalue μ_1 of $\mathcal{A}_\beta$ was studied: using a representation (originally introduced in [65]) of the eigenvalues / eigenfunctions in terms of a family of time-inhomogeneous diffusions, it was shown that $\mu_1 \sim -c_\beta$ where

$$c_\beta := \Big(\frac{3}{2\beta}\ln\frac{1}{\pi\beta}\Big)^{2/3} ,$$

and that $-\beta\sqrt{c_\beta}(\mu_1+c_\beta)$ converges to a Gumbel law. They also conjectured in that article that the joint law of the lowest eigenvalues converges to a Poisson point process.

In [DL19c], we prove this conjecture. Actually, we obtain a complete description of the bottom of the spectrum of $\mathcal{A}_\beta$ when $\beta \downarrow 0$, which is similar to our result [DL19b] on the Anderson Hamiltonian. Indeed we show that the properly rescaled eigenvalues converge to a Poisson point process with explicit intensity, and that the eigenfunctions converge to Dirac masses localized at IID points with exponential distribution. Furthermore, we obtain a precise description of the microscopic behavior of the eigenfunctions near their localization centers.

To state precisely the results, we let E_k be the first point in $(0,\infty)$ where $|\psi_k|$ reaches its maximum. We also build probability measures on $(0,\infty)$ from the rescaled eigenfunctions:

$$m_k(dx) := \frac{1}{\beta\sqrt{c_\beta}}\psi_k^2\Big(\frac{x}{\beta\sqrt{c_\beta}}\Big)dx , \quad x\in(0,\infty) .$$

Our first main result is the following.

Theorem III.12 ([DL19c]). As $\beta \downarrow 0$, we have the following convergence in law

$$\Big(\beta\sqrt{c_\beta}(\mu_k+c_\beta), E_k\beta\sqrt{c_\beta}, m_k\Big)_{k\ge1} \Longrightarrow \Big(\Lambda_k, I_k, \delta_{I_k}\Big)_{k\ge1} ,$$

where $(\Lambda_k, I_k)_{k\ge1}$ are the atoms of a Poisson point process on $\mathbb{R}\times\mathbb{R}_+$ with intensity $e^x e^{-t} dx \otimes dt$.

A natural question was then to determine the length scale of localization, together with the behavior of the eigenfunctions near their localization centers. This is the content of the next result, which relies on the following notations. We set for $x \in \mathbb{R}$

$$h_{k,\beta}(x) := \frac{\sqrt{2}}{c_\beta^{1/4}} \Big|\psi_k\Big(E_k + \frac{x}{\sqrt{c_\beta}}\Big)\Big| ,$$
$$b_{k,\beta}(x) := \frac{(\beta/4)^{1/6}}{\sqrt{c_\beta}} \Big(W\Big(E_k + \frac{x}{\sqrt{c_\beta}}\Big) - W\Big(E_k\Big)\Big) ,$$

where $W(x) := \int_0^x \xi(dy)$. We also define $h(x) = 1/\cosh x$ and $b(x) = -2\tanh(x)$ for all $x \in \mathbb{R}$.

Theorem III.13 ([DL19c]). For every $k \geq 1$, the random processes $h_{k,\beta}, b_{k,\beta}$ converge to h, b in probability locally uniformly on $\mathbb{R}$.

The situation is very similar to the bottom of the spectrum of the Anderson Hamiltonian, presented in Subsection III.2.3. To explain the connection between these two operators, let us first say that instead of $\mathcal{A}_\beta$ we consider the equivalent operator

$$\mathcal{L}_\beta f = -f'' + \frac{\beta}{4} x f' + \xi f , \quad x \in (0,\infty) ,$$

with $f(0) = 0$. It is indeed equivalent to consider this operator since its eigenvalues/eigenfunctions $(\lambda_k, \varphi_k)_k$ can be coupled with those of $\mathcal{A}_\beta$ in the following way:

$$\lambda_k = (\beta/4)^{2/3} \mu_k , \quad \varphi_k(x) = (\beta/4)^{1/6} \psi_k(x(\beta/4)^{1/3}) , \quad x \in (0,\infty) .$$

This transformation was already introduced in [4]. Let us now explain the connection with the Anderson Hamiltonian.

If one splits the domain $[0,\infty)$ into (appropriately chosen) disjoint intervals $[t_j, t_{j+1})$ and considers the restricted operators

$$\mathcal{L}_\beta^j f = -f'' + \frac{\beta}{4} x f' + \xi f , \quad x \in (t_j, t_{j+1}) ,$$

endowed with Dirichlet b.c., then we show that the bottom of the spectrum of $\mathcal{L}_\beta$ can essentially be read off the bottoms of the spectra of the $\mathcal{L}_\beta^j$'s. Now, as $\beta \downarrow 0$, we see that the term $\frac{\beta}{4}x$ is roughly constant (if the interval (t_j, t_{j+1}) is small enough compared to $1/\beta$) so that the operator $\mathcal{L}_\beta^j$ looks very much like the Anderson Hamiltonian shifted by the constant $\frac{\beta}{4} t_j$.

Our proof exploits this fact extensively. As for the Anderson Hamiltonian, it is carried out at the level of the Riccati transform which, in the case of $\mathcal{L}_\beta$, yields the following family of diffusions

$$dZ_a(t) = \Big(a + \frac{\beta}{4} t - Z_a^2(t)\Big) dt + dB(t) , \quad Z_a(0) = +\infty .$$

This family of diffusions was already thoroughly studied in [65, 4]. However, an important tool in our analysis - as in the case of the Anderson Hamiltonian - is the so-called backward diffusions. Indeed, we show that there exists a unique process satisfying

$$d\hat{Z}_a(t) = \Big(a + \frac{\beta}{4}t - \hat{Z}_a^2(t)\Big)dt + dB(t)\ , \quad Z_a(+\infty) = -\infty\ ,$$

and that $-a$ is an eigenvalue of $\mathcal{L}_\beta$ if and only if $\hat{Z}_a(0) = +\infty$. We refer to [DL19c, Sec 3].

Building on these backward diffusions, we adapted the strategy of [DL19b] to control the Riccati transforms of the eigenfunctions both forward and backward in time. Technically, the diffusions Z_a and $\hat{Z}_a$ are more difficult to deal with than the diffusions X_a and $\hat{X}_a$ since they are time-inhomogeneous and therefore do not admit a stationary measure. However, by setting up an appropriate approximation scheme one can obtain controls on the former using the knowledge on the latter from [DL19b].

Bibliography

[1] M. Aizenman and S. Molchanov, Localization at large disorder and at extreme energies: an elementary derivation, Comm. Math. Phys., 157 (1993), pp. 245–278.

[2] R. Allez and K. Chouk, The continuous Anderson hamiltonian in dimension two, ArXiv e-prints, (2015).

[3] R. Allez and L. Dumaz, From sine kernel to Poisson statistics, Electron. J. Probab., 19 (2014), pp. no. 114, 25.

[4] ——, Tracy-Widom at high temperature, J. Stat. Phys., 156 (2014), pp. 1146–1183.

[5] G. Amir, I. Corwin, and J. Quastel, Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions, Comm. Pure Appl. Math., 64 (2011), pp. 466–537.

[6] P. W. Anderson, Absence of Diffusion in Certain Random Lattices, Physical Review, 109 (1958), pp. 1492–1505.

[7] C. Bahadoran, Hydrodynamics and hydrostatics for a class of asymmetric particle systems with open boundaries, Comm. Math. Phys., 310 (2012), pp. 1–24.

[8] C. Bardos, A. Y. le Roux, and J.-C. Nédélec, First order quasilinear equations with boundary conditions, Comm. Partial Differential Equations, 4 (1979), pp. 1017–1034.

[9] F. Benaych-Georges and S. Péché, Poisson statistics for matrix ensembles at large temperature, J. Stat. Phys., 161 (2015), pp. 633–656.

[10] I. Benjamini, N. Berger, C. Hoffman, and E. Mossel, Mixing times of the biased card shuffling and the asymmetric exclusion process, Trans. Amer. Math. Soc., 357 (2005), pp. 3013–3029 (electronic).

[11] L. Bertini and G. Giacomin, Stochastic Burgers and KPZ equations from particle systems, Comm. Math. Phys., 183 (1997), pp. 571–607.

[12] R. Bürger, H. Frid, and K. H. Karlsen, On the well-posedness of entropy solutions to conservation laws with a zero-flux boundary condition, J. Math. Anal. Appl., 326 (2007), pp. 108–120.

[13] S. Cambronero, B. Rider, and J. Ramírez, On the shape of the ground state eigenvalue density of a random Hill's equation, Comm. Pure Appl. Math., 59 (2006), pp. 935–976.

[14] G. Cannizzaro, P. K. Friz, and P. Gassiat, Malliavin calculus for regularity structures: the case of gPAM, J. Funct. Anal., 272 (2017), pp. 363–419.

[15] R. Catellier and K. Chouk, Paracontrolled distributions and the 3-dimensional stochastic quantization equation, Ann. Probab., 46 (2018), pp. 2621–2679.

[16] A. Chandra and M. Hairer, An analytic BPHZ theorem for regularity structures, ArXiv e-prints, (2016).

[17] K. Chouk and W. van Zuijlen, Asymptotics of the eigenvalues of the Anderson Hamiltonian with white noise potential in two dimensions, arXiv e-prints, (2019), p. arXiv:1907.01352.

[18] I. Corwin and J. Quastel, Crossover distributions at the edge of the rarefaction fan, Ann. Probab., 41 (2010).

[19] I. Corwin and H. Shen, Open asep in the weakly asymmetric regime, Communications on Pure and Applied Mathematics, 71 (2016).

[20] I. Corwin, H. Shen, and L.-C. Tsai, Asep(q, j) converges to the kpz equation, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 54 (2018), pp. 995–1012.

[21] G. Da Prato and A. Debussche, Strong solutions to the stochastic quantization equations, Ann. Probab., 31 (2003), pp. 1900–1916.

[22] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensions, vol. 44 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1992.

[23] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, Z. Wahrsch. Verw. Gebiete, 57 (1981), pp. 159–179.

[24] R. Dobrushin and O. Hryniv, Fluctuations of shapes of large areas under paths of random walks, Probab. Theory Related Fields, 105 (1996), pp. 423–458.

[25] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, J. Math. Phys., 43 (2002), pp. 5830–5847.

[26] H. Elad Altman and L. Zambotti, Bessel SPDEs and renormalised local times, arXiv e-prints, (2018), p. arXiv:1811.00518.

[27] W. Fischer, H. Leschke, and P. Müller, Spectral localization by Gaussian random potentials in multi-dimensional continuous space, J. Statist. Phys., 101 (2000), pp. 935–985.

[28] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, Constructive proof of localization in the Anderson tight binding model, Comm. Math. Phys., 101 (1985), pp. 21–46.

[29] J. Fröhlich and T. Spencer, Absence of diffusion in the Anderson tight binding model for large disorder or low energy, Comm. Math. Phys., 88 (1983), pp. 151–184.

[30] M. Fukushima and S. Nakao, On spectra of the Schrödinger operator with a white Gaussian noise potential, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 37 (1976/77), pp. 267–274.

[31] M. Gerencsér and M. Hairer, Singular SPDEs in domains with boundaries, Probab. Theory Related Fields, 173 (2019), pp. 697–758.

[32] I. J. Gol'dšeĭd, S. A. Molčanov, and L. A. Pastur, A random homogeneous Schrödinger operator has a pure point spectrum, Funkcional. Anal. i Priložen., 11 (1977), pp. 1–10, 96.

[33] P. Gonçalves, N. Perkowski, and M. Simon, Derivation of the stochastic Burgers equation with Dirichlet boundary conditions from the WASEP, arXiv e-prints, (2017), p. arXiv:1710.11011.

[34] M. Gubinelli, Controlling rough paths, J. Funct. Anal., 216 (2004), pp. 86–140.

[35] M. Gubinelli and M. Hofmanová, Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space, Communications in Mathematical Physics, 368 (2019), pp. 1201–1266.

[36] M. Gubinelli, P. Imkeller, and N. Perkowski, Paracontrolled distributions and singular PDEs, Forum Math. Pi, 3 (2015), pp. e6, 75.

[37] M. Gubinelli, B. Ugurcan, and I. Zachhuber, Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions, ArXiv e-prints, (2018).

[38] M. Hairer, A theory of regularity structures, Invent. Math., 198 (2014), pp. 269–504.

[39] M. Hairer and J. Mattingly, The strong Feller property for singular stochastic PDEs, Ann. Inst. Henri Poincaré Probab. Stat., 54 (2018), pp. 1314–1340.

[40] M. Hairer and E. Pardoux, A Wong-Zakai theorem for stochastic PDEs, J. Math. Soc. Japan, 67 (2015), pp. 1551–1604.

[41] M. Hairer, E. Pardoux, and A. Piatnitski, Random homogenisation of a highly oscillatory singular potential, Stoch. Partial Differ. Equ. Anal. Comput., 1 (2013), pp. 571–605.

[42] M. Hairer and J. Quastel, A class of growth models rescaling to KPZ, Forum Math. Pi, 6 (2018), pp. e3, 112.

[43] B. I. Halperin, Green's functions for a particle in a one-dimensional random potential, Phys. Rev. (2), 139 (1965), pp. A104–A117.

[44] C. Kipnis and C. Landim, Scaling limits of interacting particle systems, vol. 320 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1999.

[45] C. Kipnis, S. Olla, and S. R. S. Varadhan, Hydrodynamics and large deviation for simple exclusion processes, Comm. Pure Appl. Math., 42 (1989), pp. 115–137.

[46] E. Kritchevski, B. Valkó, and B. Virág, The scaling limit of the critical one-dimensional random Schrödinger operator, Comm. Math. Phys., 314 (2012), pp. 775–806.

[47] H. Kunz and B. Souillard, Sur le spectre des opérateurs aux différences finies aléatoires, Comm. Math. Phys., 78 (1980/81), pp. 201–246.

[48] H. Lacoin, The cutoff profile for the simple exclusion process on the circle, Ann. Probab., 44 (2016), pp. 3399–3430.

[49] ——, Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion, Ann. Probab., 44 (2016), pp. 1426–1487.

[50] D. A. Levin and Y. Peres, Mixing of the exclusion process with small bias, J. Stat. Phys., 165 (2016), pp. 1036–1050.

[51] D. A. Levin, Y. Peres, and E. L. Wilmer, Markov chains and mixing times, American Mathematical Society, Providence, RI, 2017. Second edition of [MR2466937], With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson.

[52] P.-L. Lions, Generalized solutions of Hamilton-Jacobi equations, vol. 69 of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[53] T. J. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoamericana, 14 (1998), pp. 215–310.

[54] H. P. McKean, A limit law for the ground state of Hill's equation, J. Statist. Phys., 74 (1994), pp. 1227–1232.

[55] N. Minami, Local fluctuation of the spectrum of a multidimensional Anderson tight binding model, Comm. Math. Phys., 177 (1996), pp. 709–725.

[56] S. A. Molčanov, Structure of the eigenfunctions of one-dimensional unordered structures, Izv. Akad. Nauk SSSR Ser. Mat., 42 (1978), pp. 70–103, 214.

[57] ——, The local structure of the spectrum of the one-dimensional Schrödinger operator, Comm. Math. Phys., 78 (1980/81), pp. 429–446.

[58] J.-C. Mourrat and H. Weber, Convergence of the two-dimensional dynamic ising-kac model to ϕ_2^4, Communications on Pure and Applied Mathematics, (2014).

[59] ——, The dynamic Φ_3^4 model comes down from infinity, Comm. Math. Phys., 356 (2017), pp. 673–753.

[60] ——, Global well-posedness of the dynamic Φ^4 model in the plane, Ann. Probab., 45 (2017), pp. 2398–2476.

[61] C. Mueller, On the support of solutions to the heat equation with noise, Stochastics Stochastics Rep., 37 (1991), pp. 225–245.

[62] F. Nakano and K. D. Trinh, Gaussian beta ensembles at high temperature: eigenvalue fluctuations and bulk statistics, J. Stat. Phys., 173 (2018), pp. 295–321.

[63] D. Nualart and E. Pardoux, White noise driven quasilinear SPDEs with reflection, Probab. Theory Related Fields, 93 (1992), pp. 77–89.

[64] C. Pakzad, Poisson statistics at the edge of gaussian beta-ensembles at high temperature, ALEA Lat. Am. J. Probab. Math. Stat., 16 (2019), pp. 871–897.

[65] J. A. Ramírez, B. Rider, and B. Virág, Beta ensembles, stochastic Airy spectrum, and a diffusion, J. Amer. Math. Soc., 24 (2011), pp. 919–944.

[66] D. Randall and P. Winkler, Mixing points on an interval, in Proceedings of the Second Workshop on Analytic Algorithms and Combinatorics, Vancouver, 2005, 2005, pp. 216–221.

[67] F. Rezakhanlou, Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$, Comm. Math. Phys., 140 (1991), pp. 417–448.

[68] B. Rifkind and B. Virág, Eigenvectors of the 1-dimensional critical random Schrödinger operator, Geom. Funct. Anal., 28 (2018), pp. 1394–1419.

[69] H. Triebel, Interpolation theory, function spaces, differential operators, vol. 18 of North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam-New York, 1978.

[70] J. B. Walsh, An introduction to stochastic partial differential equations, in École d'été de probabilités de Saint-Flour, XIV–1984, vol. 1180 of Lecture Notes in Math., Springer, Berlin, 1986, pp. 265–439.

[71] D. B. Wilson, Mixing times of Lozenge tiling and card shuffling Markov chains, Ann. Appl. Probab., 14 (2004), pp. 274–325.

[72] L. C. Young, An inequality of the Hölder type, connected with Stieltjes integration, Acta Math., 67 (1936), pp. 251–282.

[73] L. Zambotti, A reflected stochastic heat equation as symmetric dynamics with respect to the 3-d Bessel bridge, J. Funct. Anal., 180 (2001), pp. 195–209.

Résumé

Ce mémoire d'HDR s'articule en trois parties :

- les systèmes de particules. Le coeur de cette partie consiste en une étude systématique du processus d'exclusion simple sur le segment à N sites (limites hydrodynamiques, fluctuations, temps de mélange) en fonction de l'asymétrie imposée sur les taux de sauts.
- les EDP stochastiques singulières. Cette partie présente des résultats analytiques sur les espaces fonctionnels mis en jeu dans la théorie des structures de régularité d'Hairer, ainsi que des résultats sur la construction d'EDP stochastiques singulières en volume infini à l'aide de cette théorie.
- le modèle d'Anderson continu. On s'intéresse dans cette partie à l'étude d'opérateurs de Schrödinger aléatoires continus, notamment aux propriétés de leurs spectres (lois des valeurs propres, localisation/délocalisation des fonctions propres).

Abstract

This habilitation thesis is made of three chapters:

- Particle systems. The main contribution consists in a comprehensive study of the simple exclusion process on a linear segment of N sites (hydrodynamic limits, fluctuations, mixing times) according to the asymmetry imposed on the jump rates.
- Singular stochastic PDEs. This chapter presents analytical results on the functional spaces that are involved in the theory of regularity structures of Hairer, as well as constructions of singular stochastic PDEs in infinite volume using this theory.
- The continuous Anderson model. We study random continuous Schrödinger operators, with a particular emphasis on the properties of their spectrum (laws of the eigenvalues, localisation/delocalisation of the eigenfunctions).

Mots Clés

Processus d'exclusion, équation de Burgers, équation KPZ, Temps de mélange, Cutoff, structures de régularité, théorèmes d'Embedding, modèle d'Anderson parabolique, localisation d'Anderson, renormalisation.

Keywords

Exclusion process, Burgers' equation, KPZ equation, Mixing times, Cutoff, Regularity structures, Embedding theorems, parabolic Anderson model, Anderson localisation, Renormalisation.