Gibbs measures on negatively curved manifolds

Yves Coudène

Preprint from July, the 5th, 2001.

Published in
Journal of Dynamical and Control Systems 9, (2003), No. 1 (January) 89–101.

Ecole Normale Supérieure, 45 rue d'Ulm, 75230 Paris cedex 05, France
coudene@dma.ens.fr

Some typing mistakes corrected on 24/10/2023, layout improved.

Abstract. We define the concept of a Gibbs measure on the unitary bundle of any negatively curved complete manifold. As a consequence we show that if the manifold is geometrically finite, there always exists an ergodic probability measure invariant by the geodesic flow and fully supported on the non-wandering set of the flow. [1]

1 Introduction

The notion of Gibbs measure has been developed in the context of hyperbolic dynamics in the seventies by Y. G. Sinai, R. Bowen, D. Ruelle and others [Si72] [BR75], and as such applies to the case of the geodesic flow on a negatively curved manifold if its non-wandering set is compact. In this article, we give a construction of Gibbs measures in the non-compact case, and then study the ergodic properties of these measures.

There are several motivations in doing so. First the Bowen-Margulis measure has been the object of increasing attention over the last ten years. F. Dal'bo, M. Peigné and J. P. Otal showed that this measure may not be finite nor even ergodic on a geometrically finite manifold [DOP00]. In contrast, we will show that such a manifold always admits an ergodic finite Gibbs measure, which may be used as a substitute in related problems. Also, many results obtained for the Bowen-Margulis measure extend in a natural way to general Gibbs measures and thus can be applied to the study of the Liouville measure or the harmonic measure. Finally, they may be used as a test for some questions of hyperbolic theory in a non-compact setting.

The first part is dedicated to the construction of quasi-invariant measures on the limit set. This method originated in the work of Patterson and Sullivan [Pa76][Su79] and was used by F. Ledrappier in the compact case [Led95]. We then define the concept of a Gibbs measure and deduce its basic properties from the fact that they are associated with quasi-products. The last part is devoted to finding conditions under which the measure is finite.

We use the following notations: let M be a connected complete Riemannian manifold whose sectional curvatures lie between two negative constants. Its fundamental group will be denoted by Γ and its universal cover by $\tilde{M}$, so that there is an isomorphism $M \simeq \tilde{M}/\Gamma$. The distances on M and $\tilde{M}$ are denoted by d_M and

[1] AMS classification: 37C40, 20H10, 53D25.
Keywords: ergodic theory, geodesic flow, fuchsian group, Gibbs measure.

d. Let $\partial\tilde{M}$ be the visual boundary of $\tilde{M}$, which consists of equivalence classes of asymptotic rays in $\tilde{M}$. The limit set $\Lambda\Gamma$ of Γ is the set of accumulation points in $\tilde{M} \cup \partial\tilde{M}$ of some orbit Γx. It is a compact subset of $\partial\tilde{M}$ and does not depend on $x \in \tilde{M}$. We suppose that its cardinal is infinite. The group Γ acts on the boundary $\partial\tilde{M}$, and the unitary tangent bundle S^1M of M can be identified with $\big((\partial\tilde{M} \times \partial\tilde{M} - \Delta) \times \mathbf{R}\big)/\Gamma$ whereas the non-wandering set Ω of the geodesic flow corresponds to $\big((\Lambda\Gamma \times \Lambda\Gamma - \Delta) \times \mathbf{R}\big)/\Gamma$, where Δ is the diagonal $\{(\xi, \eta) \mid \xi = \eta\}$. The projection of $S^1\tilde{M}$ on $\tilde{M}$ is denoted by π. The convex hull $Conv(\Lambda\Gamma)$ of $\Lambda\Gamma$ is the smallest convex set in $\tilde{M}$ which contains the points on geodesics with both ends in $\Lambda\Gamma$.

Let x_0 be some fixed point in $Conv(\Lambda\Gamma) \subset \tilde{M}$.

For all $r \geq 0$, $\Lambda_r\Gamma$ is the set of endpoints of geodesics starting from x_0 and meeting the closed balls $\bar{B}(\gamma x_0, r)$ of radius r centred at x_0, for infinitely many $\gamma \in \Gamma$.

The *conical limit set* $\Lambda_c\Gamma$ is the union of the sets $\Lambda_r\Gamma$, for all $r \geq 0$, and does not depend on x_0. It is a subset of $\Lambda\Gamma$.

The *shadow* $\mathcal{O}_{x_0}(y, r) \subset \partial\tilde{M}$ is the set of endpoints of geodesics starting at x_0 and meeting the open ball $B(y, r)$. It is an open subset of $\partial\tilde{M}$, and its closure is denoted by $\bar{\mathcal{O}}_{x_0}(y, r)$.

Finally, since $\tilde{M}$ is a negatively curved manifold, it satisfies the *visibility property*: given a positive number $\varepsilon > 0$, there is a $C > 0$ such that, in any geodesic triangle, if the angle at an edge is greater than ε, then the distance from the edge to the opposite side of the triangle is bounded by C (see, e.g. [BGS85]§4.14).

2 Measures on the limit set

We begin by defining measures on the limit set $\Lambda\Gamma$ using a method due to S. J. Patterson [Pa76].

Let x_0 be a reference point in $Conv(\Lambda\Gamma) \subset \tilde{M}$ and F a Borel locally bounded non-negative Γ-invariant function defined on $\pi^{-1}(Conv(\Lambda\Gamma)) \subset S^1\tilde{M}$. We will be interested in quantities of the form

$$\rho^F_{x,w}(y) = \int_y^w F - \int_x^w F$$

with $x, y, w \in Conv(\Lambda\Gamma)$. The integral $\int_y^w F$ is to be taken along the geodesic from y to w, lifted to the unitary tangent bundle $S^1\tilde{M}$. With that convention, we have the relation $\int_y^w F = \int_w^y F'$, where $F'(x, v) = F(x, -v)$ is the composition of F with the involution of the tangent bundle of $\tilde{M}$. The Γ-invariance of F implies the relation $\rho^F_{\gamma x, \gamma w}(\gamma y) = \rho^F_{x,w}(y)$ for all $\gamma \in \Gamma$.

Let us introduce the constants

$$C^F_r = sup\{|\rho^F_{x_0,w}(y)|,\ y \in \bar{B}(x_0, r) \cap Conv(\Lambda\Gamma), w \in Conv(\Lambda\Gamma)\ \}$$

$$\bar{C}^F_r = inf\{C^F_{r'} \mid r' > r\}$$

These constants are finite, for example, if the function F satisfies some Hölder condition. Following S. J. Patterson, we consider the Poincaré series

$$P_{x_0}(s) = \sum_{\gamma \in \Gamma} e^{-s \int_{x_0}^{\gamma x_0} F}$$

of exponent of convergence δ_F, and try to define a conformal density using the series

$$\mu^F_{x,s} = \frac{\sum_{\gamma\in\Gamma} h^F(e^{\int_x^{\gamma x_0} F}) e^{-s\int_x^{\gamma x_0} F} Dirac(\gamma x_0)}{\sum_{\gamma\in\Gamma} h^F(e^{\int_{x_0}^{\gamma x_0} F}) e^{-s\int_{x_0}^{\gamma x_0} F}}$$

where h^F is a continuous positive non decreasing function added so that the denominator of the series diverges in δ_F. That function may be chosen to satisfy the asymptotic $h^F(e^{r+t})/h^F(e^r) \to 1$ as $r \to \infty$, uniformly for t in a compact subset of the real line.

From now on, we make the assumption that the exponent δ_F is finite, and note that it implies that

$$\int_{x_0}^{\gamma x_0} F \to \infty \text{ as } \gamma x_0 \to \infty.$$

We also remark that F and F' have the same Poincaré series, so that $\delta_F = \delta_{F'}$.

Using Helly theorem, we can find a sequence s_i decreasing towards δ_F such that $\mu^F_{x_0,s_i}$ converges to a Borel probability measure $\mu^F_{x_0}$ supported by the limit set $\Lambda\Gamma$.

Now suppose $C^F_r < \infty$, $C^{F'}_r < \infty$, for all $r \geq 0$ and choose for each $x \in Conv(\Lambda\Gamma)$ a subsequence s'_i of s_i such that μ^F_{x,s'_i} converges to a finite Borel measure μ^F_x. The function F is Γ-invariant, so for all $\gamma \in \Gamma$ and all Borel set $A \subset \Lambda\Gamma$, we have $\mu_{x_0,s}(\gamma^{-1}A) = \mu_{\gamma x_0,s}(A)$. This implies the relation

$$\mu_{x_0}(\gamma^{-1}A) = \mu_{\gamma x_0}(A).$$

Lemma 1 *For all $x \in Conv(\Lambda\Gamma)$, the measures μ^F_x, $\mu^F_{x_0}$ are absolutely continuous with respect to each other. Moreover, the critical exponent δ_F is positive.*

Proof First recall that inequalities behave well under weak convergence: let $\mu_{1,s},\ \mu_{2,s}$ be two bounded sequences of Borel measures on a metric space, converging weakly to μ_1, μ_2. If there is a constant C such that $\mu_{1,s} \leq C\mu_{2,s}$ on an open set U, then the measures $\mu_1,\ \mu_2$ satisfy the same inequality on U. This is first obtained for open subsets of U by approaching their characteristic functions by uniformly bounded sequences of continuous functions, and then is extended to all Borel subsets of U by using the regularity of μ_1, μ_2.

Given an arbitrary finite subset G of Γ, the following equality holds for $w \in \tilde{M}$:

$$d\mu^F_{x,s}(w) \ = d\mu_{G,s}(w) + h_G(w) e^{-s\rho^F_{x_0,w}(x)} d\mu^F_{x_0,s}(w)$$

with $\mu_{G,s} = 1/P_{x_0}(s) \sum_{\gamma\in G} \big(h^F(\int_x^{\gamma x_0} F) - h^F(\int_{x_0}^{\gamma x_0} F)\big) exp(-s\int_x^{\gamma x_0} F)\ Dirac_{\gamma x_0}$,

$$h_G(w) = h^F\Big(\int_{x_0}^{w} F + \rho^F_{x_0,w}(x)\Big) / h^F\Big(\int_{x_0}^{w} F\Big) \text{ if } w \in \Gamma x_0 - G x_0,$$

$$h_G(w) = 1 \text{ otherwise.}$$

It follows from the properties of h^F that G can be chosen so that h_G is uniformly close to 1. The divergence of the Poincaré series P_{x_0} then implies that $\mu_{G,s}(\partial\tilde{M})$ converges to zero when s goes to δ_F. Moreover the function $|\rho^F_{x_0,w}(x)|$ is bounded by $C^F_{d(x_0,x)}$. This shows the absolute continuity. This also shows that $\mu^F_x = \mu^F_{x_0}$ if $\delta_F = 0$, and the Borel probability measure $\mu^F_{x_0}$ would be invariant under the action of Γ. But it is known that such a measure does not exist (see e.g. [Yu96] 2.3.1). †

So there exist functions $\zeta \mapsto \rho^F_{x_0,\zeta}(x)$ in $L^\infty(\Lambda\Gamma, \mu^F_{x_0})$, for all $x \in Conv(\Lambda\Gamma)$, satisfying the relations

$$\frac{d\mu^F_x}{d\mu^F_{x_0}}(\zeta) = e^{-\delta_F \rho^F_{x_0,\zeta}(x)}, \quad \mu^F_{x_0} - ae\ \zeta \in \Lambda\Gamma.$$

Some properties of the functions $\zeta \mapsto \rho^F_{x_0,\zeta}(x)$ are studied in the next two lemmas.

Lemma 2

$$For\ \mu_{x_0} - a.e.\ \zeta \in \Lambda\Gamma, \quad \underline{\lim}_{w\to\zeta}\ \rho^F_{x_0,w}(x) \leq\ \rho^F_{x_0,\zeta}(x)\ \leq \overline{\lim}_{w\to\zeta}\ \rho^F_{x_0,w}(x).$$

Proof Let U be some open subset of $Conv(\Lambda\Gamma) \cup \Lambda\Gamma$ and C a real number. If the upper bound $\rho^F_{x_0,w}(x) \leq C$ holds for all $w \in U \cap Conv(\Lambda\Gamma)$, then for almost all $\zeta \in U$, we have $\rho^F_{x_0,\zeta}(x) \leq C$. This is shown by putting the upper bound in the equality of the previous lemma and then letting s decrease to δ_F.

It follows that for almost all $\zeta \in \Lambda\Gamma$, for all open set U belonging to a countable basis for the topology, and for all rational number C, if $\rho^F_{x_0,w}(x) \leq C$ for all $w \in U$ and if ζ belongs to U, then $\rho^F_{x_0,\zeta}(x) \leq C$. The right inequality is now a consequence of the definition of the upper limit. The left inequality follows in the same way. †

Lemma 3 (Triangular lemma)

$$\forall r \geq 0,\ \forall \gamma \in \Gamma,\ \mu^F_{x_0} a.e.\ \zeta \in \mathcal{O}_{x_0}(\gamma x_0, r), \quad \left| \int_{x_0}^{\gamma x_0} F + \rho^F_{x_0,\zeta}(\gamma x_0) \right| \leq \bar{C}^F_r + \bar{C}^{F'}_r.$$

Proof Let $r' > r$ and $w \in Conv(\Lambda\Gamma)$. If γ is such that the distance from γx_0 to the geodesic arc from x_0 to w is less than r', and y is the point which realises that distance, we have the identity

$$-\rho^F_{x_0,w}(\gamma x_0) - \int_{x_0}^{\gamma x_0} F = \rho^{F'}_{\gamma x_0, x_0}(y) + \rho^F_{\gamma x_0,w}(y).$$

Thus the first term is bounded by $C^F_{r'} + C^{F'}_{r'}$. Let ζ be a point in $\mathcal{O}_{x_0}(\gamma x_0, r)$ satisfying the bounds of the previous lemma. There is a small neighbourhood of ζ in $\tilde{M} \cup \partial\tilde{M}$ so that any point w of $\tilde{M}$ in that neighbourhood is such that the distance from γx_0 to the geodesic going from x_0 to w is less than r'. For these points w, we have $\rho^F_{x_0,w}(\gamma x_0) \leq -\int_{x_0}^{\gamma x_0} F + C^F_{r'} + C^{F'}_{r'}$ and the previous lemma gives the upper bound

$$\rho^F_{x_0,\zeta}(\gamma x_0) \leq -\int_{x_0}^{\gamma x_0} F + C^F_{r'} + C^{F'}_{r'}.$$

The lower bound is obtained in the same way. †

A first consequence of the triangular lemma is the fact that the measure $\mu^F_{x_0}$ has no atom on the conical set $\Lambda_c\Gamma$. Indeed, for almost all points $\zeta \in \Lambda_c\Gamma$, the lemma implies $\rho^F_{x_0,\zeta}(\gamma_i x_0) \to -\infty$ as $i \to \infty$, for a sequence γ_i of elements of Γ. On the other hand, if $\mu^F_{x_0}(\{\zeta\}) > 0$, then

$$e^{-\delta_F \rho^F_{x_0,\zeta}(\gamma_i x_0)} \mu^F_{x_0}(\{\zeta\}) = \mu^F_{\gamma_i x_0}(\{\zeta\}) = \mu^F_{x_0}(\{\gamma_i^{-1}\zeta\}).$$

The right member of this equality is bounded by 1 since $\mu^F_{x_0}$ is a probability measure, whereas the left member goes to infinity, leading to a contradiction.

The next two results are well known for constant function F. They are due to D. Sullivan in the constant curvature case [Su79]. The proofs extend to the variable negative curvature case, as noticed by M. Coornaert [Co93], M. Bourdon [Bou95], C. B. Yue [Yu96]. They also generalise to our context, with minor modifications, which amount to replacing triangular inequalities for the Riemannian metric by the triangular lemma.

Lemma 4 (Shadow lemma) $\exists C > 0,\ \exists r_0 \geq 0,\ \forall r \geq r_0,\ \forall \gamma \in \Gamma,$

$$C^{-1} e^{-\delta_F \int_{x_0}^{\gamma x_0} F - \delta_F (\bar{C}_r^F + \bar{C}_r^{F'})} \leq \mu_{x_0}^F \big(\mathcal{O}_{x_0}(\gamma x_0, r)\big) \ \leq C e^{-\delta_F \int_{x_0}^{\gamma x_0} F + \delta_F (\bar{C}_r^F + \bar{C}_r^{F'})}$$

Theorem 1 *The Poincaré series of F diverges at the exponent δ_F if and only if $\mu_{x_0}^F(\Lambda_c \Gamma) > 0$.*

The next results show that the behaviour of the measure on the conical limit set is determined by its behaviour on the shadows.

Theorem 2 (Besicovitch Covering Theorem) *Let $r > 0$, $\lambda > 0$, and μ be a Borel probability measure on $\partial \tilde{M}$, that satisfies the inequality*

$$\mu(\bar{\mathcal{O}}_{x_0}(\gamma x_0, 6r)) \leq \lambda \mu(\bar{\mathcal{O}}_{x_0}(\gamma x_0, r)),\ \forall \gamma \in \Gamma.$$

Let A be a Borel subset of $\Lambda_r \Gamma$ and U an open subset of $\partial \tilde{M}$ containing A. Then there exists a sequence $\{\gamma_i\}$ such that the shadows $\bar{\mathcal{O}}_{x_0}(\gamma_i x_0, r)$ are disjoint, included in U and cover almost all of A.

Proof This is a consequence of a general covering theorem that can be found in [GMT69], Theorem 2.8.7. In that theorem, the "delta" function giving the size of the shadows should be taken equal to

$$\delta(\bar{\mathcal{O}}_{x_0}(\gamma x_0, r)) = e^{-d(x_0, \gamma x_0)}.$$

The enlargement condition is then a consequence from the following geometric fact. Given $\gamma, \gamma' \in \Gamma$, if $d(x_0, \gamma' x_0) \geq d(x_0, \gamma x_0) - r$ and $\mathcal{O}_{x_0}(\gamma' x_0, r) \cap \mathcal{O}_{x_0}(\gamma x_0, r)$ is non empty, then $\bar{\mathcal{O}}_{x_0}(\gamma' x_0, r) \subset \bar{\mathcal{O}}_{x_0}(\gamma x_0, 6r)$. †

Corollary 1 *Two Borel probability measures μ_1, μ_2 on $\partial \tilde{M}$ satisfying the shadow lemma are absolutely continuous with respect to each other, in restriction to $\Lambda_c \Gamma$.*

Proof It is sufficient to show the absolute continuity on $\Lambda_r \Gamma$, for all $r \geq 0$. Let A be a Borel subset of $\Lambda_r \Gamma$. Since μ_2 is a regular measure, one can find an open set U containing A such that $\mu_2(U \setminus A) < \varepsilon$, for all $\varepsilon > 0$. The previous theorem gives a family of disjoint shadows $\{\bar{\mathcal{O}}_{x_0}(\gamma_i x_0, r)\}$ included in U and covering μ_1-almost all of A. Together with the shadow lemma, one obtains the following inequalities.

$$\begin{aligned}
\mu_1(A) &\leq \sum \mu_1(\bar{\mathcal{O}}_{x_0}(\gamma_i x_0, r)) \\
&\leq C^2 e^{2\delta_F (\bar{C}_r^F + \bar{C}_r^{F'})} \sum \mu_2(\bar{\mathcal{O}}_{x_0}(\gamma_i x_0, r)) \\
&\leq C^2 e^{2\delta_F (\bar{C}_r^F + \bar{C}_r^{F'})} \mu_2(U) \\
&\leq C^2 e^{2\delta_F (\bar{C}_r^F + \bar{C}_r^{F'})} (\mu_2(A) + \varepsilon).
\end{aligned}$$

This is true for all $\varepsilon > 0$, so there is a constant C_r such that $\mu_1(A) \leq C_r \mu_2(A)$, for all Borel subsets A of $\Lambda_r \Gamma$. †

3 Definition of the Gibbs measures

We now define the Gibbs measure associated to the function F. It is constructed as the weak limit of the following sequence of measures on $(Conv(\Lambda\Gamma) \cup \Lambda\Gamma) \times (Conv(\Lambda\Gamma) \cup \Lambda\Gamma) - \Delta$.

$$\mu_s^F = \frac{\sum_{\gamma,\gamma'\in\Gamma} h^F\big(e^{\int_{x_0}^{\gamma x_0} F}\big) h^{F'}\big(e^{\int_{x_0}^{\gamma' x_0} F'}\big) e^{-s\int_{\gamma' x_0}^{\gamma x_0} F} Dirac(\gamma x_0) \otimes Dirac(\gamma' x_0)}{\Big(\sum_{\gamma\in\Gamma} h^F\big(e^{\int_{x_0}^{\gamma x_0} F}\big) e^{-s\int_{x_0}^{\gamma x_0} F}\Big)^2}$$

We first show that these measures admit convergent subsequences. Let

$$\beta_{x_0}(w, w') = \int_{w'}^{x_0} F + \int_{x_0}^{w} F - \int_{w'}^{w} F = -\rho^F_{x_0,w}(y) - \rho^{F'}_{x_0,w'}(y)$$

where y is any point on the geodesic from w to w'. Let r be a positive constant, K_r the closed subset of $Conv(\Lambda\Gamma) \times Conv(\Lambda\Gamma) - \Delta$ consisting of points (w, w') such that the distance from x_0 to the geodesic going from w to w' is less than or equal to r, and $\bar{K}_r$ the closure of K_r in $(\tilde{M} \cup \partial\tilde{M}) \times (\tilde{M} \cup \partial\tilde{M}) - \Delta$. From the definition of C_r^F, $C_r^{F'}$, we have the upper bound

$$|\beta_{x_0}(w, w')| \leq C_r^F + C_r^{F'}$$

as soon as $(w, w') \in K_r$. On the other hand, a short computation gives the identity

$$d\mu_s^F(w, w') = e^{s\beta_{x_0}(w,w')} d\mu^F_{x_0,s}(w) d\mu^{F'}_{x_0,s}(w').$$

So, the measures μ_s^F are uniformly bounded on the compact set $\bar{K}_r$ and a diagonal extraction argument gives a sequence s'_i such that $\mu^F_{s'_i}$ converges weakly on $\cup_{n\in\mathbf{N}} \bar{K}_n$, as s'_i decreases to δ_F.

The limit μ^F is supported by $\Lambda\Gamma \times \Lambda\Gamma - \Delta$, is absolutely continuous with respect to the product $\mu^F_{x_0} \otimes \mu^{F'}_{x_0}$. So there exists a function

$$(\zeta, \eta) \mapsto \beta_{x_0}(\zeta, \eta)$$

on $\Lambda\Gamma \times \Lambda\Gamma - \Delta$, defined μ^F-almost everywhere, such that

$$d\mu^F(\zeta, \eta) = e^{\delta_F \beta_{x_0}(\zeta,\eta)} d\mu^F_{x_0}(\zeta) d\mu^{F'}_{x_0}(\eta)$$

and the function β_{x_0} is essentially bounded on each of the compacts $\bar{K}_r - K_r$. Finally, one checks using the properties of the function h^F, that the measure μ^F is invariant under the diagonal action of Γ on $\Lambda\Gamma \times \Lambda\Gamma - \Delta$, $\gamma(\zeta, \eta) = (\gamma\zeta, \gamma\eta)$.

Let $d\tilde{m}^F = d\mu^F dt$, the measure obtained from $\tilde{m}^F$ on $\Omega \subset S^1M$ by the identification of Ω with $\big((\Lambda\Gamma \times \Lambda\Gamma - \Delta) \times \mathbf{R}\big)/\Gamma$ is denoted by m^F. The support of m^F is equal to Ω, this is a consequence of the shadow lemma and the density of $\Lambda_c\Gamma$ in $\Lambda\Gamma$.

The next theorem is a consequence of a general result of V. Kaimanovich, valid for any quasi-product geodesic current on Gromov hyperbolic spaces [K94]. His proof is a generalisation of the Hopf argument [Ho71]. The use of the Chacon-Ornstein theorem is replaced by an induction on balls of increasing radius.

Theorem 3 *Let F be a Borel locally bounded non-negative Γ-invariant function defined on the set $\pi^{-1}(Conv(\Lambda\Gamma)) \subset S^1\tilde{M}$, such that $C_r^F < \infty$, $C_r^{F'} < \infty$, $\forall r \geq 0$, and the exponent δ_F of the Poincaré series of F is finite.*

Then, either $\mu^F_{x_0}(\Lambda_c\Gamma) = \mu^{F'}_{x_0}(\Lambda_c\Gamma) = 0$ and the measure m^F is totally dissipative with respect to the geodesic flow on S^1M,

or $\mu^F_{x_0}(\Lambda_c\Gamma) = \mu^{F'}_{x_0}(\Lambda_c\Gamma) = 1$, and the measure m^F is conservative and ergodic with respect to the geodesic flow.

Corollary 2 *Under the assumption of the previous theorem, the divergence of the Poincaré series of F at δ_F is equivalent to the conservativity and ergodicity of the measure m^F with respect to the geodesic flow. In that case, the sequence μ_s^F converges weakly to μ^F as s decreases to δ_F and defines m^F uniquely. We call m^F the Gibbs measure associated to the function F.*

Proof It remains to show that there is a unique accumulation point to the sequence μ_s^F, if the measure m^F is ergodic. Given such a point μ^F, one can extract subsequences so that $\mu_{x_0,s}^F$ and $\mu_{x_0,s}^{F'}$ converge to measures $\mu_{x_0}^F$ and $\mu_{x_0}^{F'}$, in such a way that μ^F is absolutely continuous with respect to $\mu_{x_0}^F \otimes \mu_{x_0}^{F'}$. Now, the shadow lemma implies that the limit points of $\mu_{x_0,s}^F$ are absolutely continuous with respect to each other, in restriction to the conical set $\Lambda_c\Gamma$. So the limit points of μ_s^F are absolutely continuous with respect to each other, in restriction to $\Lambda_c\Gamma \times \Lambda_c\Gamma - \Delta$, which is of full measure. By ergodicity, they are equal.

An alternative proof is to apply an argument due to F. Ledrappier ([Led95], Lemma 3) to the induced flows used by V. Kaimanovich ([K94], Theorem 2.5). One can also deduce the convergence of the measures μ_s^F from the fact that

$$\mu_{x_0}^F \otimes \mu_{x_0}^{F'}(\Lambda\Gamma \times \Lambda\Gamma - \Delta) = \mu_{x_0}^F \otimes \mu_{x_0}^{F'}(\Lambda\Gamma \times \Lambda\Gamma)$$

since $\mu_{x_0}^F$, $\mu_{x_0}^{F'}$ are non-atomic on $\Lambda_c\Gamma$, hence on $\Lambda\Gamma$. †

Finally we recall that if the measure m^F is finite, then it is conservative by the Poincaré recurrence theorem, and hence ergodic by the Hopf argument. Moreover if the geodesic flow is topologically mixing, an argument due to M. Babillot shows that it is mixing with respect to the measure m^F [B01].

4 Finiteness of Gibbs measures

We now give conditions for the finiteness of the Gibbs measures. Let $\mathcal{P}$ be a subgroup of Γ. The exponent of convergence of the series $\sum_{p\in\mathcal{P}} exp(-s\int_{x_0}^{px_0} F)$ is denoted by $\delta_F^{\mathcal{P}}$.

An element in Γ is called *parabolic* if it has a unique fixed point on $\partial\tilde{M}$. A parabolic subgroup of Γ is a group containing only parabolic isometries (and the identity).

A point $p \in \Lambda\Gamma$ is a bounded parabolic point if the maximal parabolic subgroup of Γ fixing p has a compact fundamental domain for its action on $\Lambda\Gamma - \{p\}$.

The group Γ is said to be *geometrically finite* if the limit set is composed of conical limit points and bounded parabolic points. Other characterisations of these groups can be found in an article of B. H. Bowditch [Bo95].

Theorem 4 *Let F be a Borel locally bounded non-negative Γ-invariant function defined on the set $\pi^{-1}(Conv(\Lambda\Gamma)) \subset S^1\tilde{M}$, such that $C_r^F < \infty$, $C_r^{F'} < \infty$ for all $r \geq 0$, and the exponent δ_F of the Poincaré series of F is finite. Suppose that Γ is geometrically finite.*

If for any maximal parabolic subgroup $\mathcal{P}$ of Γ we have $\delta_F > \delta_F^{\mathcal{P}}$, then the measure m^F is conservative and ergodic with respect to the geodesic flow.

If the measure m^F is conservative and ergodic, then m^F is finite if and only if, for all maximal parabolic subgroups $\mathcal{P}$ of Γ, the following series is convergent for one (hence for all) x:

$$\sum_{p\in\mathcal{P}} d(x,px)\ exp(-\delta_F \int_x^{px} F).$$

The proofs given by F. Dal'bo, J. P. Otal and M. Peigné [DOP00] in the case of a constant F generalises almost verbatim to our setting. Angle lower estimates are converted into distance upper estimates using the visibility property and the triangular lemma replaces the triangular inequality for the distance.

Theorem 5 *Let Γ be a geometrically finite group with cusp. Then the geodesic flow admits an invariant finite mixing Gibbs measure fully supported on the non-wandering set Ω.*

Proof The goal is to find an F satisfying the conditions of the previous theorem. The presence of cusps insures the topological mixing of the flow [Da99].

– If F is bounded below by a positive constant C, then δ_F is less than $C^{-1}\delta_{\mathbf{1}}$, which is finite since $\delta_{\mathbf{1}}$ is the exponent of the group Γ.

– If F is a function on $Conv(\Lambda\Gamma)$ satisfying a Hölder condition, then $C_r^{F\circ\pi} < \infty$. Indeed if c_1, c_2 are two geodesics on $\tilde{M}$ starting from x_0, there exists constants A_1, K_1 such that for any $T \geq 0$ satisfying $d(c_1(T), c_2(T)) \leq r$, we have

$$d(c_1(T), c_2(T)) \geq A_1 e^{K_1(T-t)} d(c_1(t), c_2(t)).$$

This follows from a direct computation in the constant curvature case, and then is obtained in the variable curvature case by using Alexandrov Topogonov comparison theorems.

So, if there exist $C > 0$ and a positive number $\alpha \leq 1$ such that $|F(x) - F(y)| \leq Cd(x,y)^\alpha$ for all $x, y \in Conv(\Lambda\Gamma)$, then for all $w \in \tilde{M}$ and $y' \in \tilde{M}$ such that $d(y', w) = d(x_0, w)$, the following upper bound holds:

$$|\rho^F_{x_0,w}(y')| \leq \ CA_1^{-1}\ d(x_0, y')^\alpha \int_0^\infty \ e^{-K_1\alpha u} du.$$

Let y be a point in $\bar{B}(x_0, r)$, and take for y' the point at distance $d(w, x_0)$ from w on the half geodesic going from w to y. Then $d(y, y') \leq r$ and the inequality

$$|\rho^F_{x_0,w}(y)| \leq |\rho^F_{x_0,w}(y')| + |\int_y^{y'} \ F|$$

shows that $|\rho^F_{x_0,w}(y)|$ is bounded by a constant depending on F, r and the pinching of the curvatures.

– Let F_{S^1M} be the function on S^1M obtained from F by taking the quotient modulo Γ. If F_{S^1M} is proper, then $\delta_F^{\mathcal{P}} < \delta_F$: in fact $\delta_F^{\mathcal{P}} = 0$. Given any constant $C \geq 0$, there is a convex neighbourhood of the cusp associated to $\mathcal{P}$ on which F_{S^1M} is greater than C. The condition $C_r^F < \infty$, $\forall r \geq 0$ implies that the exponent $\delta_F^{\mathcal{P}}$ of the series $\sum_{p\in\mathcal{P}} exp(-s\int_x^{px} F)$ does not depend on x, so the projection of x in S^1M can be taken in that neighbourhood. Each of the geodesics going from x to px has its projection in the neighbourhood, and we have

$$\int_x^{px} F \geq C \int_x^{px} \mathbf{1},$$

$$\delta_F^{\mathcal{P}} \leq C^{-1}\delta_{\mathbf{1}}^{\mathcal{P}} \leq C^{-1}\delta_{\mathbf{1}} \text{ for all } C > 0.$$

– If F is bounded below by a positive constant, the convergence of the series $\sum d(x, px)\ exp(-\delta_F \int_x^{px} F)$ follows from the condition $\delta_F^{\mathcal{P}} < \delta_F$, since for any $\varepsilon > 0$, $d(x, px) \leq exp(\varepsilon \int_x^{px} F)$ for all $p \in \mathcal{P}$ but a finite number.

Now consider the function on M defined by $y \mapsto 1 + d_M(x, y)$, where x is some point in M. Its Γ-invariant extension to $S^1\tilde{M}$ satisfies the previous conditions and the associated measure is finite. †

We now show that the function F can be taken uniformly close to $\mathbf{1}$, by using the following criteria. The hypothesis are stronger than those given by F. Dal'bo, J.P. Otal, M. Peigné and we explain why.

Criteria *Let F be a Borel locally bounded non-negative Γ-invariant function defined on the set $\pi^{-1}(Conv(\Lambda\Gamma)) \subset S^1\tilde{M}$, such that $C_r^F < \infty$, $C_r^{F'} < \infty$, for all $r \geq 0$, and the exponent δ_F of the Poincaré series of F is finite. Let G a subgroup of Γ such that the action of G on $\Lambda\Gamma - \Lambda G$ admits a non empty fundamental domain $\mathcal{G}$ relatively compact in $\Lambda\Gamma - \Lambda G$.*

If the series $\sum_{p\in G} exp(-s\int_{x_0}^{px_0} F)$ diverges at δ_F^G, then $\delta_F^G < \delta_F$.

Proof The set $\Lambda\Gamma - \Lambda G$ is open in $\Lambda\Gamma$ and so of positive measure for $\mu_{x_0}^F$. The equality $\amalg g\mathcal{G} = \Lambda\Gamma - \Lambda G$ implies that $\mu_{x_0}^F(\mathcal{G}) > 0$. We claim that, upon discarding a finite number of $g \in G$, the angle between the two geodesics starting from x_0 and going to gx_0 and ζ is bounded below, for $g \in G$ and $\zeta \in \mathcal{G}$.

Indeed if this is not the case, there is an accumulation point of the set gx_0 in the closure of $\mathcal{G}$ and so this closure and ΛG wouldn't be disjoint. Using the visibility property, this implies that the distance from x_0 to the geodesic going from gx_0 to ζ is bounded, independently of $g \in G$ and $\zeta \in \mathcal{G}$. The distance from $g^{-1}x_0$ to the geodesic going from x_0 to $g^{-1}\zeta$ is then bounded, and it follows from the triangular lemma that the quantities

$$|\rho_{x_0,g^{-1}\zeta}^F(g^{-1}x_0) + \int_{x_0}^{g^{-1}x_0} F|$$

are bounded. There is a constant C so that

$$\mu_{x_0}^F(\Lambda\Gamma) \geq \sum \mu_{x_0}^F(g^{-1}\mathcal{G}) = \sum \int_{\mathcal{G}} e^{-\delta_F \rho_{x_0,\zeta}^F(gx_0)} d\mu_{x_0}^F(\zeta) \geq C\mu_{x_0}^F(\mathcal{G}) \sum e^{-\delta_F \int_{x_0}^{g^{-1}x_0} F}$$

where the sums are taken over the elements of G. The last sum converges and the divergence of G implies $\delta_F^G < \delta_F$. †

As a consequence, the Gibbs measure associated to some function F is finite if the Poincaré series associated to each parabolic subgroup of Γ diverges at its critical exponent. We show how to build such a function F.

A geometrically finite manifold M can be decomposed into a compact part and a finite number of cusp regions of the form $\mathcal{H}_i/\mathcal{P}_i$, where $\mathcal{H}_i$ are disjoints horoballs in $\tilde{M}$ and $\mathcal{P}_i$ are parabolic groups whose elements fix all horospheres contained in the related $\mathcal{H}_i$. Moreover every maximal parabolic subgroup of Γ is conjugate to one of these groups $\mathcal{P}_i$. The function F will be obtained as the pullback on $S^1\tilde{M}$ of a function on M, constant on the compact part, and slowly increasing in each cusp region, so that the Poincaré series of each cusp diverges at its critical exponent.

We write $F = \mathbf{1} - H$, and define H in a horoball $\mathcal{H}$ associated to a cusp region of the form $\mathcal{H}/\mathcal{P}$. The point x_0 is chosen in $\mathcal{H}$. Since $\mathcal{H}$ is convex, the geodesics going from x_0 to px_0 lie in $\mathcal{H}$, for all $p \in \mathcal{P}$. The *height* of a point x in $\mathcal{H}$ is the Lipschitz function defined by

$$z(x) = -\rho_{x_0,\infty}^{\mathbf{1}}(x),$$

where ∞ denotes the common fixed point of the elements of $\mathcal{P}$.

The level curves of this function are the horospheres contained in $\mathcal{H}$ and so are left globally invariant by the elements of $\mathcal{P}$. The height $z(p)$ of a parabolic element $p \in \mathcal{P}$ is the maximum of the height of the points on the geodesic going from x_0 to px_0.

Since there is only a finite number of $p \in \mathcal{P}$ such that the endpoint of the geodesic from x_0 to px_0 is outside of a given neighbourhood of ∞, the elements of $p \in \mathcal{P}$ can be ordered in such a way that the height goes to infinity as p goes to infinity. Given any decreasing sequence of positive numbers ε_n converging to 0, it is then possible to choose an increasing sequence of numbers Y_n so that $Y_1 = 0$, $Y_{n+1} - Y_n \geq \varepsilon_{n-1} - \varepsilon_n$ and the following bound holds:

$$\sum_{p \in z^{-1}(]Y_n, Y_{n+1}])} e^{-\delta_{\mathbf{1}}^{\mathcal{P}}(1-\varepsilon_n)d(x_0, px_0)} \geq 1.$$

The function H is defined in the following way:

$$\begin{array}{ll} H(u) = \varepsilon_{n-1} - z(u) + Y_n & on\ z^{-1}([Y_n, Y_n + \varepsilon_{n-1} - \varepsilon_n]), \\ H(u) = \varepsilon_n & on\ z^{-1}([Y_n + \varepsilon_{n-1} - \varepsilon_n, Y_{n+1}]). \end{array}$$

This is a positive Lipschitz $\mathcal{P}$-invariant function on $\mathcal{H}$, bounded by ε_0. It remains to prove that the Poincaré series $\sum exp\big(-s\int_{x_0}^{px_0}(\mathbf{1} - H)\big)$ diverges at $\delta_{\mathbf{1}-H}^{\mathcal{P}}$.
The Poincaré series of $\mathbf{1} - H$ diverges at $\delta_{\mathbf{1}}^{\mathcal{P}}$, since the sums

$$\sum_{p \in z^{-1}(]Y_n, Y_{n+1}])} exp\big(-\delta_{\mathbf{1}}^{\mathcal{P}}\int_{x_0}^{px_0}(\mathbf{1} - H)\big)$$

are greater than 1. On the other end, we have

$$\delta_{\mathbf{1}-H}^{\mathcal{P}} = \delta_{\mathbf{1}}^{\mathcal{P}}.$$

This is seen as follows. We claim that for any positive integer n, there is a constant $C > 0$ such that, for all $p \in \mathcal{P}$, $exp\big(\int_{x_0}^{px_0} H - \varepsilon_n d(x_0, px_0)\big) < C$. Indeed, when p goes to ∞, the two parts of the geodesic going from x_0 to px_0 which lie in $z^{-1}([0, Y_n])$ converge to the two geodesics going from x_0 to ∞ and from px_0 to ∞, and the length of the part of these two geodesics lying in $z^{-1}([0, Y_n])$ is bounded by Y_n.

Whereas the other part of the geodesic going from x_0 to px_0 lies in $z^{-1}([Y_n, +\infty[)$, where we have $H \leq \varepsilon_n$. This shows the claim, which in turn implies the inequality $\delta_{\mathbf{1}-H}^{\mathcal{P}} \leq \delta_{\mathbf{1}}^{\mathcal{P}}/(1 - \varepsilon_n)$. The other inequality $\delta_{\mathbf{1}}^{\mathcal{P}} \leq \delta_{\mathbf{1}-H}^{\mathcal{P}}$ follows from the positivity of H. †

Remark The function $h^{\mathbf{1}}$ used by S. J. Patterson, in order to obtain a divergent Poincaré series, was of the form $h^{\mathbf{1}} = exp(\int_{x_0}^{\gamma x_0} H)$, with H constant on coronas in $\tilde{M}$ centred on x_0. That function $h^{\mathbf{1}}$ is not invariant by the group Γ. We have just shown that this invariance can be restored in a neighbourhood of a cusp by taking horoballs instead of balls, and this fact is sufficient to get a finite measure.

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Yves Coudene
coudene@dma.ens.fr