The Hopf argument

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Abstract

Let T be a measure preserving transformation of a metric space X. Assume T is conservative and X can be covered by a countable family of open sets, each of finite measure. Then any eigenfunction is invariant with respect to the stable distribution of T.

1 Introduction

We consider a metric space X, endowed with a Borel measure μ. Let $T : X \to X$ be a measure preserving measurable transformation.

Definition *The stable distribution of T is defined by:*

$$W^{ss}(x) = \{y \in X \mid d(T^n x, T^n y) \to 0 \text{ when } n \to +\infty\}$$

A measurable function $g : X \to \mathbf{R}$ is W^{ss}-invariant if there is a set $\Omega \subset X$ with $\mu(X \backslash \Omega) = 0$, such that for all $x, y \in \Omega$, $\quad y \in W^{ss}(x)$ implies $g(y) = g(x)$.

If T is invertible, we may also define the *unstable distribution* $W^{su}(x)$ of a point $x \in X$. This is just the stable distribution of x with respect to the transformation T^{-1}.

E. Hopf proved the ergodicity of the geodesic flow on a negatively curved surface of finite volume in 1936 [H36]. The proof relied on the following fact, which is now called *the Hopf argument*: any measurable set invariant under the geodesic flow is in fact invariant by the stable and unstable distributions of the flow. This argument was then applied by D. V. Anosov to the systems that now bear his name [A69] and the Hopf argument is now a standard tool of hyperbolic theory.

In particular, it has been extended to the infinite measure setting. This extension was the main motivation for giving a generalisation of the ergodic theorem for σ-finite measures, the so-called Hopf ratio ergodic theorem [H71].

It should be emphasized that, in the case of infinite measure preserving transformations, most of the current versions of the Hopf argument make use of some explicit estimates on the measure of big balls, which follow from geometric considerations. These estimates are needed to build a positive integrable function p satisfying:

$$\left| \frac{p(x) - p(y)}{p(y)} \right| \xrightarrow[d(x,y) \to 0]{} 0$$

This function is plugged in the Hopf ratio ergodic theorem and the proof can then proceed as in the finite measure case.

In the context of geodesic flows on negatively curved manifolds, V. Kaimanovich gave a proof of the ergodicity of invariant quasi-product measures which does not rely on the Hopf ratio ergodic theorem [K94]. The main idea is to induce on balls of increasing radius.

This short note shows how to combine the ratio approach of E. Hopf together with the induction method of V. Kaimanovich, in order to bypass the need of explicit estimates on the size of balls. As a consequence, we obtain a result valid for any space admitting a countable basis of open sets with finite measure.

2 The Hopf argument and weak mixing

Recall that a *T-eigenfunction* f is a measurable function that satisfies $f \circ T = \lambda f$ for some complex number λ of modulus 1. T-invariant functions are examples of eigenfunctions that are associated to the eigenvalue 1. We show that all bounded eigenfunctions are W^{ss}-invariant:

Theorem *Let X be a metric space, μ a Borel measure on X, $T : X \to X$ a measure preserving conservative transformation of X. We assume that there is a countable family of open sets of finite measure which covers almost all of X. Then any bounded T-eigenfunction is W^{ss}-invariant.*

Remarks
– If the transformation is invertible, then the eigenfunctions are both W^{ss} and W^{su}-invariant. Indeed a T-eigenfunction is a T^{-1}-eigenfunction and we can apply the theorem to both transformations T and T^{-1}.
– The theorem extends to measurable flows in a straightforward way: an eigenfunction for the flow is an eigenfunction for the time 1 map associated to the flow. So it is invariant with respect to the stable distribution of the time 1 map, which contains the stable distribution of the flow.

We now embark in the proof of the theorem, which will be divided in several steps.

Birkhoff means and Hopf Ratio sums
Let F be a closed subset of X, and define the δ-neighborhood of F by:

$$V_\delta(F) = \{x \in X \mid d(x, F) < \delta\}$$

We suppose that F admits such a neighborhood of finite measure, which is denoted by U. Consider some bounded measurable function $f : X \to \mathbf{R}$ which is zero on F^c. We begin by inducing T on F. Let T_F be the induced transform on F:

$$T_F(x) = T^{n_F(x)}(x), \quad \text{with} \quad n_F(x) = min\{n \in \mathbf{N} - \{0\} \mid T^n(x) \in F\}$$

This integer $n_F(x)$ is finite for almost every x because we assume that T is conservative. Let us apply the Birkhoff ergodic theorem to T_F:

$$\frac{1}{k}\sum_{i=1}^{k} f(T_F^i(x)) \longrightarrow E_{\mu_{|F}}(f \mid \mathcal{I} \cap F)(x)$$

where $\mathcal{I}$ the σ-algebra of T-invariant subsets of X, $E_{\mu_{|F}}$ is the conditional expectation with respect to the finite measure $\mu_{|F}$. We have used the fact that the σ-algebra of T_F-invariant subsets of F coincides with $\mathcal{I} \cap F$. At that point, we could show that the limit of the Birkhoff means is invariant with respect to the stable distribution of T_F. But it is in no way clear how this distribution relates to the stable distribution of T.

However, we can relate the Birkhoff means of T_F with the Hopf Ratio sums of T. First define for all k:

$$n_F^k(x) = \sum_{i=0}^{k-1} n_F(T_F^i(x))$$

so that for all $N \in \mathbf{N}$ such that $n^k(x) \leq N < n^{k+1}(x)$,

$$T_F^k(x) = T^{n_F^k(x)}(x) \quad \text{and} \quad \sum_{j=1}^{N} \mathbf{1}_F(T^j(x)) = k$$

The function f is zero on F^c, so $f(T^j(x)) = 0$ if $T^j(x) \notin F$. This gives:

$$\sum_{i=1}^{k} f\big(T_F^k(x)\big) = \sum_{j=1}^{N} f\big(T^j(x)\big)$$

We have obtained:

$$\frac{1}{k}\sum_{i=1}^{k} f\big(T_F^i(x)\big) = \frac{\sum_{j=1}^{N} f\big(T^j(x)\big)}{\sum_{j=1}^{N} \mathbf{1}_F\big(T^j(x)\big)}$$

We could have applied the Hopf ratio theorem to T with ratio function $\mathbf{1}_F$ instead of the usual Birkhoff theorem to T_F; the limit is then identified with the Radon-Nikodym derivative $\frac{d\ (f\mu_{|\mathcal{I}})}{d\ (\mathbf{1}_E\mu_{|\mathcal{I}})}$. Note that the Hopf ratio theorem can be proven by inducing; this was done by R. Zweimüller [Z04].

Monotonicity of the limit

Let us assume that F admits a δ-neighborhood $U_\delta = \{x \in X \mid d(x,F) < \delta\}$ of finite measure. For sake of notation, we drop the subscript δ, until further notice. We also restrict our attention to bounded Lipschitz functions $f : X \to \mathbf{R}_+$ which are zero on F^c. Inducing both on F and U, we get:

$$\frac{\sum_{j=1}^{N} f\big(T^j(x)\big)}{\sum_{j=1}^{N} \mathbf{1}_F\big(T^j(x)\big)} \longrightarrow E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(x), \quad \frac{\sum_{j=1}^{N} f\big(T^j(y)\big)}{\sum_{j=1}^{N} \mathbf{1}_U\big(T^j(y)\big)} \longrightarrow E_{\mu_{|U}}(f \mid \mathcal{I}\cap U)(y).$$

These convergence hold almost everywhere on F and in L^2. Note also that, from the conservativity of T, for a.e. $x \in F$, $\sum_{j=1}^{N} \mathbf{1}_F\big(T^j(x)\big) \to \infty$. This implies that we may start the sums in the ratios at $j = N_0$, N_0 arbitrary, instead of $j = 1$, without modifying the convergence given above.

If $x \in F$ and $y \in W^{ss}(x)$, then there is an N_0 such that for all $N \geq N_0$, $T^j(y) \in U$ as soon as $T^j(x) \in F$. This shows that:

$$\sum_{j=N_0}^{N} \mathbf{1}_F\big(T^j(x)\big) \leq \sum_{j=N_0}^{N} \mathbf{1}_U\big(T^j(y)\big)$$

If $T^j(x)$ and $T^j(y)$ do not belong to F, $f(T^j(x)) = f(T^j(y)) = 0$. Let K be the Lipschitz constant of f. We get:

$$\sum_{j=N_0}^{N} f\big(T^j(x)\big) - f\big(T^j(y)\big) \leq K sup_{j\geq N_0} d(T^j(x), T^j(y)) \sum_{j=N_0}^{N} \mathbf{1}_F(T^j(x)) + \mathbf{1}_F(T^j(y))$$

This last sum is less than $2\sum_{j=N_0}^{N} \mathbf{1}_U\big(T^j(y)\big)$. If N_0 is big enough, the quantity $sup_{j\geq N_0} d(T^j(x),T^j(y))$ is less than some fixed $\varepsilon>0$. We now pass to the limit in the expression:

$$\frac{\sum_{j=N_0}^{N} f\big(T^j(x)\big)}{\sum_{j=N_0}^{N} \mathbf{1}_F\big(T^j(x)\big)} = \frac{\sum_{j=N_0}^{N} \mathbf{1}_U\big(T^j(y)\big)}{\sum_{j=N_0}^{N} \mathbf{1}_F\big(T^j(x)\big)} \left(\frac{\sum_{j=N_0}^{N} f\big(T^j(y)\big)}{\sum_{j=N_0}^{N} \mathbf{1}_U\big(T^j(y)\big)} + \frac{\sum_{j=N_0}^{N} f\big(T^j(x)\big)-f\big(T^j(y)\big)}{\sum_{j=N_0}^{N} \mathbf{1}_U\big(T^j(y)\big)} \right)$$

to obtain: $\quad E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(x) \geq E_{\mu_{|U}}(f \mid \mathcal{I}\cap U)(y) - 2\varepsilon.$

Let us summarize what has been proven so far: F is a closed set which admits a δ-neighborhood U of finite measure and f is a nonnegative Lipschitz bounded function which is zero outside F. Then there exists a set $\Omega\subset F$, $\mu(F\backslash\Omega)=0$, such that for all $x,y\in\Omega$, $y\in W^{ss}(x)$ implies $E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(x) \geq E_{\mu_{|U}}(f \mid \mathcal{I}\cap U)(y)$.

W^{ss}-Invariance on F

We now consider the open sets $U_\delta=\{x\in X \mid d(x,F)<\delta\}$. Fix $\delta_0>0$ such that U_{δ_0} is of finite measure. From the general properties defining the conditional expectation, we have the following relation for all $A\subset U_{\delta_0}$ and $f\in L^2(U_{\delta_0})$:

$$E_{\mu_{|A}}(f_{|A} \mid \mathcal{I}\cap A) = E_{\mu_{|U_{\delta_0}}}(f \mid \mathcal{I}_A)_{|A}, \quad \text{a.e. on } \ A$$

where the σ-algebra $\mathcal{I}_A$ is defined by $\mathcal{I}_A=\{(I\cap A)\cup B \mid \ I\in\mathcal{I},\ B\subset\ U_{\delta_0}\backslash A\ \}$. The intersection of the $\mathcal{I}_{U_\delta}$ coincide with $\mathcal{I}_F$ and the L^2 increasing Martingale convergence theorem shows that:

$$E_{\mu_{|U_{\delta_0}}}(f \mid \mathcal{I}_{U_\delta}) \longrightarrow E_{\mu_{|U_{\delta_0}}}(f \mid \mathcal{I}_F) \quad \text{in} \quad L^2(F)$$

Extracting an almost convergent subsequence, we obtain the following inequality: $E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(x) \geq E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(y)$. Inversing the role of x and y, we finally get a set $\Omega\subset F$ of full measure in F such that:

$$\forall x,y\in\Omega, \quad y\in W^{ss}(x) \ \text{ implies } \ E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(x) = E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(y).$$

We now approach in norm $L^2(F,\mu)$ any L^2 function defined on F by a sequence of (restriction of) Lipschitz bounded functions on X which are zero on F^c. This is possible because F has negligible boundary. Now, extracting an almost converging subsequence, we get the inequality: $E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(x) = E_{\mu_{|F}}(f \mid \mathcal{I}\cap F)(y)$ for $f\in L^2(F,\mu)$. In particular, if f is the restriction of a T-invariant function, then there is a set of full measure $\Omega\subset F$, such that for all $x,y\in\Omega$ with $y\in W^{ss}(x)$, $f(x)=f(y)$.

The case of T-invariant functions

To finish the proof, we show that almost all of X may be written as a countable increasing union of closed sets F_i, each admitting some δ_i-neighborhood of finite measure, and each with negligible boundary.

Indeed any open set U of finite measure is the increasing union of the closed sets $F_\delta=\{x\in X \mid d(x,U^c)\geq\delta\}$. These sets have disjoint boundaries, so there exists a decreasing sequence $\delta_i\to 0$ such that the boundaries of F_{δ_i} are negligible. Moreover the δ_i-neighborhood of F_{δ_i} is a subset of U, hence of finite measure. Since almost all of X can be written as a countable union of open sets of finite measure, it is also a countable union of closed sets satisfying the required properties. This union can be made increasing, since for any sets $F_1, F_2 \subset X$, we have $\partial(F_1 \cup F_2) \subset \partial F_1 \cup \partial F_2$.

Finally, let f be a T-invariant function. We know that each F_i contains a subset Ω_i of full measure such that for all $x, y \in \Omega_i$ with $y \in W^{ss}(x)$, we have $f(x) = f(y)$. The set $\cap_{i \geq k} \Omega_i$ is a full measure subset of F_k. Define $\Omega' = \cup_{k \in \mathbf{N}} \cap_{i \geq k} \Omega_i$. The complement of Ω' in X is negligible, and, for $x, y \in \Omega'$, $y \in W^{ss}(x)$ implies $f(x) = f(y)$. This shows that T-invariant functions are W^{ss}-invariant.

The case of T-eigenfunctions
Eigenfunctions associated to the eigenvalue $\lambda \in S^1$ are fixed points of the operator: $f \to \lambda^{-1} f \circ T$. We consider the skew-product $\tilde{T} : X \times S^1 \longrightarrow X \times S^1$ given by $\tilde{T}(x, y) = (T(x), \lambda^{-1} y)$. This new transform acts as an isometry on the fibers and preserves the measure $\mu \otimes d\lambda_{S^1}$. Its stable sets $\tilde{W}^{ss}(x, y)$ are related to the stable sets of T by the formula:

$$\tilde{W}^{ss}(x, y) = W^{ss}(x) \times \{y\}$$

The transform $\tilde{T}$ is conservative because T is conservative [Aa96], prop 1.2.4. Let $f : X \longrightarrow \mathbf{R}$ be a bounded measurable T-eigenfunction associated to the eigenvalue λ. Then the function $\tilde{f} : X \times S^1 \longrightarrow \mathbf{R}$ defined by $\tilde{f}(x, y) = y\, f(x)$ is $\tilde{T}$-invariant. Hence, it is $\tilde{W}^{ss}$-invariant: there exists some subset $\tilde{\Omega} \subset X \times S^1$ of full measure such that if (x_1, y_1) and (x_2, y_2) belongs to $\tilde{\Omega}$ and x_2 is in $W^{ss}(x_1)$, $y_1 = y_2$, then $\tilde{f}(x_1, y_1) = \tilde{f}(x_2, y_2)$. So, taking some y such that $\Omega_y = \{x \in X \mid (x, y) \in \tilde{\Omega}\}$ is of full measure in X, we see that f is W^{ss}-invariant. This ends the proof of the main theorem.

As a sidenote, we could have proceeded as in the T-invariant case, replacing the operator $f \mapsto f \circ T$ by $f \to \lambda^{-1} f \circ T$, and considering the induced contraction on $L^2(F, \mu_{|F})$:

$$\mathbf{U}_F f(x) = \lambda^{-n_F(x)} f\big(T_F(x)\big).$$

The convergence a.e. of its Birkhoff sum follows from the convergence a.e. of the Birkhoff sum of the extension $\tilde{T}(x, y) = (T_F(x), y\ \lambda^{-n_F(x)})$. The connection with the Hopf ratio sum would have been given by the formula:

$$\frac{1}{k} \sum_{i=1}^{k} \mathbf{U}_F^i(f)(x) = \frac{\sum_{j=1}^{N} \lambda^{-j} f\big(T^j(x)\big)}{\sum_{j=1}^{N} \mathbf{1}_F\big(T^j(x)\big)}$$

The conditional expectation is replaced by the abstract L^2-projector on the set of $\mathbf{U}_F$-invariant functions and the proof proceeds as above.

Some applications
There are a number of systems, for which ergodicity is deduced from the triviality of the W^{ss} and W^{su}-invariant functions. Our result shows without further effort that these systems are in fact *weak-mixing*: there are no bounded nonconstant T-eigenfunctions.

We recover, for example, the weak-mixing property for the 2 dimensional Lorentz gas, a result first obtained by F. Pène [P00]. This infinite measure-preserving system can be seen as a skew-product over the Sinai billiard. The stable and unstable distributions of the system are not regular, hence the need for a result which does not rely on smoothness assumptions.

We also obtain the weak-mixing property for topologically weak-mixing, conservative abelian skew-products over hyperbolic systems. Indeed, ergodicity of the stable distribution was proven in [C03]. This result is new, and next section makes it a bit surprising.

3 Invariant distributions and mixing

For finite measure preserving transformations, it can be shown that all (weak, strong or a.e.) limits of subsequences of $f \circ T^n$, for any measurable bounded f, are in fact W^{ss}-invariant, and also W^{su}-invariant if T is invertible; see [C06].

As pointed out by S. Gouëzel, this result is false in the infinite category. Indeed J. Aaronson, M. Lin, B. Weiss [ALW80] show that an invertible transformation preserving an infinite measure that is non-atomic and ergodic is not *mild mixing*. Using a characterization of the mild mixing property due to K. Schmidt and P. Walters [SW82](first part of the proof of th 2.3), this implies that there is a measurable set C with $\mu(C) > 0$, $\mu(C^c) > 0$, and an increasing sequence of integers $\{n_k\}$ such that:

$$\mathbf{1}_C\big(T^{n_k}(x)\big) \longrightarrow \mathbf{1}_C(x), \quad a.e.\ x \in X$$

On the other hand, there exist examples of such transformations such that W^{ss} is ergodic. Skew-products over Bernoulli shifts with ergodic stable distribution were constructed in [C01]. In that case, the set C cannot be W^{ss}-invariant. In the terminology of H. Furstenberg, although the eigenfunctions are W^{ss}-invariant, this is not the case of the rigid functions if the measure is infinite.

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