# On invariant distributions and mixing

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**Abstract**

We show that any probability preserving transformation of a metric space is mixing as soon as they are no non constant $L^2$-functions which are invariant under both the stable and unstable distributions. This generalizes a result of M. Babillot. [1]

## 1 Introduction

The original proof of mixing of Anosov diffeomorphisms and flows, given by Anosov and Sinai [A69], relied on the ergodicity of the strong stable foliation of the system. This ergodicity, together with the existence of a measurable partition subordinated to the foliation, allowed them to show that Anosov systems are in fact $K$-mixing, as soon as they are topologically mixing.

The joint ergodicity of the stable and unstable distributions of any $C^2$ diffeomorphism with non-zero exponents is in fact sufficient to obtain $K$-mixing. This surprisingly general result, due to F. Ledrappier and L.S. Young [LY85] relies on Pesin theory to bypass the lack of hyperbolicity.

It has been remarked recently that, if we are only interested in strong mixing, then a short proof can be obtained, which does not make use of entropy and Pesin theory. The elegant argument of M. Babillot [B02] relies on a trick from spectral theory, which connects weak accumulation points of the positive iterates of a function with accumulation points for its negative iterates.

We propose an elementary proof of a generalisation of M. Babillot result, which does not use spectral theory. The proof is modelled on the famous Hopf argument [H71], which originally dealt with geodesic flows on negatively curved manifolds.

In order to make the proof of the mixing property more transparent, we first recall the Hopf argument, in a generality which may not have been known before. Here are some motivations for such a generality : billiards maps are examples of transformations which are not smooth; still they retain sufficient hyperbolicity to carry out the argument of M. Babillot, as was pointed out by F. Pène. Another example is given by hyperbolic systems on infinite dimensional manifolds. Here, the phase space is not locally compact, still the Hopf argument can be applied. The simplest example of such system is the geodesic flow on the constant negatively curved Hilbert ball.

[1] 37B10, 37D40, 34C28

## 2 The Hopf Argument

We consider a metric space $X$, endowed with a Borel probability measure $\mu$. Let $T : X \to X$ be a measure preserving measurable transformation.

**Definition** *The stable distribution of $T$ is defined by:*

$$W^{ss}(x) = \{y \in X \mid d(T^n x, T^n y) \to 0 \text{ as } n \to +\infty\}.$$

*A measurable function $g : X \to \mathbf{R}$ is $W^{ss}$-invariant if there is a set $\Omega \subset X$ of full measure, such that for all $x, y \in \Omega$, $\quad y \in W^{ss}(x)$ implies $g(y) = g(x)$.*

If $T$ is invertible, we may also define the *unstable distribution* $W^{su}(x)$ of a point $x \in X$. This is just the stable distribution of $x$ with respect to the transformation $T^{-1}$.

Finally recall first that bounded Lipschitz functions are dense in $L^2(X, \mu)$; this follows from the exterior regularity of $\mu$. And second, that any $L^2$ convergent sequence admits a subsequence that converges almost everywhere.

**Theorem** *Let $X$ be a metric space, $\mu$ a Borel probability measure on $X$, $T : X \to X$ a measure preserving transformation of $X$. Then any $T$-invariant $L^2$ function is $W^{ss}$-invariant.*

**Proof** We denote by $\mathcal{I}$ the $\sigma$-algebra of $T$-invariant measurable sets; let us apply the $L^2$ ergodic theorem : for each bounded Lipschitz function $f : X \to \mathbf{R}$, with Lipschitz constant $C$, there exists a set $\Omega_f \subset X$ of full measure, and integers $n_i \to \infty$ such that

$$\forall x \in \Omega_f, \quad \frac{1}{n_i} \sum_{k=0}^{n_i - 1} f(T^k(x)) \underset{i \to +\infty}{\longrightarrow} E(f|\, \mathcal{I})(x)$$

Let $y \in \Omega_f \ \cap \ W^{ss}(x)$. We have:

$$\Big| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) - \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(y)) \Big| \ \leq \ \frac{1}{n}\, C \ \sum_{k=0}^{n-1} \ d(T^k x, T^k y) \xrightarrow[n \to +\infty]{} 0$$

This implies: $E(f|\, \mathcal{I})(x) = E(f|\, \mathcal{I})(y)$.

Let $f$ be any $T$-invariant $L^2$-function. There exists a sequence $f_n$ of bounded Lipschitz functions such that $f_n \to f$ in $L^2$ norm. This implies: $E(f_n|\, \mathcal{I}) \to E(f|\, \mathcal{I}) = f$ in $L^2$ norm, and passing to a subsequence $n_\ell$, for all $x$ in a set $\Omega_0$ of full measure. So, for all points $x, y \in \Omega_0 \bigcap \Omega_{f_{n_\ell}}$, and $y \in W^{ss}(x)$,

$$f(x) = \lim E(f_{n_\ell}|\, \mathcal{I})(x) = \lim E(f_{n_\ell}|\, \mathcal{I})(y) = f(y).$$

This ends the proof of the theorem.

**Corollary** *Let $X$ be a metric space, $\mu$ a Borel probability measure on $X$, $T : X \to X$ an invertible measure preserving transformation of $X$. Assume that any function which is both $W^{ss}$-invariant, $W^{su}$-invariant and $T$-invariant, is constant almost everywhere. Then $T$ is ergodic.*

The proof of the corollary is straightforward : the preceding theorem, applied to $T$ and $T^{-1}$, shows that any $T$-invariant function is $W^{ss}$ and $W^{su}$-invariant.

As a final remark, we note that we did not need any continuity assumption on $T$, regularity property on $W^{ss}$ or separability hypothesis on $X$.

# 3 Mixing

The proof of the mixing property is based on the following two results concerning Hilbert spaces. cf F. Riesz [RN90] for short elementary proofs.

– (Banach-Alaoglu) The unit ball of a Hilbert space is weakly sequentially compact.
– (Banach-Saks) If $f_n \to f$ weakly, then there is a subsequence $n_k$ such that we have the convergence $\frac{1}{n}\sum_{k=0}^{n-1} f_{n_k} \to f$ in $L^2$-norm.

**Theorem** *Let $X$ be a metric space, $\mu$ a Borel probability measure on $X$, $T : X \to X$ a measure preserving transformation of $X$ and $f \in L^2(X, \mu)$. Then any weak limit of a subsequence of $f \circ T^n$ is $W^{ss}$-invariant.*

**Proof** Let $g$ be a weak limit of $f \circ T^{n_i}$. First assume $f$ is bounded and Lipschitz. The Banach-Saks lemma gives subsequences $m_\ell$, $n_{i_k}$ such that:

$$\Psi_\ell(x) := \frac{1}{m_\ell}\sum_{k=1}^{m_\ell} f \circ T^{n_{i_k}} \xrightarrow[a.e.]{} g$$

If $y \in W^{ss}(x)$, $\quad |\Psi_\ell(x) - \Psi_\ell(y)| \le C\frac{1}{m_\ell}\sum_{k=1}^{m_\ell} d\big(T^{n_{i_k}}(x), T^{n_{i_k}}(y)\big) \xrightarrow[\ell \to \infty]{} 0.$

Hence, $g$ is $W^{ss}$-invariant.

Let $f \in L^2$. For all $\varepsilon > 0$, there exists $f_0$ bounded and Lipschitz such that $||f - f_0|| < \varepsilon$. Upon passing to a subsequence, we may assume that the sequence $f_0 \circ T^{n_i}$ weakly converges to some function $g_0$, which is $W^{ss}$-invariant. This gives: $(f - f_0) \circ T^{n_i} \to g - g_0$ weakly, which in turn implies:

$$||g - g_0|| \;\le\; \underline{\lim}\; ||(f - f_0) \circ T^{n_i}|| \;\le\; ||f - f_0|| \;<\; \varepsilon.$$

So there exists a sequence of $W^{ss}$-invariant functions which converges to $g$ in $L^2$ norm, and, taking a subsequence, almost everywhere. This shows that $g$ is $W^{ss}$-invariant.

Recall that the stable distribution $W^{ss}$ is *ergodic* if all $W^{ss}$-invariant functions are constant almost everywhere. On the other hand, $T$ is *strongly mixing* if the sequence $f \circ T^n$ converges weakly to $\int f \, d\mu$. So we get the following corollary:

**Corollary** *If $W^{ss}$ is ergodic, then $T$ is strongly mixing.*

We now come to the main result. In contrast to the Hopf argument, the invertible case is not a straightforward consequence of the non-invertible case.

**Theorem** *Let $X$ be a metric space, $\mu$ a Borel probability measure on $X$, $T : X \to X$ an invertible measure preserving transformation of $X$ and $f \in L^2(X, \mu)$. Then any weak limit of a subsequence of $f \circ T^n$, $n \in \mathbf{N}$, is $W^{ss}$-invariant and $W^{su}$-invariant.*

**Proof** Let $g$ be a weak limit of $f \circ T^{n_i}$. We first show that there is a function $g'$ which is both $W^{ss}$ and $W^{su}$-invariant, and that satisfies $< f, g' >=< g, g >$.

We apply the preceding lemma to $T^{-1}$ and to the sequence $g \circ T^{-n_i}$. There is a subsequence $n_{i_k}$ and a $W^{su}$-invariant function $g'$ such that $g \circ T^{-n_{i_k}} \to g'$ weakly. The function $g$ is $W^{ss}$-invariant, so for all $n \in \mathbf{Z}$, the functions $g \circ T^{-n}$ are $W^{ss}$-invariant. Hence $g'$ is $W^{ss}$-invariant. Now we get:

$$< f, g' >= \lim_{k\to+\infty} < f, g \circ T^{-n_{i_k}} >= \lim_{k\to+\infty} < f \circ T^{n_{i_k}}, g >=< g, g >$$

We denote by $I$ the closed subspace of $L^2$ containing all functions which are both $W^{ss}$ and $W^{su}$-invariant. This subspace is $U$-invariant. Now write $f$ as a sum $f = f_1 + f_2$, with $f_1 \in I$ and $f_2 \in I^{\perp}$. Extracting subsequences, we can find two functions $g_1$ and $g_2$ and a sequence $n'_i$ such that $f_1 \circ T^{n'_i} \to g_1$ weakly and $f_2 \circ T^{n'_i} \to g_2$ weakly. This shows that $g_1 \in I$, $g_2 \in I^{\perp}$ and $g_1 + g_2 = g$. But there is a function $g'_2 \in I$ such that $< f_2, g'_2 >=< g_2, g_2 >$. Hence $g_2 = 0$ and we are done.

**Remarks**

– Any $T$-invariant function $g$ is a weak limit of a sequence $f \circ T^n$ (just take $f = g$). Hence this gives a proof of the Hopf argument which does not use the ergodic theorem. Note that the $L^2$ ergodic theorem can be deduced from the Banach-Alaoglu and Banach-Saks theorem.

– Any eigenvalue $f$ is also a weak limit of a subsequence of $f \circ T^n$. Hence it is both $W^{ss}$ and $W^{su}$-invariant.

– The results of this paper extend to flows without difficulty.

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