Hyperbolic systems on nilpotent covers

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Abstract

We study ergodicity of weak and strong stable foliations of hyperbolic systems on nilpotent covers. Subshifts of finite type and geodesic flows on negatively curved manifolds are also considered.

1 Introduction

This article is devoted to the study of the ergodicity of the stable foliation of an hyperbolic flow defined on a nilpotent cover of a manifold.

The geodesic flow on a compact two dimensional manifold of constant negative curvature is one of the simplest example of hyperbolic flow. E. Hopf [Ho36], G. A. Hedlund [Hed36][Hed39] showed the ergodicity of the associated horospheric flow with respect to the Liouville measure. H. Furstenberg obtained its unique ergodicity [F73]. This was generalised to the case of the stable foliation of an hyperbolic flow by D. V. Anosov [An67] and R. Bowen and B. Marcus [BM77]. In the context of infinite measure ergodic theory, it was natural to ask whether ergodicity of the foliation is again true on a regular cover.

This question has been investigated by many authors in the last decade. M. Babillot, F. Ledrappier studied the case of the geodesic flow on a compact manifold of constant negative curvature [BL96]. The extension of the flow to an abelian cover admits a number of invariant measures ; they were able to show that the stable foliation of the extension is ergodic with respect to these measures. Their proof relies on counting estimates obtained by transfer operator techniques. Using results from harmonic analysis, V. Kaimanovich was able to generalise their result to nilpotent extensions for the Liouville measure [Ka00]. In another direction, M. Pollicott considered homologically full Anosov flows and obtained ergodicity of the foliation on an abelian cover for the measure of maximal entropy [Po00]. The method makes use of a symbolic model. We studied the case of Axiom A flows [C01] ; for these systems, the stable foliation may be non ergodic and we gave necessary and sufficient conditions in term of periodic orbits in order to get that ergodicity. Very few hypothesis were made on the measure, in order to treat the case of the Liouville measure, measure of maximal entropy and harmonic measure. Following M. Pollicott, we were working in a symbolic setting, so that two restrictive hypothesis were needed: the non-wandering set of the flow on the basis of the cover had to be compact and the cover was abelian. R. Solomyak [So01], U. Hamenstadt [Ha01] and then J. Aaronson, R. Solomyak and O. Sarig [ASS01], also obtained several results, using methods of independent interest.

The goal of this article is to provide an unified approach to these problems and to recover these results using just the hyperbolicity of the systems. No compactness assumption on the wandering set is made and the cover can be taken nilpotent. As a consequence, we obtain a number of theorems, conjectured from the beginning, but which resisted all previous attempts, like the finite volume case.

We first state a general result concerning the ergodicity of the stable foliation of systems admitting local product structures. The second part of the article is devoted to the proof of that result. It is then applied to hyperbolic flows, subshifts of finite type and geodesic flows on negatively curved manifolds.

2 Main result

Let G be a Polish group acting continuously on a Polish space $\hat{X}$ and $\hat{\phi}_t$ a continuous flow on $\hat{X}$ commuting with the G action. The quotient of $\hat{X}$ by G is denoted by X, the projection from $\hat{X}$ to X by π, and the quotient flow on X by ϕ_t. Let us assume that the quotient topology on X is given by a metric for which π is Lipschitz. Strong stable sets are defined by :

$W^{ss}(x) := \{\ y \in X \mid \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0\ \}$;
$W^{ss}_\varepsilon(x) := \{\ y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \leq \varepsilon \text{ for all } t \geq 0\ \}$;
$\hat{W}^{ss}(\hat{x}) := \{\ \hat{y} \in \hat{X} \mid \lim_{t\to\infty} d(\hat{\phi}_t(\hat{x}), \hat{\phi}_t(\hat{y})) = 0\ \}$;

$\hat{W}^{ss}_{\varepsilon}(\hat{x}) := \{\ \hat{y} \in \hat{W}^{ss}(\hat{x}) \mid d(\hat{\phi}_t(\hat{x}), \hat{\phi}_t(\hat{y})) \leq \varepsilon \text{ for all } t \geq 0\ \}$.
The weak stable sets $\hat{W}^{ws}$, W^{ws} of $\hat{\phi}_t$ and ϕ_t are equal to $\hat{\phi}_{\mathbf{R}}(\hat{W}^{ss})$ and $\phi_{\mathbf{R}}(W^{ss})$. One also defines the strong unstable sets $\hat{W}^{su}$, $\hat{W}^{su}_{\varepsilon}$, W^{su} and W^{su}_{ε} of $\hat{\phi}_t$ and ϕ_t ; these are the stable sets of $\hat{\phi}_{-t}$ and ϕ_{-t}.

Let τ be a periodic orbit of ϕ_t on X and $\hat{x} \in \hat{X}$ a preimage of a point on the orbit τ. The period of τ is denoted by $l(\tau)$. The *Frobenius element* $[\tau]$ in G is the translation in the fiber along τ: $\hat{\phi}_{l(\tau)}(\hat{x}) = [\tau](\hat{x})$. Its conjugacy class does not depend on the chosen preimage. If it belongs to the center of G, it is independent of the chosen preimage.

The next three uniformity assumptions are satisfied, for example, if the G-action on $\hat{X}$ is isometric and the distance on X is locally equivalent to the distance $d'(x, y) = d_{\hat{X}}(\pi^{-1}x, \pi^{-1}y)$.

• For all $g \in G$, the function $\hat{x} \mapsto g\hat{x}$ is uniformly continuous on $\hat{X}$. In particular, this implies the relation $g\ \hat{W}^{ss}(\hat{x}) = \hat{W}^{ss}(g\hat{x})$ for all $\hat{x}$.

• For all $v \in X$, $\varepsilon > 0$ and $T \geq 0$, there is a $\delta > 0$ such that for all $\hat{v} \in \pi^{-1}(v)$, $d(\hat{x}, \hat{v}) < \delta$ implies $d(\hat{\phi}_t(\hat{x}), \hat{\phi}_t(\hat{v})) < \varepsilon$ for all $|t| \leq T$.

• Let v some point in X. If $\{\hat{x}_n\}$ is a sequence in $\hat{X}$, $d(\pi(\hat{x}_n), v) \to 0$ implies $d(\hat{x}_n, \pi^{-1}v) \to 0$.

The strong unstable foliation $\hat{W}^{su}$ is said to be *locally contracting uniformly in fibers* if for any $v \in X$, there is an $\varepsilon > 0$ such that for any $\delta_0 > 0$, one can find a $t_0 \in \mathbf{R}$ so that $d(\hat{\phi}_{-t}(\hat{x}), \hat{\phi}_{-t}(\hat{y})) \leq \delta_0$ whenever $t \geq t_0$, $d(\hat{x}, \pi^{-1}v) < \varepsilon$, $d(\hat{y}, \pi^{-1}v) < \varepsilon$ and $y \in \hat{W}^{su}_{\varepsilon}(x)$.

The flow $\hat{\phi}_t$ is said to admit a *local product structure* if, for all points $v \in X$, and all $\varepsilon > 0$, there exists an $\varepsilon_0 > 0$, a neighborhood V of $\pi^{-1}v$ with $\varepsilon_0 < d(\pi^{-1}v, V^c)$, $d(x, \pi^{-1}v) < \varepsilon \quad \forall x \in V$, and positive constants δ_1, δ_2 less than ε, such that for all $x, y \in V$ with $d(x, y) \leq \delta_1$, there is a point $< x, y > \in X$, a real number t with $|t| \leq \delta_2$, and an element $g \in G$ with $d(g, id) \leq \delta_2$, so that:

$$< x, y > \in\ \hat{W}^{ss}_{\delta_2}(g\hat{\phi}_t(x)) \cap \hat{W}^{su}_{\delta_2}(y).$$

The maps sending (x, y) to $< x, y >$, g and t are furthermore supposed to be Borel maps; in practice, they will even be continuous on V.
In fact, this property will only be used for v periodic; since periodic orbits are compact, the constants ε_0, δ_1, δ_2 are uniform on the orbit of v.

The spaces $\hat{X}$ and X are endowed with two Borel measures, $\hat{\mu}$ and μ, in such a way that the projection $\pi : \hat{X} \to X$ is non-singular: inverse images of sets of zero measure are of zero measure. These measures are supposed to be invariant by the flows $\hat{\phi}_t$ and ϕ_t ; the space $\hat{X}$ admits a countable cover by open sets of finite $\hat{\mu}$-measure, whereas μ is a probability measure ergodic with respect to the flow ϕ_t. The measure $\hat{\mu}$ satisfies the following conditions: there is a neighborhood of the identity in G and a constant C such that $g_*\hat{\mu} \leq C\hat{\mu}$ for g in the neighborhood. We also suppose:

Given a periodic point v in X, let V be a neighborhood of $\pi^{-1}v$ coming from the local product structure. Consider the set A of points $\hat{x}$ in V, for which $[v]^{-1}\hat{\phi}_{l(v)}(\hat{x})$ is again in V and δ_1-close to $\hat{x}$, and the transformation T on A which associate to the point $\hat{x}$ in A the point $< [v]^{-1}\hat{\phi}_{l(v)}(\hat{x}), \hat{x} >$. Then we suppose that there is a constant C depending only on v, such that $\hat{\mu}(T^{-1}B \cap A) \leq C\hat{\mu}(B)$ for all Borel set B in $\hat{X}$.

This condition may seem awkward ; in practice, it follows easily from the absolute continuity of the measure with respect to the holonomy of the foliations. However, since no regularity on the foliations is assumed, one has to give a more technical statement.

Finally, recall that the partition of $\hat{X}$ by the stable sets $\hat{W}^{ss}$ is said to be ergodic if any union of elements of the partition that forms a Borel set is of measure 0 or with complement of measure 0. In the following, this partition will be called abusively a foliation, and the sets $\hat{W}^{ss}$ will be called the leaves of the foliation.

We can now state the main result:

Theorem 1

Let G be a Polish group acting continuously on a Polish space $\hat{X}$ and $\hat{\phi}_t$ a continuous flow on $\hat{X}$ commuting with the G action. The system satisfies the previous hypothesis: three uniformity conditions, local contraction of $\hat{W}^{su}$ uniformly in fibers, and local product structure. Moreover X and $\hat{X}$ are equipped with measures μ and $\hat{\mu}$ as defined above.

Suppose that G is a nilpotent group admitting a central series composed of closed subgroups $\{1\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ so that, for all $i \geq 1$, the subgroup of G_i/G_{i-1} generated by the Frobenius elements in G_i of periodic orbits of ϕ_t is dense in G_i/G_{i-1}:

$$\overline{< \{[\tau] \mid \tau \textit{ periodic orbit of } \phi_t \textit{ with } [\tau] \in G_i\} >} = G_i/G_{i-1},$$

then the weak stable foliation $\hat{W}^{ws}$ is ergodic with respect to $\hat{\mu}$.
If moreover the following is satisfied :

$$\{0\} \times G_i/G_{i-1} \subset \overline{< \{(l(\tau), [\tau]) \mid \tau \textit{ periodic orbit of } \phi_t \textit{ with } [\tau] \in G_i\} >},$$

$$\overline{< \{l(\tau) \mid \tau \ periodic\ orbit\ of\ \phi_t\} >} = \mathbf{R},$$

then the strong stable foliation $\hat{W}^{ss}$ is ergodic with respect to $\hat{\mu}$.

Remarks

- The periodic orbits of ϕ_t with Frobenius elements in G_i are precisely the orbits whose lifts in $\hat{X}/G_i$ are again periodic.
- The conditions for the ergodicity of the strong stable foliation can be rewritten as $\overline{< \{[\tau]' \mid [\tau]' \in G_i'\} >} = G_i'/G_{i-1}'$ for all $i \geq 1$, with the convention $G' = \mathbf{R} \times G$, $G_i' = \{0\} \times G_i'$ if $i < n$, and $[\tau]' = (l(\tau), [\tau])$.
- There are systems satisfying the equalities $\overline{\{[\tau] \mid [\tau] \in G_i\}} = G_i/G_{i-1}$ and $\overline{< \{l(\tau) \mid [\tau] \in G_i\} >} = \mathbf{R}$, for all $i \geq 1$, and whose strong stable foliation is not ergodic. An example is provided in the section dealing with subshifts of finite type. In that example, the flow $\hat{\phi}_t$ has no periodic orbits on $\hat{X}$.
- If the subgroup generated by the lengths of the periodic orbits of $\hat{\phi}_t$ on $\hat{X}$ is dense in $\mathbf{R}$, then the conditions $\overline{< \{[\tau] \mid [\tau] \in G_i\} >} = G_i/G_{i-1}$, for all i, are easily seen to imply the ergodicity of both foliations on $\hat{X}$.

A similar result can be proven for transformations instead of flows; here are the modifications to introduce in that case: the group $\mathbf{R}$ should be replaced by $\mathbf{Z}$ whenever it occurs; in the definition of the local product structure, the constant δ_2 is removed and $t(x, y)$ is equal to zero. The statement of the theorem now translates verbatim, and there is essentially no change in the proof.

3 Proof of the ergodicity

This section is devoted to the proof of the main theorem. The central series of G gives a sequence of systems:

$$\hat{X} \to \hat{X}/G_1 \to \ldots \to \hat{X}/G_{i-1} \to \hat{X}/G_i \to \ldots \to \hat{X}/G_{n-1} \to \hat{X}/G = X$$

Ergodicity of the strong stable foliation is proven by a recurrence on this series. Assuming that the strong stable foliation of the quotient flow on $\hat{X}/G_i$ is ergodic, one has to show that it is again ergodic on $\hat{X}/G_{i-1}$. This amounts to showing that Borel functions $F : \hat{X}/G_{i-1} \to \mathbf{R}$ invariant by the strong stable foliation of the quotient flow on $\hat{X}/G_{i-1}$ are invariant by the action of G_i/G_{i-1} ; indeed this implies that F factorises through an invariant function defined on $\hat{X}/G_i$, and is thus constant by the recurrence hypothesis.

The following set is easily seen to be a closed subgroup of $\mathbf{R} \times G/G_{i-1}$:

$\{(t,g) \mid for\ all\ \hat{W}^{ss} invariant\ F\ in\ L^\infty(\hat{X}/G_{i-1}),\ F(\hat{\phi}_{-t}gx) = F(x)\}$
The invariant functions will factorise through $\hat{X}/G_i$ if $\{0\} \times G_i/G_{i-1}$ is included in this set. In order to obtain this inclusion, it will be shown that the lengths and the Frobenius elements $(l(\tau), [\tau])$ of the periodic orbits τ of ϕ_t are included in this set, if $[\tau]$ belongs to G_i.

Lemma 1 *The closed group*
$\{(t,g) \in \mathbf{R} \times G/G_{i-1} \mid \forall F\ \hat{W}^{ss}-invariant\ on\ \hat{X}/G_{i-1},\ F(\hat{\phi}_{-t}gx) = F(x)\}$
contains the elements of the form $(l(\tau), [\tau])$*, where* τ *is any periodic orbit of* ϕ_t *on* X*, whose Frobenius element belongs to the center of* G/G_{i-1}.

This will imply the theorem. Indeed, G_i/G_{i-1} is included in the center of G/G_{i-1}, since G is nilpotent. The hypothesis of Theorem 1 imply that the closed group mentioned in Lemma 1 contains $\{0\} \times G_i/G_{i-1}$. Hence, functions on $\hat{X}$ invariant by $\hat{W}^{ss}$, are lifts of functions on X invariant by W^{ss}. Moreover, if the condition $\overline{<\{l(\tau)\}>} = \mathbf{R}$ is satisfied, then functions on X, which are W^{ss}-invariant, are invariant by the flow: this is the case $G_{i-1} = G$ in Lemma 1. So, these functions are constant, by ergodicity of ϕ_t.

The case of the foliation $\hat{W}^{ws}$ is similar; the $\mathbf{R}$-component of the group mentioned in Lemma 1 can be dropped, since functions invariant by the weak stable foliation are invariant by the flow $\hat{\phi}_t$ itself.

In order to simplify the notations, the space $\hat{X}/G_{i-1}$ will be denoted by $\tilde{X}$ and the quotient flow by $\tilde{\phi}_t$. The central series then gives the sequence:

$$\tilde{X} \to \tilde{X}/(G_i/G_{i-1}) \to \tilde{X}/(G/G_{i-1}) \ = \ X.$$

3.1 Deformation along periodic orbits

Let $v \in X$ be a periodic point of ϕ_t, whose Frobenius element $[v]$ belongs to G_i. Since the group G_i/G_{i-1} is abelian, the image of $[v]$ in this quotient is uniquely defined. Given some point $\tilde{x} \in \tilde{X}$, we explain how to build a point $\tilde{y}$ belonging to $\tilde{W}^{ss}(\tilde{x})$ so that the point $\tilde{\phi}_{l(v)}[v]^{-1}(\tilde{y})$ is arbitrarily close to $\tilde{x}$. This will be enough to insure the ergodicity of $\tilde{W}^{ss}$.

Let $\varepsilon > 0$. By conservativity and ergodicity of ϕ_t, almost every point $\tilde{x} \in \tilde{X}$ has a projection $\pi(\tilde{x})$ in X which has a positively transitive orbit. In particular, the orbit of $\pi(\tilde{x})$ comes arbitrarily close to v. This means that the orbit of $\tilde{x}$ comes close to some preimage $\tilde{v} \in \tilde{X}$ of v. By continuity of the flow, it stays close to the orbit of $\tilde{v}$ during a time equal to, say, twice the period of v. So, for almost all $\tilde{x} \in \tilde{X}$, and any $\varepsilon' > 0$, there exists an

$n \in \mathbf{N}$ and a $\tilde{v} \in \pi^{-1}(\phi_{[0,l(v)]}(v))$ such that $d\Big(\tilde{\phi}_{nl(v)+t}(\tilde{x}), \tilde{\phi}_t(\tilde{v})\Big) < \varepsilon'$, for all positive t less than or equal to $l(v)$. The set of such elements $\tilde{x}$ is denoted by $\tilde{D}_n$; These Borel sets cover almost all of $\tilde{X}$. For the points in $\tilde{D}_n$, we have:

$$d\Big(\tilde{\phi}_{nl(v)}(\tilde{x}), \tilde{v}\Big) < \varepsilon', \qquad d\Big(\tilde{\phi}_{nl(v)+l(v)}(\tilde{x}), \tilde{\phi}_{l(v)}(\tilde{v})\Big) < \varepsilon'.$$

By definition of the Frobenius element, $\tilde{\phi}_{l(v)}(\tilde{v}) = [v](\tilde{v})$. Moreover, $[v]$ acts uniformly on the fiber of $\tilde{v}$; given some $\delta_1 > 0$, one can choose ε' such that $d\Big(\tilde{\phi}_{nl(v)}(\tilde{x}), [v]^{-1}\tilde{\phi}_{l(v)}\tilde{\phi}_{nl(v)}(\tilde{x})\Big) \leq \varepsilon$. The local product structure can now be used to get a real number t close to 0, an element g of G close to identity, and a point $\tilde{z}$ close to $\tilde{v}$ which lies both in $\tilde{W}^{su}_{\varepsilon}(\tilde{\phi}_{nl(v)}(\tilde{x}))$ and in $\tilde{W}^{ss}_{\varepsilon}([v]^{-1}\tilde{\phi}_{l(v)}\tilde{\phi}_{nl(v)}g\tilde{\phi}_t(\tilde{x}))$. One then take $\tilde{y} = [v]\tilde{\phi}_{-l(v)}g^{-1}\tilde{\phi}_{-nl(v)-t}(\tilde{z})$. The point $\tilde{y}$ is in $\tilde{W}^{ss}(\tilde{x})$ and satisfies:

$$[v]^{-1}\tilde{\phi}_{l(v)}g\tilde{\phi}_t(\tilde{y}) \in \tilde{W}^{su}(\tilde{x}), \quad d\Big(g\tilde{\phi}_t[v]^{-1}\phi_{l(v)}(\tilde{y}), \tilde{x}\Big) \leq \varepsilon.$$

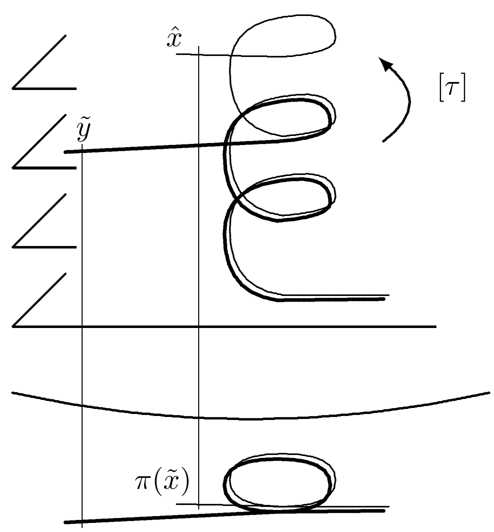


Figure 1: Deformation of the orbit

3.2 Ergodicity

We now suppose that the element $(l(v), [v])$ does not belong to the subgroup $\{(t, g) \in \mathbf{R} \times G/G_{i-1} \mid \forall F \in L^{\infty}(\tilde{X}), F \ invariant, \ \ F(\tilde{\phi}_{-t}gx) = F(x)\}$ and try to get a contradiction.

We first note that we may restrict our attention to functions F satisfying the following continuity assumption :

$\forall\ \eta > 0,\ \exists\ r > 0,\ \forall\ (t,g) \in \mathbf{R} \times G\ with\ d((t,g),(0,id)) < r,$

$$\forall \tilde{x} \in \tilde{X},\ |F(g\tilde{\phi}_t(\tilde{x})) - F(\tilde{x})| < \eta/2.$$

Indeed, any bounded measurable function F may be approximated by such a function, using the formula :

$$F * \theta(\tilde{x}) = \int_{\mathbf{R}\times G} F(h\phi_s\tilde{x})\theta(s,h)dsdh$$

where θ is some continuous positive function on $\mathbf{R}\times G$ with integral one and support in a small neighborhood of $(0, id)$. It is elementary to show that the function $F*\theta$ satisfies the continuity assumption above; one also checks that the invariance of F with respect to the foliation implies the same invariance for $F * \theta$. Now, if the functions of the form $F * \theta$ are invariant by the flow and the group action, the same is true for F.

Lemma 2 *There exists functions θ_n such that the sequence $F*\theta_n$ converges almost everywhere to F.*

Proof. First note that since $\tilde{X}$ can be covered by countably many open sets of finite measure, $\tilde{X}$ may be written as a union of a set with zero measure and of countably many compact sets, each of them admitting a neighborhood of finite measure. Let K be one of these compact sets. Then, we have

$$\int_K |F * \theta - F| \to 0$$

as the support of θ shrinks to $(0, id)$. This is seen by approximating F by a continuous function on a neighborhood of K and using the inequality : $g_*\tilde{\mu} < C\tilde{\mu}$ for g close to the identity. One then chooses a sequence of functions θ with support decreasing to zero, and extracts subsequences.

So, in order to get a contradiction, we suppose that there exists some function $F \in L^\infty(\tilde{X})$ satisfying the above continuity assumption, invariant by the stable foliation, and that there also exists a Borel set B of positive measure, and some constant $\eta > 0$ such that $F(\tilde{\phi}_{-l(v)}[v](\tilde{x})) > F(\tilde{x}) + \eta$, for all $\tilde{x} \in B$. This implies that:

$$F(\tilde{\phi}_{-l(v)}[v](\tilde{x})) > F(\tilde{y}) + \eta, \quad for\ all\ \tilde{x} \in B, \quad for\ all\ \tilde{y} \in \tilde{W}^{ss}(\tilde{x}).$$

Considering $F \circ \phi_{-l(v)}[v]$ instead of F, $\tilde{\phi}_{l(v)}[v]^{-1}(B)$ instead of B, one gets $F(\tilde{x}) > F(\tilde{\phi}_{l(v)}[v]^{-1}\tilde{y}) + \eta$, for all $\tilde{x} \in B$, for all $\tilde{y} \in \tilde{W}^{ss}(\tilde{x})$. Since $\tilde{X}$ can be covered by a countable cover of open sets of finite measure, the measure $\tilde{\mu}$ is regular and B can be taken compact, with a δ-neighborhood of finite measure. Restricting B slightly and applying Lusin's theorem, one can suppose that F is uniformly continuous on B: there is an $\varepsilon > 0$ such that if $\tilde{x}, \tilde{x}' \in B$, $d(\tilde{x}, \tilde{x}') < \varepsilon$, then $|F(\tilde{x}) - F(\tilde{x}')| < \eta/2$.

So if there exists $\tilde{x} \in B$, $\tilde{y} \in \tilde{W}^{ss}(\tilde{x})$ such that $[v]^{-1}\tilde{\phi}_{l(v)} g\tilde{\phi}_t(\tilde{y})$ is in B and at distance less than ε from $\tilde{x}$, one can bound $|F(\tilde{x}) - F([v]^{-1}\tilde{\phi}_{l(v)}(\tilde{y}))|$ by

$$|F(\tilde{x}) - F(g\tilde{\phi}_t[v]^{-1}\tilde{\phi}_{l(v)}(\tilde{y}))| + |F(g\tilde{\phi}_t[v]^{-1}\tilde{\phi}_{l(v)}(\tilde{y})) - F([v]^{-1}\tilde{\phi}_{l(v)}(\tilde{y}))|$$

The first term is bounded by $\eta/2$ by uniform continuity, and the second term is bounded by $\eta/2$ by the continuity assumption on F. This gives a contradiction.

The point $\tilde{y}$ is built by deformation along the periodic orbit, using the ε giving the contradiction in the previous argument. It remains to show that $[v]^{-1}\tilde{\phi}_{l(v)} g\tilde{\phi}_t(\tilde{y})$ can be taken in B. This is the goal of the next section.

3.3 Return times

Recall that the sets $\tilde{D}_k$, $k \in \mathbf{N}$, are the sets of points $\tilde{x} \in \tilde{X}$ which enter a sufficiently small neighborhood of the fiber of $\tilde{v}$ at time $kl(v)$. Note also that there is an $\varepsilon' > 0$ such that each of the sets $\pi^{-1}(\phi_t^{-1}(B(v, \varepsilon')))$, $t > 0$, is contained in some $\tilde{D}_k$. The sets $\tilde{D}_k$ cover almost all of $\tilde{X}$. The point $[v]^{-1}\tilde{\phi}_{l(v)} g\tilde{\phi}_t(\tilde{y})$ associated to $\tilde{x}$ via deformation is denoted by $T_k(\tilde{x})$. It is equal to $\tilde{\phi}_{-kl(v)}(\tilde{z})$, where $\tilde{z}$ is the point near $\tilde{v}$ obtained by using the local product structure. In order to complete the proof of the theorem, it is sufficient to find an $\tilde{x} \in B$ and an integer k such that $\tilde{x}$ is in $\tilde{D}_k$ and $T_k(\tilde{x})$ is in B. This is the goal of the next two lemmas.

Lemma 3

$$\tilde{\mu}(\{\tilde{x} \in B \mid \tilde{x} \in \tilde{D}_k,\ T_k(\tilde{x}) \notin B\}) \to 0 \textit{ as } k \to \infty.$$

Proof. Recall that $V_\delta B$ is the δ-neighborhood of B. Since B is a closed set, the quantities $\tilde{\mu}(V_\delta B \backslash B)$ tend to zero as δ tends to zero. So, given some $\eta > 0$, there exists a $\delta_0 > 0$ such that $\tilde{\mu}(V_{\delta_0} B \backslash B) < \eta/C$. Moreover, $T_k(\tilde{x})$ is on $W^{su}(\tilde{x})$ and the images of these two points get ε-close after time k. Since the unstable foliation is locally contracting uniformly in fibers, there is a k_0 such that for all $k \geq k_0$, and all $\tilde{x} \in \tilde{D}_k$, $d(\tilde{x}, T_k(\tilde{x})) < \delta_0$. The transformation T_k is obtained by composition of $\tilde{\phi}_{kl(v)}$ with the deformation process in the local product neighborhood of some preimage of v. The measure $\tilde{\mu}$ is invariant by the flow and satisfies a uniform bound under the deformation; this gives:

$$\tilde{\mu}(T_k^{-1}(V_{\delta_0}B\backslash B)\ \cap \tilde{D}_k) \leq C\tilde{\mu}(V_{\delta_0}B\backslash B) \leq \eta.$$

Finally the set $\{\tilde{x} \in B \mid \tilde{x} \in \tilde{D}_k,\ T_k(\tilde{x}) \notin B\}$ is contained in the set $T_k^{-1}(V_{\delta_0}B\backslash B) \cap \tilde{D}_k$ if $k \geq k_0$.

Lemma 4

$$\tilde{\mu}(\{\tilde{x} \in B \mid \tilde{x} \in \tilde{D}_k\}) \not\to 0 \text{ } as \text{ } k \to \infty.$$

Proof. If the quantities $\tilde{\mu}(B \cap \pi^{-1}\phi_{-t}B(v,\varepsilon'))$ tend to zero as t tends to infinity, the same is true of the integral:

$$\frac{1}{t}\int_0^t \int_B \mathbf{1}_{B(v,\varepsilon)} \circ \phi_s \circ \pi \ d\tilde{\mu}\ ds.$$

But this integral converges to $\mu(B(v,\varepsilon'))\tilde{\mu}(B)$. This is a direct application of the Fubini's theorem, the L^2-ergodic theorem and the Lebesgue dominated convergence theorem. This proves the lemma.

4 Hyperbolic systems

In this section, the main result is applied to hyperbolic flows and diffeomorphisms.

Definition 1
Let ϕ_t be a C^1 flow on a C^∞ Riemannian manifold M and Λ a compact set invariant by ϕ_t. The flow is said to be hyperbolic on Λ if Λ contains no fixed points of the flow, periodic points are dense in Λ, there exists an orbit dense in Λ, and there is an open set U containing Λ such that $\cap\phi_t U = \Lambda$, the intersection being on all $t \in \mathbf{R}$ and, moreover, if there exist positive constants C, λ, such that for all $x \in \Lambda$ there is a decomposition of the tangent bundle $T_xM = E^0_x \oplus E^{ss}_x \oplus E^{su}_x$ satisfying:
- *E^0 is one-dimensional tangent to the flow;*
- *$D\phi_t E^{ss}_x = E^{ss}_x$, $D\phi_t E^{su}_x = E^{su}_x$;*
- *$\forall t > 0$, $||D\phi_{t|E^{ss}_x}|| \leq Ce^{-\lambda t}$ and $||D\phi_{-t|E^{su}_x}|| \leq Ce^{-\lambda t}$.*

There is a similar definition for diffeomorphisms; the E^0 part has to be removed in the previous statement; see, e.g. [Shu87].

Transitive Anosov flows correspond to flows hyperbolic on the whole manifold. Axiom A flows are a specific class of flows which are hyperbolic on the components of the non-wandering set.

Hyperbolic flows admit many invariant ergodic probability measures. Given some Hölder continuous function f on Λ, there is a unique measure maximising the quantity $h_\mu + \int f d\mu$ among the set of invariant Borel probability measures μ supported on Λ, where h_μ is the entropy of μ with respect to ϕ_1 [BR75]. This measure is called the *Gibbs measure* associated to the potential f. We will check below that it satisfies the conditions of Theorem 1.

Let ϕ_t a hyperbolic flow on a compact set $\Lambda \subset M$, $\hat{M}$ a G-fiber bundle over M and $\hat{\phi}_t$ an extension of ϕ_t on $\hat{M}$ commuting with the G-action. In local trivialisations $U \times G$, $\hat{\phi}_t$ takes the form $(x, g) \to (\phi_t(x), A_t(x)g)$, where A_t is an map from U to G. We say that $\hat{\phi}_t$ is a *Hölder extension* of ϕ_t if there is a covering of Λ with trivialisations charts such that the maps $A_t(x)$ are Hölder continuous in x for small t. Note that when G is discrete, there exists a unique extension, which corresponds to $A_t \equiv 0$. In the case of diffeomorphisms, an extension is said to be Hölder if it is of the form $(x, g) \to (\phi(x), f(x)g)$ in local charts $U \times G$, with $f : U \to G$ Hölder.

Theorem 2
Let ϕ_t (resp. ϕ) an hyperbolic flow (resp. diffeomorphism) on some compact subset Λ of a manifold M, G a locally compact second countable group carrying a bi-invariant distance, $\hat{M}$ a Hölder principal G-fiber bundle over M, $\hat{\phi}_t$ (resp. $\hat{\phi}$) a Hölder extension of ϕ_t (resp. ϕ) to $\hat{M}$ commuting with G, μ a Gibbs measure on Λ. The manifold $\hat{M}$ is endowed with a measure $\hat{\mu}$ which coincides with $\mu \times Haar$ on local trivialisations of the bundle. Suppose that G is a nilpotent group admitting a central series composed of closed subgroups $\{1\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ such that:

$$\overline{< \{[\tau] \mid \tau \textit{ periodic orbit on } \Lambda \textit{ with } [\tau] \in G_i\} >} = G_i/G_{i-1},$$

then the weak stable foliation $\hat{W}^{ws}$ is ergodic with respect to $\hat{\mu}$. If moreover:

$$\{0\} \times G_i/G_{i-1} \subset \overline{< \{(l(\tau), [\tau]) \mid \tau \textit{ periodic orbit on } \Lambda \textit{ with } [\tau] \in G_i\} >},$$

$$\overline{< \{l(\tau) \mid \tau \textit{ periodic orbit on } \Lambda\} >} = \mathbf{R} \; (\textit{resp. } \mathbf{Z}),$$

then the strong stable foliation $\hat{W}^{ss}$ is ergodic with respect to $\hat{\mu}$.

Remark
There are similar statements for $\hat{W}^{su}$, $\hat{W}^{ws}$, $\hat{W}^{wu}$, which follow from Theorem 1. Note that the condition $\overline{<\{l(\tau)\}>} = \mathbf{R}$ (resp. $\mathbf{Z}$) is equivalent to the flow (resp. the diffeomorphism) being topologically mixing on Λ. It is always satisfied for hyperbolic diffeomorphisms if Λ is connected; on the other hand, it fails for flows which are constant suspension of diffeomorphisms.

In order to prove this result, we have to show that the hypothesis of Theorem 1 are satisfied for these systems; this is checked in three steps.

The case $G = \{0\}$
In the context of Axiom A flows, local product structures were introduced by S. Smale [Sm67]. The next result is taken from R. Bowen [Bo72].

Proposition 1
Let ϕ_t be a hyperbolic flow on some compact Λ ; there are δ_1, δ_2 for which the following is true: if $x, y \in \Lambda$ and $d(x,y) \leq \delta_1$, then there is a unique $t = t(x,y)$ with $|t| \leq \delta_2$ so that

$$W^{ss}_{\delta_2}(\phi_t(x)) \cap W^{su}_{\delta_2}(y) =< x, y > .$$

This set consists of a single point, which is denoted by $< x, y >$. The maps t and $<,>$ are continuous on the set $\{(x,y) \mid d(x,y) \leq \delta_1\} \subset X \times X$.

A rectangle is a compact set, equal to the closure of its interior, such that the local product of two points in the rectangle is again a rectangle. Each point of Λ has a basis of neighborhoods composed of rectangles. This gives the local product structure we need.

In the case of diffeomorphisms, there is a similar result; in fact, spaces admitting maps with local product structures are called Smale spaces; their study is developed in [Ru78], where it is shown that they are essentially equivalent to subshifts of finite type.

Now a Gibbs measure μ can be written as a product of a family μ^{su} of transverse measures to the strong unstable foliation by a family of measures μ^s supported by the strong unstable leaves; moreover the modulus of absolute continuity of these families with respect to the holonomy of the foliations is explicitly known:

$$\frac{d\zeta_* \mu^{su}_A}{d\mu^{su}_B}(y) = e^{[\zeta^{-1} y, y]}$$

where A, B are any pairs of transversals to W^{su}, ζ any one-to-one measurable map such that $\zeta(x) \in W^{su}(x)$ for almost all $x \in A$, and $[x, y]$ is defined by

$$[x, y] = \lim_{t \to -\infty} \int_0^t f(\phi_s x) ds \ - \ \int_0^t f(\phi_s y) ds, \text{ for all } x \in W^{su}(y),$$

where f is a Hölder function on M called the potential of the measure μ. Note that this limit is converging because f is Hölder and ϕ_{-t} is exponentially contracting. The bracket $[x, y]$ is thus bounded by a constant depending on the Hölder constants of f, the contraction exponent of the flow, and the diameter of Λ, but not on x, y. There are similar estimates for the modulus of absolute continuity of μ^s with respect to the action of ϕ_l and the holonomy of the weak stable foliation. We refer to [BL96], lemma 1, for explicit formulas. The point is that these modulus are uniformly bounded on Λ.

The estimate needed on the transformation T, which sends the point x to $< \phi_l(x), x >$, follows from these properties. Given some point x in a rectangle, T sends the local weak stable leaf $W^{ws}_{\varepsilon}(x)$ to $W^{ws}(\phi_l(x))$; in restriction to this leaf, it coincides with the holonomy map ζ between the two transversals $W^{ws}_{\varepsilon}(x)$ and $W^{ws}(\phi_l(x))$. Hence, T has a bounded modulus of absolute continuity on $W^{ws}_{\varepsilon}(x)$ with respect to μ^{su}, and this modulus does not depend on x. On the other hand, the fibers $W^{su}(x)$ are sent to themselves by T; the image of a point y in $W^{su}(x)$ is obtained first by applying ϕ_l from $W^{su}(x)$ to $W^{su}(\phi_l(x))$, then by following the weak stable leaves to return to $W^{su}(x)$. The modulus of absolute continuity is again bounded independently of x, so we are done.

The case G discrete

Principal bundles with discrete structure group correspond to regular covers. The data on X lift to $\hat{X}$: given some point v, one considers a rectangle contained in some neighborhood U trivializing the covering; the inverse image of this neighborhood is of the form $U \times G$ and the local product structure is defined by $< (x, g), (y, g) >= (< x, y >, g)$. The properties of the measure $\tilde{\mu}$ then follows from the one of μ, since $\tilde{\mu}$ is equal to μ on each level of the cover.

The general case

In order to pull back the local product structure from Λ to $\pi^{-1}\Lambda$, we show how to find for any two close points x, y on Λ such that $x \in W^{ss}(y)$, an element $g(x, y)$ with $(x, g(x, y)) \in \hat{W}^{ss}((y, id))$.

The compact Λ is first covered with a finite number of flow boxes trivializing the fiber bundle. There is an ϵ such that for any point x in Λ, $\phi_{[-\varepsilon,\varepsilon]}(B(x,\varepsilon))$ is contained in one of the previous flow boxes. The orbit of some point $\hat{x}$ in the fiber of x can be described by using a succession of such sets: the point first travels during time ε through the first trivialisation; one then changes from trivialisation to travel through another during time ε, and so on. The expression of $\hat{\phi}_{n\epsilon}(x,g)$ takes the form $(\phi_{n\varepsilon}(x), \prod_{k=0}^{n} g_k(\phi_{k\varepsilon}(x))A_\varepsilon^k(\phi_{k\varepsilon}(x)\ g)$, where the g_k correspond to the change of trivialisation at time $k\varepsilon$ and the A^k refer to the expression of the flow in that trivialisation. Note that the $g_k{}'s$ and the $A^k{}'s$ are commuting with each others.

If y belongs to $W_\varepsilon^{ss}(x)$, the displacement in the fibers between the orbits of $\hat{\phi}_{n\varepsilon}(x,id)$ and $\hat{\phi}_{n\varepsilon}(y,id)$ is equal to

$$\prod_{k=0}^{n} g_k(\phi_{k\varepsilon}(y))A_\varepsilon^k(\phi_{k\varepsilon}(y))\Big(\prod_{k=0}^{n} g_k(\phi_{k\varepsilon}(x))A_\varepsilon^k(\phi_{k\varepsilon}(x))\Big)^{-1}$$

The bi-invariance of the distance on G implies the inequality $d(g_1g_2, g_3g_4) \leq d(g_1,g_3) + d(g_2,g_4)$. Since there are a finite number of flow boxes, the $g_k{}'s$ and the $A^k{}'s$ are Hölder continuous, and the flow is exponentially contracting, the series $\sum d(g_k(\phi_{k\varepsilon}(x))A_\varepsilon^k(\phi_{k\varepsilon}(x)), g_k(\phi_{k\varepsilon}(y))A_\varepsilon^k(\phi_{k\varepsilon}(y)))$ is uniformly convergent, and bounded by a constant times $d(x,y)$. This shows that the translation in the fiber converges to a continuous function $g(x,y)$. This function tends to *zero* as $d(x,y)$ tends to zero. There is a similar construction for the strong unstable foliation which gives a function $g'(x,y)$ so that $(x,g'(x,y)) \in \hat{W}^{su}((y,id))$.

The local product structure can now be defined by the following formula: $<(x,g_1),(y,g_2)>=(<x,y>, g(<x,y>,y)'g_2)$. This point belongs to $\hat{W}^{su}((y,g_2))$; it also belongs to $\hat{W}^{ss}(\tilde{g}\hat{\phi}_t(x,g_1))$, where $\tilde{g}$ is equal to $\tilde{g} = A_t^{-1}g(<x,y>,\phi_t(x))g(<x,y>,y)'g_2g_1^{-1}$, and t given by the local product structure on Λ. Note that the element $\tilde{g}$ is close to the identity, as needed. Moreover, the local product structure acts as a translation in each fiber; since the measure $\hat{\mu}$ coincides with the Haar measure in the fiber, the estimate needed for the measure $\hat{\mu}$ follows from the one for μ and the Fubini's theorem.

5 Subshifts of finite type

Shifts on subshifts of finite type are essentially the same as hyperbolic diffeomorphisms. On the one hand, hyperbolic diffeomorphisms can be coded by subshifts (see, e.g. [Shu87]), whereas any subshift can be realised as the non-wandering set of some hyperbolic system, the shift being conjugated to the diffeomorphism. They both stand in the category of Smale spaces [Ru78], for which Theorem 1 applies.

A subshift of finite type is given by a finite alphabet I with n elements and a $n \times n$ transition matrix A with coefficients in $\{0, 1\}$. The space of sequences of the subshift is defined by

$$\Sigma_A := \{\ \{x_k\}_{k \in \mathbf{Z}} \in I^{\mathbf{Z}} \mid \forall\ k \in \mathbf{Z},\ A_{x_k x_{k+1}} = 1\}$$

and the shift is the homeomorphism on Σ_A given by $\sigma(\{x_i\})_i = x_{i+1}$. The distance between two words x_i, y_i is equal to

$$d(\{x_i\}, \{y_i\}) = exp(-min\{|i|\ with\ x_i \neq y_i\}).$$

The local product structure is defined on words with same zero coordinate as: $< (\ldots x_{-1}, x_0, x_1 \ldots), (\ldots y_{-1}, y_0, y_1 \ldots) >= (\ldots y_{-2}, y_{-1}, y_0, x_1, x_2, \ldots)$.

The notion of Gibbs measure is defined in the same way as for hyperbolic diffeomorphisms: given a Hölder function f on Σ_A, there is a unique invariant Borel probability measure maximising the quantity $h_\mu + \int f d\mu$, where h_μ denotes the entropy of the measure with respect to the shift. This measure is ergodic, of full support.

Repeating the arguments of the previous section, one gets the following result; V. Kaimanovich and K. Schmidt announced a similar statement under slightly different hypothesis, in [KS01]:

Theorem 3
Let (Σ_A, σ) a transitive subshift of finite type, μ a Gibbs measure on Σ_A, G a locally compact second countable group admitting a bi-invariant metric, $f : \Sigma_A \to G$ a Hölder continuous function depending only on positive coordinates, and T the transformation on $\Sigma_A \times G$ which sends (x, g) to $(\sigma x, f(x)g)$. Suppose that G is a nilpotent group admitting a central series composed of closed subgroups $\{1\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ such that:

$$\overline{< \{[\tau] \mid \tau\ periodic\ point\ of\ \sigma\ with\ [\tau] \in G_i\} >} = G_i/G_{i-1},$$

then the weak stable foliation of T is ergodic with respect to $\mu \times Haar$. If moreover the following is satisfied:

$$\{0\} \times G_i/G_{i-1} \subset \overline{< \{(l(\tau), [\tau]) \mid \tau\ periodic\ point\ of\ \sigma\ with\ [\tau] \in G_i\} >},$$

$$< \{l(\tau) \mid \tau\ periodic\ point\ of\ \sigma\} >= \mathbf{Z},$$

then the strong stable foliation of T is ergodic with respect to $\mu \times Haar$.

Remarks

• The condition $< \{l(\tau)\} >= \mathbf{Z}$ is equivalent to the shift being topologically mixing.

• The ergodicity of the weak (resp. strong) stable foliation implies the equality $\overline{< \{[\tau]\} >} = G$ (resp. $\overline{< \{(l(\tau), [\tau])\} >} = \mathbf{Z} \times G$); this is a consequence of the Livsic theorem [Li72]; the argument can be found in [C01]. In particular, the conditions on the periodic orbits are necessary if G is abelian.

• If the set $\{[\tau]\}$ is invariant by the map $g \to g^{-1}$ and the subshift is a Bernoulli shift, which means that $A_{i,j} = 1, \ \forall i, j$, then the conditions in term of periodic orbits break down to $\overline{< \{[\tau]\} >} = G$ for the weak stable foliation; indeed a product of Frobenius elements of periodic orbits is the Frobenius element of a single orbit, obtained by concatenating the words associated to each orbit, hence commutators of Frobenius elements are again Frobenius elements.

• Since we have made the assumption that f depends only on positive coordinates, the stable foliation may be described as follows: one writes $f^m(x)$ for the product $f(\sigma^{m-1}x)..f(x)$; then two points $(\{x_i\}, u)$, $(\{y_i\}, v)$ are on the same weak (resp. strong) stable leaf if and only if there are two positive integers m, n (resp. $m = n$) with $x_{i+m} = y_{i+n}$, $\forall i \geq 0$, and $f^m(x)u = f^n(y)v$.

Examples

Let $f : \{0,1\}^{\mathbf{Z}} \to \mathbf{Z}$ be the map which takes value 1 on the set $\{x \mid x_0 = 0\}$ and -1 on its complementary set. Consider the system (σ, f) on $\{0,1\}^{\mathbf{Z}} \times \mathbf{Z}$; its weak stable foliation is ergodic, whereas its strong stable foliation is not. Indeed, $< [\tau] >= \mathbf{Z}$ but $l(\tau) + [\tau]$ is even, so that $< \{(l(\tau), [\tau])\} > \neq \mathbf{Z} \times \mathbf{Z}$.

Moreover let α be an irrational real number, $r : \{0,1\}^{\mathbf{Z}} \to \mathbf{R}$ the map which takes value 1 on the set $\{x \mid x_0 = 0\}$ and α on its complementary set. The space $\{0,1\}^{\mathbf{Z}} \times \mathbf{R}$ can be quotiented by the action $(x, t) \to (\sigma x, t - r(x))$ to obtain a suspension over the subshift $\{0,1\}^{\mathbf{Z}}$. The $\mathbf{R}$-action on the suspension can be conjugated to a hyperbolic flow on some compact set homeomorphic to the suspension space.

One gets an extension of that system by quotienting $\{0,1\}^{\mathbf{Z}} \times \mathbf{R} \times \mathbf{Z}$ by $(x, t, n) \to (\sigma x, t - r(x), n + f(x))$. Note that ergodicity of the stable foliation of the $\mathbf{R}$-action on the whole space $\{0,1\}^{\mathbf{Z}} \times \mathbf{R} \times \mathbf{Z}$ is the same as ergodicity on the quotient, since two points with the same image in the quotient are on the same stable leaf. We now have:

$$\overline{< \{(l(\tau), [\tau])\} >} = \mathbf{Z}(1,1) \oplus \mathbf{Z}(\alpha, -1), \quad \overline{< \{l(\tau)\} >} = \mathbf{R}, \quad \{[\tau]\} = \mathbf{Z}.$$

This gives an example of a mixing hyperbolic flow ϕ_t with extension $\hat{\phi}_t$ satisfying $\{[\tau]\} = G$, for which the strong stable foliation is not ergodic.

6 Geodesic flow on negatively curved manifolds

Let H be a connected simply connected Riemannian manifold with sectional curvature bounded between two negative constants. Such a manifold is homeomorphic to a ball and can be compactified by adding a sphere at infinity, which is denoted by ∂H; see, e.g. [Eb72]. This boundary may be defined as equivalence classes of asymptotic rays in H. Given two different points on ∂H, there is exactly one geodesic going from one point to the other. The boundary can be used to get a parametrisation of the unitary tangent bundle S^1H of H, called the Hopf coordinates, from $(\partial H \times \partial H - \Delta) \times \mathbf{R}$ to S^1H, where Δ denotes the diagonal $\{(\xi, \eta) \mid \xi = \eta\}$. It sends the point (ξ, η, t) to the unitary vector tangent to the oriented geodesic from ξ to η, and whose projection on H is at distance t from an origin on the geodesic; one can take as origin the closest point on the geodesic to a fixed point $x_0 \in H$, for example.

The geodesic flow on S^1H translates vectors along the geodesic they generate. In the Hopf coordinates, this is just the translation on the **R**-factor. The stable foliations of the flow may be described using the Hopf coordinates:

$$W^{ws}(\xi, \eta, t) = \{(\xi', \eta', t') \mid \eta' = \eta\}, W^{ss}(\xi, \eta, t) = \{(\xi', \eta', t') \mid \eta' = \eta,\ t' = t\}.$$

Any connected complete Riemannian manifold M whose sectional curvatures lie between two constants can be written as H/Γ, where H is a manifold as above, and Γ is a subgroup of the isometry group of H, which can be identified with the fundamental group $\pi_1(M)$ of M. The action of Γ extends naturally to the boundary ∂H. The elements of Γ have one or two fixed points on ∂H; in the first case they are called parabolic, in the second case hyperbolic. An hyperbolic element h fixes the geodesic joining its two fixed points. This geodesic projects on a closed geodesic in M whose Frobenius element is equal to h, up to conjugacy.

The limit set $\Lambda\Gamma$ of Γ is the set of accumulation points in ∂H of the orbit under Γ of some point in H. It does not depend on the point chosen. The group Γ is said to be elementary if this set has finitely many elements. This is equivalent to Γ being virtually nilpotent.

The concept of Gibbs measure can be generalised to this setting [C02], following a construction due to S. J. Patterson. These measures are supported by the non-wandering set of the flow. A finite Gibbs measure is ergodic; geometrically finite manifolds always admit finite Gibbs measures.

We will only use some mild properties of the measure in the Hopf coordinates, which are explained below, so that we do not give details of the construction.

Theorem 4
Let M be a connected complete Riemannian manifold M whose sectional curvatures lie between two negative constants, and with non-elementary fundamental group. Let μ be a finite Gibbs measure on S^1M, and $\hat{M}$ a connected regular cover of M with nilpotent deck transformation group. Then the weak stable foliation $\hat{W}^{ws}$ on $S^1\hat{M}$ is ergodic with respect to $\mu \times Haar$.
Moreover, if the lengths of the periodic geodesics on $\hat{M}$ generate a dense subgroup of $\mathbf{R}$, then the horosphere foliation $\hat{W}^{ss}$ is ergodic with respect to $\mu \times Haar$.

Remarks
• Bowen-Margulis measure and harmonic measure are examples of Gibbs measures; they may be finite measures, depending on the geometry of M.
• The topological mixing of the flow $\hat{\phi}_t$ on its non-wandering set in $\hat{M}$ is equivalent to the condition $\overline{< \{l(\tau) \mid [\tau] = 0\} >} = \mathbf{R}$, and to the transitivity of $\hat{W}^{ss}$ on $\hat{M}$ [Da01]. It is known, for example, if there is a cusp on $\hat{M}$, if M is of finite volume, or if the curvature is constant [Da99]. It is conjectured to be always satisfied.
• In constant curvature, the theorem can also be stated in term of group actions. The horospheres are the orbits of a unipotent subgroup of the isometry group of the manifold.
• the result can be generalised to CAT(-1) spaces. Finally, it is true if the deck transformation group is virtually nilpotent; one just applies the result with M replaced by the finite extension over which $\hat{M}$ is a nilpotent cover.
• In [Ha01], U. Hamenstädt studied the ergodicity of $\hat{W}^{ws}$ on $\hat{M}$, when the manifold M is compact ; her approach is quite different from the one described here, and makes use of leaf-wise Laplacians, harmonic measures and diffusion processes. These methods seem difficult to generalise to the non-compact case. One problem comes from the fact that the harmonic measure may be infinite, non-ergodic, even on a finite volume manifold.

Proof. The product structure may be defined globally on the universal cover H of the manifold M, and then pushed down on the cover. The product of two points (ξ_1, η_1, t_1) and (ξ_2, η_2, t_2), in Hopf coordinates, is given by the formula:

$$< (\xi_1, \eta_1, t_1), (\xi_2, \eta_2, t_2) >= (\xi_2, \eta_1, t_2)$$

and belongs to $\tilde{W}^{ss}(\tilde{\phi}_{t_2-t_1}(\xi_1, \eta_1, t_1)) \cap \tilde{W}^{su}(\xi_2, \eta_2, t_2)$ as needed. This formula commutes with the action of the isometry group, so it is uniform in the fibers of the cover.

A Gibbs measure takes the following form in Hopf coordinates on H:

$$d\mu(\xi, \eta, t) = e^{\delta \beta_{x_0}(\xi,\eta)} d\mu_{x_0}(\xi) d\mu'_{x_0}(\eta) dt$$

where x_0 is some point in H chosen as origin. The function β is bounded below and above on compact sets, by constants depending only on the distance from x_0 to the geodesic with endpoints ξ and η; whereas the modulus of absolute continuity of μ_{x_0} with respect to some element γ of the isometry group of H is known, and bounded on ∂H by a constant depending only on the distance from γx_0 to x_0. We refer to [C02] for an explicit formula, which is not needed here.

Let $\tilde{v}$ be a periodic point on H and $[\tilde{v}]$ the hyperbolic element which fixes the geodesic starting from $\tilde{v}$ and for which $\phi_{l(\tilde{v})}(\tilde{v}) = [\tilde{v}]\tilde{v}$. The transformation T on H which associates the point $< [\tilde{v}]^{-1}\phi_{l(\tilde{v})}(x), x >$ to x is given in the Hopf coordinates by the simple formula: $(\xi, \eta, t) \to (\xi, [\tilde{v}]^{-1}\eta, t)$. Taking x_0 as the base point of the vector v, we see that the image of μ under T is bounded by a constant times μ on balls centered on x_0, the constant depending only on the radius of the ball and on $\tilde{v}$. Moreover the constant is the same for $\tilde{v}$ and its images under the isometry group of H. This gives the estimate on the measure needed to apply Theorem 1.

It remains to check the condition on the periodic orbits.

Lemma 5
Let Γ be a non-elementary subgroup of the group of isometry of a connected Riemannian manifold with sectional curvatures lying between two negative constants. Then Γ is generated by its hyperbolic elements.

Proof. One has to show that parabolic elements are products of hyperbolic elements. Let p be a parabolic element and p_∞ its fixed point in ∂H. The action of p on $\Lambda\Gamma - \{p_\infty\}$ is free and properly discontinuous. So there is an open set $U \subset \Lambda\Gamma$, whose images under p are disjoint. Let V be an open ball whose closure is contained in U. The ends of periodic geodesics in M are dense in $\Lambda\Gamma \times \Lambda\Gamma$ [Eb72]. So, there is a hyperbolic element in Γ whose fixed points are contained in V.
Taking some power of that element, we have found an hyperbolic element h and two disjoint compact neighborhoods $V_1, V_2 \subset V$ of the fixed points of h, which satisfy $h(V_1^c) \subset V_2$. The group $< p, h >$ generated by p and h is called

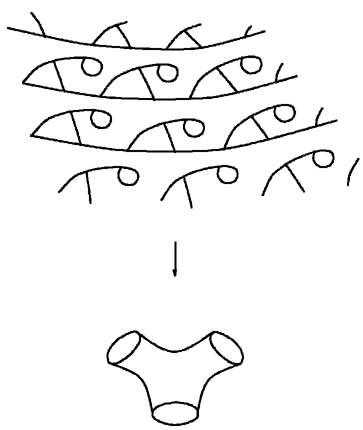
an extended Schottky group by F. Dal'bo and M. Peigné [DP98]. They show that it is a free group with generators p and h. In particular the abelianized of $< p, h >$ is equal to $\mathbf{Z}p \oplus \mathbf{Z}h$. It is moreover a geometrically finite group with one cusp, which implies that all parabolic elements are conjugated to a power of p [Bow95]. So, the projection of a parabolic element in the abelianized group ends in the first factor $\mathbf{Z}p$. We deduce that the product of p by h is hyperbolic; this ends the proof of the lemma.

We return to the proof of the theorem. There is a periodic orbit associated to each hyperbolic element in $\pi_1(M)$; the projections of hyperbolic elements of $\pi_1(M)$ in the deck transformation group of the cover can thus be realised as Frobenius elements of periodic geodesics. So, the lemma gives the equality $< \{[\tau] \mid [\tau] \in G_i\} >= G_i/G_{i-1}$, for any central series associated to the nilpotent group. If the flow is topologically mixing on the cover, we also have $\overline{< \{l(\tau) \mid [\tau] = 0\} >} = \mathbf{R}$. In particular, $\mathbf{R} \times \{0\}$ belongs to the set $\overline{< \{(l(\tau), [\tau]) \mid [\tau] \in G_i\} >}$ for all i, hence $\overline{< \{(l(\tau), [\tau]) \mid [\tau] \in G_i\} >} = \mathbf{R} \times G_i/G_{i-1}$. This ends the proof.

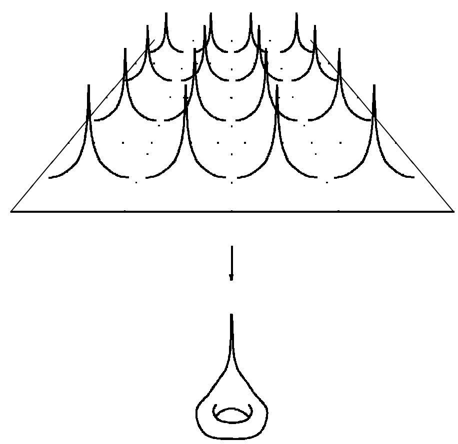
Corollary 1
Let M be a connected finite volume manifold whose sectional curvatures lie between two negative constants, and with first partial derivatives of the curvatures bounded on M. Let $\hat{M}$ be a connected nilpotent cover of M. Then the strong stable foliation on $\hat{M}$ is ergodic with respect to the Liouville measure.

Remark
On a compact manifold, the absolute continuity of the Liouville measure with respect to the foliations is due to D. V. Anosov [An67]. The potential $f(x) = -d/dt_{|t=0}(\ln|det(D_x\phi_{t_{|T_xW^{su}}})|)$, associated to the Liouville measure, is Hölder continuous, so that it is a Gibbs measure. The proof extends straightforwardly to the finite volume case if the first derivatives of the curvatures are bounded; this remark has been made by several authors, for example M. Brin [Br99].

Examples: homology covers.



(a) Thick part of the thrice punctured sphere



(b) Punctured torus

The corollary may be applied to homology covers. The thick part of the thrice punctured sphere is on the left of the picture; the once punctured torus is on the right. These are finite volume manifolds and their horospheric foliations are ergodic on the cover with respect to the Liouville measure.

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