Generic measures for hyperbolic flows on non-compact spaces

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Abstract

We consider the geodesic flow on a complete connected negatively curved manifold. We show that the set of invariant Borel probability measures contains a dense G_δ-subset consisting of ergodic measures fully supported on the non-wandering set. We also treat the case of non-positively curved manifolds and provide general tools to deal with hyperbolic systems defined on non-compact spaces. [1]

1   Introduction

In the context of hyperbolic dynamics, Axiom A flows admit many ergodic invariant probability measures supported on each connected component X of the non-wandering set : the measure of maximal entropy and the SRB measure are two examples which provide many informations on the asymptotic behaviour of trajectories. K. Sigmund [Si72] showed even more: the ergodic probability measures with full support on X form a G_δ dense subset of the set $\mathcal{M}^1(X)$ of all probability invariant measures.

Our goal is to extend these results to hyperbolic systems defined on non-compact spaces. We build our study on two remarkable properties of hyperbolic systems : the existence of a product structure and the closing lemma. Neither of these two properties relies on the compactness of the ambient space. Yet they are sufficient to obtain a general density theorem, which shows that ergodic probability measures of full support are indeed abundant.

The most famous example of hyperbolic system with noncompact phase space is given by the geodesic flow on a complete connected negatively curved manifold. Our general density theorem implies the following result :

Theorem 1.1 *Let M be a negatively curved, connected, complete riemannian manifold. We assume that the non wandering set Ω of the geodesic flow contains more than two periodic orbits. Then the G_δ subset of ergodic measures fully supported on Ω is dense in the set of all probability Borel measures on T^1M invariant by the geodesic flow.*

As the construction ultimately relies on the Baire category theorem, the measures obtained are not explicit. This result was previously known for geodesic flows on

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[1] 37B10, 37D40, 34C28

geometrically finite negatively curved manifolds. In that setting, there is an explicit construction of an ergodic invariant probability Gibbs measure [C03].

On the other hand, the theorem does not hold in the context of non-positive curvature. We give a counterexample, and proceed to show that ergodic probability measures of full support are dense, if we assume that the non-wandering set is everything and there is a hyperbolic closed geodesic.

Other examples of hyperbolic systems on non-compact manifolds are obtained by lifting an Anosov or Axiom A flow to some non-compact cover of the phase space. Our density result applies verbatim and gives the density of ergodic measures of full support, if we restrict the system to a connected component of the non-wandering set on the cover.

The set of all Borel probability measures invariant by a dynamical system will be denoted by $\mathcal{M}^1(X)$ in the sequel. It is endowed with the weak topology: a sequence μ_n of probability measures converges to some measure μ if $\int f d\mu_n \to \int f d\mu$ for all bounded continuous f. We refer to the classical treatise of Billingsley [Bi99] for a detailed study of that topology. In particular, we will use the fact that we may restrict ourselves to bounded *Lipschitz* functions in the definition of weak convergence.

In the first two sections, using first the closing lemma, and then the local product structure of a hyperbolic flow, we prove the density of the Dirac measures supported on periodic orbits in the set of all invariant probability measures. In section 4 we prove theorem 1.1. We then extend this result to rank one manifolds, and to hyperbolic systems on covers. We end by an appendix containing a proof of the closing lemma for rank one vectors on non-positively curved manifolds.

2   The closing lemma and ergodic measures

The geodesic flow on the unit tangent bundle of a negatively curved manifold satisfies a property known as a closing lemma. Roughly speaking, a piece of orbit that comes back close to its initial point is in fact close to a periodic orbit. Here is a more formal statement :

Definition 2.1 *A flow ϕ_t on a metric space X satisfies the* closing lemma *if for all points $v \in X$, there exists a neighborhood V of v which satisfies :*
for all $\varepsilon > 0$, there exists a $\delta > 0$ and a $t_0 > 0$ such that for all $x \in V$ and all $t > t_0$ with $d(x, \phi_t x) < \delta$ and $\phi_t x \in V$, there exists x_0 and $l > 0$, with $|l - t| < \varepsilon$, $\phi_l x_0 = x_0$, and $d(\phi_s x_0, \phi_s x) < \varepsilon$ for $0 < s < \min(t, l)$.

Note that for such flow, the non-wandering set and the closure of the recurrent points coincide with the closure of the set of periodic points.

The first closing lemma was proven by Hadamard in 1898, for negatively curved surfaces embedded in ambient space. D. V. Anosov generalized the closing lemma to the systems that now bear his name [An67] ; for this reason, it is often called the Anosov closing lemma. A proof of the closing lemma valid for rank one manifolds and higher rank locally symmetric spaces can be found in [Eb96]. We recall its argument at the end of the paper.

Lemma 2.2 *Let X be a metric space, ϕ_t a flow on X which satisfies the closing lemma. Then the set of Dirac probability measures supported on periodic orbits of the flow is dense in the set of all invariant ergodic Borel probability measures μ on X.*

Proof : Let f_i be finitely many bounded Lipschitz functions on X, with Lipschitz constants bounded by K. From the Poincaré recurrence theorem and the Birkhoff ergodic theorem, we know that for almost all $x \in X$, x is recurrent and the Birkhoff averages of f_i evaluated at x converges to $\int f_i \, d\mu$. Fix $\varepsilon > 0$ and let t_0, δ be given by the closing lemma. Choose $t > 0$ large enough so that $t \geq t_0$, $d(\phi_t x, x) < \delta$ and for all i, $\left| \frac{1}{t} \int_0^t f_i \circ \phi_s(x) ds - \int_X f_i \, d\mu \right| \leq \varepsilon$. Let x_0 be the periodic point of period l provided by the closing lemma. We obtain

$$
\begin{aligned}
\left| \int f_i \, d\mu - \frac{1}{l} \int_0^l f_i(\phi_s x_0) \, ds \right| &\leq \left| \frac{1}{t} \int_0^t f_i(\phi_s(x)) \, ds - \frac{1}{l} \int_0^l f_i(\phi_s x_0) \, ds \right| + \varepsilon \\
&\leq \frac{1}{t} \int_0^t K \, d(\phi_s x, \phi_s x_0) \, ds + 2|t - l| \, ||f_i||_\infty + \varepsilon \\
&\leq (K + 2||f_i||_\infty + 1) \, \varepsilon
\end{aligned}
$$

This shows that the Dirac measure on that periodic orbit is close to the measure μ.
□

If the space X is *Borel standard*, that is, a Borel subset of a separable complete metric space, the decomposition theorem of Choquet implies that the convex set generated by the invariant ergodic Borel probability measures is dense in the set of all invariant probability measures. We have proven:

Corollary 2.3 *Assume moreover that X is Borel standard. Then the convex subset generated by the normalized Dirac measures on periodic orbits is dense in the set of all invariant Borel probability measures $\mathcal{M}^1(X)$.*

There is another path that leads to the previous result: first prove a Livsic type theorem [Li71]; that is, all functions whose integrals on periodic orbit vanish are coboundaries. For these functions, deduce that the integrals with respect to all invariant probability measures is zero; this requires additional work since the function given by the coboundary equation cannot be expected to be integrable when the underlying space is not compact. Then end the argument by applying a Hahn-Banach theorem adapted to the weak topology on the space of measures.

3 Product structure and Dirac measures

Our next goal is to show that Dirac measures on periodic orbits are in fact dense in the set of all invariant measures. This will require an additional assumption. Let X be a metric space, ϕ_t a continuous flow on X. The strong stable sets of the flow are defined by :
$W^{ss}(x) := \{ \, y \in X \mid \lim_{t \to \infty} d(\phi_t(x), \phi_t(y)) = 0 \, \}$;
$W_\varepsilon^{ss}(x) := \{ \, y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \leq \varepsilon \text{ for all } t \geq 0 \, \}$.
One also defines the strong unstable sets W^{su} and W_ε^{su} of ϕ_t ; these are the stable sets of ϕ_{-t}.

Definition 3.1 *The flow ϕ_t is said to admit a* local product structure *if all points $v \in X$ have a neighborhood V which satisfies : for all $\varepsilon > 0$, there exists a positive constant δ, such that for all $x, y \in V$ with $d(x, y) \leq \delta$, there is a point $< x, y > \in X$, a real number t with $|t| \leq \varepsilon$, so that:*

$$
< x, y > \in \ W_\varepsilon^{su}\big(\phi_t(x)\big) \cap W_\varepsilon^{ss}(y).
$$

This property is summarized by the following picture :
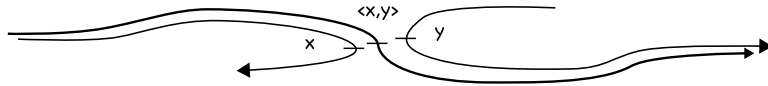
This assumption allows us to "glue" orbits together. Note that we can glue an arbitrary finite number of pieces of orbits, just by using the product structure recursively. In the next picture, we first glue the trajectory of x_1 with the trajectory of x_2. The resulting dotted trajectory is then glued with x_3. We end up with an orbit which follows first the orbit of x_1, then the orbit of x_2 and finally the orbit of x_3.
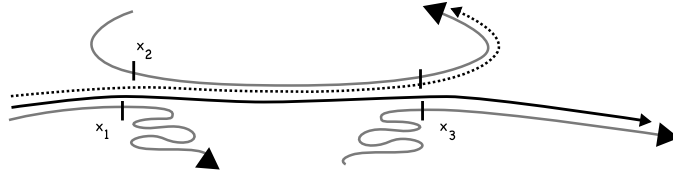
Geodesic flows on negatively curved manifolds admit a local product structure. In fact, there is even a global product structure on the unit tangent bundle of the universal cover of the manifold.

We are now ready to prove :

Proposition 3.2 *Let X be a metric space, ϕ_t a transitive flow on X admitting a local product structure and satisfying the closing lemma. Then the set of normalized Dirac measures on single periodic orbits is dense in its convex closure.*

From this proposition and corollary 2.3, we deduce that the set of normalized Dirac measures on periodic orbits is dense in the set of all invariant probability measures $\mathcal{M}^1(X)$.

Proof : We show that any convex combination of Dirac measures on periodic orbits can be approached by a Dirac measure on a single periodic orbit. So, let x_1, x_3,...,x_{2n-1} be periodic points with period l_1, l_3,...,l_{2n-1}, and c_1,c_3,...,c_{2n-1} positive real numbers with $\Sigma c_{2i+1} = 1$. Let us denote the Dirac measure on a periodic orbit of some point x by δ_x. We want to find a periodic point x such that δ_x is close to the sum $\Sigma c_{2i+1} \delta_{x_i}$. The numbers c_{2i+1} may be assumed to be rational numbers of the form p_{2i+1}/q.

By transitivity, for each integer i less than n, we can find a point x_{2i} close to x_{2i-1} whose trajectory becomes close to x_{2i+1}, say, after time t_{2i}. We can also find a point x_{2n2} close to x_{2n-1} whose trajectory becomes close to x_1 after some time.
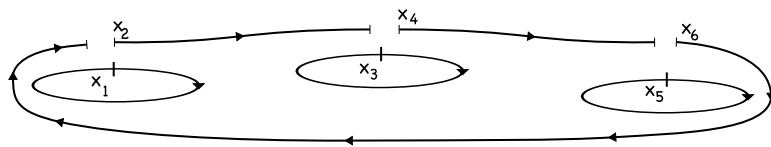

Now these trajectories are glued together. We fix an integer N, that will be chosen big at the end. First glue the piece of orbit starting from x_1, of length Nl_1p_1, together with the orbit of x_2, of length t_2. The resulting orbit ends in a neighborhood of x_3, and that neighborhood does not depend on the value of N. This orbit is glued with the trajectory starting from x_3, of length Nl_2p_2, and so on. The reader

may want to write down the sequence of ε obtained through the successive gluings. This is done in [C04], where a similar argument was used.

We end up with a trajectory starting close to x_1, doing N turns around the first periodic orbit, then following the trajectory of x_2 until it reaches x_3 ; then it turns N times around the second periodic orbit, and so on, until it reaches x_{2n} and goes back to x_1.
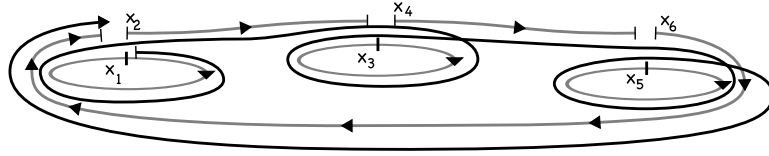
For sake of clarity, the trajectory only makes one turn around the periodic orbits on the pictures. Finally, we use the closing lemma to obtain the periodic orbit we are looking for.
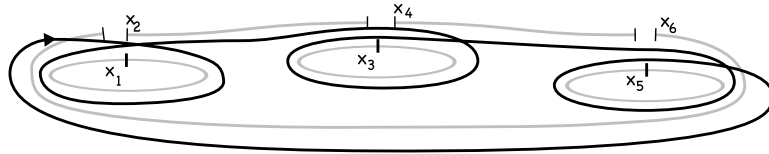
Integrating a function on that closed orbit gives a result close to what we would have obtained if we integrate first Np_1 times around the first closed orbit, then on the piece of orbit leading to x_3, followed by the second periodic orbit, Np_2 times, and so on. This amounts to integrating with respect to the measure:

$$\frac{\Sigma\, N\, p_{2i+1}\, \delta_{x_{2i+1}}\ +\ \Sigma\, \delta_{x_{2i}}}{\Sigma\, N\, p_{2i+1}\ +\ \Sigma\, t_{2i}}$$

If N is big enough, the contribution of the pieces of orbits associated to the x_{2i} is small, and this measure is close to $\Sigma\, \frac{p_{2i+1}}{q}\, \delta_{x_{2i}}$; so we are done $\qquad\square$

Remark:
Under the hypothesis of the previous proposition, transitivity is equivalent to the connectedness of X together with the density of the periodic orbits on X [C04].

4   Ergodic measures with full support

We first prove that there are always probability measures with full support, as soon as there are enough periodic orbits. Recall that a *polish space* is a topological space homeomorphic to a separable complete metric space. Locally compact separable metric spaces are polish and the set of Borel probability measures on a polish space is again a polish space with respect to the weak topology. Hence, the Baire property holds in that context.

Lemma 4.1 *Let X be a polish space, ϕ_t a flow defined on X. We assume that the periodic orbits of the flow form a dense subset in X. Then the set of Borel probability measures of full support is a dense G_δ subset of the set of all invariant Borel probability measures.*

Proof : Let $\{U_i\}_{i\in\mathbf{N}}$ be a countable basis of open sets for the topology of X. Let F_i be the closed set of all probability measures which are zero on U_i. The complement of the union of all the F_i is exactly the set of Borel probability measures with full support. Let us show that the F_i have empty interiors. Take $\nu \in F_i$ and x_0 a periodic point in U_i; the Dirac measure on the orbit of x_0 is denoted by δ_{x_0}. Then the sequence $(1 - \frac{1}{n})\nu + \frac{1}{n}\delta_{x_0}$ is not in F_i and converges to ν. The Baire theorem can now be applied and the lemma is proven. $\qquad\square$

We can now prove a genericity result concerning ergodic invariant measures with full support.

Theorem 4.2 *Let X be a polish space, ϕ_t a transitive continuous flow on X admitting a local product structure and satisfying the closing lemma. Then the set of invariant ergodic Borel probability measures with full support is a dense G_δ subset of the set of all invariant Borel probability measures.*

Theorem 1.1 is a particular case of this result.

Proof : As noted above, the periodic orbits are dense in X. From the previous lemma, we know that the set of probability measures with full support is a dense G_δ subset of the set of all invariant Borel probability measures $\mathcal{M}^1(X)$. Proposition 3.2 shows that the set of ergodic invariant probability measures is also dense in $\mathcal{M}^1(X)$. It remains to check that this set is a G_δ set, since the intersection of two dense G_δ sets is again a dense G_δ set.

Let $\{f_i\}_{i\in\mathbf{N}}$ be some countable algebra of Lipschitz bounded functions which separate points on X. This set is dense in $L^1(m)$, for all Borel probability measure m ; this fact goes back to Rohlin; we refer to [C02] for a short proof. We can know write the complement of the set of ergodic probability measures as the union of the following closed sets:

$$F_{k,l,i} = \{m \in \mathcal{M}^1(X) \mid \exists \mu, \nu \in \mathcal{M}^1(X), \alpha \in \left[\tfrac{1}{k}, 1 - \tfrac{1}{k}\right], \; s.t. \; m = \alpha\mu + (1-\alpha)\nu$$
$$and \; \int f_i d\mu \geq \int f_i d\nu + \tfrac{1}{l} \}$$

The condition $\int f_i d\mu \geq \int f_i d\nu + \frac{1}{l}$ insures that the two measures μ and ν are not equal. This is a closed condition. Now let m_n be a sequence of measures that can be written as barycenters $\alpha_n \mu_n + (1 - \alpha_n)\nu_n$. If the sequence m_n is converging in $\mathcal{M}^1(X)$, then it is tight. Up to a subsequence, we can assume that μ_n and ν_n converge to probability measures μ and ν of mass ≤ 1, and α_n converges to $\alpha \in [\frac{1}{k}, 1 - \frac{1}{k}]$. The relation $m = \alpha\mu + (1-\alpha)\nu$ implies that the limit m of m_n is a barycenter, and $F_{k,l,i}$ is closed. This ends the proof. $\qquad\square$

5 Non-positively curved manifolds

We have just proven that a geodesic flow on a negatively curved manifold always admits an ergodic invariant measure fully supported on the non-wandering set of the flow. We now give generalisations of this result to non-positively curved manifolds.

5.1 Closing lemma and local product

Let M be a riemannian manifold and v a vector belonging to the unit tangent bundle T^1M of M. The vector v is a *rank one vector*, if the only parallel Jacobi fields along the geodesic generated by v are proportional to the generator of the geodesic flow. A connected complete non-positively curved manifold is said to be

a *rank one manifold* if its tangent bundle admits a rank one vector. In that case, the set of rank one vectors is an open subset of T^1M. Rank one vectors generating closed geodesics are precisely the hyperbolic periodic points of the geodesic flow.

Negatively curved manifolds are rank one manifolds. There are also examples of rank one manifolds for which the sectional curvatures vanish on an open subset of the manifold. The concept of rank one manifold was introduced by W. Ballmann, M. Brin, P. Eberlein [BBE85]; we refer to the survey of G. Knieper [Kn02] for an overview of the properties of rank one manifolds.

The goal is now to find a big subset Ω_1 of the non wandering set Ω, so that the geodesic flow ϕ_t has a local product structure and satisfies the closing lemma in restriction to that subset.

Let us denote by $T^1\tilde{M}$ the unitary bundle of the universal cover of M and $\tilde{\phi}_t$ the lift of the geodesic flow to $T^1\tilde{M}$. Let v be a non-wandering rank one vector in T^1M and $\tilde{v}$ a lift of v to $T^1\tilde{M}$. The vector v belongs to the set Ω_1^+ if it satisfies :

For all $\tilde{w} \in T^1\tilde{M}$ such that the distance $d(\tilde{\phi}_t(\tilde{w}), \tilde{\phi}_t(\tilde{v}))$ stays bounded for $t \geq 0$, there is a $t_0 \in \mathbf{R}$ such that $\tilde{\phi}_{t_0}(\tilde{w}) \in W^{ss}(\tilde{v})$.

In other words, any non-wandering half-geodesic whose lift stays at a bounded distance from the lift of a geodesic in Ω_1^+ is in fact asymptotic to that geodesic. Typical examples of vectors which are not in Ω_1^+ are vectors whose associated geodesic is asymptotic to a closed geodesic belonging to a flat half-cylinder.

We define a set Ω_1^- in the same way, just by replacing the flow $\tilde{\phi}_t$ by its inverse $\tilde{\phi}_{-t}$ in the previous definition. We now take $\Omega_1 = \Omega_1^+ \cap \Omega_1^-$. This is the set of non-wandering rank one vectors that satisfy the above property both for negative and positive time.

Proposition 5.1 *Let M be a rank one manifold. Then the restriction of the geodesic flow to Ω_1 admits a local product structure and satisfies the closing lemma.*

Proof : The geodesic flow, in restriction to the set of non-wandering rank one vectors, satisfies the closing lemma. This is proven by P. Eberlein in [Eb96], prop 4.5.15, in the general setting of non-positive curvature. We recall the proof in an appendix at the end of the paper, since the rank one hypothesis allows for some simplification.

We also know that recurrent rank one vectors belong to Ω_1 [Kn02], prop. 4.4. Hence, rank one periodic orbits are in Ω_1. Now, if we apply the closing lemma to a non-wandering rank one vector, the closed orbit obtained is close to the rank one vector. Hence, as the set of rank one vectors is open, it is a closed rank one orbit and belongs to Ω_1. So the closing lemma is satisfied in restriction to Ω_1.

Let us consider two vectors v, w in T^1M which are close to each other. Then there is a point w' close to v and w, such that $d(\phi_t(w), \phi_t(w'))$ is bounded for negative times, and $d(\phi_t(v), \phi_t(w'))$ is bounded for positive times [Kn02], lemma 1.3. So, if v and w belong to Ω_1, then w' is a rank one vector whose associated geodesic is positively asymptotic trajectory of v, and negatively asymptotic to the trajectory of w. This shows that w' is in Ω_1 and that there is a local product structure in restriction to Ω_1. $\qquad\square$

Proposition 5.2 *Let M be a rank one manifold. We assume that there are more than two rank one periodic geodesics on T^1M. Then the restriction of the geodesic flow to Ω_1 is transitive.*

Proof : Let U_1 and U_2 be two open sets containing points in Ω_1. We show that there is a trajectory in Ω_1 that starts from U_1 and ends in U_2. This will prove transitivity on Ω_1. From the previous proposition, we have seen that rank one periodic vectors are dense in Ω_1. So there exists rank one periodic vectors $v_1 \in U_1$ and $v_2 \in U_2$. Let us assume that v_2 is not opposite to v_1 or an iterate of v_1 ; $-v_2 \notin \cup_{t \in \mathbf{R}} g_t(\{v_1\})$. Then there is a rank one vector v_3 whose trajectory is negatively asymptotic to the trajectory of v_1 and positively asymptotic to the trajectory of v_2, cf [Kn02] lemma 1.4 ff.

Let us show that v_3 is non-wandering. First note that there is also a trajectory negatively asymptotic to v_2 and positively asymptotic to the trajectory of v_1. That is, the two periodic orbits v_1, v_2 are heteroclinically related.
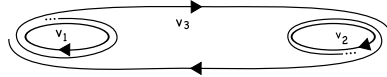
This implies that the two connecting orbits are non-wandering : indeed, using the local product structure, we can glue the two connecting orbits to obtain a trajectory that starts close to v_3, follows the second connecting orbit, and then follows the orbit of v_3, coming back to the vector v_3 itself. Hence v_3 is in Ω. Since it is asymptotic to v_2, it belongs to Ω_1 and we are done.

If v_1 and v_2 generate opposite trajectories, then we take a third rank one periodic vector w and connect first v_1 to w then w to v_2. Using again the product structure, we can glue the connecting orbits to create a non-wandering trajectory from U_1 to U_2. $\qquad\square$

We want to apply Theorem 4.2 in restriction to Ω_1. It is not at all obvious that Ω_1 is a polish space. However, it contains a polish space, which is of full measure with respect to all measures in $\mathcal{M}^1(\Omega_1)$. Indeed, by the Poincaré recurrence theorem, the finite measures on the separable metric space Ω_1 are supported by the set of recurrent rank one vectors. Now, the set of recurrent trajectories of a continuous flow on a polish space is a G_δ-subset of the ambient space, hence a polish space. This shows that $\mathcal{M}(\Omega_1)$ is polish and that the Choquet theorem holds on this space. The proofs given above apply verbatim in that setting and we get :

Corollary 5.3 *Let M be a non-positively curved connected complete riemannian manifold. We assume that the geodesic flow on T^1M has more than two hyperbolic periodic orbits. Then there exists an ergodic finite Borel measure on T^1M, invariant with respect to the geodesic flow, whose support contains all hyperbolic periodic orbits.*

5.2 Rank one manifolds

There are examples of rank one manifolds for which Ω_1 is not dense in the non-wandering set Ω.
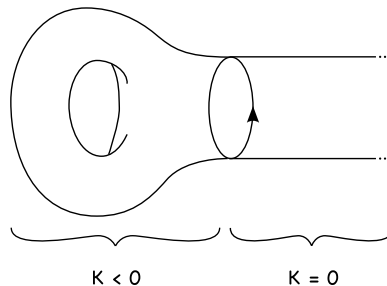
For example, let us consider a non-positively curved surface, with a geodesic γ bounding an half euclidean cylinder. This corresponds to the right part on the picture above. The remainder of the surface is compact, negatively curved. This is depicted on the left hand side of the picture. There are three types of geodesics on the surface :

– Closed geodesics in the cylinder part, which are parallel to γ.

– Geodesics which cross γ. They are associated to rank one wandering vectors.

– Geodesics that stay in the negatively curved part of the surface. They are associated to rank one vectors.

The non-wandering set Ω extends arbitrarily far on the right whereas the set Ω_1 is contained in the left part of the picture. The only ergodic invariant probability measures giving positive measure to the cylinder part are the Dirac measures on the periodic orbits. This gives a rank one example with no ergodic invariant finite measure supported by Ω.

There is however a simple condition that insures both the density of Ω_1 in Ω and the transitivity on Ω_1: if $\Omega = T^1M$, then the rank one vectors form an open dense subset of T^1M; moreover we have seen that the geodesic flow is transitive; this was first proven by Eberlein, see e.g. [Eb96], prop. 4.7.3 and 4.7.4. This implies the existence of a rank one vector with dense orbit. As mentioned earlier, such a vector belongs to Ω_1; we refer to G. Knieper, [Kn02] prop 4.4. We have established:

Theorem 5.4 *Let M be a rank one manifold. We assume that the non-wandering set of the geodesic flow coincides with T^1M. Then there exists an ergodic Borel probability measure on T^1M, which is invariant by the geodesic flow, and of full support.*

6   Additional examples

Theorem 4.2 can be applied to other examples:

● *Irreducible Markov chains defined on a countable alphabet.*

We recover the density of Dirac measures on periodic points, in the set of all probability invariant measures, a result first proven by Oxtoby [0x63] in the case of a finite alphabet.

● *Skew-products over Markov chains.*

The fiber has to be discrete, so that both the product structure and the closing lemma lift to the skew-product. The transitivity on the skew-product was studied, for example, in [C04]. When the fiber is a non-amenable group, the lift of a probability measure on the basis cannot be ergodic; this result is due to Zimmer [Z78]. So the ergodic measures on the skew-product must be singular with respect to the action of the group extension.

● *Lift of hyperbolic systems to coverings of the base space.*

For these systems, lifts of Gibbs measures are known to be ergodic only if the deck tranformation group is virtually $\mathbf{Z}$ or $\mathbf{Z}^2$ [G91]. From this viewpoint, even the existence of an infinite ergodic measure of full support is surprising. Since these systems have received much attention recently, we rephrase Theorem 4.2 in that context.

Theorem 6.1 *Let X be a compact manifold, $\hat{X}$ a covering of X and $\hat{\phi}_t$ a transitive flow on $\hat{X}$ which is the lift of an Anosov flow defined on X. Then the set of invariant ergodic Borel probability measures with full support in $\hat{X}$ is a dense G_δ-set in the set of all invariant Borel probability measures.*

An analogous result holds for Axiom A systems, in restriction to preimages of basic pieces.

Finally, we remark that our results are again true if we look at transformations instead of flows. The proofs apply verbatim.

7 Appendix : the rank one closing lemma

We give a proof of the closing lemma for the geodesic flow on a non-positive complete manifold, in restriction to the rank one non-wandering vectors. There are two motivations for providing a proof: first the rank one hypothesis allows for some simplification, as compared with the classical proof given by P. Eberlein [Eb96]. Second, in the context of non-positive curvature, it seems that it is still unknown whether the closing lemma holds for all non-wandering vectors.

Theorem 7.1 *Let $\tilde{M}$ be a Hadamard rank one manifold, and $M = \tilde{M}/\Gamma$, with Γ a discrete subgroup of $Isom(\tilde{M})$. Let $C \subset T^1 M$ be a compact set of rank one vectors and $\varepsilon > 0$. Then there exists $\delta > 0$ and $T > 0$ with the following property:*
for all $t \geq T$ and $v \in C$ such that $D(g^t v, v) < \delta$, there exists $t' \in]t - \varepsilon, t + \varepsilon[$ and v' a rank one vector such that $g^{t'} v' = v'$ and $D(g^s v, g^s v') \leq \varepsilon$ for all $0 \leq s \leq \min(t, t')$.

Here, we denote by D the natural distance on $T^1 M$ (and $T^1 \tilde{M}$) induced by the riemannian metric on M.

Proof : Assume by contradiction that there exists a compact set $C \subset T^1 M$ of rank one vectors, $\varepsilon > 0$, and a sequence $(v_n)_{n \in \mathbf{N}}$ of vectors of C, together with a sequence $t_n \to \infty$, such that $D(v_n, g^{t_n} v_n) \to 0$, without periodic orbits w_n of period ω_n ε-close to t_n staying in their ε-neighbourhood during the time t_n. By compactness, we can assume that v_n converges to some $v \in C$, as n goes to infinity.

Lift the situation to $\tilde{M}$. There exists a compact set $\tilde{C} \subset T^1 \tilde{M}$, $\varepsilon > 0$, a sequence $(\tilde{v}_n)_{n \in \mathbf{N}} \in (\tilde{C})^{\mathbf{N}}$ of rank one vectors converging to $\tilde{v}$, $t_n \geq n$ and $\varphi_n \in \Gamma$, such that $D(\tilde{v}_n, d\varphi_n \circ g^{t_n} \tilde{v}_n) \leq \frac{1}{n}$.

Moreover, there exists no vector $\tilde{w}_n$ such that $D(\tilde{w}_n, \tilde{v}_n) \leq \varepsilon/2$ and $d\varphi_n \circ g^{\omega_n} \tilde{w}_n = \tilde{w}_n$ for some $\omega_n \in]t_n - \varepsilon, t_n + \varepsilon[$. Here is a key geometric point: the fact that on $T^1 M$, the orbits $g^s w_n$ and $g^s v_n$ are ε-close during the time t_n is expressed by the fact that the isometry φ_n which sends $g^{\omega_n} \tilde{w}_n$ on $\tilde{w}_n$ is the same which sends $g^{t_n} \tilde{v}_n$ close to $\tilde{v}_n$.

Indeed, as $\omega_n = t_n \pm \varepsilon$ and φ_n is an isometry, the distance between $g^{\omega_n} \tilde{w}_n$ and $g^{t_n} \tilde{v}_n$ is very close from the distance between $\tilde{w}_n$ and $\tilde{v}_n$, which is very small. By convexity of the riemannian distance on the Hadamard manifold $\tilde{M}$, we deduce that $\gamma_{\tilde{v}_n}(s)$ and $\gamma_{\tilde{w}_n}(s)$ stay close one another for $0 \leq s \leq \min(t_n, s_n)$. It also implies that $D(g^s \tilde{v}_n, g^s \tilde{w}_n)$ is small.

Now, the idea of the proof is to show that for n big enough, φ_n is an axial isometry, and to find on its axis a periodic vector $\tilde{w}_n$ converging to v, in contradiction with the above assumption.

Denote by $\gamma_{\tilde{v}}$ the geodesic determined by $\tilde{v}$. Let p (resp. p_n, q_n) be the base point of $\tilde{v}$ (resp. $\tilde{v}_n$, $g^{t_n} \tilde{v}_n$). As $\varphi_n^{-1}(p_n)$ is close to $\gamma_{\tilde{v}_n}(t_n)$, and $p_n \to p$, $v_n \to v$, it is easy to check that $\varphi_n^{-1}(p) \to \gamma_{\tilde{v}}(+\infty)$. Similarly, one gets $\varphi_n(p) \to \gamma_{\tilde{v}}(-\infty)$. We need the following lemma.

Lemma 7.2 *[Ba95, lemma 3.1] Let $\tilde{M}$ be a Hadamard rank one manifold, $\tilde{v} \in T^1 \tilde{M}$ a regular vector, $x = \gamma_{\tilde{v}}(+\infty)$ and $y = \gamma_{\tilde{v}}(-\infty)$. For all $\eta > 0$, there exist neighbourhoods $V_\eta(x)$ and $V_\eta(y)$ of x and y respectively, such that for all $x' \in V_\eta(x)$, $y' \in V_\eta(y)$, there exists a geodesic γ' joining y to x such that $d(\gamma_{\tilde{v}}(0), \gamma') \leq \eta$.*

This lemma is easy to prove on negatively curved manifolds with curvature bounded by above by $-a^2 < 0$. A similar result is also true [Eb96] for regular vectors on higher rank symmetric spaces. If $\tilde{v}$ is a singular vector on a rank one manifold, this lemma is probably almost always false. However it does not mean that a closing lemma could not be proven in another way for such vectors.

Let us now finish the proof of the closing lemma. Let $V_k(x)$ and $V_k(y)$ be the neighbourhoods given by the above lemma for $\eta = \frac{1}{k}$. We can also assume that they are homeomorphic to balls of $\mathbf{R}^{d-1}$ and their closures are homeomorphic to closed balls of $\mathbf{R}^{d-1}$ (with $d = \dim \tilde{M}$). (Endowed with the cone topology, the boundary $\partial \tilde{M}$ is homeomorphic to the $d-1$-dimensional sphere, [Eb-O73].)

Using the fact that $\varphi_n(p) \to y$ and $\varphi_n^{-1}(p) \to x$ one proves easily that for n big enough we have $\varphi_n^{-1}(\overline{V_k(x)}) \subset V_k(x)$ and $\varphi_n(\overline{V_k(y)}) \subset V_k(y)$. By the Brouwer fixed point theorem, we deduce that φ_n has two fixed points $x_n \in V_k(x)$ and $y_n \in V_k(y)$. Consider a geodesic joining y_n to x_n given by the above lemma. It is invariant by φ_n, which acts by translation on it. Let w_n be the vector of this geodesic minimizing the distance to $\tilde{v}$, and p'_n be its basepoint. This vector is close to $\tilde{v}$, hence close to $\tilde{v}_n$, if n has been chosen big enough.

Denote by $\omega_n = d(p'_n, \varphi_n(p'_n))$ the period associated to w_n. Since p'_n is close to p_n, ω_n is close to $d(p_n, \varphi_n(p_n))$, and so close to t_n for n big enough. Thus, we get the desired contradiction. $\qquad\square$

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