A short proof of the unique ergodicity of horocyclic flows

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Abstract

We give a short dynamical proof of the unique ergodicity of the horocyclic flow associated to an Anosov flow with one dimensional orientable strong stable distribution. This proof extends to the partially hyperbolic setting. [1]

The unique ergodicity of the horocyclic flow on a compact surface of constant negative curvature was proven by H. Furstenberg in [Fu73]. The proof was based on the study of the linear action of $PSL_2(\mathbf{R})$ on $\mathbf{R}^2$, and used techniques from harmonic analysis. This result was then generalised in different directions by B. Marcus [Ma75], R. Bowen [BoMa77], W. Veech [Ve77], M. Ratner [Ra92] and others.

We propose a proof of the unique ergodicity of the horocyclic flow shorter than the previous one, along the lines of the dynamical proof of B. Marcus. It holds in the context of Anosov flows with one dimensional orientable strong stable distribution. Our proof does not make use of any contraction estimate along the weak unstable distribution. Hence we obtain a result valid in the partially hyperbolic setting (Compare [EP78]).

Definition
Let X be a compact metric space, g_t a continuous flow on X. Given $\varepsilon > 0$ and $x \in X$, the *local weak unstable distribution* of g_t is defined by :

$$W_\varepsilon^{wu}(x) = \{y \in X \mid \forall\ t \geq 0,\ d(g_{-t}(x), g_{-t}(y)) < \varepsilon\}$$

Let h_s be a continuous flow on X; a Borel probability measure μ invariant under h_s, is said to be *absolutely continuous with respect to* W^{wu} if the following conditions are satisfied :

– For all $x_0 \in X$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that :
for all $y \in W_\varepsilon^{wu}(x_0) \cap B(x_0, \delta)$ and $s \in]-1, 1[$, the intersection $W_\varepsilon^{wu}(h_s(x_0)) \cap h_{]-2,3[}(y)$ consists in a single point x. The map $(y, s) \mapsto x$, which is defined from $\big(W_\varepsilon^{wu}(x_0) \cap B(x_0, \delta)\big) \times]-1, 1[$ into X, is a homeomorphism onto a neighbourhood of x_0.

– In these coordinates, the measure μ, after renormalisation, is of the form $d\nu_s(y) \otimes ds$, with ν_s probability measures, which depend measurably on s.

The first condition says that W^{wu} is transverse to the flow h_s; the second condition says that the Lebesgue measure on h_s-orbits is invariant under the holonomy given by W^{wu}. This is a compatibility condition between μ and the parameterisation h_s.

[1] 37B10, 37D40, 34C28

Theorem
Let X be a compact metric space, g_t and h_s two continuous flows on X which satisfy the relation : $g_t \circ h_s = h_{se^{-t}} \circ g_t$. Let μ a Borel probability measure invariant under both flows, which is absolutely continuous with respect to W^{wu}, and with full support. Finally assume that the flow h_s admits a dense orbit. Then h_s is uniquely ergodic.

Remarks
• These assumptions are satisfied if X is a compact negatively curved surface, g_t is the geodesic flow on the unit tangent bundle, h_s is the horocyclic flow with the Margulis parametrisation, and μ is the Bowen-Margulis measure.
• More generally, it holds if g_t is a topologically mixing Anosov flow with one-dimensional strong stable distribution, h_s is a well chosen parameterisation of that distribution and μ is the measure of maximal entropy [Ma75].
• The simplest example of a partially hyperbolic flow satisfying the hypothesis of the theorem is given by the suspension of a linear automorphism of the torus with a single eigenvalue of modulus strictly smaller than one, and with eigenvalues on the unit circle. Other examples are given by the action of an Anosov flow on the frame bundle of the manifold.
• The unique ergodicity of the flow h_s implies the ergodicity of h_s with respect to μ. This in turn implies the mixing of g_t with respect to μ, hence the ergodicity of g_t with respect to μ. This does not follow from the classical Hopf argument [Ho39][Ho71], since we didn't ask for contraction along W^{wu}.

Proof
Let f be a continuous function defined on X. The Birkhoff sums of f with respect to the flow h_s are denoted by $S_t(f)$. The relation between h_s and g_t gives :

$$\frac{1}{t}S_t(f)(x) = \frac{1}{t}\int_0^t f(h_s(x))\,ds = \int_0^1 f(h_{st}(x))\,ds = \int_0^1 f(g_{-\ln(t)} \circ h_s \circ g_{\ln(t)}(x))\,ds$$

We write: $M_t(f)(x) = \int_0^1 f\big(g_{-\ln(t)}(h_s(x))\big)ds$, so that $\frac{1}{t}S_t(f)(x) = M_t(f)\big(g_{\ln(t)}(x)\big)$.

Lemma *The family $\{M_t(f)(x)\}_{t\in\mathbf{R}_+}$ is equicontinuous.*

Proof of the lemma
The modulus of uniform continuity of f is denoted by $\omega_f(\varepsilon)$:

$$\omega_f(\varepsilon) = sup\{|f(x) - f(y)| \mid x, y \in X \text{ with } d(x,y) < \varepsilon\}$$

We fix $x_0 \in X$, $\varepsilon > 0$ and consider x close to x_0.
In the coordinates system associated to x_0, x can be written as $(y(x), s(x))$; we define $V_x = B(y(x),\delta)\times]s(h_{-1}(x)), s(h_1(x))[$ in these coordinates. We see that $\mathbf{1}_{V_x}(z)$ converges to $\mathbf{1}_{V_{x_0}}(z)$ pointwise for $z \in (\partial V_{x_0})^c$, when x tends to x_0. This convergence also holds in the L^2 topology, if V_{x_0} has been chosen such that $\mu(\partial V_{x_0}) = 0$.

We now work in the coordinates system associated to x. In these coordinates, V_x can be written as $K_x\times]-1,1[$, with $K_x = W^{wu}_\varepsilon(x) \cap h_{]-2,3[}\big(B(y(x),\delta)\cap W^{wu}_\varepsilon(x_0)\big)$, and the measure $\mu/\mu(V_x)$ can be decomposed as $d\nu_s \otimes\ ds$.

$$\begin{aligned}
&|M_t(f)(x) - \tfrac{1}{\mu(V_x)}\textstyle\int_{V_x} f(g_{-\ln(t)}(z))d\mu(z)| \\
&\qquad\leq\ |\textstyle\int_0^1 f\big(g_{-\ln(t)}(h_s(x))\big)ds - \int_0^1\int_{K_x} f\circ g_{-\ln(t)}(y,s)d\nu_s(y)ds\ | \\
&\qquad\leq\ \textstyle\int_0^1\int_{K_x} |f\circ g_{-\ln(t)}(0,s) - f\circ g_{-\ln(t)}(y,s)|\ d\nu_s(y)\ ds
\end{aligned}$$

The points $(s,0)$ and (s,y) are on the same local weak unstable leaf $W^{wu}_\varepsilon(h_s(x))$. Since for all $z \in X$, $g_{-t}\big(W^{wu}_\varepsilon(z)\big) \subset W^{wu}_\varepsilon(g_{-t}(z))$, we see that the quantity $|f\circ g_{-\ln(t)}(0,s) - f\circ g_{-\ln(t)}(y,s)|$ is bounded by $\omega_f(\varepsilon)$.

$$\begin{aligned}|M_t(f)(x_0) - M_t(f)(x)| &\le 2\omega_f(\varepsilon) + |\int_{V_{x_0}} f(g_{-\ln(t)}(z))\frac{d\mu(z)}{\mu(V_{x_0})} - \int_{V_x} f(g_{-\ln(t)}(z))\frac{d\mu(z)}{\mu(V_x)}| \\ &\le 2\omega_f(\varepsilon) + ||f||_2 \, ||\frac{1}{\mu(V_{x_0})}\, \mathbf{1}_{V_{x_0}} - \frac{1}{\mu(V_x)}\, \mathbf{1}_{V_x}||_2\end{aligned}$$

This ends the proof of the lemma.

The Ascoli theorem now asserts that the family $\{M_t(f)(x)\}_{t\in\mathbf{R}_+}$ has a compact closure with respect to the uniform topology. Let us denote by $\bar{f}$ one of its accumulation points. If we manage to show that $\bar{f}$ is constant, then the family $\{M_t(f)(x)\}$ will converge uniformly to that constant. Thus, $\frac{1}{t}S_t(f)$ will also converge uniformly to a constant; this fact implies the unique ergodicity of h_s.

Let $t_k \to \infty$ and $\bar{f}$ a continuous function such that : $||\ M_{t_k}(f) - \bar{f}\ ||_\infty \longrightarrow 0$. This convergence also holds in L^2-norm, so the quantity $||\ \frac{1}{t_k}S_{t_k}(f)(x) - \bar{f}\circ g_{\ln(t_k)}\ ||_2$ goes to 0 with k. Let us apply the Von Neumann ergodic theorem to h_s: there is an h_s-invariant L^2-function Pf such that $\quad ||\ \frac{1}{t}S_t(f) - Pf\ ||_2 \longrightarrow 0$.

From these two facts, and the g_t-invariance of μ, we get:

$$||\ \bar{f} - Pf\circ g_{-\ln(t_k)}\ ||_2 = ||\ \bar{f}\circ g_{\ln(t_k)} - Pf\ ||_2 \longrightarrow 0 \quad \text{with } k.$$

We know that Pf is h_s-invariant almost everywhere. From the commutation relation between h_s and g_t, we see that $Pf\circ g_t$ is h_s-invariant, for all s, t. The function $\bar{f}$, as an L^2 limit of h_s-invariant functions, is also h_s-invariant almost everywhere: $\bar{f}\circ h_s = \bar{f}\ \mu - a.e.$ So, $\bar{f}$ is an h_s-invariant continuous function, and the flow h_s has a dense orbit. This implies that $\bar{f}$ is constant.

The non-compact case

Can we drop the compactness assumption in the previous result ? We cannot expect the flow to be uniquely ergodic in that case; still, we can show that μ is the unique invariant probability measure amongst the ergodic invariant measures ν that satisfy:

$$\nu\big(\ \{x \in X \mid \omega(x) = \phi\}\ \big) = 0.$$

The set of accumulation points of $g_t(x)$, $t \ge 0$, has been denoted by $\omega(x)$. In other words, ergodic h_s-invariant probability measures, different from μ, must be supported by the set of points going to infinity under the action of the flow g_t. Note that this set is both h_s- and g_t-invariant.

The previous argument uses compactness at one point, in order to apply the Ascoli theorem. If X is not compact, we can still use that theorem, if we endow the space of continuous functions with the compact-open topology, instead of the uniform topology; cf [Du73] 7.6.4. Convergence for this topology is equivalent to uniform convergence on compact subsets. This implies simple convergence, and in our case, L^2-convergence, since all the quantities considered are bounded by the uniform norm of f. So the previous argument applies verbatim and gives uniform convergence on compact subsets of $M_t(f)$ to the constant $\int f d\mu$, for all f bounded uniformly continuous. This shows that μ is ergodic with respect to h_s but this is not enough to get unique ergodicity.

We now assume that ν is an h_s-invariant probability measure that satisfies the condition: $\nu\big(\ \{x \in X \mid \omega(x) = \phi\}\ \big) = 0$. Applying the Birkhoff ergodic theorem to h_s and ν, we find an h_s-invariant function Pf with $\int Pf\, d\nu = \int f\, d\nu$, such that $\frac{1}{t}S_t(f)$ converges to Pf, ν-a.e. Let $x \in X$ such that $\frac{1}{t}S_t(f)(x)$ converges to $Pf(x)$, and $g_{\ln t}(x)$ has an accumulation point: there is a sequence t_k such that $g_{\ln(t_k)}(x)$ is converging. Let K be a compact set containing that subsequence (e.g.

$K = \overline{\{g_{\ln(t_k)}(x), k \in \mathbf{N}\}})$. The quantity $M_t(f)$ converges uniformly on K to $\int f d\mu$, so we have:

$$\frac{1}{t_k} S_{t_k} f(x) = M_{t_k}(f)(g_{\ln(t_k)}(x)) \to \int f d\mu$$

Hence $Pf(x) = \int f d\mu$, for ν-a.e.x, and integrating with respect to ν, $\int f d\nu = \int f d\mu$. We have proven:

Theorem
Let X be a separable metric space, g_t and h_s two continuous flows on X which satisfy the relation : $g_t \circ h_s = h_{se^{-t}} \circ g_t$. Let μ be a Borel probability measure invariant under both flows, which is absolutely continuous with respect to W^{wu}, and with full support. Finally assume that the flow h_s admits a dense orbit. Then μ is ergodic with respect to h_s, and this is the only h_s-invariant probability measure that satisfies:

$$\mu\big(\ \{x \in X \mid \omega(x) = \phi\} \ \big) = 0.$$

As an application, we recall that, on a finite volume negatively curved manifold, a point that goes to infinity under the action of the geodesic flow is on a closed horocycle. So we recover a famous theorem of Dani: on a finite volume surface with negative constant curvature, the only ergodic h_s-invariant probability measures are the volume and the Dirac masses on closed horocycles.

Finally, we note that in the variable curvature setting, the measure μ is not equal to the Riemannian volume, and there are examples of surfaces with finite volume but for which the natural candidate for μ is infinite (M. Peigné, personal communication).

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