Cocycles and stable foliations of Axiom A flows

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Published in *Ergodic Theory and Dynamical Systems*, 2001, **21**.

Abstract

We show that the strong stable foliation of the canonical extension of an Axiom A flow to an Abelian cover of a compact manifold is ergodic with respect to the product of a Gibbs measure on the manifold times the Haar measure on the fiber, as soon as it is transitive on the support of the measure. This generalizes previous results of M. Babillot, F. Ledrappier, M. Pollicott and V. Kaimanovich.

1 Introduction

This article is devoted to the study of the ergodicity of the stable foliation of an Axiom A flow defined on an Abelian cover of a compact manifold.

The most famous example of an Axiom A flow is the geodesic flow on a compact hyperbolic manifold. The stable foliation then corresponds to the horospheric foliation. E. Hopf [Ho36] and G.A. Hedlund [Hed36] [Hed39] showed its ergodicity at the end of the thirties. In the seventies, H. Furstenberg [F73] obtained a stronger property : the horocyclic flow is uniquely ergodic. This was soon generalized to the case of the stable foliation of an Axiom A flow by R. Bowen and B. Marcus [M75] [BM77].

Since then, the infinite volume case has attracted some attention. M. Babillot and F. Ledrappier [BL96] [L97] studied the case of a flow obtained by extending to an Abelian cover an Axiom A flow defined on a compact manifold. They were able to exhibit a whole family of transverse measures invariant by the foliation and asked whether there were any others. The first step toward some “uniqueness” theorem is to show the ergodicity of these measures. Using counting estimates due to M. Pollicott and

R. Sharp [PS94], they managed to prove it for the geodesic flow in the constant curvature case. This result was generalized by M. Pollicott [Po00] to the Anosov setting for the measure of maximal entropy. In another direction, V. Kaimanovich [V00] obtained the ergodicity of the horocyclic flow for the Liouville measure when the covering group is polycyclic.

We show here the ergodicity of the stable foliation with respect to any Gibbs measure assuming the covering is Abelian. Most interesting examples of such measures are the Liouville measure, the Bowen-Margulis measure, the harmonic measure and the measures constructed by M. Babillot and F. Ledrappier in [BL96].

Recently R. Solomyak [S00] gave an alternative proof of the ergodicity for homologically full Anosov flows based on works of Y. Guivarch [Gui89] and R. Sharp [Sh93]. Also V. Kaimanovich and K. Schmidt announced further generalizations for coverings with nilpotent or polycyclic deck transformation groups.

2 Ergodicity of the stable foliation

In the following, the groups G under consideration will always be locally compact second countable. They are endowed with their Haar measure, which is denoted by λ_G.

We begin by recalling some definitions and notations. Let M be a compact riemannian manifold. A flow on M is said to satisfy Axiom A if the tangent bundle of M splits over the non-wandering set of the flow into three invariant sub-bundles respectively tangent to the flow, uniformly contracted and uniformly dilated by the differential of the flow. Periodic orbits are moreover assumed to be dense in the non-wandering set and fixed points are isolated in that set. Axiom A flows are ergodic with respect to a family of invariant measures supported by the connected components of the non-wandering set and called Gibbs or equilibrium measures. Such a measure depends on a Hölder potential f and maximize the quantity $h_\mu + \int f d\mu$ among the set of Borel invariant measures μ supported by a given component, where h_μ is the entropy of μ [BR75]. We will mainly use the fact that there are precise estimates for the volume of balls for these measures.

Let $\pi : \hat{M} \to M$ a covering of M which is supposed regular : the deck transformation group of the covering, denoted by G, acts transitively on the fibers. Let ϕ_t a flow on M and $\hat{\phi}_t$ its unique extension to $\hat{M}$. The strong stable foliations are defined by

$W^{ss}(x) := \{ \ y \in M \mid \ \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0 \ \}$;

$\hat{W}^{ss}(\hat{x}) := \{ \ \hat{y} \in \hat{M} \mid \lim_{t\to\infty} d(\hat{\phi}_t(\hat{x}), \hat{\phi}_t(\hat{y})) = 0 \ \}$.
Let τ be a periodic orbit of ϕ_t on M and $\hat{x} \in \hat{M}$ a preimage of a point on the orbit τ. We denote the period of τ by $l(\tau)$. The Frobenius element $[\tau]$ in G is the translation in the fiber along τ : $\hat{\phi}_{l(\tau)}(\hat{x}) = [\tau](\hat{x})$. Note that if G is not Abelian, there is still a Frobenius element defined up to conjugacy. The foliation $\hat{W}^{ss}$ is said to be ergodic if any union of leaves that forms a Borel set is of measure 0 or with complement of measure 0.

Theorem 1
Let M be a compact manifold, ϕ_t an Axiom A flow defined on M, Λ a connected component of its non-wandering set, G a locally compact second countable Abelian group, $\pi : \hat{M} \to M$ a regular covering of M with deck transformation group G, $\hat{\phi}_t$ the natural extension of ϕ_t to $\hat{M}$, μ a Gibbs measure on Λ. The following propositions are equivalent :
- *Transitivity of the foliation : $\exists\ \hat{x} \in \pi^{-1}(\Lambda)$ s.t. $\pi^{-1}(\Lambda) \subset \overline{\hat{W}^{ss}(\hat{x})}$.*
- *Density of the subgroup generated by the periods and the Frobenius elements of the periodic orbits of ϕ_t on Λ :*

$$\overline{< \{ \ (l(\tau), [\tau]) \ , \ \tau \text{ a periodic orbit on } \Lambda \ \} >} = \mathbf{R} \times G$$

- *Ergodicity of the foliation : the stable foliation $\hat{W}^{ss}$ of $\hat{\phi}_t$, in restriction to $\pi^{-1}(\Lambda)$, is ergodic with respect to $\mu \times \lambda_G$.*

Remarks :

The hypothesis on the orbits implies that the group generated by the periods $l(\tau)$ is dense in $\mathbf{R}$. This fact is equivalent to the flow ϕ_t being topologically mixing on Λ (see e.g. [PP90]).

The hypothesis on the orbits can be broken in two parts for some particular flows. The flow $\hat{\phi}_t$ is said to be reversible if it can be conjugated to $\hat{\phi}_{-t}$ via a G-equivariant map of $\hat{M}$. In this case $(l(\tau), [\tau])$, $(l(\tau), -[\tau])$ and their sum $(2l(\tau), 0)$ are in the group generated by the periods and the Frobenius elements. If the flow ϕ_t is mixing, the condition reduces to the density in G of the group generated by the Frobenius elements $[\tau]$: $\overline{< \{ \ [\tau] \ \} >} = G$. Examples of non reversible mixing Anosov flows for which that condition is not sufficient can be built by suspending an automorphism of the torus.

The geodesic flow defined on the unitary bundle of the quotient of a connected simply connected complete manifold with negative pinched curvature by a convex cocompact group of isometry acting properly discontinuously, is close to be an Axiom A flow (see e.g. [Eb72]). The only lacking property is the compactness of the ambient manifold. However this is not needed to build the symbolic model which is used in the proof of Theorem 1. Compactness and local maximality of the non-wandering set are sufficient [An95] and are easily checked. For such system, Theorem 1 is valid and the hypothesis on the orbits is satisfied as soon as the flow is mixing and the covering is connected. This follows from the reversibility of the flow and from the fact that there is a closed geodesic in each free homotopy class.

The rest of the article is devoted to the proof of Theorem 1. We first recall briefly standard facts about symbolic dynamic and Axiom A flows. The proof of the ergodicity can then be reduced to the study of essential values of a certain cocycle. A simple geometric argument followed by a careful analysis from the measure theoretic viewpoint bring the desired result.

3 Symbolic dynamics

The Axiom A flow ϕ_t is studied with the help of symbolic dynamics.
Using Markov partitions, one can exhibit a semi-conjugation between the flow $\hat{\phi}_t$ in restriction to the preimage of Λ in $\hat{M}$ and the suspension of a subshift of finite type. The symbolic model is thus given by a finite alphabet I, a finite dimensional (0-1) matrix A, a Hölder continuous function r defined on Σ_A depending only on positive coordinates, and a deck function $g : \Sigma_A \to G$ depending only on the first two positive coordinates,

$$\Sigma_A := \{\ \{x_k\}_{k\in\mathbf{Z}} \in I^{\mathbf{Z}} \mid \forall\ k \in \mathbf{Z},\ A_{x_k x_{k+1}} = 1\}$$

$${}^{g}\Sigma_A^r :=\ \Sigma_A \times \mathbf{R} \times G\ /\ < (x,s,u) \sim (\sigma(x), s - r(x), g(x).u) >$$

so that the flow $(x,s,u) \to (x, s+t, u)$ on ${}^{g}\Sigma_A^r$ is semi-conjugated to $\hat{\phi}_t$. These results are due to R. Bowen [Bo73], M. Ratner [Ra75]. We refer to [PP90] for more details. We now consider the space :

$$\Sigma_A^+ := \{\ \{x_k\}_{k\in\mathbf{N}} \in I^{\mathbf{N}} \mid \forall\ k \in \mathbf{N},\ A_{x_k x_{k+1}} = 1\}$$

Since we are interested in what happens for positive time, we can restrict our attention to $\Sigma_A^+ \times \mathbf{R} \times G$. Proving the theorem now amounts to proving the ergodicity of the following equivalence relation on $\Sigma_A^+ \times \mathbf{R} \times G$ for the measure $\tilde{\mu} \times dt \times \lambda_G$, $\tilde{\mu}$ being a Gibbs measure on Σ_A^+ :

$$(x,t,u) \sim (y,s,v) \leftrightarrow \exists\, m,n \geq 0\ \ s.t. \qquad \begin{array}{c} \sigma^m(x) = \sigma^n(y) \\ t - r^m(x) = s - r^n(y) \\ u + g^m(x) = v + g^n(y) \end{array}$$

The shift is denoted by σ as usual. Moreover we use the following notations $r^m(x) := r(x) + r(\sigma(x)) + ... + r(\sigma^{m-1}(x))$, $g^m(x) := g(x) + ... + g(\sigma^{m-1}(x))$.

4 Cocycles

In order to study that equivalence relation, we use the concept of essential values of a cocycle, due to K. Schmidt, J. Feldman and C. Moore.

Let $(X, \mathcal{B}, \tilde{\mu})$ be a standard Borel space, $\sim$ an equivalence relation on X such that the set of pairs of equivalent points is a Borel subset of $X \times X$ and such that the number of elements in each equivalence classe is at most countable ; let $\tilde{G}$ be a locally compact second countable group. A cocycle defined on X with value in $\tilde{G}$ is a measurable application from the set of pairs of equivalent points in X to $\tilde{G}$, such that if $x \sim y$ and $y \sim z$, $c(x,y)c(y,z) = c(x,z)$. An element λ in $\tilde{G}$ is an essential value of c if for all neighborhoods V of λ, for all Borel subsets B of X and for almost all element $x \in B$, one can find an element $y \in B$ equivalent to x such that $c(x,y)$ belongs to V.

Proposition 1 ([Sch77],[FM77],[Sch89])
Suppose that $\sim$ is ergodic on $(X, \mathcal{B}, \tilde{\mu})$ and that the subgroup of $\tilde{G}$ generated by the essential values of c is dense in $\tilde{G}$. Then the following equivalence relation is ergodic on $(X \times \tilde{G},\ \tilde{\mu} \times \lambda_{\tilde{G}})$:

$$(x,u) \sim (y,v) \leftrightarrow \quad x \sim y\ ,\ v = u\ c(x,y).$$

The problem can be now reformulated in term of essential values ; the group G is supposed to be abelian. Take $X := \Sigma_A^+$, $\tilde{\mu}$ a Gibbs measure on X, $\tilde{G} := \mathbf{R} \times G$ and define for $x, y \in \Sigma_A^+$:

$$x \sim y \leftrightarrow \exists\ m, n \geq 0 \quad s.t. \quad \sigma^m(x) = \sigma^n(y).$$

$$c(x,y) := (\ r^n(y) - r^m(x),\ g^m(x) - g^n(y)\ ).$$

This cocycle is a symbolic counterpart of the Buseman cocycle, which was studied by V. Kaimanovich in the constant curvature case [V00].

Let us define $f := (r, -g)$. The weight $[\tilde{\tau}]$ of a periodic orbit $\tilde{\tau}$ of σ of period $l(\tilde{\tau}) = p \in \mathbf{N}$ with respect to f is equal to $f^p(x) := f(x) + ... + f(\sigma^{p-1}(x))$, where x is some point on the orbit. In the Axiom A case, this is just the quantity $(l(\tau), [\tau])$. The ergodicity of the relation defined on Σ_A^+ follows from the ergodicity of the shift, which is a consequence of the transitivity of ϕ_t on Λ. In fact the shift is even mixing on Σ_A^+ ([Bo73]). So in view of Proposition 1, Theorem 1 is a consequence of the following symbolic result.

Theorem 2
Let (Σ_A^+, σ) a transitive one-sided subshift of finite type, $\tilde{\mu}$ a Gibbs measure on Σ_A^+, $\tilde{G}$ a locally compact second countable abelian group, $f : \Sigma_A^+ \to \tilde{G}$ a Hölder continuous function. We define a relation on Σ_A^+ and a cocycle :

$$x \sim y \leftrightarrow \exists\ m, n \geq 0 \quad s.t. \quad \sigma^m(x) = \sigma^n(y).$$

$$c(x, y) := (f^n(y) - f^m(x))$$

Then for all periodic orbit $\tilde{\tau}$ of σ, the weight $[\tilde{\tau}]$ of the orbit is an essential value of c. Moreover the following propositions are equivalent :
- *The relation defined on $\Sigma_A^+ \times \tilde{G}$ by the cocycle admits a dense orbit.*
- *the subgroup generated by the weights of the periodic orbits is dense in $\tilde{G}$.*
- *The relation on $\Sigma_A^+ \times \tilde{G}$ defined by the cocycle is ergodic with respect to $\tilde{\mu} \times \lambda_{\tilde{G}}$.*

5 Deformation of orbits

We now fix a periodic orbit $\tilde{\tau}$ and embark in the proof of Theorem 2.
Let $\bar{v}$ be a point on the orbit. $\bar{v} = (v_0, ...v_M, v_0, ...v_M, ...), v = (v_0, ..v_M)$, $[\tilde{\tau}] = f^{M+1}(\bar{v})$. The following lemma is inspired by works of D. Maharam [Mah64] as presented, for example, in [Aa96], and M. Pollicott [Po00].

Topological Lemma :
Let $Q, k \in \mathbf{N}$. Let D_k be the set of elements x of Σ_A^+ such that, beginning from the k^{th} coordinate, v occurs $Q+1$ times in a row in the development of x. Let $T_k(x)$ the element of Σ_A^+ whose coordinates are obtained by removing the occurrence of v at rank k in the developpement of x : $(T_k(x))_i = x_i$ if $i < k$, $(T_k(x))_i = x_{i+M+1}$ if $i \geq k$. Then,

$$\forall\ q > k,\ |f^{q+M}(x) - f^q(T_k(x)) - [\tilde{\tau}]\ | \leq \parallel f \parallel 2^{-MQ+2}.$$

Proof :
The symbol $|.|$ denote some invariant distance on $\tilde{G}$. $\parallel f \parallel$ is the Lipschitz norm of f. Recall that if two points coincide on their first i^{th} coordinates, their distance is less than 2^{-i}.
For $i = 0, ..., k$, $\sigma^i(x)$ and $\sigma^i(T_k(x))$ coincide on $k - i + QM$ coordinates,
$|\sum_{i=0}^{k} f(\sigma^i(x)) - \sum_{i=0}^{k} f(\sigma^i(T_k(x)))| \leq \parallel f \parallel \sum_{i=0}^{k} 2^{i-k-QM} \leq \parallel f \parallel 2^{-QM+1}$
For $j = 0, ..., M$, $\sigma^{j+k}(x)$ and $\sigma^k(\bar{v})$ coincide on $M - j + QM$ coordinates,
$|\sum_{j=0}^{M} f(\sigma^{k+j}(x)) - \sum_{j=0}^{M} f(\sigma^j(\bar{v}))| \leq \parallel f \parallel 2^{-QM+1}$
Finally, for $q > k$, $f(\sigma^{q+M}(x)) = f(\sigma^q(T_k(x)))$.

Let B be a Borel subset of Σ_A^+, $\varepsilon > 0$ some positive real number. For $x \in B$ we search for an element $y \in B$ equivalent to x such that $c(x, y)$ is near $[\tilde{\tau}]$, say in a ε-neighborhood. The element $T_k(x)$ is a good candidate for y since $T_k(x)$ is equivalent to x, $c(x, T_k(x)) = (f^{q+M}(x) - f^q(T_k(x)))$ is close to $[\tilde{\tau}]$ by the lemma, and $T_k(x)$ tends to x when k tends to infinity. Let Q be chosen such that $\| f \| 2^{-MQ+2}$ is less than ε. In order to conclude, we only need to estimate $\tilde{\mu}(B \cap \cup_{k=0}^{\infty} T_k^{-1}(B))$ and show this is equal to $\tilde{\mu}(B)$. This is the goal of the next four lemmas.

6 Computation of the essential values

Let $\tilde{F}$ be an essentially bounded function on Σ_A^+ . We will eventually take for $\tilde{F}$ the indicator function of the set B.

Lemma 1 : $\exists C \geq 1$ such that $\forall k \in \mathbf{N}$, $\forall B$ Borel sets in D_k,

$$C^{-1}\tilde{\mu}(B) \leq \tilde{\mu}(T_k(B)) \leq C\tilde{\mu}(B).$$

Proof :
$\tilde{\mu}$ is a Gibbs measure [Bo75] [PP90] so there is a Hölder continuous function ψ on Σ_A^+ and constants P and $C_0 > 1$ such that $\forall q \in \mathbf{N}$, $\forall x \in \Sigma_A^+$,

$$C_0^{-1} e^{\psi^q(x) - Pq} \leq \tilde{\mu}\big(\{y \mid y_i = x_i \quad \forall i \in \{0, ..., q-1\}\}\big) \leq C_0 e^{\psi^q(x) - Pq}.$$

We apply these inequalities for some $x \in D_k$ and a $q > k + (M+1)(Q+1)$ then for $T_k(x)$ and $q - M$. The topological lemma says that the difference between $f^q(x)$ and $f^{q-M}(T_k(x))$ is bounded independently of x and k. The same reasoning shows that the difference between $\psi^q(x)$ and $\psi^{q-M}(T_k(x))$ is also bounded independently of x and k. This prove the lemma for cylinder sets of length greater than $k + (M+1)(Q+1)$ in D_k and, by approximation, for all Borel sets in D_k.

Lemma 2 : $\forall\, n \in \mathbf{N}, \forall\, B \subset \Sigma_A^+$, $\exists\, K_n \in \mathbf{N}$ *such that* $\forall l \in \mathbf{N}$,

$$\tilde{\mu}\left(B \cap \bigcup_{k=l}^{l+K_n} D_k\right) \geq (1 - 2^{-n-1})\; \tilde{\mu}(B).$$

Proof :
The shift σ is ergodic on Σ_A^+ for $\tilde{\mu}$, so there is a K_n depending on n and B,

$$\tilde{\mu}\Big(\bigcup_{k=0}^{K_n} D_k\Big) \geq 1 - 2^{-n-1}\tilde{\mu}(B).$$

$$\tilde{\mu}\Big(B \cap \bigcup_{k=l}^{l+K_n} D_k\Big) \geq \tilde{\mu}(B) + \tilde{\mu}\Big(\sigma^{-l}\big(\bigcup_{k=0}^{K_n} D_k\big)\Big) - 1 \geq (1 - 2^{-n-1})\tilde{\mu}(B).$$

Lemma 3 : $\forall\, \eta_1 > 0, \forall\, \eta_2 > 0, \exists\, N_{\eta_1,\eta_2} \in \mathbf{N}$ *such that* $\forall\, k > N_{\eta_1,\eta_2}$,

$$\tilde{\mu}\left(\left\{x \in D_k \; s.t. \; |\tilde{F}(x) - \tilde{F}(T_k(x))| > \eta_1\right\}\right) < \eta_2.$$

Proof :
There exists $\tilde{F}_p$ a function depending only on the first p coordinates such that $\tilde{F}_p - \tilde{F}$ converges to 0 in probability when p tends to infinity.
$\forall\ \eta_1, \eta_2 > 0, \exists\ N_{\eta_1,\eta_2},\ \forall k > N_{\eta_1,\eta_2},\ \tilde{\mu}\left(x\ s.t.\ |\tilde{F}_k(x) - \tilde{F}(x)| > \eta_1/2\right) < \frac{\eta_2}{2C}$
and using Lemma 1, $\quad \tilde{\mu}\left(x\ s.t.\ |\tilde{F}_k(T_k(x)) - \tilde{F}(T_k(x))| > \eta_1/2\right) < \frac{\eta_2}{2}$.
moreover $\tilde{F}_k(T_k(x)) = \tilde{F}_k(x)$, because $T_k(x)$ and x have the same first k coordinates.

Lemma 4 : $\forall\ n \in \mathbf{N}, \forall\ B \in \Sigma_A^+,\ \exists\ n_1, n_2 \in \mathbf{N}, n_1 < n_2,$
$\tilde{\mu}(\{x \in B | \exists k \in \{n_1..n_2\}, x \in D_k, |\tilde{F}(x) - \tilde{F}(T_k(x))| \leq \frac{1}{n}\}) \geq (1 - 2^{-n})\tilde{\mu}(B)$

Proof :
Given an integer n we get a K_n using Lemma 2. Let $\eta_1 = 1/n$ and $\eta_2 = 2^{-n-1}\tilde{\mu}(B)/K_n$. Lemma 3 gives a N_{η_1,η_2} and we take $n_1 = N_{\eta_1,\eta_2}$, $n_2 = n_1 + K_n$. Now,

$$\begin{aligned}
&\tilde{\mu}\left(\left\{x \in B \mid \exists\ k \in \{n_1, ..., n_2\}, x \in D_k, |\tilde{F}(x) - \tilde{F}(T_k(x))| \leq \tfrac{1}{n}\right\}\right) \\
&\quad \geq \quad \tilde{\mu}\left(\{x \in B \mid \exists\ k \in \{n_1, ..., n_2\}, x \in D_k\}\right) \\
&\quad\quad -\tilde{\mu}\left(\bigcup_{k=n_1}^{n_2} \left\{x \in D_k\ s.t.\ |\tilde{F}(x) - \tilde{F}(T_k(x))| \geq \tfrac{1}{n}\right\}\right) \\
&\quad \geq (1 - 2^{-n-1})\tilde{\mu}(B) - K_n 2^{-n-1}\tilde{\mu}(B)/K_n \\
&\quad \geq (1 - 2^{-n})\tilde{\mu}(B).
\end{aligned}$$

End of the proof of theorem 2 :

Taking for $\tilde{F}$ the indicator function of B and letting n goes to infinity, Lemma 4 just says that $\tilde{\mu}\left(\{x \in B \mid \exists k \in \mathbf{N}, x \in D_k,\ T_k(x) \in B\}\right) = \tilde{\mu}(B)$, so that $[\tilde{\tau}]$ belongs to the set of essential values of c.

Here are the constants used in the proof. A given constant depends on all the previous ones : $\tilde{\tau}$, B, ε, Q, $\{T_k\}$, n, K_n, η_1, η_2, N_{η_1,η_2}, n_1, n_2.

The condition on the weights of periodic orbits is necessary : if the subgroup generated by the weights is not dense, $\tilde{G}$ can be quotiented by the closure of this subgroup ; all the weights are now equal to the identity and so, by the Livsic theorem [Li72], the application $\bar{f}\ : \Sigma_A^+ \to \tilde{G}/ < \{[\tilde{\tau}]\} >$ obtained by composition of f with the projection is a coboundary : there exists a Hölder application $\bar{b}$ such that $\bar{f} = \bar{b} \circ \sigma - \bar{b}$. The map $(x, u) \mapsto \bar{b}(x) - \bar{u}$ defined from $\Sigma_A^+ \times \tilde{G}$ to $\tilde{G}/ < \{[\tilde{\tau}]\} >$ is continuous, non-constant and invariant by the relation associated to the cocycle ; so this relation admits no dense orbit and is not ergodic.

Remark :
The fact that G is Abelian was used in only one place during the proof , namely to compute $c(x, T_k(x)) = (r^{q+M}(x) - r^q(T_k(x)), [\tau])$. If G is not Abelian, one obtains $c(x, T_k(x)) = (r^{q+M}(x) - r^q(T_k(x)), g^{-k}(x)[\tau]g^k(x))$, where $[\tau] := g^{M+1}(\bar{v})$ depends on the choice of a point on τ. The cocycle is now moving in the conjuguacy class of $[\tau]$ and the proof doesn't bring any essential values.

7 The discrete case

One can prove similar results for an Axiom A diffeomorphism. There is no canonical G-equivariant extension of the diffeomorphism however. The condition needed in term of periodic orbits is the density in $\mathbf{Z} \times G$ of the subgroup generated by the periods and the weights of the periodic orbits of the diffeomorphism. This can be deduced from the following theorem, which generalizes a result due to Y. Guivarc'h [Gui89] :

Theorem 3
Let (Σ_A^+, σ) a transitive one-sided subshift of finite type, $\tilde{\mu}$ a Gibbs measure on Σ_A^+, $\tilde{G}$ a locally compact second countable abelian group, $f : \Sigma_A^+ \to \tilde{G}$ a Hölder continuous function. We define a relation on Σ_A^+ and a cocycle :

$$x \sim y \leftrightarrow \exists\ m \geq 0 \ \ s.t.\ \ \sigma^m(x) = \sigma^m(y).$$
$$c(x, y) := (f^m(y) - f^m(x))$$

The following propositions are equivalent :
- *The relation defined on $\Sigma_A^+ \times \tilde{G}$ by the cocycle admits a dense orbit.*
- *the subgroup generated by the periods and the weights of the periodic orbits $\tilde{\tau}$ of σ is dense in $\mathbf{Z} \times \tilde{G}$: $\overline{< (l(\tilde{\tau}), [\tilde{\tau}]) >} = \mathbf{Z} \times \tilde{G}$.*
- *The relation on $\Sigma_A^+ \times \tilde{G}$ defined by the cocycle is ergodic with respect to $\tilde{\mu} \times \lambda_{\tilde{G}}$.*

Proof :
Theorem 2 is applied to the function $(\mathbf{1}, f)$. The condition $\overline{< (l(\tilde{\tau}), [\tilde{\tau}]) >} = \mathbf{Z} \times \tilde{G}$ is now equivalent to the ergodicity of the equivalence relation on $\Sigma_A^+ \times \mathbf{Z} \times \tilde{G}$ given by the cocycle of Theorem 2 with respect to $\tilde{\mu} \times \lambda_{\mathbf{Z}} \times \lambda_G$. This in turn is equivalent to the ergodicity of the restriction of that relation to $\Sigma_A^+ \times \{0\} \times \tilde{G}$, which is the one we are interested in.

Acknowledgements :
I wish to thank M. Babillot, F. Dal'bo and F. Ledrappier for their help and encouragements. I also thank V. Kaimanovich and K. Schmidt for their comments.

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