The weak mixing property on negatively curved manifolds

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Abstract. Given a complete manifold of negative curvature, we show that weak mixing is a generic property in the set of all probability measures invariant by the geodesic flow, as soon as the flow is topologically weakly mixing in restriction to its non-wandering set. [1]

1 Introduction

This article is dedicated to the question of measure-theoretic weak mixing of dynamical systems displaying hyperbolic behavior, when the underlying phase space fails to be compact. We solve a longstanding question in the ergodic theory of the geodesic flow defined on complete negatively curved manifolds by proving that weak mixing is a generic property in the space of all invariant probability measures by the flow, assuming topological weak mixing.

Given a complete separable metric space X and a continuous flow φ_t defined on X, we denote by $\mathcal{M}^1(X)$ the set of Borel probability measures invariant by φ_t, endowed with the weak topology [Bi99]. Weak convergence may be characterised as follows. A sequence of measures $\mu_n \in \mathcal{M}^1(X)$ is converging to a measure μ for this topology if for all open sets $U \subset X$,

$$\mu(U) \leq \varliminf_{n\to\infty} \mu_n(U).$$

Endowed with this topology, the space $\mathcal{M}^1(X)$ is a complete separable metrisable space. A subset is said to be G_δ-dense if it is a countable intersection of open dense sets. A property shared by a set of measures is said to be *generic* if it is satisfied on a G_δ-dense subset of $\mathcal{M}^1(X)$.

What are the generic properties satisfied by measures invariant by the flow?

In the context of hyperbolic dynamics, this question was first investigated by K. Sigmund, who showed that ergodicity is indeed a generic property when the flow is the restriction of an Axiom A flow to a basic set [Si72]. An early example of an Axiom A flow is the geodesic flow defined on the unit tangent bundle T^1M of a connected compact manifold of negative curvature M. The Axiom A hypothesis also covers the case of a convex cocompact manifold. Despite its name, this assumption

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is weaker than compactness of the space and only asks for compactness of the non-wandering set $\Omega \subset T^1M$ of the flow.

There are two natural generalisations to Sigmund's result in the geometrical setting. The assumption on the curvature may be weaken so as to encompass the non-positively curved manifolds. In a joint work with B. Schapira [CS11], we showed that on a nonpositively curved nonflat compact surface with a flat cylinder, the set of ergodic measures is not dense in the set of all probability measures invariant by the geodesic flow. Another generalisation deals with negatively curved complete but non compact manifolds. We were able to prove that ergodicity is a generic property on all these manifolds [CS10]. We also showed that in contrast to the ergodic measures, the strong mixing measures form a meager set in the set of all invariant probability measures [Par61], [Si72], [CS14].

In this article, we are interested in the *weak mixing property.* K. Parthasarathy proved that this is a generic property for irreducible aperiodic Markov chains [Par62]. Using Markov partitions or the specification property, the theorem extends to topologically weakly mixing Axiom A diffeomorphisms [Si74] and flows. As a consequence, the result holds for geodesic flows defined on convex-compact negatively curved manifolds. Does it remain valid without any compactness hypothesis? Kamel Belarif [Be17] showed that weak mixing is a generic property for geodesic flows defined on complete *constant* negative curvature manifolds. He also succeeded in proving the genericity of weak mixing for geodesic flows defined on geometrically finite manifolds with pinched negative curvature, if the flow is topologically weakly mixing in restriction to the nonwandering set of the flow.

Our goal is to show that the weak mixing property is generic on all complete negatively curved manifolds under the sole assumption of topological weak mixing of the flow, by proving the following theorem.

Theorem 1 *Let M be a connected pinched negatively curved complete Riemannian manifold with bounded first derivatives of sectional curvatures. We assume that the geodesic flow defined on the unit tangent bundle T^1M of M is topologically weakly mixing in restriction to its non-wandering set $\Omega \subset T^1M$. Then the weak mixing probability measures form a dense G_δ-subset of the set of all Borel probability measures on T^1M invariant by the flow.*

In the hyperbolic setting, topological weak mixing is equivalent to topological mixing of the flow [C04]. It is conjectured that on a complete negatively curved manifold, the geodesic flow is topologically mixing in restriction to its nonwandering set as soon as that set contains more than two periodic orbits. The conjecture is proven for example in constant curvature, in dimension two or when there is a cusp, as shown by F. Dal'Bo in [Da99]. If it were known to hold in full generality, then Theorem 1 could be deduced from the convex-cocompact case by working on a convex-cocompact cover of the manifold. Unfortunately, the conjecture seems to be pretty elusive and is one of the outstanding questions concerning the topological dynamics of the geodesic flow in negative curvature.

The proof of Theorem 1 follows the strategy of K. Parthasarathy and K. Sigmund, together with arguments from K. Belarif and methods developed by S. Gouezel, F. Paulin, M. Pollicott, B. Schapira and S. Tapie. It is known that invariant measures supported by periodic orbits are dense in $\mathcal{M}^1(T^1M)$. These measures can be approximated by mixing measures using thermodynamical formalism. Building equilibrium measures on non-compact spaces is an unsolved problem in full generality. In the context of geodesic flows in negative curvature, a construction going back to Patterson and Sullivan gives sufficient information to build mixing probability measures close to Dirac measures supported by periodic orbits.

2 Equilibrium measures on metric spaces

We start by recalling a general result of K. Belarif concerning equilibrium measures defined on metric spaces.

Let X be a metric space endowed with a continuous flow $\varphi_t : X \to X$ and $F : X \to \mathbf{R}$ be a bounded Borel function. The entropy of the time one map φ_1 of the flow φ_t with respect to an invariant probability measure μ is denoted by $h_\mu(\varphi_1)$. We define the *pressure* of F with respect to the flow φ_t by

$$P_F(\varphi_t) = \sup\Big\{h_\mu(\varphi_1) + \int_X F\, d\mu \mid \mu \in \mathcal{M}^1(X)\Big\}.$$

This supremum may be infinite. When it is finite, an *equilibrium measure* associated to F is an invariant probability measure attaining this supremum. The function F is the *potential* associated to the measure. There may be no equilibrium measure associated to a continuous potential even when X is compact, see e.g. [Wal82] §8.3. The following abstract result concerns the behavior of an equilibrium measure in the vicinity of the set of points where the potential attains its maximum, assuming that such potential exists.

Theorem 2 *[Be17] Let X be a metric space endowed with a continuous flow $\varphi_t : X \to X$ and $F : X \to \mathbf{R}$ be a bounded Borel function. We assume that there is a compact set $K \subset X$ invariant by φ_t, a neighborhood V of K and constants c, c' with $c' < c$ such that*

- *for all $x \in K$, $F(x) = c$,*
- *for all $x \in X$, $F(x) \leq c$,*
- *for all $x \in X \setminus V$, $F(x) \leq c'$.*

Then for any equilibrium measure μ associated to F,

$$\mu(X \setminus V) \leq \frac{h_\mu(\varphi_1)}{c - c'}.$$

Since the proof is short, we recall it for the reader's convenience.

Proof
Since K is a compact invariant set, there is an invariant probability measure μ_K supported on K. From the variational principle,

$$h_\mu(\varphi_1) + \int_X F\, d\mu \geq h_{\mu_K}(\varphi_1) + \int_X F\, d\mu_K \geq \int_X F\, d\mu_K = c.$$

Moreover,

$$\int_X F\, d\mu = \int_V F\, d\mu + \int_{X\setminus V} F\, d\mu \leq c\, \mu(V) + c'\mu(X \setminus V).$$

These two inequalities imply

$$h_\mu(\varphi_1) \geq c - \int_X F\, d\mu \geq c - c\, \mu(V) - c'\mu(X \setminus V) = (c - c')\, \mu(X \setminus V).$$

The theorem is proven.

When K is a periodic orbit of the flow, Theorem 2 shows that an equilibrium measure associated to a potential reaching its maximum on K and fast decreasing away from K is close to the Dirac measure on the periodic orbit.

3 Mixing measures in negative curvature

The existence of mixing probability measures on non-compact negatively curved manifolds was investigated in [C03]. It was shown that on the unit tangent bundle of any geometrically finite negatively curved manifold with cusp, there exists probability measures invariant by the geodesic flow that are mixing and of full support on the wandering set of the flow. Combined with the previous theorem, this led K. Belarif to the proof of Theorem 1 in the geometrically finite setting.

On a geometrically finite manifold of constant curvature, it was already known that the measure obtained by the Patterson-Sullivan method gives a probability measure that is mixing. This measure is in fact the unique measure of maximal entropy built by Bowen and Margulis for Anosov and Axiom A flows if the nonwandering set of the flow is compact. Examples by F. Dal'Bo, M. Peigné, J.C. Picaud and A. Sambusetti showed that this measure may be infinite when the nonwandering set of the flow is non-compact and the curvature is variable, in which case there is no probability invariant measure maximising the entropy [DPPS19].

In order to bypass this difficulty, we introduced a potential in the Patterson-Sullivan construction, in the spirit of the thermodynamical methods used in hyperbolic dynamics [C03]. A criterion on the finiteness of the Bowen-Margulis measure by F. Dal'Bo, J.P. Otal and M. Peigné [DOP00] generalises to that setting and shows that there are potentials for which the construction leads to a finite measure. Mixing follows from the topological mixing of the geodesic flow by an argument of M. Babillot [Ba02].

A general thermodynamical formalism for the geodesic flow in negative curvature was developed by F. Paulin, M. Pollicott and B. Schapira in [PPS15]. Moreover, a criterion for finiteness of equilibrium measures was devised by V. Pit and B. Schapira in [PS18] and further extended in [GSTR23], so that all ingredients are now available to build probability measures with specific local behavior on any complete negatively curved manifold. We will use the next theorem, whose proof follows from [PPS15] Theorem 6.1, Theorem 8.1 and [GSTR23] Corollary 1.5, Corollary 4.12.

Theorem 3 *Let M be a connected complete Riemannian manifold of negative pinched curvature and bounded first derivatives of the curvatures, such that the associated geodesic flow is topologically weakly mixing in restriction to its nonwandering set Ω. Let $A : T^1M \to \mathbf{R}_+$ be a nonnegative Hölder continuous function with compact support, which is not identically zero on Ω.*

Then for $\lambda > 0$ large enough, the pressure of the function λA is finite and there is a unique equilibrium measure associated to λA. Moreover, this measure is mixing with respect to the geodesic flow.

In principle, such result should hold for any expansive flow with the specification property and finite entropy, under suitable versions of these properties when the phase space is noncompact. The proof of the previous theorem makes heavy use of the geometric features on the geodesic flow though and does not generalize to that setting.

4 Genericity and the weak mixing property

We now prove Theorem 1.

Given a continuous flow on a Polish space, it was shown in [Par61], [CS14] that the set of weakly mixing invariant probability measures is a G_δ-subset of the set of all invariant probability measures. In order to prove Theorem 1, we only need to show that this set is dense in $\mathcal{M}^1(T^1M)$. We will show that mixing probability measures are in fact dense in $\mathcal{M}^1(T^1M)$.

Recall that given a periodic vector v, the *Dirac measure* δ_v associated to v is the only invariant probability measure whose support is equal to the orbit of v. In view of the next theorem, it is sufficient to approximate Dirac measures on periodic orbits by mixing measures with respect to the weak topology to prove Theorem 1.

Theorem 4 *[CS10] Let M be a connected complete pinched negatively curved manifold. Then the set of Dirac measures supported by periodic orbits on T^1M are dense in $\mathcal{M}^1(T^1M)$ for the weak topology.*

Given a periodic vector $v \in T^1M$, let us denote its orbit by K and let V_n be the set of points of T^1M at distance less than $1/n$ from K.

$$V_n = \{w \in T^1M \mid d(w, K) \leq 1/n\}.$$

For $n > 0$, we consider a Hölder continuous function $F_n : T^1M \to \mathbf{R}_+$ bounded by 1 on T^1M, equal to 1 on K and vanishing on the complement of V_n. We remark that such a function can be taken to be C^∞ since K is a smooth subvariety of T^1M. By Theorem 3, we can choose $c_n \geq n$ such that there is an mixing equilibrium state μ_n associated to the function c_nF_n.

J.P. Otal and M. Peigné showed that the entropy of a probability measure invariant by the geodesic flow is always bounded by the topological entropy of the flow and that this entropy is finite. It is actually equal to the critical exponent of the fundamental group of M [OP04]. By Theorem 2,

$$\mu_n(T^1M \setminus V_n) \leq \frac{h_{\mu_n}(\varphi_1)}{c_n} \leq \frac{h_{top}(\varphi_1)}{n}$$

which is small when n is large.

This shows that the sequence $\{\mu_n\}_{n\in\mathbf{N}}$ is tight. Moreover any converging subsequence μ_{n_k} has a limit μ supported by K since, for all $m > 0$,

$$\mu(V_m^c) \leq \varliminf_{n_k \geq m} \mu_{n_k}(V_m^c) \leq \varliminf_{n_k \geq m} \mu_{n_k}(V_{n_k}^c) \leq \varliminf_{n_k \geq m} \frac{h_{top}(\varphi_1)}{n_k} = 0.$$

Such limit μ is invariant by the flow so it has to be equal to δ_v. Convergence of $\{\mu_n\}$ to δ_v now follows from Prokhorov's compactness theorem [Bi99]. Theorem 1 is proven.

In retrospect, the proof is quite indirect and relies on several geometric features of the geodesic flow. Fifty years since the original article of K. Sigmund, one may still hope for a general dynamical argument establishing the genericity of weak mixing for topologically weakly mixing uniformly hyperbolic systems defined on non-compact spaces.

References

[Ba02] M. Babillot. On the mixing property for hyperbolic systems. *Israel J. Math.* **129**, (2002), 61–76.

[Be16] K. Belarif. Genericity of weak-mixing measures on geometrically finite manifolds. Preprint https://arxiv.org/abs/1610.03641

[Be17] K. Belarif. Propriétés génériques des mesures invariantes en courbure négative. *Thesis*, Université de Bretagne Occidentale, Brest 2017.

[Bi99] P. Billingsley. Convergence of probability measures. 2nd ed. Wiley Series in Probability and Statistics. Chichester: Wiley, (1999), 277 p.

[C03] Y. Coudene. Gibbs measures on negatively curved manifolds. *Journal of Dynamical and Control System*, vol.**9**, (2003), No. 1 (January) 89–101.

[C04] Y. Coudene. Topological dynamics and local product structure. *J. London Math. Soc.* (2) **69** (2004), No. 2, 441–456.

[CS10] Y. Coudene, B. Schapira. Generic measures for hyperbolic flows on noncompact spaces. *Israel J. Math.*, (2010), No 179, 157–172.

[CS11] Y. Coudene, B. Schapira. Counterexamples in nonpositive curvature. *Discrete Contin. Dyn. Syst.* **30**, (2011), no. 4, 1095–1106.

[CS14] Yves Coudene, Barbara Schapira. Generic measures for geodesic flows on nonpositively curved manifolds. *J. Éc. polytech. Math.*, (2014), No. 1, 387–408.

[Da99] F. Dal'Bo. Remarques sur le spectre des longueurs d'une surface et comptages. *Bol. Soc. Brasil. Mat. (N.S.)* **30**, (1999), no. 2, 199–221

[DOP00] F. Dal'Bo, J. P. Otal, M. Peigné, Séries de Poincaré des groupes géométriquement finis. *Israel J. Math.* **118** (2000), 109–124.

[DPPS19] F. Dal'Bo, M. Peigné, J.C. Picaud, A. Sambusetti. Asymptotic geometry of negatively curved manifolds of finite volume. *Ann. Sci. Éc. Norm. Supér.* (4) **52**, (2019), no.6, 1459–1485.

[GSTR23] S. Gouëzel, B. Schapira, S. Tapie, F. Riquelme. Pressure at infinity and strong positive recurrence in negative curvature. *Comment. Math. Helv.* **98**, (2023), no. 3, 431-508.

[OP04] Principe variationnel et groupes kleiniens. *Duke Math. J.*, **125**, (2004), 15–44.

[Par61] K. R. Parthasarathy. On the category of ergodic measures. *Illinois Journal of Mathematics* **5**, (1961), 648–655.

[Par62] K. R. Parthasarathy. A note on mixing processes. *Sankhya. The Indian Journal of Statistics* Ser. A **24**, (1962), 331–332.

[PPS15] F. Paulin, M. Pollicott, B. Schapira. Equilibrium states in negative curvature. *Astérisque* **373**, Société Mathématique de France, Paris (2015).

[PS18] V. Pit, B. Schapira. Finiteness of Gibbs measures on noncompact manifolds with pinched negative curvature. *Annales de l'Institut Fourier*, **68**, (2018) no. 2, 457–510.

[Si72] K. Sigmund. On the space of invariant measures for hyperbolic flows, *Amer. J. Math.* **94**, (1972), 31–37.

[Si74] K. Sigmund. On dynamical systems with the specification property. *Trans. A.M.S.* **190**, (1974), 285–299.

[Wal82] Peter Walters. An Introduction to Ergodic Theory. *Springer*, GTM **79**, (1982), 250 p.