A note on horospheric points

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Abstract

We define the concept of horospheric points for flows admitting a local product structure, and show that these points have a dense strong stable set. This generalizes a classical result from G.A.Hedlund, in the context of geodesic flows on negatively curved manifolds. [1]

1 Introduction

We study the topological dynamics of the stable foliation of hyperbolic flows. We generalize the concept of horospheric point, first defined for geodesics flows on negatively curved manifolds, to flows admitting a local product structure. We then show how it is related to the transitivity and the minimality of the stable foliation.

Let us consider a complete surface of constant negative curvature ; this surface can be seen as the quotient of the hyperbolic unit disk $\mathbf{D}$, by the action of a subgroup $\mathbf{\Gamma}$ of the isometry group of the disk. Any geodesic on the surface can be lifted to the disk , and ends on a well defined point on the boundary of the disk $\partial\mathbf{D}$.

A point x of the boundary $\partial\mathbf{D}$ is said to be a *conical point*, if there is a sequence of points in a cone based on x, and in the orbit of the origin under the action of $\mathbf{\Gamma}$, that approaches x. Let us recall that an *horosphere* based on x is a circle in the disk $\mathbf{D}$, tangent to the boundary $\partial\mathbf{D}$ at the point x. A point x of the boundary $\partial\mathbf{D}$ is said to be an *horospheric point*, if there is a sequence of points in the orbit of the origin under the action of $\mathbf{\Gamma}$, which converges to x, and enters any horosphere based on x.

G. A. Hedlund showed in [Hed36] that a point on the unit tangent bundle of the negatively curved surface has a dense horosphere, if and only if the lift of the geodesic starting from x ends on a horospheric point of the boundary $\partial\mathbf{D}$. This result was generalized to the setting of variable negative curvature by F. Dal'bo [Da01]. If the surface is compact, this result implies that the horospheric flow is minimal : every horosphere on the surface is dense in the surface.

In order to generalize this result, we must first interprete the concept of conical and horospheric points in term of the dynamics only. Let X be a locally compact metric space, ϕ_t a continuous flow on X. With a slight abuse in terminology, we say that a point $x \in X$ is of "conical type" if its ω-limit set :

$$\omega(x) = \{y \in X \mid \exists\ t_i \to \infty \quad s.t. \quad \phi_{t_i}(x) \to y\}$$

[1] 37B10, 37D40, 34C28

is non empty. If X is the unit tangent bundle of a negatively curved surface and ϕ_t is the geodesic flow on X, this is equivalent to the fact that the lift of the geodesic starting from x ends on a conical point of the boundary $\partial\mathbf{D}$ (see, for example, the book of P. J. Nicholls [Ni89] §8.3, where the points with non empty ω-limit set are called "conservative"). On a compact manifold, all points are of conical type.
The strong stable sets of the flow ϕ_t are defined by :
$W^{ss}(x) := \{ \ y \in X \mid \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0 \ \}$;
$W^{ss}_\varepsilon(x) := \{ \ y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \leq \varepsilon$ for all $t \geq 0 \ \}$.
The strong unstable sets W^{su} and W^{su}_ϵ of ϕ_t are the strong stable sets of the inverse of the flow ϕ_{-t}.

On negatively curved manifolds, the set W^{ss} corresponds to the horosphere passing through the point x. We say that a point $x \in X$ is of *"horospheric type"* if the ω-limit set of its stable set :

$$\omega(W^{ss}(x)) = \{y \in X \mid \exists\ t_i \to \infty,\ \exists\ x_i \in W^{ss}(x),\ s.t.\ \phi_{t_i}(x_i) \to y\}$$

is non-empty. The reader should convince himself that on a negatively curved surface, this amounts to say that the lift of the geodesic starting from x ends on a horospheric point of the boundary $\partial\mathbf{D}$. Note also that a point of conical type is a point of horospherical type : $\omega(x) \subset \omega(W^{ss}(x))$.

Our goal is to prove the following result :

Theorem 1 *Let X be a metric space, ϕ_t a continuous positively and negatively transitive flow defined on X, which admits a local product structure. We suppose that the relation induced by the stable sets W^{ss} is open. Let us moreover assume that the periods of the periodic trajectories of the flow generate a dense subgroup of $\mathbf{R}$: $\overline{< l(\gamma) >} = \mathbf{R}$.*
Then a point x of X is of horospheric type if and only if it has a dense strong stable set : $\overline{W^{ss}(x)} = X$.

Note that no compactness assumption on X is needed. On the other hand, if X is compact, the theorem implies that every point has a dense strong stable set.

We first recall what is a local product structure. Since X is not assumed to be compact, our definition is somewhat weaker than the classical definition by S. Smale [Sm67]. We then prove that periodic points have a dense strong stable set. The last part is dedicated to the proof of the theorem.

2 Terminology

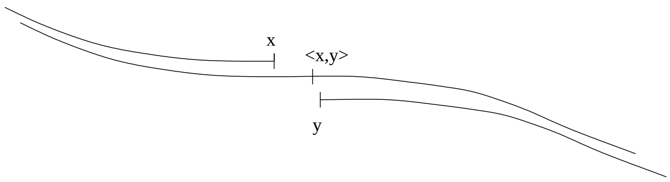
First recall that a flow is said to be *transitive* if it admits a dense orbit. It is said to be positively (resp. negatively) transitive if there exists a point with dense forward (resp. backward) orbit : $\overline{\phi_{\mathbf{R}_+}(x)} = X$ (resp. $\overline{\phi_{\mathbf{R}_-}(x)} = X$). If X has the Baire property, positive transitivity is equivalent to negative transitivity ; this is explained for example in [Wa82]§5.6. Finally, if the flow is transitive, the space X is the closure of an orbit ; so it is connected.

An equivalence relation is said to be *open* if the saturation of any open set is open. In our case, this means that the set $\bigcup_{x\in U} W^{ss}(x)$ is open if U is an open subset of X. This implies the following fact : if $x \in \overline{W^{ss}(y)}$, then $W^{ss}(x) \subset \overline{W^{ss}(y)}$. This regularity hypothesis on the stable sets should be seen as a counterpart of the non-singularity assumption in the context of measurable dynamics : saturation of sets of zero measures have zero measure. Our hypothesis is satisfied, for example, if the stable sets form a continuous foliation.

Finally, the flow is said to admit a *local product structure* if all points $v \in X$ have a neighborhood V which satisfies : for all $\varepsilon > 0$, there exist a positive constant δ, such that for all $x, y \in V$ with $d(x, y) \leq \delta$, there is a point $< x, y > \in X$, a real number t with $|t| \leq \varepsilon$, so that:

$$< x, y > \in \ W^{su}_{\varepsilon}\big(\phi_t(x)\big) \cap W^{ss}_{\varepsilon}(y).$$

This property is summarized by the following picture :



Examples of flows admitting a local product structure include Axiom A flows in restriction to their non-wandering sets, suspension of Markov chains, and geodesic flows on CAT(-1) spaces. Other examples are obtained by lifting these flows to a regular cover. That situation was studied in [C04], and motivated this work.

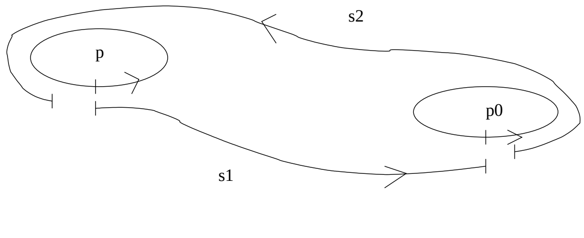
3 Stable sets of periodic points

Lemma 1 *The periodic points of ϕ_t have a dense strong stable set.*

Proof
Let p be a periodic point of ϕ_t ; its period is denoted by $l(p)$.We show that for all $s \in \mathbf{R}$, $\phi_s(p)$ belongs to the closure of $W^{ss}(p)$. Indeed, since W^{ss} defines an open relation, this will imply the inclusion $\phi_s(W^{ss}(p)) \subset \overline{W^{ss}(p)}$. Now, it was shown in [C04] that any point $p \in X$ has a dense weak stable set : $\phi_{\mathbf{R}}(W^{ss}(p))$ is dense in X.
Let us recall the argument : First note that the points on the backward orbit of the negatively transitive point, form a dense set of points with dense backward orbit. Let y a point which belongs to $\overline{\phi_{\mathbf{R}}(W^{ss}(p))}$, and U a neighborhood of y given by the local product structure. The set U contains a point whose forward orbit is asymptotic to p. It also contains a point with backward dense orbit. This backward orbit can be "glued" with the point whose forward orbit is asymptotic to p. The resulting orbit is positively asymptotic to the orbit of p, hence included in $\phi_{\mathbf{R}}(W^{ss}(p))$, and accumulates backward on every point.

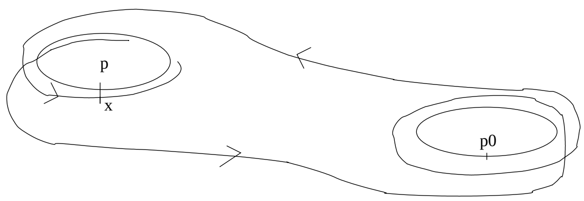
We assumed that the periods of the periodic orbits generate a dense subgroup of $\mathbf{R}$. So, given some $\varepsilon > 0$, there is another periodic point p_0, and integers $m \in \mathbf{Z}, n \in \mathbf{N}$, such that $|ml(p) + nl(p_0)| < \varepsilon$.

The flow ϕ_t is positively transitive, so there exists a piece of orbit starting arbitrarily close to p, and getting close to p_0 after time s_1. There is also an orbit starting very close to p_0 and coming close to p after time s_2.



We use the local product structure to "glue" the backward orbit of p together with the orbit going from p to p_0. We glue the resulting orbit with the periodic orbit defined by p_0. This gives an orbit negatively asymptotic to the orbit of p, which follows the piece of orbit from p to p_0, and then winds around the orbit of p_0. After this orbit has turned k times around the orbit of p_0, we glue it to the orbit going from p_0 to p. The resulting orbit is then glued to the forward orbit of p.

To summarize, we build an orbit negatively asymptotic to the orbit of p, that goes from p to p_0, then winds k times around the periodic orbit defined by p_0, then goes from p_0 to p, and winds around the periodic orbit defined by p. Note that the neighborhoods used for the gluing of the orbits at p and p_0 do not depend on k.



In other words, for all $\varepsilon_1 > 0$, there exist $s_1,\ s_2$ such that for all $k \in \mathbf{N}$, there is an $x \in W^{ss}_{\varepsilon_1}(p)$ and a $s_k \in \mathbf{R}$, with $|s_k - s_1 - s_2| < \varepsilon_1$, such that $\phi_{-s_k - kl(p_0)}(x) \in W^{su}_{\varepsilon_1}(p)$. Let $j \in \mathbf{Z}$ the greatest integer for which $jl(p) < -s_k - kl(p_0)$. Then $\phi_{jl(p)}(x)$ is on $W^{ss}(p)$ and ε_1-close to $\phi_{s_k+kl(p_0)-jl(p)}(p)$. This last point is equal to $\phi_{s_k+kl(p_0)}(p)$; it also coincides with $\phi_{s_k+kl(p_0)+m'l(p)}(p)$, for all integer m'.
So, taking for k and m' multiples of n and m, there exists points on $W^{ss}(p)$ which are very close to $\phi_{s_1+s_2+i(ml(p_0)+nl(p))}(p)$, for all positive integer i. But any point on the orbit of p is ε-close to one of these points. This ends the proof of the lemma.

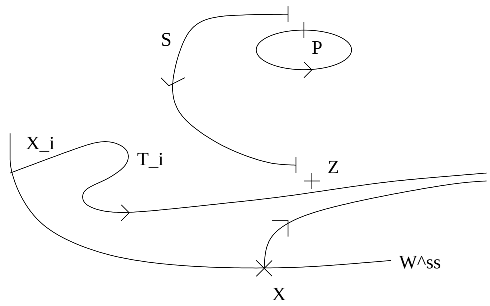
4 proof of the theorem

First, the ω-limit set of the strong stable set of a point $x \in X$ can be rewritten as :

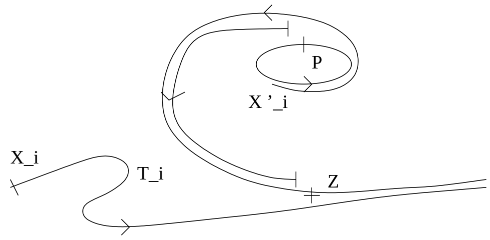
$$\omega(W^{ss}(x)) = \bigcap_{t \geq 0} \overline{\phi_{[t,\infty[}(W^{ss}(x))}.$$

So, if the stable set of x is dense, then $\omega(W^{ss}(x)) = X$.

Let $x \in X$ be a point of horospheric type, and $z \in \omega(W^{ss}(x))$. There exists points $x_i \in W^{ss}(x)$, and a sequence $t_i \to \infty$, such that $\phi_{t_i}(x_i) \to z$ as $i \to \infty$. Let p be some periodic point ; the flow is positively transitive, so there exists a piece of orbit starting arbitrarily close to p and getting close to z after a some time s. Let us choose $i \in \mathbf{N}$ such that $\phi_{t_i}(x_i)$ is very close to z and t_i is greater than s.



We use the local product structure to "glue" the backward orbit of p together with the orbit going from p to z. We glue the resulting orbit with the forward orbit of $\phi_{t_i}(x_i)$. The orbit obtained is positively asymptotic to the orbit of x_i, hence to the orbit of x. It is also negatively asymptotic to the orbit of p.



So there is a point x'_i on this orbit, which is on the strong stable set of x_i. And since $t_i > s$, it is on the first part of the orbit, that is, close to the orbit of p.

In other words, we have found a subsequence $x'_i \in W^{ss}(x)$ such that $d(x'_i, \phi_{\mathbf{R}}(p)) \to 0$. The orbit of p is compact, so upon extracting a subsequence, we can assume that the x'_i converge to some point $\phi_{t_0}(p)$ on the orbit of p. This means that $\phi_{t_0}(p)$ belongs to the closure of $W^{ss}(x)$. Since W^{ss} defines an open relation, the following inclusion holds : $W^{ss}(\phi_{t_0}(p)) \subset \overline{W^{ss}(x)}$. The point $\phi_{t_0}(p))$ is periodic, so by the lemma, its strong stable set is dense and the theorem is proven.

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