Topological dynamics and local product structure

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Abstract

We study topological properties of dynamical systems admitting a local product structure. The results obtained are applied to hyperbolic systems on covers. Our main theorem caracterizes transitivity on solvable covers in term of Frobenius elements of periodic orbits. [1]

1 Introduction

In this article, we study the topological properties of dynamical systems admitting a local product structure. Geodesic flows on negatively curved manifolds, subshifts of finite type and Axiom A flows are examples of such systems. First recall that a flow is said to be *transitive* (positively transitive) if it admits a dense (forward) orbit.

This study is motivated by the following problem : consider an hyperbolic flow on a compact manifold X, and $\pi : \hat{X} \to X$ a regular cover of M with deck transformation group G. The flow may be extended to $\hat{X}$ in a G-equivariant way. What are the properties of ϕ_t which can be lifted to $\hat{X}$? Yves Guivarch [Gu89] showed that in general the system is no longer ergodic on the cover. R. Sharp [Sh93] obtained counting estimates concerning periodic orbits on $\hat{X}$. They both used an hypothesis, which says that one can translate from a given quantity in the fiber, just by turning around a periodic orbit on X. That hypothesis was then used by several people [KS01] [So01], although its dynamical meaning remained unclear.

Our main result shows that this hypothesis is in fact equivalent to the transitivity of the flow on $\hat{X}$. Whereas the flow on $\hat{X}$ is totally dissipative with respect to the lift of any Gibbs measure if $G \neq \mathbf{Z}, \mathbf{Z^2}$, it still admits dense orbits. Its topological behaviour is thus quite different from its behaviour in measure. During our work, we realized that many topological properties of Anosov flows can be proven without any finiteness assumptions (measure, volume, compactness) and can be applied directly to the flow on the cover. The only hypothesis needed is the existence of a weak form of local product structure.

We first begin by explaining what is a local product structure and give examples of flows admitting such a structure. We then show how to derive classical properties of Anosov systems using just this property, before turning to the problem of transitivity on covers.

[1] 37B10, 37D40, 34C28

2 Product structure : definition and examples

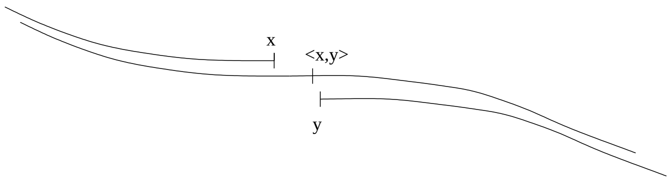
We first define the concept of a local product structure for a flow, and give several examples of such systems.

Let X be a metric space, ϕ_t a continuous flow on X. The strong stable sets of the flow are defined by :
$W^{ss}(x) := \{ \ y \in X \mid \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0 \ \}$;
$W^{ss}_\varepsilon(x) := \{ \ y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \leq \varepsilon$ for all $t \geq 0 \ \}$.
The weak stable sets W^{ws} of ϕ_t are equal to $\phi_{\mathbf{R}}\big(W^{ss}\big)$. One also define the strong unstable sets W^{su} and W^{su}_ε of ϕ_t ; these are the stable sets of ϕ_{-t}. The sets W^{ws} (W^{ss}) will be called the leaves of the weak (strong) stable foliation in the sequel. This abuse of language is motivated by the geometric examples given below.

The flow ϕ_t is said to admit a *local product structure* if all points $v \in X$ have a neighborhood V which satisfies : for all $\varepsilon > 0$, there exist a positive constant δ, such that for all $x, y \in V$ with $d(x, y) \leq \delta$, there is a point $< x, y > \in X$, a real number t with $|t| \leq \varepsilon$, so that:

$$< x, y > \in \ W^{su}_\varepsilon\big(\phi_t(x)\big) \cap W^{ss}_\varepsilon(y).$$

This property is summarized by the following picture :



There is a similar definition for transformations, instead of flows. In this case, there is no shift in time, and the point $< x, y >$ is both on the strong unstable leaf of x and on the strong stable leaf of y.

Subshifts of finite type

Let I be some finite alphabet with n elements, and A a $n \times n$ matrix with elements in $\{0, 1\}$. The subshift with transition matrix A is defined by :

$$\Sigma_A := \{ \ \{x_k\}_{k\in\mathbf{Z}} \in I^{\mathbf{Z}} \mid \forall \ k \in \mathbf{Z}, \ A_{x_k x_{k+1}} = 1\}$$

The shift σ on Σ_A is defined by $\sigma(\{x_i\}) := \{x_{i+1}\}$. In order to obtain a flow, we take some Hölder function $r : \Sigma_A \to \mathbf{R}_+$ depending only on positive coordinates, and build the suspension :

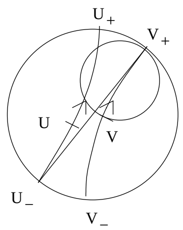
$$\Sigma^r_A := \Sigma_A \times \mathbf{R} / < (x, t) \sim (\sigma(x), t - r(x)) >$$

The flow is the translation on the $\mathbf{R}$-factor ; the product structure is given by :
$$< (\{x_i\}, s), (\{y_i\}, t) > := (\{...x_{-2}x_{-1}x_0, y_1y_2...\}, t)$$
This formula is valid as soon as $x_0 = y_0$ and $|s - t| \ < \ inf \ r$, that means, as soon as the two points are sufficiently close to each other.

Geodesic flows on negatively curved manifolds

In that case, the product structure is even globally defined on the universal cover. Recall that the universal cover of a complete connected manifold, with sectional curvatures bounded away from zero, is a simply connected manifold homeomorphic to a ball. An ideal boundary may be added to this ball, in such a way that there is exactly one geodesic joining two distinct points of the boundary. This is explained by P. Eberlein in [Eb72].

Now, let u and v be two unit vectors in the tangent bundle of the universal cover. They join points of the boundary, say u_- and u_+, v_- and v_+ respectively. The geodesic which goes from u_- to v_+ intersects the stable leaf of v in a single vector. The vector thus obtained is the product of u and v.



The geodesic flow on a two dimensional manifold of curvature -1 can be described algebraically. The unit tangent bundle of the universal cover may be identified with $PSL_2(\mathbf{R})$, and the action of the flow is given by the matrix $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

The results in this article are well known for geodesic flows on negatively curved manifolds. Some go back to E.Hopf [Ho36], others are quite recent (F. Dal'bo [Da01]). The usual proofs rely heavily on the properties of the ideal boundary. Here we avoid their use.

Anosov flows

Our last example is given by Anosov flows, or more generally Axiom A flows. A flow on a compact differentiable manifold is an *Anosov flow* if there exists a splitting of the tangent bundle in two invariant subbundles, transverse to the direction of the flow, that are respectively uniformly contracted and dilated by the differential of the flow. A flow is *Axiom A* if such a splitting exists in restriction to the non-wandering set, and if moreover the periodic points are dense in the non-wandering set. The notion of Anosov and Axiom A diffeomorphisms is defined similarly ; the stable and unstable bundles now span the whole tangent space.

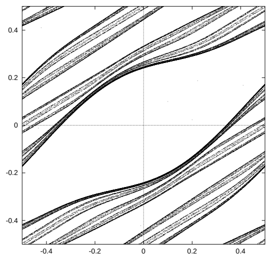
These flows satisfies the shadowing property, which is stronger than the local product structure. The shadowing property, which allows the "gluing" of infinitely many pieces of orbits, was introduced by D. V. Anosov in [An67]. The term "local product structure" appears in the seminal paper of S. Smale [Sm67]. A modern presentation can be found in the book [KH95].

One example of Anosov flow is given by the geodesic flow on the unit tangent bundle of a *compact* negatively curved manifold. Other examples are obtained by suspending Anosov diffeomorphisms, as we did for subshifts. The simplest example of an Anosov diffeomorphism is the action of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ on the quotient space $\mathbf{R}^2/\mathbf{Z}^2$.

Examples of Axiom A flows include flows which are gradient of Morse functions. A less trivial example is obtained by suspending an Axiom A diffeomorphism "derived from Anosov". Such a diffeomorphism is built by perturbing an automorphism of the torus $\mathbf{R}^2/\mathbf{Z}^2$:

$$f(x,y) := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ y/0.4 \times k(x/0.4) \times k(y/0.4) \end{pmatrix}$$

with $k(z) := (1-z^2)^2 \mathbf{1}_{[-1,1]}(z)$ used as a C^1 bump function. The non-wandering set of f is composed of the repelling point $(0,0)$ and of an attractor. The trajectory of any point different from $(0,0)$ accumulates on the attractor. The picture on the right shows the first million iterates of the point $(0.1, 0.1)$.



3 Product structure : first results

We show that a few classical results concerning the topological dynamics of Anosov flows may be deduced from the product structure alone. We first need a lemma which allows to paste finitely many pieces of orbits:

Lemma 1 *Let X be a metric space, ϕ_t a flow on X admitting a local product structure. Let us consider a finite number of points $x_1, x_2 ... x_n$ in X. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all positive numbers $t_1, t_2 ... t_{n-1}$ satisfying $d(\phi_{t_i}(x_i), x_{i+1}) < \delta$, one can find t and x with*

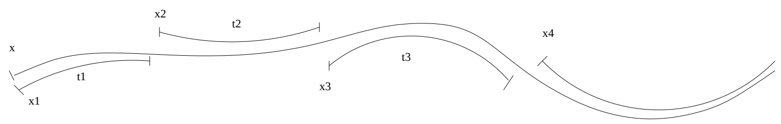
$$|t| < \varepsilon, \quad x \in W^{su}_{\varepsilon}(\phi_t(x_1)), \quad \phi_{\Sigma t_i}(x) \in W^{ss}_{\varepsilon}(x_n).$$

Moreover there are numbers $t'_2 ... t'_{n-1}$ with $|t'_i| < \varepsilon$ such that, for all $k = 2..n-1$,

$$d\big(\phi_{t_1+...+t_{k-1}+t}(x), \phi_{t'_k+t}(x_k)\big) < \varepsilon, \quad \forall\ 0 < t < t_k,$$

$$and \quad d\big(\phi_t(x), \phi_{t'_1+t}(x_1)\big) < \varepsilon, \quad \forall\ 0 < t < t_1.$$

We say that the point x is obtained by gluing together *the pieces of orbits $[x_1, \phi_{t_1} x_1]$, $[x_2, \phi_{t_2} x_2]$,... $[x_{n-1}, \phi_{t_{n-1}} x_{n-1}]$, and the positive orbit of x_n :*



Proof
The local structure around the points x_i is used recursively. One first builds a point y_1 on the strong stable set of x_2 and on the weak stable set of $\phi_{t_1}(x_1)$. The point $\phi_{t_2}(y_1)$ is close to $\phi_{t_2}(x_2)$, hence close to x_3. So there is a point y_2 on the strong stable set of x_3 and on the weak stable set of $\phi_{t_2}(y_1)$. In the same way, we build $y_3 ... y_{n-1}$, such that y_k is on the weak stable set of $\phi_{t_k}(y_{k-1})$, and on the strong stable set of x_{k+1}. The point we are looking for is $x = \phi_{-\Sigma t_i}(y_{n-1})$.

Let us explain in details how to choose the size of the local product structures. At each point x_i, one has to take some ϵ_i and this gives a δ_i.

We begin by taking $\varepsilon_n = \varepsilon/2$. The product structure at x_n gives a δ_n. For the value of ϵ_{n-1}, we take a number less than $\delta_n/2$ and $\varepsilon_n/2$. This gives a δ_{n-1}. Recursively, one chooses ε_k less than $\delta_{k+1}/2$ and $\varepsilon_{k+1}/2^{n-k}$. The δ in the theorem is chosen less than the minimum of the $\delta_i/2$.
The points y_k can now be constructed :
The condition $d(\phi_{t_1}(x_1), x_2) < \delta_1$ gives t''_1, $y_1 \in W^{su}_{\varepsilon_1}(\phi_{t''_1+t_1}(x_1)) \cap W^{ss}_{\varepsilon_1}(x_2)$.
We then have $d(\phi_{t_2}(y_1), x_3) < d(\phi_{t_2}(y_1), \phi_{t_2}(x_2)) + d(\phi_{t_2}(x_2), x_3)$.
The relation $y_1 \in W^{ss}_{\varepsilon_1}(x_2)$ shows that $d(\phi_{t_2}(y_1), \phi_{t_2}(x_2))$ is less than ε_1. So it is less than $\delta_2/2$, as is $d(\phi_{t_2}(x_2), x_3)$. This means that we can apply the local product structure at x_3, and so on. At the end, we obtain numbers t''_k, points y_k such that :

$$|t''_k| < \varepsilon_k, \quad y_k \in W^{su}_{\varepsilon_k}(\phi_{t''_k+t_k}(y_{k-1})) \cap W^{ss}_{\varepsilon_k}(x_{k+1}), \quad with\ \varepsilon_k < \varepsilon/2^{n-k}.$$

The following relations are then an easy consequence of the previous ones :

$$\phi_{-t_k-..-t_{n-1}}(y_{n-1}) \in W^{su}_{\varepsilon_k+..+\varepsilon_{n-1}}(\phi_{t''_k+..+t''_{n-1}} y_{k-1}), \quad 2 \le k \le n-1.$$

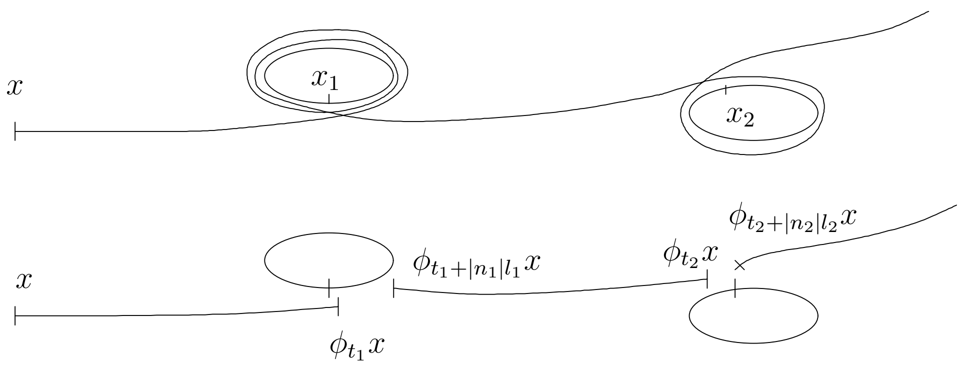
Finally we put $x = \phi_{-t_1-t_2..-t_{n-1}}(y_{n-1})$, $t = t''_1 + t''_2 + .. + t''_{n-1}$ and $t'_i = t''_i + .. + t''_{n-1}$.
The last relations in the theorem follow from the identities :

$$\phi_{t_1+..+t_k}(x) \in W^{su}_{\varepsilon_{n-1}+..+\varepsilon_k}(\phi_{t''_{n-1}+..+t''_k+t_k}(y_{k-1})), \quad y_{k-1} \in W^{ss}_{\varepsilon_{k-1}}(x_k).$$

Theorem 1 *Let X be a metric space, ϕ_t a flow on X admitting a local product structure. Let us suppose that the subgroup of $\mathbf{R}$ generated by the lengths of the periodic orbits of the flow is dense in $\mathbf{R}$: $\overline{< l(\gamma) >} = \mathbf{R}$. Then the strong stable sets of the points with dense forward orbit are dense in X.*

Proof
Given some point $x \in X$ with $\overline{\phi_{\mathbf{R}_+}(x)} = X$, we show that for any $t \in \mathbf{R}$, the strong stable foliation of x accumulates on $\phi_t(x)$. Given some $\varepsilon > 0$, there are periodic points x_i, $i = 1..m$, with period l_i and integers n_i such that $\mid t - \sum n_i l_i| < \varepsilon$.

The trajectory of x is dense ; so there is a time $t_1 > 0$ such that $\phi_{t_1}(x)$ is δ-close to x_1, and stays δ-close to the orbit of x_1 during a time equal to $|n_1|l_1$. In other words, the orbit of x winds n_1 times around the periodic orbit of x_1. Since the orbit of x is dense, after some time, say t_2, it comes δ-close to x_2, and stays δ-close to the orbit of x_2 during a time greater than $|n_2|l_2$. So we can find an increasing sequence of times t_i, such that $\phi_{t_i}(x)$ stays δ-close to x_i during time $|n_i|l_i$. The orbit of x winds successively around the periodic orbits defined by the points x_i, as illustrated below :



The goal is to shorten or lengthen the orbit of x, by adding or removing $|n_i|$ occurences of the periodic orbit of x_i, in the orbit of x. We first remark that the orbit of x is very close to a piecewise orbit, in which the parts of the orbits of x winding around x_i is replaced by a multiple of the periodic orbit x_i.

Consider the pieces of orbits with endpoints :
$[x, \phi_{t_1}(x)]$, $[x_1, \phi_{|n_1|l_1}(x_1)]$, $[\phi_{t_1+|n_1|l_1}(x), \phi_{t_2}(x)]$, $[x_2, \phi_{|n_2|l_2}(x_2)]$, ...
... $[\phi_{t_{i-1}+|n_{i-1}|l_{i-1}}(x), \phi_{t_i}(x)]$, $[x_i, \phi_{|n_i|l_i}(x_i)]$, ...
$[\phi_{t_{m-1}+|n_{m-1}|l_{m-1}}(x), \phi_{t_m}(x)]$, $[x_m, \phi_{|n_m|l_m}(x_m)]$,
and the positive orbit of $\phi_{t_m+|n_m|l_m}x$.
Here, the segment $[x_i, \phi_{|n_i|l_i}(x_i)]$ represents the periodic orbit of x_i, with multiplicity n_i.

These pieces of orbits follow closely the trajectory of x. We may paste these pieces of orbits together, by applying Lemma 1, and obtain a point whose orbit is ε-close to the orbit of x, and asymptotic to the orbit of x, both for negative and positive time. The point on the strong stable set of x, which lies on this orbit, is close to x. In fact, if we had assumed expansivity, this point would have been equal to x ; but the expansivity property is not needed here.

We glue instead the pieces :
$[x, \phi_{t_1}(x)]$, $[x_1, \phi_{(|n_1|+n_1)l_1}(x_1)]$, $[\phi_{t_1+|n_1|l_1}(x), \phi_{t_2}(x)]$, $[x_2, \phi_{(|n_2|+n_2)l_2}(x_2)]$, ...
... $[\phi_{t_{i-1}+|n_{i-1}|l_{i-1}}(x), \phi_{t_i}(x)]$, $[x_i, \phi_{(|n_i|+n_i)l_i}(x_i)]$, ...
$[\phi_{t_{m-1}+|n_{m-1}|l_{m-1}}(x), \phi_{t_m}(x)]$, $[x_m, \phi_{(|n_m|+n_m)l_m}(x_m)]$,
and the positive orbit of $\phi_{t_m+|n_m|l_m}x$.

This means that the piece corresponding to the periodic orbit of x_i is removed, or replaced by twice the orbit of x_i, depending on the sign of n_i.

Lemma 1 gives a point y and a time t', $|t'| < \varepsilon$, such that :

$$y \in W^{su}_{\varepsilon}(\phi_{t'}(x)), \quad \phi_p(y) \in W^{ss}_{\varepsilon}(\phi_{t_n+|n_m|l_m}(x)),$$

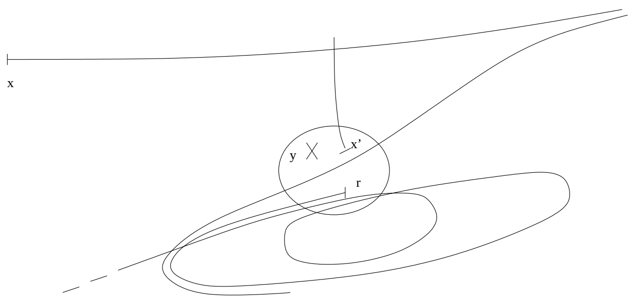
with $p = t_1+(|n_1|+n_1)l_1+\sum\Big(t_i - (t_{i-1} + |n_{i-1}|l_{i-1}) + (|n_i| + n_i)l_i\Big)+t_m+|n_m|l_m$, that is, $p = \sum n_i l_i \ + \ t_m + |n_m|l_m$.
The relations $y \in W^{su}_{\varepsilon}(\phi_{t'}(x))$ and $|\ t - \sum n_i l_i| < \varepsilon$ show that $\phi_{\Sigma n_i l_i}(y)$ is close to $\phi_t(x)$, whereas the relation $\phi_p(y) \in W^{ss}(\phi_{t_m+|n_m|l_m}(x))$ shows that $\phi_{\Sigma n_i l_i}(y)$ belongs to $W^{ss}(x)$. So we have shown that there is a point on $W^{ss}(x)$ arbitrary close to $\phi_t(x)$.

Note that when gluing the point $\phi_{n_i l_i}(x_i)$ $(= x_i)$ with the point $\phi_{t_i+n_i l_i}(x)$, one should use the product structure at x_i, not at $\phi_{t_i+n_i l_i}(x)$, so that the $\varepsilon_i,\ \delta_i$ do not depend on $t_i,\ n_i$. This ends the proof.

Recall now that the weak stable foliation W^{ws} of the flow is said to be *minimal* if for all $x \in X$, the leaf $W^{ws}(x)$ is dense in X.

Theorem 2 *Let X be a metric space, ϕ_t a flow on X admitting a local product structure. Let us suppose that the recurrent points are dense in X. Then for all points $x \in X$, the closure of the weak stable leaf of x, $\overline{W^{ws}(x)}$, is both open and closed in X. So, if X is connected, the weak stable foliation is minimal.*

Proof

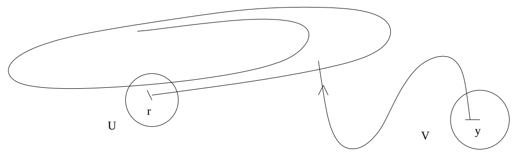


Given $y \in \overline{W^{ws}(x)}$, we choose a neighborhood of local product structure around y. There is a point y' in this neighborhood, which is on $W^{ws}(x)$. It is sufficient to prove that the recurrent points close to y belong to $W^{ws}(x)$. Let r be a recurrent point (for negative times) in the product structure neighborhood of y. There is an orbit asymptotic to the orbit of y' for positive times, and asymptotic to the orbit of r for negative times. The closure of this orbit belongs to $\overline{W^{ws}(x)}$, and contains r, since the orbit comes arbitrarily close to r for some negative sequence of times.

The next result is a topological version of the Hopf argument.

Theorem 3 *Let X be a Baire metric space, ϕ_t a flow on X admitting a local product structure. Then the flow ϕ_t is positively transitive if and only if the recurrent points are dense in X and X is connected.*

Proof
First note that the Baire property implies that a flow is positively transitive if and only if two arbitrary open sets U and V are positively connected by the flow. This fact is proved, for example, in the book of Walters [Wa82] §5.6. Since a flow is invertible, it is also negatively transitive. So it admits a dense set of positive or negative recurrent points. In fact, the points on the orbit of a point with dense forward orbit, are all positively recurrent points.



Let r be a positively recurrent point in U. Its weak stable leaf is dense, by Theorem 2. So there is a t and a $y \in V$ which belongs to $W^{ss}(\phi_t(r))$. The iterates of y come arbirarily close to the iterates of r, hence close to r. This shows that there is an iterate of V intersecting U.

4 The Anosov closing lemma

We want to give a converse to Theorem 1. In order to do so, we need an additional assumption, which allows to glue an orbit with itself. This property was introduced by D. V. Anosov in [An67].

Definition 1 *A flow ϕ_t on a metric space X satisfies the* Anosov closing lemma *if for all points $v \in X$, there exists a neighborhood V which satisfies :*
for all $\varepsilon > 0$, there exists a $\delta > 0$ and a $t_0 > 0$ such that for all $x \in V$ and $t > t_0$ with $d(x, \phi_t x) < \delta$ and $\phi_t x \in V$, there exists x_0 and l, with $|l - t| < \varepsilon$, $\phi_l x_0 = x_0$, and $d(\phi_s x_0, \phi_s x) < \varepsilon$ for $0 < s < min(t, l)$.

For such a flow, the closure of the recurrent points coincides with the closure of the periodic points. The next result relates the behaviour of the lengths of periodic orbits with the weak mixing property.

Theorem 4 *Let X be a locally complete metric space, ϕ_t a positively transitive flow on X satisfying the Anosov closing lemma. Then the density of the subgroup generated by the lengths of periodic orbits is equivalent to the weak-mixing property : There is no non-constant S^1-valued continuous function F and number θ satisfying $F \circ \phi_s = e^{2\pi i \theta s} F$.*

Proof
If F, θ exist, then applying the relation to the periodic points shows that the lengths belong to $\theta^{-1}\mathbf{Z}$.

If the lengths belong to $\theta^{-1}\mathbf{Z}$, one builds F as in the Livsic theorem [Li71] : the function F is defined on the orbit of a positively transitive point y_0 by the relation $F(\phi_t y_0) = e^{2\pi i \theta t}$. It remains to extend F to X.
Let v be a point on the forward orbit of y_0, we show that F is uniformly continuous on the part of the orbit included in the neighborhood V given by the closing lemma. Given some $\varepsilon > 0$, the closing lemma gives a t_0 and a δ. If $\phi_{t_1} y_0$, $\phi_{t_2} y_0$ are in V and $\delta/2$-close, there exists t_3 larger than $t_0 + t_1$, $t_0 + t_2$ satisfying $d(\phi_{t_1} y_0, \phi_{t_3} y_0) < \delta/2$.
The closing lemma now gives the inequalities :

$$|(t_3 - t_1) - l| < \varepsilon, \text{ for some } l \in \theta^{-1}\mathbf{Z},$$
$$|(t_3 - t_2) - l'| < \varepsilon, \text{ for some } l' \in \theta^{-1}\mathbf{Z}.$$

Together with the definition of F, we finally obtain :

$$|F(\phi_{t_1} y_0) - F(\phi_{t_2} y_0)| < |e^{2\pi i \theta((t_3 - t_1) - l + (t_3 - t_2) - l')} - 1| \; < \; \pi\theta\varepsilon$$

So the function F may be extended on V. From these local extensions, we obtain a globally defined continuous function satisfying the required relation. We note that this function is locally uniformly continuous. This ends the proof of the theorem.

Lemma 2 *Let X be a locally complete metric space, ϕ_t a flow on X. If there is a point x with a dense strong stable leaf, and whose forward orbit admits an accumulation point, then there is no locally uniformly continuous non-constant function F and number θ satisfying. $F \circ \phi_s = e^{2\pi i\theta s} F$.*

Proof
Let us assume that there exists a locally uniformly continuous function F and a θ satisfying $F \circ \phi_s = e^{2\pi i\theta s} F$.

Let y be the accumulation point of the forward orbit of x, and let U be a neighborhood of y, on which F is uniformly continuous. There is a sequence s_i going to infinity, such that $\phi_{s_i} x$ converges to the accumulation point. Let y be a point on the strong stable leaf of x. If i is large enough, $\phi_{s_i} y$ belongs to U and is close to $\phi_{s_i} x$. so $|F(\phi_{s_i} y) - F(\phi_{s_i} x)| = |F(y) - F(x)|$ is close to zero. This means that F is constant on the strong stable foliation of x, hence everywhere constant.

The two preceding results now give a converse to Theorem 1. First recall that the strong stable foliation W^{ss} is *transitive* if there exists a point with a dense strong stable leaf.

Theorem 5 *Let X be a locally complete metric space, ϕ_t a positively transitive flow on X, which satisfies the Anosov closing lemma. If the strong stable foliation of ϕ_t is transitive, then the lengths of the periodic orbits generate a dense subgroup of* $\mathbf{R}$.

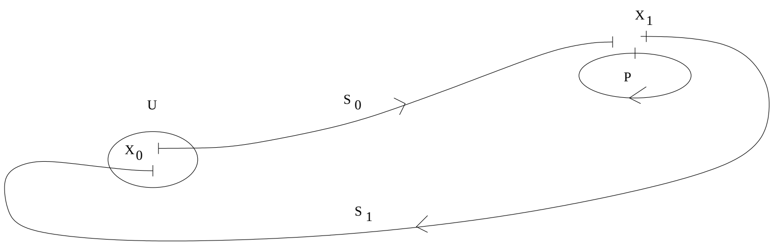
proof
A locally complete metric space has the Baire property. So, the set of points with a dense strong stable leaf is a G_δ-dense set, by transitivity of W^{ss}. The set of points with a dense forward orbit is also a G_δ-dense set. So there exists a point with a dense strong stable leaf, and with a forward orbit accumulating everywhere. Lemma 2 shows that the flow is weak mixing, and the proof of Theorem 4 implies that the the lengths of the periodic orbits generate a dense subgroup of $\mathbf{R}$.

Topological mixing

This raises the question of the mixing of the flow : a flow is said to be topologically mixing if for all non empty open sets U, V, there is some $t_0 > 0$ such that $\phi_t(U)$ intersects V for all $t > t_0$. Does the weak mixing property imply the mixing property ? We show that if the flow has a local product structure and satisfies the Anosov closing lemma, this is indeed the case.

Lemma 3 *Let X be a metric space, ϕ_t a positively transitive flow on X admitting a local product structure and satisfying the Anosov closing lemma. If the lengths of the periodic orbits of the flow generate a dense subgroup of $\mathbf{R}$: $\overline{< \{l(\gamma)\} >} = \mathbf{R}$, then the lengths of periodic orbits intersecting some fixed open set U also generate a dense subgroup of* $\mathbf{R}$.

Proof
Given some periodic point p of period l, and some $\varepsilon > 0$, we show that there is an $s > 0$ and periodic orbits l_n intersecting U with periods ε-close to $s + nl$, for all integers n. This will imply that the group generated by the lengths of the periodic orbits intersecting U contains elements 2ε-close to any element of $\{l(\gamma)\}$.



Let U' a small local product neighborhood at p. The flow is positively transitive, so there is a point $x_0 \in U$ and a number $s_0 > 0$ such that $\phi_{s_0}(x_0)$ belongs to U'. There is also a point x_1 in U' and a number $s_1 > 0$ with $\phi_{s_1}(x_1) \in U$. We consider the orbit of x_0, followed by the orbit of p, of period nl, and finally the orbit of x_1. This gives pieces of orbits :

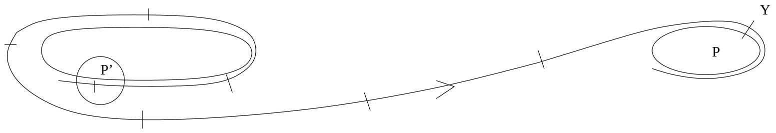
$$[x_0, \phi_{s_0}(x_0)], [p, \phi_{nl}(p)], [x_1, \phi_{s_1}(x_1)]$$

which may be glued together using Lemma 1. We can now apply the Anosov closing lemma to the orbit thus obtained, to create a periodic orbit intersecting U, with length close to $s_0 + s_1 + nl$.

Theorem 6 *Let X be a metric space, ϕ_t a positively transitive flow on X admitting a local product structure and satisfying the Anosov closing lemma. Then the flow ϕ_t is mixing as soon as it is weak mixing.*

Proof
Let p be a periodic point of the flow, with period l ; we show that the strong stable set of p is dense in X; given some ball $B(x,\varepsilon) \subset X$, the strong stable set of p intersects this ball.

First note that if a point y belongs to the strong stable set of p, then all the points $\phi_{-nl}(y)$, $n \in \mathbf{N}$, are on the strong stable set of p.

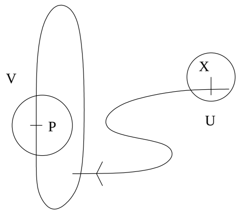


Recall that if E is a subset of $\mathbf{R}$, then its projection in $\mathbf{R}/l\mathbf{Z}$ generates a dense subgroup of $\mathbf{R}/l\mathbf{Z}$ if and only if $\cup_{l' \in E} \mathbf{N}l'$ is dense in $\mathbf{R}/l\mathbf{Z}$. We take for E the set of lengths of periodic orbits intersecting the ball $B(x,\varepsilon)$. By Lemma 3, it generates a dense subgroup of $\mathbf{R}$. Hence, there is a periodic point p' of period l' in $B(x,\varepsilon)$ so that the distance (as subsets of $\mathbf{R}$) between $\mathbf{Z}l'$ and $\mathbf{N}_+ l$ is less than $\varepsilon/2$.

Consider an orbit which is backward asymptotic to the orbit of p' and forward asymptotic to the orbit of p. This orbit can be obtained as follows : by transitivity, there exists a piece of orbit starting close to p' and ending close to p. We now use Lemma 1 to glue this orbit with the backward orbit of p' and the forward orbit of p.

Let y be a point on the orbit obtained, which belongs to the strong stable set of p. The points $\phi_{-nl}(y)$, $n \in \mathbf{N}$, wind around the periodic orbit of p', and come ε-close to any point on the orbit of p'. This shows that one of them belongs to $B(x,\varepsilon)$. The proof of the mixing property ends as in [Da01]. We recall the argument for sake of completeness.

If the flow is not mixing, there exist open sets U, V and a sequence $t_i \to \infty$ such that $\phi_{t_i}(U) \cap V = \emptyset$. Given some periodic orbit p in V of length l, the sequence t_i can be written $t_i = r_i l + s_i$ with s_i, $r_i \in \mathbf{N}$, $0 < s_i < l$. Taking some subsequence, we can suppose that s_i converge to s. The strong stable set of $\phi_{-s}(p)$ intersects U in a point x. This means that $d(\phi_{t_i}(x), \phi_{t_i - s}(p)) \to 0$ as $i \to \infty$. On the other end, $\phi_{t_i - s}(p) = \phi_{r_i l + s_i - s}(p) = \phi_{s_i - s}(p)$. That point comes close to p as i tends to infinity. Hence $\phi_{t_i}(U)$ intersects V.



5 Fiber bundles and covering spaces

We now study systems which are defined on covering spaces. Let ϕ_t be a flow on a metric space X, and $\pi : \hat{X} \to X$ a regular cover of X, with deck transformation group G. We suppose that the flow may be lifted on $\hat{X}$ in an equivariant way. What can be said about the transitivity of the flow on $\hat{X}$, and the transitivity of the stable foliation on the cover ?

We first note that if (X, ϕ_t) is a suspension of a subshift, an hyperbolic flow or a geodesic flow on a negatively curved manifold, then the product structure and the closing lemma lift to the cover : the flow $\hat{\phi}_t$ admits a local product structure and satisfies the Anosov closing lemma. For the subshift or the hyperbolic flow, this is a consequence of the compactness of X. For the geodesic flow, this comes from the fact that the product structure was obtained from the universal cover ; moreover the closing lemma is true on any CAT(-1)-space.

The theory of covering spaces usually deals with connected covers on locally path-connected spaces and proceeds by showing that these covers are classified by the fundamental group of the base space. We want to study coverings defined over suspensions of subshifts of finite type. These spaces are not locally path-connected, so that the fundamental group approach cannot be used here. Still there is a classification result that admits a simple proof. If σ is a homeomorphism over a topological space M, we set $M_\sigma := M \times \mathbf{R}/(x,t) \sim (\sigma(x), t-1)$.

Theorem 7 *Let M be a paracompact space with zero first Cech cohomology group, and σ be an homeomorphism defined on M. Let M_σ be the suspension of M associated to σ, and G a polish group.*
Then the principal G-fiber bundles over M_σ, up to isomorphism, are in one-to-one correspondance with continuous functions from M to G, up to σ-cohomology.

$$\begin{array}{ccl} C(M,G) \ / \ (f \sim h \circ \sigma \ f \ h^{-1}) & \to & \{G - fiber \ bundles \ over \ M_\sigma\} \\ f & \to & M \times \mathbf{R} \times G \ / \ (x,t,g) \sim (\sigma(x), t-1, f(x)g) \end{array}$$

Remarks
The preceding theorem deals with constant suspensions over a paracompact space ; there is a similar correspondance when the suspension is not constant. Changing the suspension function does not change the topology of the base space.

A subshift of finite type is totally disconnected ; so its first Cech cohomology group is zero. Regular covers correspond to principal bundles with discrete group G. Hence, coverings over suspensions of subshifts of finite type are classified by G-valued locally constant functions on the subshift, up to cohomology.

We can now explain what is the relation between a skew-product over a subshift Σ_A and a regular cover over a suspension of the subshift. Set $T(x,g) := (\sigma(x), f(x)g)$, $\tilde{T}(x,t,g) := (\sigma(x), t-1, f(x)g)$. The action of the flow on the cover defined by f is given by translation on the second factor of :

$$\Sigma_A \times \mathbf{R} \times G \ / \ <\tilde{T}>$$

whereas the skew product action is given by the action of T on :

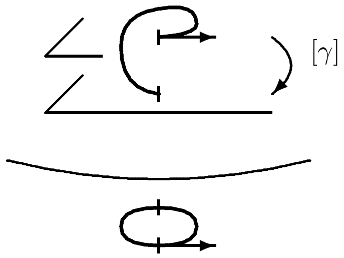
$$\Sigma_A \times G$$

These two actions are dual with respect to each other. In practice, it is more natural to deduce properties of the skew-product from results concerning $\mathbf{Z}$-actions on covers.

Proof
Set $\hat{M}_f := M \times \mathbf{R} \times G \ / \ (x,t,g) \sim (\sigma(x), t-1, f(x)g)$. The map $(x,t,g) \to (x,t,h(x)g)$ gives an isomorphism between the fiber bundles $\hat{M}_f$ and $\hat{M}_{h \circ \sigma \ f \ h^{-1}}$. Cohomologous functions give isomorphic bundles. Note also that $\hat{M}_f$ and $\hat{M}'_f$ are isomorphic if and only if f and f' are cohomologous, since maps of the form $(x,t,g) \to (x,t,h(x)g)$ are the only one compatible with the projection on M_σ and the right action of G.

Let $\hat{M}$ be a fibre bundle over M_σ ; it can be pulled back to $M \times [0,1]$ via the projection from $M \times [0,1]$ to M_σ. The annulation of the first cohomology group of M implies that all fiber bundles over $M \times [0,1]$ are trivial. So the bundle $\hat{M}$ is obtained from the trivial bundle on $M \times [0,1]$ by gluing the fiber over $M \times \{1\}$ to the fiber over $M \times \{0\}$ via some G-valued continuous function on M. This ends the proof.

In order to state the main result, a definition is needed. Let $\pi : \hat{X} \to X$ be a regular cover over a metric space X. The lift of a periodic orbit γ is not necessarily periodic. If $\hat{x}$ is a point on $\hat{X}$, whose trajectory projects on γ, there is an element of G denoted by $[\hat{x}]$ such that $\hat{\phi}_{l(\gamma)}(\hat{x}) = [\hat{x}](\hat{x})$. The Frobenius element of γ is the conjuguacy class of $[\hat{x}]$, and is denoted by $[\gamma]$.



Theorem 8 *Let X be a Baire metric space, $\pi : \hat{X} \to X$ a regular cover of X with deck transformation group G. We suppose that G is a solvable group of finite type. Moreover the space $\hat{X}$ carries a G-equivariant flow $\hat{\phi}_t$, which admits a local product structure, and satisfies the Anosov closing lemma. The quotient flow on X is denoted by ϕ_t. Then the following statements are equivalent:*

- *$\hat{\phi}_t$ is positively transitive ;*
- *$\hat{X}$ is connected and the periodic points of $\hat{\phi}_t$ are dense in $\hat{X}$;*
- *$\{[\gamma]\} = G$ and ϕ_t is positively transitive.*

If any of these statements holds, the weak stable foliation is minimal. If moreover the lengths of periodic orbits of $\hat{\phi}_t$ on $\hat{X}$ generate a dense subgroup of $\mathbf{R}$, then the strong stable foliation of $\hat{\phi}_t$ is transitive, and the flow $\hat{\phi}_t$ is mixing.

Remark The strong stable foliation $\hat{W}^{ss}$ can be transitive even if the flow $\hat{\phi}_t$ is not transitive (see [C02]).

Proof
A solvable cover is just a succession of abelian covers. So it suffices to prove the theorem for G abelian.

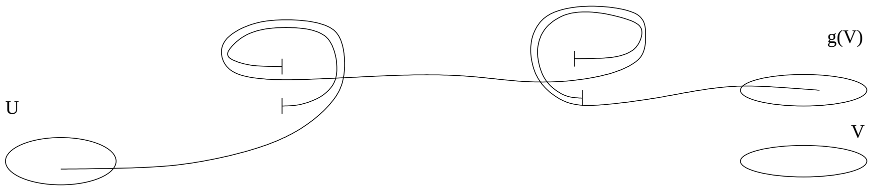
The equivalence between the two first statements is given by Theorem 3 and the Anosov closing lemma, which allows to find periodic points arbitrary close to any recurrent point.

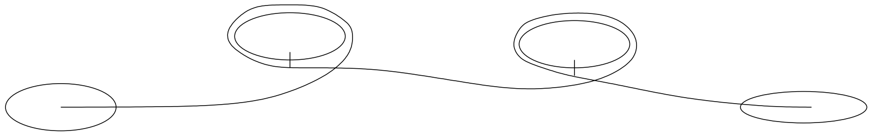
Assume that the flow $\hat{\phi}_t$ is transitive ; given any element $g \in G$ and any point y_1 on $\hat{X}$, we can find a piece of orbit starting close to y_1 and ending close to $g(y_1)$. Consider the projection on X of this piece of orbit ; the Anosov closing lemma may be used to close this piece of orbit, and thus gives a periodic orbit on X with Frobenius element equal to g. Hence, $\{[\gamma]\} = G$.

So, we must show that if ϕ_t is transitive and $\{[\gamma]\} = G$, then $\hat{\phi}_t$ is transitive. We show the following : if there exists finitely many periodic orbits γ_i on X such that $\sum \mathbf{N}_+[\gamma_i] = G$, then the flow is positively transitive. Given two open sets U, V in $\hat{X}$, we must find a trajectory going from U to V.

The proof is similar as the one of Theorem 1 : we consider an orbit that turns around the periodic orbits γ_i, and we add additional copies of the periodic orbits to get the correct translation along the fiber.

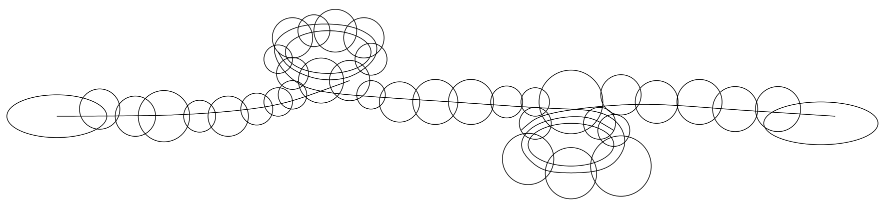
Consider the following orbits on X : first, an orbit starting in $\pi(U)$ which is positively asymptotic to γ_1 ; then an orbit negatively asymptotic to γ_1 and positively asymptotic to γ_2 ; an orbit negatively asymptotic to γ_2 and positively asymptotic to γ_3, and so on. The last one is negatively asymptotic to γ_n and crosses $\pi(V)$. The existence of these orbits is guaranted by the transitivity of ϕ_t and the projected local product structure on X.





Using Lemma 1, these orbits may be glued so as to give an orbit starting in $\pi(U)$, getting very close to γ_1, then γ_2, ...γ_n and finally ending in $\pi(V)$. This orbit may be lifted to an orbit starting in U, ending in $g(V)$, for some element $g \in G$. The element $-g$ may be written as $-g = \sum n_i[\gamma_i]$, n_i positive integers. We may now glue n_i copies of γ_i, $i = 1..n$ to the orbit on X, as we did in the proof of Theorem 1, to obtain an orbit which turns n_i times more around γ_i, $i = 1..n$. The lift of this new orbit goes from U to V.

It remains to check that an orbit on X obtained by gluing some piecewise orbit, has a lift that translates from the same amount as the lift of the piecewise orbit. The orbits used in the pasting scheme, namely the orbits negatively asymptotic to a periodic orbit γ_i and positively asymptotic to another, only cross finitely many local trivializations charts. So there is an ε such that any curve ε-close to any of these orbits has a lift which translates from the same amount as the orbit it ε-shadows.



This ends the proof.

These results have a counterpart for $\mathbf{Z}$ or $\mathbf{R}^d$-actions. One can also look at the case of general fiber bundles. Some hölder condition on the bundle is then needed to control the displacement in the fiber. In that direction, we state a result concerning skew products over subshifts of finite type ; the proof is an easy adaptation of the previous methods.

Theorem 9 *Let G be a polish solvable group, $f : \Sigma_A \to G$ a Holder function defined on an irreductible subshift of finite type. The shift on Σ_A is denoted by σ and we set $f^p(x) := f(\sigma^{p-1}(x))...f(\sigma(x))f(x),\ x \in \Sigma_A$.*
Then the transformation $T(x,g) := (\sigma(x), f(x)g)$ on $\Sigma_A \times G$ is transitive if and only if the set $\{f^p(x) \mid x\ p-periodic\ for\ \sigma\}$ is dense in G.

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