BSDEs of Counterparty Risk

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Abstract

We study a BSDE with random terminal time that appears in the modeling of counterparty risk in finance. We model the terminal time as an invariant time, i.e. a time such that local martingales with respect to a reduced filtration and a possibly changed probability measure, once stopped right before that time, stay local martingales with respect to the original model filtration and probability measure. Using an Azéma supermartingale characterization of invariant times, we establish the equivalence between the original and a reduced BSDE.

Keywords: BSDE, Progressive enlargement of filtration, Counterparty risk.

Mathematics Subject Classification: 60H10, 60G44, 91G40.

1 Introduction

We study a backward stochastic differential equation (BSDE) with random terminal time $\vartheta = T \wedge \theta$, where T is a positive constant and the stopping time θ has an intensity. This BSDE, introduced in Crépey (2015) for the modeling of bilateral counterparty risk in finance, is a generalization of various pricing formulas in the credit risk literature. One can mention as seminal references Duffie, Schroder, and Skiadas (1996) or Collin-Dufresne, Goldstein, and Hugonnier (2004). See also the recovery valuation formula in the classical reduced-form approach to credit risk, where θ is modeled as doubly stochastic (or Cox) time or, more broadly, as a pseudo-stopping time in the sense of Nikeghbali and Yor (2005) (cf. Bielecki and Rutkowski (2001, Corollary 5.1.3), Bielecki, Jeanblanc, and Rutkowski (2009, Lemma 3.1.3) or Crépey, Bielecki, and Brigo (2014, Lemma 13.7.5 page 331)). In the case of counterparty risk, the fact that we deal with an equation (BSDE) rather than with an explicit formula is due to the nonlinear funding issue that accompanies bilateral counterparty risk (see Crépey et al. (2014, Chapter 4)).

Independent of the financial motivation that motivates this work, BSDEs with random terminal time (given as first exit time of a domain) were first introduced in Darling and Pardoux (1997), in order to give a BSDE formulation to a semilinear elliptic PDE. Other studies in a related spirit include Briand and Hu (1998) or Royer (2004). Toldo (2006) studies the stability in the BSDEs with random terminal time with respect to the driving noise and the random terminal time. More specifically, the problem of studying a BSDE under a enlargement of filtration has also been considered in the literature. Motivated by the question of hedging of defaultable claims, Blanchet-Scalliet, Eyraud-Loisel, and Royer-Carenzi (2010) proved existence and uniqueness of the solution to a BSDE with random terminal time under the density hypothesis of Jacod (1987) in a progressively enlarged Brownian filtration. For insider trading modeling (of American options in particular), Eyraud-Loisel and Royer (2010) treated the problem, posed in the initial enlargement of some reference filtration, by changing the measure to a probability that makes the reference filtration and the random time independent (such a changed measure always exists in the case of density times). Other work related to BSDE and change of filtration has been developed more recently by Kharroubi and Lim (2012) and Jiao, Kharroubi, and Pham (2013), who use a decomposition method of a solution to a BSDE between successive marked default times in order to reduce a multiple default risk BSDE to a family of Brownian BSDEs. By this method they are able to solve a variety of optimal investment, utility or hedging problems.

1.1 Contributions and Outline of the Paper

Our approach is to reduce the original counterparty risk BSDE to a simpler BSDE with relative to a smaller filtration and a possibly changed probability measure. Moreover, we want to achieve this under minimal assumptions on θ, so that the model stays as flexible as possible in view of applications. This requires a relaxation of the basic immersion conditions of Crépey (2015), which leads us to model θ as an invariant time in the sense of Crépey and Song (2014), i.e. a time such that local martingales with respect to a reduced filtration and a possibly changed probability measure, once stopped right before that time, stay local martingales with respect to the original model filtration and probability measure. Assuming θ invariant, we show the equivalence between the original and the reduced BSDE. Beyond its theoretical interest and its implications regarding existence and uniqueness of solutions, the reformulation of the original counterparty risk BSDE as a reduced BSDE also gives increased perspectives from the point of view of numerical solutions (see e.g. Crépey and Song (2015)). This also generalizes various credit risk recovery pricing formulas in the literature, with a recovery that is both nonpredictable and implicit—implicit or recursive in the original terminology of Duffie et al. (1996) or Collin-Dufresne et al. (2004).

The paper is organized as follows. In Sect. 2, after a presentation of the financial motivation (Sect. 2.1), the BSDE of counterparty risk is studied. The original (called "full") BSDE is rewritten in terms of an auxiliary BSDE with solution continuous at ϑ (Sect. 2.2). Both equations are posed with respect to a common stochastic basis $(\Omega, \mathbb{G}, \mathbb{Q})$. Sect. 3 starts with a short review of the theory of progressive enlargement of filtration under a condition (B) relative to a smaller filtration $\mathbb{F}$ and under a stronger condition (A) also involving a changed probability measure $\mathbb{P}$. Under the condition (B), we reduce the auxiliary $(\mathbb{G}, \mathbb{Q})$ BSDE with random terminal time ϑ to an $(\mathbb{F}, \mathbb{Q})$ BSDE with a null terminal condition at the fixed time T (Sect. 3.1). Under the condition (A), an even simpler $(\mathbb{F}, \mathbb{P})$ BSDE is obtained (Sect. 3.2) and discussed (Sect. 3.3). The equivalence between the full and the reduced BSDEs is first established in the special case where the data only depend on the value of the solution, but Sect. 4 shows that this equivalence is also valid when the data also depend on the integrands in a martingale representation of the solution, as further developed on an example in Sect. 4.1. The concluding section Sect. 5 draws the practical consequences of our study for the motivating counterparty and credit risk problems. Appendix A deals with measurability issues in relation with single step martingales that appear through the terminal condition of the original full BSDE. An index of symbols is provided after the bibliography.

1.2 Standing Assumptions and Notation

The real line, half-line and the nonnegative integers are respectively denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$; $\mathcal{B}(\mathbb{R}^k)$ is the Borel σ algebra on $\mathbb{R}^k$ ($k \in \mathbb{N}$); $\boldsymbol{\lambda}$ is the Lebesgue measure on $\mathbb{R}_+$. We work on a space Ω equipped with a σ-field $\mathcal{A}$, a probability measure $\mathbb{Q}$ on $\mathcal{A}$ and a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ of sub-σ-fields of $\mathcal{A}$, satisfying the usual conditions. We use the terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer (1975) and He, Wang, and Yan (1992). Footnotes are used for referring to comparatively standard results. We denote by $\mathcal{P}(\mathbb{F})$, $\mathcal{O}(\mathbb{F})$ and $\mathcal{R}(\mathbb{F})$ the predictable, optional and progressive σ-fields with respect to a filtration $\mathbb{F}$. For any semimartingale Y and predictable, Y integrable process L, the stochastic integral process of L with respect to Y is denoted by $\int_0^{\cdot} L_t dY_t = \int_{(0,\cdot]} L_t dY_t = L \centerdot Y$, with the usual precedence convention $KL \centerdot Y = (KL) \centerdot Y$ if K is another predictable process such that KL is Y integrable. For any càdlàg process Y, for any random time τ (nonnegative random variable), $\Delta_\tau Y$ represents the jump of Y at τ. Like Dellacherie and Meyer (1975) or He et al. (1992), we use the convention that $Y_{0-} = Y_0$ (hence $\Delta_0 Y = 0$) and we write Y^τ and $Y^{\tau-}$ for the process Y stopped at τ and at $\tau-$ ("right before τ"), i.e., respectively,

$$Y^\tau = Y\mathbb{1}_{[0,\tau)} + Y_\tau \mathbb{1}_{[\tau,+\infty)},\ \ Y^{\tau-} = Y\mathbb{1}_{[0,\tau)} + Y_{\tau-}\mathbb{1}_{[\tau,+\infty)}. \tag{1.1}$$

In particular, if τ' is another random time, one can check from the definition that

$$(Y^{\tau-})^{\tau'-} = Y^{\tau\wedge\tau'-}. \tag{1.2}$$

We also work with semimartingales on a predictable set of interval type $\mathcal{I}$ as of He et al. (1992, Sect. VIII.3) and, occasionally, with stochastic integrals on $\mathcal{I}$, where $Z = L \centerdot Y$ on $\mathcal{I}$, for semimartingales Y and Z on $\mathcal{I}$, means that $Z^{\tau_n} = L \centerdot (Y^{\tau_n})$ holds for at least one, or equivalently any, nondecreasing sequence of stopping times such that $\cup[0,\tau_n] = \mathcal{I}$ (the existence of at least one such sequence is ensured by He et al. (1992, Theorem 8.18 3))). We call compensator of a stopping time τ the compensator of $\mathbb{1}_{[\tau,\infty)}$. For $A \in \mathcal{G}_\tau$, we denote $\tau_A = \mathbb{1}_A \tau + \mathbb{1}_{A^c}\infty$, a $\mathbb{G}$ stopping time[1]. Unless otherwise stated, a function (or process) is real-valued; order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense); a time interval is random. We don't explicitly mention the domain of definition of a function when it is implied by the measurability, e.g. we write "a $\mathcal{B}(\mathbb{R}^k)$ measurable function g (or $g(x)$)" rather than "a $\mathcal{B}(\mathbb{R}^k)$ measurable function g defined on $\mathbb{R}$". For a function $g(\omega, x)$ defined on a product space $\Omega \times E$, we usually write $g(x)$ without ω (or g_t in the case of a stochastic process).

Throughout the paper ϑ denotes a finite $\mathbb{G}$ stopping time with indicator process $H = \mathbb{1}_{[\vartheta,+\infty)}$ and $J = 1 - H$, so that $Y^{\vartheta-} = YJ + Y_{\vartheta-}H$ (cf. (1.1)) and $J_- = (\mathbb{1}_{[0,\vartheta)})_- = \mathbb{1}_{0<\vartheta}\mathbb{1}_{[0,\vartheta]}$. According to He et al. (1992, Theorem 4.20), there exists $A \in \mathcal{G}_{\vartheta-}$ such that $\vartheta^a := \vartheta_A$ is accessible and $\vartheta^i := \vartheta_{A^c}$ is totally inaccessible. The compensators $\mathbf{v}^i$ of ϑ^i and $\mathbf{v}^a$ of ϑ^a are the continuous component and the pure jump component of the compensator

$$\mathbf{v} = \mathbf{v}^i + \mathbf{v}^a \tag{1.3}$$

[1] Cf. Theorem 3.9 in He et al. (1992).

of ϑ.

2 Full BSDEs

Let $g_t(\omega, x)$ be a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable function and $G(\omega, x)$ be a $\mathcal{G}_\vartheta \otimes \mathcal{B}(\mathbb{R})$ measurable function. We consider the backward martingale problem consisting of the following integrability, martingale and terminal conditions for a $(\mathbb{G}, \mathbb{Q})$ semimartingale X on $\mathbb{R}_+$:

$$\begin{cases} \int_0^\vartheta |g_s(X_{s-})| ds < \infty, \\ M_t^X := X_t^\vartheta + \int_0^{t\wedge\vartheta} g_s(X_{s-}) ds \;\; \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } \mathbb{R}_+, \\ X_\vartheta = G(X_{\vartheta-}) \end{cases} \tag{2.1}$$

2.1 Counterparty Risk Motivation

First we present the counterparty risk motivation for this problem. See also Crépey et al. (2014)[2], Crépey and Song (2015) or Brigo and Pallavicini (2014) where related equations are considered. For more background and details on counterparty risk in general, see Brigo, Morini, and Pallavicini (2013) and Crépey et al. (2014) from a more financial and mathematical perspective, respectively.

Remark 2.1 We refer to (2.1) or the related martingale problems henceforth as BSDEs. In some of the above references, the data g and G depend on an additional real vector, say u, corresponding to the integrands in a martingale representation of the martingale part of X, making (2.1) what we call in Sect. 4 a true BSDE.

We consider a bank, the perspective of which is taken, and its counterparty in a OTC derivative contract with maturity T. The two parties are default-prone, with a first-to-default time θ modeled as a $\mathbb{G}$ totally inaccessible stopping time. If $\theta < T$, then the position is closed at θ, with a corresponding exposure (loss with respect to the situation in which there would be no counterparty risk) given as a $\mathcal{G}_\theta \otimes \mathcal{B}(\mathbb{R})$ measurable function

$$\mathsf{C}(y) = \mathsf{C}_c - (y - C)^+ \Lambda, \tag{2.2}$$

where the real y represents the wealth of the bank. The random variable C_c corresponds to the credit and debit exposure of the bank to the default of its counterparty and to its own default, the constant (or random variable) Λ to the loss-given-default of a third party (external lender) funding the position of the bank and the random variable C to the value of the collateral posted by either party (depending on the sign of C) to mitigate counterparty risk. Moreover, the risky bank incurs extra funding costs with respect to a risk free setup. The instantaneous funding cost of the bank in excess over a risk free cost is modeled as an $\mathcal{R}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable funding cost coefficient $c_t(\omega, y)$.

As developed in the above references, the resulting counterparty risk and funding valuation adjustment can be modeled as a solution to a BSDE of the form (2.1), where

$$\vartheta = \theta \wedge T, \;\; g_t(x) = c_t(P_{t-} - x) - r_t x \text{ and } G(x) = \mathbb{1}_{\{\vartheta < T\}} \mathsf{C}(P_{\vartheta-} - x). \tag{2.3}$$

[2]Or Crépey (2015) in the journal version available as preprint on S. Crépey's webpage.

Here P is a $\mathbb{G}$ semimartingale that represents the risk free value of the contract, ignoring counterparty risk and assuming a $\mathcal{R}(\mathbb{G})$ measurable risk free funding rate r. The nature of C_c in (2.2) depends on several factors, including the identity of the defaulter (bank or counterparty), which is only revealed at the totally inaccessible time θ. This is the reason why C in (2.2) and in turn G in (2.3) can't be assumed more regular than $\mathcal{G}_\theta \otimes \mathcal{B}(\mathbb{R})$ measurable; in particular, G isn't $\mathcal{G}_{\theta-} \otimes \mathcal{B}(\mathbb{R})$ measurable. By passing to the predictable projection in the case of r and proceeding similarly, following up on Stricker and Yor (1978, Proposition 3), in the case of g, it is not restrictive to assume that r and g are $\mathcal{P}(\mathbb{G})$ and $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable, respectively. Hence, we effectively deal with an equation of the form (2.1) (of the special kind considered in Sect. 3).

Remark 2.2 The situation depicted above where g and G only depend on x (as opposed to an additional argument u as explained in the remark 2.1) corresponds to the case of a securely funded hedge, which covers the vast majority of hedges that are used in practice (see Crépey et al. (2014, Section 4.2.1 page 87)[3]). The case where g and G depend on u is dealt with in Sect. 4.

Of course, the above sketched BSDE modeling approach implicitly relies on the well posedness of the equation (2.1). This is the motivation for this paper, where the nonstandard BSDE (2.1) is proven equivalent to the more tractable reduced BSDE (3.20)—at least, assuming an invariant time θ with a positive Azéma supermartingale S over $\mathbb{R}_+$, which is typically satisfied in applications (see e.g. Crépey and Song (2015)).

2.2 Equivalent BSDEs

The BSDE (2.1) is nonstandard due to its terminal condition $G(X_{\vartheta-})$, which depends on the solution X right before θ (as alluded to in the introduction, related issues were already considered in Duffie et al. (1996) or Collin-Dufresne et al. (2004)) and with, for fixed x, $G(\omega, x)$ only $\mathcal{G}_\vartheta$ measurable, as opposed to $\mathcal{G}_{\vartheta-}$ measurable in standard credit risk problems (see Bielecki et al. (2009)). This is required in regard of the financial interpretation exposed in Sect. 2.1. Our approach is to reduce the full BSDE (2.1) to equivalent but simpler BSDEs relative to a reduced filtration $\mathbb{F}$ and a possibly changed probability measure $\mathbb{P}$, ultimately the BSDE (3.20) (in the simplest case where the Azéma supermartingale S of θ is positive on $\mathbb{R}_+$), with constant terminal time T and a null terminal condition. Moreover, in view of the applications of Crépey and Song (2015), we want to achieve this under minimal assumptions on θ (see Crépey and Song (2014)), less restrictive than the basic immersion conditions of Crépey (2015), where θ is modeled as a pseudo-stopping time in a classical progressive enlargement of filtration setup.

Henceforth, we suppose the existence of $\widehat{G}$, hence of $\widehat{|G|}$, where $\widehat{\cdot}$ denotes the "parameterized conditional expectation given $\mathcal{G}_{\vartheta-}$" as of Definition A.1 applied to G and ϑ here for G and θ there. In addition to M^X in (2.1), we denote

$$\begin{aligned} M^\bullet &= (G(X_{\vartheta-}) - X_{\vartheta-})H - (\widehat{G}_\cdot(X_-) - X_-) \centerdot \mathbf{v}, \\ M^\circ &= M^X - M^\bullet, \end{aligned} \tag{2.4}$$

both $(\mathbb{G}, \mathbb{Q})$ local martingales on $\mathbb{R}_+$ if X satisfies the BSDE (2.1):

[3] Or Section 2.1 in the journal version Crépey (2015, Part I).

Lemma 2.1 *If X is a solution to (2.1), then*

$$\int_0^t \widehat{|G|}_s(X_{s-})d\mathbf{v}_s < \infty,\ t \in \mathbb{R}_+ \tag{2.5}$$

and $M^\bullet$, hence M°, are $\mathbb{G}$ local martingales.

Proof. Notice that $\Delta_\vartheta M^X = \Delta_\vartheta X = G(X_{\vartheta-}) - X_{\vartheta-}$ on $\{0 < \vartheta < \infty\}$. Let $(\tau_n)_{n\in\mathbb{N}}$ denote a nondecreasing sequence of finite stopping times tending to the infinity such that each $(M^X)^{\tau_n}$ is a uniformly integrable martingale and each $(M^X)^{\tau_n}_-$ and $X^{\tau_n}_-$ are bounded. We write

$$\begin{aligned}\mathbb{E}[\textstyle\int_0^{\tau_n} \widehat{|G|}_s(X_{s-})d\mathbf{v}_s] &= \mathbb{E}[\widehat{|G|}_\vartheta(X_{\vartheta-})\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] = \mathbb{E}[|G(X_{\vartheta-})|\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]\\ &\le \mathbb{E}[|G(X_{\vartheta-}) - X_{\vartheta-}|\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] + \mathbb{E}[|X_{\vartheta-}|\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]\\ &= \mathbb{E}[|\Delta_\vartheta M^X|\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] + \mathbb{E}[|X_{\vartheta-}|\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] < \infty.\end{aligned}$$

This proves (2.5). Therefore, Lemma A.2 is applicable with G and ϑ here for G and θ there, which proves that $M^\bullet$ is a $(\mathbb{G},\mathbb{Q})$ local martingale on $\mathbb{R}_+$. ∎

We consider the following BSDE for a $(\mathbb{G},\mathbb{Q})$ semimartingale Y on $\mathbb{R}_+$:

$$\begin{cases}\int_0^\vartheta |g_s(Y_{s-})|ds + \int_0^\vartheta \widehat{|G|}_s(Y_{s-})d\mathbf{v}_s < \infty,\\ M^Y_t := Y^\vartheta_t + \int_0^{t\wedge\vartheta} g_s(Y_{s-})ds + \int_0^{t\wedge\vartheta} \widehat{G}_s(Y_{s-})d\mathbf{v}_s \ \text{ defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } \mathbb{R}_+,\\ Y_\vartheta = 0.\end{cases} \tag{2.6}$$

Lemma 2.2 *If X is a solution to the BSDE (2.1), then $Y = XJ$ is a solution to the BSDE (2.6). Conversely, if Y is a solution to the BSDE (2.6), then the process*

$$X = YJ + G(Y_{\vartheta-})H.$$

is a solution to the BSDE (2.1).

Proof. Assuming (2.1), we have (2.5). Now, the Itô formula yields, for $t > 0$,

$$\begin{aligned}dX^{\vartheta-}_t &= d(XJ)_t + d(X_{\vartheta-}H)_t\\ &= J_{t-}dX_t - X_t dH_t + X_{\vartheta-}dH_t\\ &= dM^X_t - J_{t-}g_t(X_{t-})dt - X_\vartheta dH_t + X_{\vartheta-}dH_t\\ &= dM^\circ_t + dM^\bullet_t - J_{t-}g_t(X_{t-})dt - \Delta_\vartheta X dH_t\\ &= dM^\circ_t - (\widehat{G}_t(X_{t-}) - X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt.\end{aligned}$$

Hence,

$$\begin{aligned}dY_t &= d(XJ)_t\\ &= dM^\circ_t - d(X_{\vartheta-}H)_t - (\widehat{G}_t(X_{t-}) - X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt\\ &= dM^\circ_t - (d(X_{\vartheta-}H)_t - X_{t-}d\mathbf{v}_t) - \widehat{G}_t(X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt\\ &= dM^\circ_t - X_{t-}d(H-\mathbf{v})_t - \widehat{G}_t(X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt.\end{aligned}$$

The process $(H - \mathbf{v})$ being a local martingale, this computation and (2.5) show that the process $Y = XJ$ solves (2.6).

Conversely, given Y solving (2.6), let

$$X = YJ + G(Y_{\vartheta-})H.$$

Lemma A.2 is applicable with G and ϑ here for G and θ there, which proves that

$$M^{\star} := G(Y_{\vartheta-})H - \widehat{G}.(Y_{-}) \centerdot \mathbf{v}$$

is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $\mathbb{R}_+$. The Itô formula gives, for $t > 0$,

$$\begin{aligned}
dX_t &= d(YJ)_t + d(G(Y_{\vartheta-})H)_t \\
&= J_{t-}dY_t - Y_t dH_t + dM^{\star}_t + \widehat{G}_t(Y_{t-})d\mathbf{v}_t \\
&= dM^Y_t - J_{t-}g_t(Y_{t-})dt - \widehat{G}_t(Y_{t-})d\mathbf{v}_t - Y_{\vartheta}dH_t + dM^{\star}_t + \widehat{G}_t(Y_{t-})d\mathbf{v}_t \\
&= dM^Y_t - J_{t-}g_t(Y_{t-})dt + dM^{\star}_t
\end{aligned}$$

(using the terminal condition $Y_{\vartheta} = 0$), hence

$$M^X \ = M^Y + M^{\star}, \tag{2.7}$$

which shows that X is a solution to the BSDE (2.1). ∎

By passing from (2.1) to (2.6), we got rid of the implicit terminal condition at ϑ in (2.1). But, for the application in the reduced models of Sect. 3, we need still another form of the BSDE, where, instead of a null "Dirichlet" condition $Y_{\vartheta} = 0$ in (2.6), we have a no jump "Neumann" condition that will become visible as $\Delta_{\vartheta}\mathsf{Z} = 0$ in the BSDE (2.11).

Lemma 2.3 *Given a $(\mathbb{G}, \mathbb{Q})$ semimartingale Z on $\mathbb{R}_+$, $Y = ZJ$ solves the BSDE (2.6) if and only if Z solves the following BSDE for a $(\mathbb{G}, \mathbb{Q})$ semimartingale on $\mathbb{R}_+$:*

$$\begin{cases}
\int_0^{\vartheta} |g_s(Z_{s-})|ds + \int_0^{\vartheta} \widehat{|G|}_s(Z_{s-})d\mathbf{v}_s < \infty, \\[1ex]
M^Z_t := Z^{\vartheta-}_t + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta} \left(\widehat{G}_s(Z_{s-}) - Z_{s-}\right) d\mathbf{v}_s \\
\text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } \mathbb{R}_+.
\end{cases} \tag{2.8}$$

In particular, if X solves the BSDE (2.1), then X, hence $X^{\vartheta-}$, solve the BSDE (2.8) and we have $M^Z = M^{\circ}$. When the accessible component ϑ^a of ϑ is predictable, the BSDE (2.8) becomes (cf. (1.3))

$$\begin{cases}
\int_0^{\vartheta} |g_s(Z_{s-})|ds + \int_0^{\vartheta} \widehat{|G|}_s(Z_{s-})d\mathbf{v}_s < \infty, \\[1ex]
M^Z_t = Z^{\vartheta-}_t + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta} (\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}^i_s \\
\text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } \mathbb{R}_+, \\[1ex]
\mathbb{1}_{\{0<\vartheta^a<\infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}) = 0.
\end{cases} \tag{2.9}$$

Proof. The Itô formula applied to $Y = ZJ$ yields, for $t > 0$,

$$\begin{aligned}
dY_t \ &= \ d(ZJ)_t = d(Z^{\vartheta-}J)_t = dZ^{\vartheta-}_t - Z^{\vartheta-}_t dH_t \\
&= \ dM^Z_t - J_{t-}g_t(Z_{t-})dt - \widehat{G}_t(Z_{t-})d\mathbf{v}_t + Z_{t-}d\mathbf{v}_t - Z_{\vartheta-}dH_t \\
&= \ dM^Z_t - J_{t-}g_t(Z_{t-})dt - \widehat{G}_t(Z_{t-})d\mathbf{v}_t - Z_{t-}d(H - \mathbf{v})_t,
\end{aligned}$$

which proves the first part of the lemma. If ϑ^a is predictable, then $\mathbf{v}^a = \mathbb{1}_{[\vartheta^a,\infty)}$. Hence,

$$M_t^Z = Z_t^{\vartheta-} + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta} (\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}_s^i + \mathbb{1}_{0<\vartheta^a}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-})\mathbb{1}_{\{t\geq\vartheta^a\}},$$

where $\mathbf{v}^i$ is continuous. As a consequence,

$$\mathbb{1}_{\{0<\vartheta^a<\infty\}}\Delta_{\vartheta^a}M^Z = \mathbb{1}_{\{0<\vartheta^a<\infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}). \tag{2.10}$$

Therefore (2.9) obviously implies (2.8). Conversely, assuming (2.8), so that M_t^Z is a local martingale, by taking the conditional expectation given $\mathcal{G}_{\vartheta^a-}$ both sides of (2.10), we obtain since ϑ^a is predictable:

$$\mathbb{1}_{\{0<\vartheta^a<\infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}) = 0,$$

which is the terminal condition in (2.9). ∎

The BSDE (2.8) seems to have no terminal condition so that (2.8), hence (2.1), could have multiple solutions. However, observe that (2.8) has a solution if and only if the following BSDE has a solution:

$$\begin{cases} \int_0^\vartheta |g_s(\mathsf{Z}_{s-})|ds + \int_0^\vartheta \widehat{|G|}_s(\mathsf{Z}_{s-})d\mathbf{v}_s <\infty, \\ \\ \mathsf{Z}_t + \int_0^{t\wedge\vartheta} g_s(\mathsf{Z}_{s-})ds + \int_0^{t\wedge\vartheta} \widehat{G}_s(\mathsf{Z}_{s-})d\mathbf{v}_s - \int_0^{t\wedge\vartheta} \mathsf{Z}_{s-}d\mathbf{v}_s \\ \text{ defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } \mathbb{R}_+, \\ \\ \Delta_\vartheta \mathsf{Z} = 0, \end{cases} \tag{2.11}$$

with $\mathsf{Z} = Z^{\vartheta-}$ as a solution to (2.11) if Z solves (2.8) and $Z = \mathsf{Z}$ as a solution to (2.8 if Z solves (2.11). Hence, one may say that (2.8) hides a "no-jump terminal condition" corresponding to the last line in (2.11).

Corollary 2.1 *When the accessible component ϑ^a of ϑ is predictable, if Z solves* (2.9), *then $X = ZJ + G(Z_{\vartheta-})H$ solves* (2.1) *and we have $M^X = M^Z + M^\bullet$, where*

$$\begin{aligned} M^\bullet &= (G(X_-) - X_-) \centerdot H - (\widehat{G}.(X_-) - X_-) \centerdot \mathbf{v}^i \\ &= (G(Z_-) - Z_-) \centerdot H - (\widehat{G}.(Z_-) - Z_-) \centerdot \mathbf{v}^i. \end{aligned} \tag{2.12}$$

Proof. If Z solves (2.8) (equivalent to (2.9) when the accessible component ϑ^a of ϑ is predictable), then, from the previous lemmas, $Y = ZJ$ solves (2.6), $X = ZJ + G(Z_{\vartheta-})H$ solves (2.1), $M^Z = M^\circ$ and (2.12) holds in view of the terminal condition in (2.9). ∎

The main findings of this section are summarized in the following result that immediately stems from Lemmas 2.1 through 2.3.

Theorem 2.1 *If the BSDE (2.1) has a solution X, then*

$$\int_0^t \widehat{|G|}_s(X_{s-})d\mathbf{v}_s < \infty,\ t \in \mathbb{R}_+,$$

and any $\mathbb{G}$ semimartingale Z on $\mathbb{R}_+$ such that $JZ = JX$ (e.g. $Z = X^{\vartheta-}$) solves the BSDE (2.8)*, equivalent to* (2.9) *when the accessible component ϑ^a of ϑ is predictable. Conversely, if Z is a solution to the BSDE* (2.8)*, then the process*

$$X = ZJ + G(Z_{\vartheta-})H$$

solves the BSDE (2.1). In both cases, we have $M^X = M^Z + M^\bullet$, where (2.12) *holds when the accessible component ϑ^a of ϑ is predictable.* ∎

3 Reduced BSDEs

Let θ be a (non necessarily finite) $\mathbb{G}$ stopping time. Let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$ satisfying the usual conditions and the following:

Condition (B). For any $\mathbb{G}$ predictable process L, there exists an $\mathbb{F}$ predictable process K, called the $\mathbb{F}$ predictable reduction[4] of L, such that $\mathbb{1}_{(0,\theta]}K = \mathbb{1}_{(0,\theta]}L$.

This condition is a relaxation of the classical progressive enlargement of filtration setup, where the bigger filtration $\mathbb{G}$ is simply the smaller reference filtration $\mathbb{F}$ progressively enlarged by θ ("$\mathbb{G} = \mathbb{F} \vee \mathbb{H}$" in a standard notation), which implies the condition (B). The additional flexibility offered by this condition is exploited in Crépey and Song (2015) to deal with a dynamic Marshall-Olkin copula model of counterparty risk on credit derivatives. We write $\mathsf{J} = \mathbb{1}_{[0,\theta)}$, hence $\mathsf{J}_- = \mathbb{1}_{0<\theta}\mathbb{1}_{[0,\theta]}$. Let ${}^o\cdot$ and ${}^p\cdot$ denote the $\mathbb{F}$ optional and predictable projections. In particular, $\mathsf{S} = {}^o\mathsf{J}$ represents the $\mathbb{F}$ Azéma supermartingale of θ. We recall that

$$ {}^p(\mathsf{J}_-) = \mathsf{S}_- \text{ on } (0, \infty) \tag{3.1}$$

(see Jeulin (1980, page 63)) and

$$\mathsf{S}_{\theta-} > 0 \text{ on } \{0 < \theta\} \tag{3.2}$$

(cf. Yor (1978, page 63)). Let

$$\varsigma = \inf\{s > 0; \mathsf{S}_s = 0\} = \inf\{s > 0; \mathsf{S}_{s-} = 0\} \tag{3.3}$$

(since S is a nonnegative supermartingale[5]) and

$$\varsigma_n = \inf\left\{s > 0; \mathsf{S}_s \le \frac{1}{n}\right\} \ (n > 0), \tag{3.4}$$

so that, using the definitions,

$$\varsigma = \sup_n \varsigma_n, \ \ \{\mathsf{S}_- > 0\} \cup [0] = \cup_n [0, \varsigma_n]. \tag{3.5}$$

In particular:

$$\begin{aligned} &\text{on } \{\mathsf{S}_0 > 0\} \text{ , we have } 0 \in \{\mathsf{S}_- > 0\}, \text{ hence } \cup_n [0, \varsigma_n] = \{\mathsf{S}_- > 0\}, \\ &\text{on } \{\mathsf{S}_0 = 0\} \text{ (hence } \{\mathsf{S}_- > 0\} = \emptyset), \text{ all the } \varsigma_n \text{ are equal to 0, hence } \cup_n [0, \varsigma_n] = [0]. \end{aligned} \tag{3.6}$$

The next lemma assembles the main results that we need under the condition (B) (see Crépey and Song (2014) for references).

[4]Also known as pre-default process in the credit risk literature (see e.g. Bielecki and Rutkowski (2001)).
[5]Cf. n°17 Chapitre VI in Dellacherie and Meyer (1975).

Lemma 3.1 *1) Let $(E, \mathcal{E})$ be a measurable space. Any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ (respectively $\mathcal{O}(\mathbb{G}) \otimes \mathcal{E}$) measurable function $h_t(\omega, x)$ admits a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ (respectively $\mathcal{O}(\mathbb{F}) \otimes \mathcal{E}$) reduction, i.e. a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$ (respectively $\mathcal{O}(\mathbb{F}) \otimes \mathcal{E}$) measurable function $f_t(\omega, x)$ such that $\mathbb{1}_{(0,\theta]} f = \mathbb{1}_{(0,\theta]} h$ (respectively $\mathsf{J} f = \mathsf{J} h$) everywhere. In addition, two $\mathbb{F}$ predictable (resp. optional) processes indistinguishable until (resp. before) θ are indistinguishable on $\{\mathsf{S}_- > 0\}$ (resp. $\{\mathsf{S} > 0\}$).*

2) Let M be a $\mathbb{G}$ local martingale on $\mathbb{R}_+$ with $\Delta_\theta M = 0$. For any $\mathbb{F}$ optional reduction K of M, K is an $\mathbb{F}$ semimartingale on $\{\mathsf{S}_- > 0\}$, $\mathbb{1}_{\{\mathsf{S}_- > 0\}} K_-$ is an $\mathbb{F}$ predictable reduction of M_- and $\mathsf{S}_- \centerdot K + [\mathsf{S}, K]$ is an $\mathbb{F}$ local martingale on $\{\mathsf{S}_- > 0\}$. Conversely, for any $\mathbb{F}$ semimartingale K on $\{\mathsf{S}_- > 0\}$ such that $\mathsf{S}_- \centerdot K + [\mathsf{S}, K]$ is an $\mathbb{F}$ local martingale on $\{\mathsf{S}_- > 0\}$, $K^{\theta-}$ is a $\mathbb{G}$ local martingale on $\mathbb{R}_+$.

Given a positive constant T, we say that $(\mathbb{F}, \mathbb{P})$ is a reduced stochastic basis of $(\mathbb{G}, \mathbb{Q})$ if $\mathbb{F}$ is a subfiltration of $\mathbb{G}$ satisfying the usual conditions and the condition (B) and if $\mathbb{P}$ is a probability measure equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$. Note that earlier, letters of the family "m" (e.g. M^X, M°, etc.) were used to denote $\mathbb{G}$ local martingales, which are all defined in reference to the original probability measure $\mathbb{Q}$. Regarding $\mathbb{F}$, letters of the family "q" and "p" are used to denote $(\mathbb{F}, \mathbb{Q})$ and $(\mathbb{F}, \mathbb{P})$ local martingales, respectively. We consider the following:

Condition (A). $(\mathbb{F}, \mathbb{P})$ is a reduced stochastic basis of $(\mathbb{G}, \mathbb{Q})$ such that for any $(\mathbb{F}, \mathbb{P})$ local martingale P, $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$.

The condition (A) is studied from the theoretical and practical point of view in Crépey and Song (2014) and Crépey and Song (2015), respectively. Specifically, the results of Crépey and Song (2014) reduce the condition (A) to suitable integrability conditions, checked to hold in Crépey and Song (2015) in concrete models where the basic immersion assumptions of Crépey (2015) are violated. In particular, the flexibility offered by the possibility to change the measure in the condition (A) is exploited in Crépey and Song (2015) to deal with a dynamic Gaussian copula model of counterparty risk on credit derivatives. The following result is proven in Crépey and Song (2014).

Lemma 3.2 *Under the condition (A), if θ has an intensity, then*

$$\{\mathsf{S}_- > 0\} = \{\mathsf{S} > 0\}. \tag{3.7}$$

and a process P is an $(\mathbb{F}, \mathbb{P})$ local martingale on $\{\mathsf{S}_- > 0\} \cap [0, T]$ if and only if $\mathsf{S}_- \centerdot P + [\mathsf{S}, P]$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{\mathsf{S}_- > 0\} \cap [0, T]$.

In practice, the choice of a reduced stochastic basis $(\mathbb{F}, \mathbb{P})$ is a degree of freedom of the modeler. Thus, we are interested in the stopping times θ such that the condition (A) holds for at least one reduced stochastic basis $(\mathbb{F}, \mathbb{P})$. Following Crépey and Song (2014), we call invariant a $\mathbb{G}$ stopping time θ for which there exists a reduced stochastic basis $(\mathbb{F}, \mathbb{P})$ of $(\mathbb{G}, \mathbb{Q})$ satisfying the condition (A). See the beginning of Sect. 3.3 regarding the BSDE motivation for this invariant terminology, beyond the obvious reference to the martingale invariance property defined by the condition (A).

3.1 Under the Condition (B)

Back to the BSDE (2.1), we assume $\vartheta = \theta \wedge T$, where T is a positive constant and θ is a $\mathbb{G}$ totally inaccessible stopping time with $\mathbb{G}$ compensator $\gamma \centerdot \boldsymbol{\lambda}$, for some $\mathbb{G}$ predictable intensity process γ. Hence

$$\vartheta^i = \theta_{\{\theta<T\}},\ \ \vartheta^a = T_{\{T\le\theta\}},\ \ \mathbf{v}^i = \mathbb{1}_{[0,T)}\gamma \centerdot \boldsymbol{\lambda},\ \ \mathbf{v}^a = \mathbb{1}_{\{T\le\theta\}}\mathbb{1}_{[T,\infty)}.$$

In addition, we assume a terminal function of the form $G(x) = \mathbb{1}_{\{\theta<T\}}\mathsf{G}(x)$ in (2.1), so that

$$\widehat{|G|}_t(x) = \mathbb{1}_{t<T}\widehat{|\mathsf{G}|}_t(x),\ \ \widehat{G}_t(x) = \mathbb{1}_{t<T}\widehat{\mathsf{G}}_t(x)$$

(where $\widehat{|\mathsf{G}|}$ and $\widehat{\mathsf{G}}$ can be computed relatively to the stopping time θ, which is simpler in practice as one then does not need care about T). Since $\vartheta^a = T_{\{T\le\theta\}}$ is predictable, (2.1) is equivalent to (2.9), which is rewritten as the following BSDE for a $(\mathbb{G},\mathbb{Q})$ semimartingale Z on $\mathbb{R}_+$:

$$\begin{cases} \int_0^{\theta\wedge T}\big(|g_s(Z_{s-})| + \gamma_s\widehat{|\mathsf{G}|}_s(Z_{s-})\big)ds < \infty, \\ \\ M^Z_t = Z^{\theta\wedge T-}_t + \int_0^{t\wedge\theta\wedge T}\Big(g_s(Z_{s-}) + (\widehat{\mathsf{G}}_s(Z_{s-}) - Z_{s-})\gamma_s\Big)\,ds \\ \ \text{defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } \mathbb{R}_+, \\ \\ Z_{T-}\mathbb{1}_{\{T\le\theta\}} = 0. \end{cases} \tag{3.8}$$

In addition, (2.12) is rewritten as

$$\begin{aligned} M^\bullet &= \mathbb{1}_{[0,T)}(\mathsf{G}(X_-) - X_-)\centerdot H - \mathbb{1}_{[0,T)}(\widehat{\mathsf{G}}.(X_-) - X_-)\gamma\centerdot\boldsymbol{\lambda} \\ &= \mathbb{1}_{[0,T)}(\mathsf{G}(Z_-) - Z_-)\centerdot H - \mathbb{1}_{[0,T)}(\widehat{\mathsf{G}}.(Z_-) - Z_-)\gamma\centerdot\boldsymbol{\lambda}. \end{aligned} \tag{3.9}$$

Remark 3.1 If one adds the condition that Z is stopped at $(\theta-)$ to avoid artificially multiple solutions, then the BSDE (3.8), constrained in this way, becomes equivalent to: Z is stopped at $(\theta-)$ and

$$\begin{cases} \int_0^{t\wedge T}\Big(|g_s(Z_{s-})| + \gamma_s\widehat{|\mathsf{G}|}_s(Z_{s-})\Big)\,ds < \infty,\ \ t\in\{\mathsf{J}_- > 0\}, \\ \\ Z^{T-}_t + \int_0^{t\wedge T}\Big(g_s(Z_{s-}) + (\widehat{\mathsf{G}}_s(Z_{s-}) - Z_{s-})\gamma_s\Big)\,ds \\ \ \text{defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } \{\mathsf{J}_- > 0\}, \\ \\ Z_{T-}\mathsf{J}_{T-} = 0, \end{cases} \tag{3.10}$$

the equality between the martingale parts stemming from the identity $(Z^{T-})^{\theta-} = (Z^{T-})^{\theta}$ if $\Delta_\theta Z = 0$, by (1.2).

We are now in the position to derive reduced forms of the BSDE (2.1) or, equivalently, (3.8). Let φ be the $\mathcal{P}(\mathbb{F})$ reduction of γ and let f, F and F' be respective $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})$ reductions of g, $\widehat{\mathsf{G}^+}$ and $\widehat{\mathsf{G}^-}$ (reductions that all exist by Lemma 3.1 1)). By Lemma A.1 3), $(\widehat{\mathsf{G}^+} + \widehat{\mathsf{G}^-})$ and $(\widehat{\mathsf{G}^+} - \widehat{\mathsf{G}^-})$ are respective versions of $\widehat{|\mathsf{G}|}$ and $\widehat{\mathsf{G}}$. Hence, $(\mathsf{F}+\mathsf{F}')$ and $(\mathsf{F}-\mathsf{F}')$ are respective reductions of $\widehat{|\mathsf{G}|}$ and $\widehat{\mathsf{G}}$. For any càdlàg process R on $\mathbb{R}_+$ (or on a predictable set of interval type), we write

$$\overline{R} = R + \big(f.(R_-) + ((\mathsf{F}-\mathsf{F}').(R_-) - R_-)\varphi\big)\centerdot\boldsymbol{\lambda}. \tag{3.11}$$

We consider the following BSDE for an $(\mathbb{F},\mathbb{Q})$ semimartingale U on $\{\mathsf{S}_- > 0\}$:

$$\begin{cases} I^U_t := \int_0^{t\wedge T}\big(|f_s(U_{s-})| + (\mathsf{F}+\mathsf{F}')_s(U_{s-})\varphi_s\big)ds < \infty, \ \ t\in\{\mathsf{S}_->0\}, \\ \mathsf{S}_- \centerdot \overline{U}^{T-} + [\mathsf{S},\overline{U}^{T-}] \text{ defines an } (\mathbb{F},\mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_->0\}, \\ U_{T-}\mathsf{S}_{T-} = 0. \end{cases} \tag{3.12}$$

Lemma 3.3 *Assume the condition (B). If Z is a solution to the BSDE (3.8), then the $\mathbb{F}$ optional reduction U of $Z^{\theta-}$ is a solution to the BSDE (3.12). Conversely, if U is a solution to the BSDE (3.12), then $Z = U^{\theta-}$ is a solution to the BSDE (3.8).*

Proof. Assuming the BSDE (3.8) has a solution Z and U given as the $\mathbb{F}$ optional reduction of $Z^{\theta-}$, the local martingale M^Z in the BSDE (3.8) satisfies

$$\begin{aligned} M^Z_t &= Z^{\theta\wedge T-}_t + \int_0^{t\wedge\theta\wedge T}\Big(g_s(Z_{s-}) + (\widehat{G}_s(Z_{s-}) - Z_{s-})\gamma_s\Big)\,ds \\ &= U^{\theta\wedge T-}_t + \int_0^{t\wedge\theta\wedge T}\big(f_s(U_{s-}) + ((\mathsf{F}-\mathsf{F}')_s(U_{s-}) - U_{s-})\varphi_s\big)\,ds \\ &= \overline{U}^{\theta\wedge T-}_t = (\overline{U}^{T-})^{\theta-}_t, \end{aligned} \tag{3.13}$$

by (1.2). In particular, $\overline{U}^{T-}$ is the $\mathbb{F}$ optional reduction of M^Z. Hence, by the direct part in Lemma 3.1 2) applied with $M = M^Z$ and $K = \overline{U}^{T-}$, $\mathsf{S}_- \centerdot \overline{U}^{T-} + [\mathsf{S},\overline{U}^{T-}]$ is an $(\mathbb{F},\mathbb{Q})$ local martingale on $\{\mathsf{S}_- > 0\}$. Hence, U satisfies the martingale condition in the BSDE (3.12). In addition, for any $\mathbb{F}$ predictable stopping time σ, we have by predictable projection and $\mathcal{F}_{\sigma-}$ measurability of $\mathbb{1}_{\{I^U_\sigma=\infty\}}$:

$$0 = \mathbb{E}[\mathbb{1}_{\{I^U_\sigma=\infty\}}\mathbb{1}_{\{\sigma\le\theta\}}] = \mathbb{E}[\mathbb{1}_{\{I^U_\sigma=\infty\}}\mathsf{S}_{\sigma-}].$$

Hence, by application of the predictable section theorem[6], the process $\mathbb{1}_{\{I^U=\infty\}}\mathsf{S}_-$ is indistinguable from 0, so that I^U is finite on $\{\mathsf{S}_- > 0\}$, which proves that U satisfies the integrability condition in the BSDE (3.12). Last, taking the $\mathcal{F}_{T-}$ conditional expectation of the terminal condition in the BSDE (3.8) and using (3.1) shows that U satisfies the terminal condition in the BSDE (3.12).

Conversely, if U is a solution to the BSDE (3.12) and $Z = U^{\theta-}$, the integrability condition in the BSDE (3.12) for U implies the integrability condition in the BSDE (3.8) for Z, because $\theta \in \{\mathsf{S}_- > 0\}$ (unless $\theta = 0$; cf. (3.2)). In view of the martingale condition for U, by the converse part in Lemma 3.1 2) applied with $K = \overline{U}^{T-}$, $(\overline{U}^{T-})^{\theta-} = \overline{U}^{T\wedge\theta-}$ (cf. (1.2)) is a $(\mathbb{G},\mathbb{Q})$ local martingale, hence $Z = U^{\theta-}$ satisfies the martingale condition in the BSDE (3.8). Using (3.1), $U_{T-}\mathbb{Q}[T\le\theta|\mathcal{F}_{T-}] = U_{T-}\mathsf{S}_{T-} = 0$, hence

$$\begin{cases} 0 = \mathbb{E}[U^+_{T-}\mathbb{E}[T\le\theta|\mathcal{F}_{T-}]] = \mathbb{E}[U^+_{T-}\mathbb{1}_{\{T\le\theta\}}] = \mathbb{E}[Z^+_{T-}\mathbb{1}_{\{T\le\theta\}}], \\ 0 = \mathbb{E}[U^-_{T-}\mathbb{E}[T\le\theta|\mathcal{F}_{T-}]] = \mathbb{E}[U^-_{T-}\mathbb{1}_{\{T\le\theta\}}] = \mathbb{E}[Z^-_{T-}\mathbb{1}_{\{T\le\theta\}}], \end{cases}$$

i.e. $Z_{T-}\mathbb{1}_{\{T\le\theta\}} = 0$, which is the terminal condition in the BSDE (3.8). ∎

[6] Cf. He et al. (1992, Theorem 4.8).

3.2 Under the Condition (A)

Lemma 3.3 establishes an equivalence between the $(\mathbb{G},\mathbb{Q})$ BSDE (3.8) and the $(\mathbb{F},\mathbb{Q})$ BSDE (3.12) under the condition (B). However, the martingale condition in the BSDE (3.12) is quite involved. In this section we study the reduction of the BSDE (3.8) under the condition (A) on a reduced stochastic basis $(\mathbb{F},\mathbb{P})$ of $(\mathbb{G},\mathbb{Q})$. We consider the following BSDE for an $(\mathbb{F},\mathbb{P})$ semimartingale U on $\{\mathsf{S}_- > 0\}$:

$$\begin{cases} \int_0^{t\wedge T} \big(|f_s(U_{s-})| + (\mathsf{F}+\mathsf{F}')_s(U_{s-})\varphi_s\big)ds < \infty,\ t\in\{\mathsf{S}_- > 0\}, \\ \overline{U}_t^{T-} \text{ defines an } (\mathbb{F},\mathbb{P}) \text{ local martingale on } \{\mathsf{S}_- > 0\}, \\ U_{T-}\mathsf{S}_{T-} = 0. \end{cases} \tag{3.14}$$

Lemma 3.4 *Under the condition (A), the BSDEs (3.12) and (3.14) are equivalent.*

Proof. The integrability and terminal conditions in (3.12) and (3.14) are the same (they don't depend on the filtration or probability measure). Since θ has an intensity, the martingale conditions in (3.14) and in (3.12) are equivalent by Lemma 3.2. ∎

Next, we consider the following BSDE for an $(\mathbb{F},\mathbb{P})$ semimartingale V on $\{\mathsf{S}_- > 0\}$:

$$\begin{cases} \int_0^{t\wedge T} \big(|f_s(V_{s-})| + (\mathsf{F}+\mathsf{F}')_s(V_{s-})\varphi_s\big)ds < \infty,\ t\in\{\mathsf{S}_- > 0\} \\ \overline{V}_t^{T} \text{ defines an } (\mathbb{F},\mathbb{P}) \text{ local martingale on } \{\mathsf{S}_- > 0\}, \\ V_T\mathsf{S}_{T-} = 0. \end{cases} \tag{3.15}$$

Note that this BSDE is the same as (3.14), except that we stop at $(T-)$ instead of T in the martingale and terminal conditions (the integrability conditions are the same)

Lemma 3.5 *Assume the condition (A). If U is a solution to the BSDE (3.14), then $V = U^{T-}$ is a solution to the BSDE (3.15). Conversely, if V is a solution to the BSDE (3.15), then $U = V$ is a solution to the BSDE (3.14).*

Proof. The direct part is obvious. We show the converse part. Let us suppose (3.15). To show the terminal condition in (3.14), it is enough to prove that $V_{T-} = 0$ holds on $\{T \le \varsigma_n\}$ for each $n \in \mathbb{N}$, so that $V_{T-} = 0$ on $\{\mathsf{S}_{T-} > 0\}$, by (3.5). By definition of a local martingale on $\{\mathsf{S}_- > 0\}$ in (3.15), $\overline{V}^{\varsigma_n\wedge T}$ is an $(\mathbb{F},\mathbb{P})$ local martingale on $\mathbb{R}_+$. Note that

$$\Delta_T\big(\overline{V}^{\varsigma_n\wedge T}\big) = \mathbb{1}_{\{T\le\varsigma_n\}}\Delta_T V = -\mathbb{1}_{\{T\le\varsigma_n\}}V_{T-}, \tag{3.16}$$

by the terminal condition in (3.15). As T is predictable and $\overline{V}^{\varsigma_n\wedge T}$ is an $(\mathbb{F},\mathbb{P})$ local martingale on $\mathbb{R}_+$, taking conditional expectation with respect to $\mathcal{F}_{T-}$ in (3.16) yields $\mathbb{1}_{\{T\le\varsigma_n\}}V_{T-} = 0$ for each $n \in \mathbb{N}$, as wanted, hence $V_{T-} = 0$ on $\{\mathsf{S}_{T-} > 0\}$. In view of this, $\overline{V}_t^{T}$ and $\overline{V}_t^{T-}$ coincide on $\{\mathsf{S}_- > 0\}$, therefore the martingale condition (3.15) implies the one in (3.14). ∎

Summarizing:

Theorem 3.1 *Under the condition (A), the BSDEs (3.8) and (3.15) are equivalent. Specifically, if Z solves (3.8), then the $\mathbb{F}$ optional reduction U of $Z^{\theta-}$ solves (3.14) and $V = U^{T-}$ solves (3.15). Conversely, if V solves (3.15), then $U = V$ solves (3.14) and $Z = U^{\theta-} = V^{\theta-}$ solves (3.8). In both cases, V is the $\mathbb{F}$ optional reduction of $Z^{\vartheta-}$ with respect to the time ϑ and we have*

$$M^Z = \overline{V}^{\vartheta-}. \tag{3.17}$$

Proof. Everything but (3.17) directly follows from lemmas 3.3 through 3.5. Moreover, in both cases, Z solves (2.8) and it's easy to check that $Z^{\vartheta-} = V^{\vartheta-}$, hence the $(\mathbb{G}, \mathbb{Q})$ canonical Doob-Meyer local martingale parts M^Z of $Z^{\vartheta-}$ (cf. the martingale condition in (2.9)) and $\overline{V}^{\vartheta-} = \overline{U}^{\vartheta-} = (\overline{U}^{T-})^{\theta-}$ (a $(\mathbb{G}, \mathbb{Q})$ local martingale by the condition (A)) of $V^{\vartheta-}$ coincide. ∎

Theorem 3.2 *Under the condition (A), the BSDE (2.1) with the data $\vartheta = \theta \wedge T, g$ and $G = \mathbb{1}_{\theta<T}\mathsf{G}$ and the BSDE (3.15) are equivalent. Specifically, if X solves the BSDE (2.1) with the data $\vartheta = \theta \wedge T, g$ and $G = \mathbb{1}_{\theta<T}\mathsf{G}$, then any $\mathbb{G}$ semimartingale Z on $\mathbb{R}_+$ such that $JZ = JX$ (e.g. $Z = X^{\vartheta-}$) solves (3.8) and all the consequences of the direct part in Theorem 3.1 follow (with also $Z^{\vartheta-} = X^{\vartheta-}$). Conversely, if V solves (3.15), then all the consequences of the converse part in Theorem 3.1 follow and*

$$X = JV + H\mathbb{1}_{\{\theta<T\}}\mathsf{G}(V_{\theta-}) \tag{3.18}$$

solves the BSDE (2.1) with the data $\vartheta = \theta \wedge T, g$ and $G = \mathbb{1}_{\theta<T}\mathsf{G}$. In both cases, we have (3.18)*, V is the $\mathbb{F}$ optional reduction of $X^{\vartheta-}$ with respect to the time ϑ and we have $M^X = M^Z + M^\bullet$, where*

$$M^Z = \overline{V}^{\vartheta-}, \quad M^\bullet = \mathbb{1}_{[0,T)}(\mathsf{G}(V_-) - V_-) \centerdot H - \mathbb{1}_{[0,T)}(\widehat{\mathsf{G}}_{\cdot}(V_-) - V_-)\gamma \centerdot \boldsymbol{\lambda}. \tag{3.19}$$

Proof. This follows by combining Theorems 2.1 and 3.1, using (3.17) and the form (3.9) of (2.12) to obtain (3.19) (by Corollary 2.1) . ∎

In applications (see Crépey and Song (2015)), we use the following BSDE for an $(\mathbb{F}, \mathbb{P})$ semimartingale W on $\mathbb{R}_+$:

$$\begin{cases} \int_0^T (|f_s(W_{s-})| + (\mathsf{F} + \mathsf{F}')_s(W_{s-})\varphi_s)\, ds < \infty, \\ \overline{W}_t^T \text{ defines an } (\mathbb{F}, \mathbb{P}) \text{ local martingale on } \mathbb{R}_+, \\ W_T = 0, \end{cases} \tag{3.20}$$

Obviously:

Theorem 3.3 *Under the condition (A), if W solves the BSDE (3.20), then $V = W$ solves the BSDE* (3.15)*. The two BSDEs are the same if S is a (strictly) positive process.*

By definition (3.11), $\overline{W}_t^T = W_t^T + \int_0^{t\wedge T} \bar{f}_s(W_{s-})ds$, where

$$\bar{f}_s(x) = f_s(x) + ((\mathsf{F} - \mathsf{F}')_s(x) - x)\varphi_s. \tag{3.21}$$

Under mild regularity and growth assumptions on the reduced coefficient $\bar{f}_s(x)$, the BSDE (3.20) is well-posed, in various senses. For instance, assuming a square integrable coefficient $\bar{f}_{\cdot}(x)$ Lipschitz in x on $[0, T]$, (3.20) admits a unique square integrable solution W (see Crépey and Song (2015) for precise statements).

3.3 Discussion

Formally, the BSDE (3.14) is the BSDE (3.10) with J_- replaced by S_-, its predictable projection (on $(0, +\infty)$; cf. (3.1)). Recall that, under the nonrestrictive constraint that Z is stopped at $(\theta-)$, (3.10) is equivalent to (3.8), i.e. (2.9), i.e. (2.1). Hence, from the financial point of view, Lemma 3.4 can be interpreted as an invariance principle, stating a consistency relation between a non-arbitrable reduced pricing model $(\mathbb{F}, \mathbb{P})$ and a non-arbitrable full pricing model $(\mathbb{G}, \mathbb{Q})$, under the condition (A). Now, from a practical point of view, one might feel that the reduced $\mathbb{F}$ BSDEs (the ones beyond (3.10)) are not so useful since our final recipe is to solve (3.20) in order to get a solution to (3.14), hence to (3.8), i.e. (2.9), i.e. (2.1). Indeed, one may object that a shortcut would be to get a solution to (cf. (3.10))

$$\begin{cases} \int_0^T \Big(|g_s(Z_{s-})| + \gamma_s \widehat{|\mathsf{G}|}_s(Z_{s-}) \Big)\, ds < \infty, \\ Z_t^T + \int_0^{t\wedge T} \Big(g_s(Z_{s-}) + (\widehat{\mathsf{G}}_s(Z_{s-}) - Z_{s-})\gamma_s \Big)\, ds \\ \text{ defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } \mathbb{R}_+, \\ Z_T = 0, \end{cases} \tag{3.22}$$

which is basically of the same difficulty as (3.20) and implies (3.10) in the same way as (3.20) implies (3.8). This shortcut would also be more general since it does not need the condition (A). The caveat, which in a sense justifies this paper, is that the solution to (3.10) obtained in this way does not need to satisfy the constraint that Z is stopped at $(\theta-)$, whereas it's only under this condition that (3.10) yields a solution to (3.8), i.e. (2.9), i.e. (2.1). By contrast, passing by (3.20) always yields a solution $Z = W^{\theta-}$ to (3.8) (stopped at $(\theta-)$, i.e. a solution Z stopped at $(\theta-)$ to (3.10)). We add that, in practical cases, S is typically positive on $\mathbb{R}_+$ (see e.g. Crépey and Song (2015)) and (3.20) is in fact equivalent to (3.8), whereas there is always a gap between (3.22) and (3.10). One can establish a parallel between this comparison regarding (3.22) and (3.20) and the comparison between a seminal defaultable cashflow pricing formula of Duffie et al. (1996, Proposition 1), where a conditional expectation is taken given the full model σ algebra $\mathcal{G}_t$ but the formula is not practical unless some nontrivial no-jump condition is satisfied, and what is now known as the key lemma in the reduced-form approach to credit risk (see e.g. Bielecki et al. (2009, Lemma 3.1.2 and Corollary 3.1.1 page 88-89)), where the conditional expectation is taken given a reference σ algebra $\mathcal{F}_t$, in a classical progressive enlargement of filtration setup.

4 True BSDEs

The situation where g and G, hence $\bar{f}$ in (3.21), only depend on x covers the vast majority of the counterparty risk applications that motivate this paper. With respect to the setup of Sect. 2.1, the situation where g and G depend on an additional real vector u, interpreted as integrands in a martingale representation of the solution, would correspond to the case of an unsecurely funded hedge (see the remark 2.2 and Sect. 4.1). In this section we deal with the corresponding true BSDEs in the case of a marked stopping time $\theta = \min_{e\in E} \theta^e$ as of Sect. A.1 (with E finite for the sake of the simple martingale representation properties postulated below), for a terminal function (cf. (A.2))

$$\mathsf{G}(\omega, x, u) = \Gamma_{\theta(\omega)}(\omega, \epsilon(\omega), x, u), \;\; u = (\psi, \delta) \in \mathbb{R}^d \times \mathbb{R}^{|E|}, \tag{4.1}$$

for some $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}\otimes\mathcal{B}(\mathbb{R}^{1+d+|E|})$ measurable function $\Gamma_t(\omega,e,x,u)$, where $\mathcal{E}$ is a σ algebra on a finite set of marks E and where the random variable ϵ yields the mark of θ, i.e. $e\in E$ such that $\theta=\theta^e$. Hence, by Lemma A.3,

$$\widehat{|\mathsf{G}|}_t(\omega,u)=\sum_{e\in E}q^e_t|\Gamma_t(e,u)|,\ \ \widehat{\mathsf{G}}_t(\omega,u)=\sum_{e\in E}q^e_t\Gamma_t(e,u), \tag{4.2}$$

where, for every $e\in E$, q^e is a $\mathbb{G}$ predictable process such that $q^e_\theta=\mathbb{Q}[\{\theta^e=\theta\}|\mathcal{G}_{\theta-}]$ on $\{\theta<\infty\}$. We assume $q^e\gamma>0$, $e\in E$, and a $(\mathbb{G},\mathbb{Q})$ local martingale unique representation property for a collection of processes M and M^e, $e\in E$, where M is a d-variate process without jump at θ and $\mathsf{M}^e=\mathbb{1}_{\{\theta=\theta^e\}}\mathbb{1}_{[\theta,+\infty)}-q^e\gamma\centerdot\boldsymbol{\lambda}, e\in E$. Recall that an integrand in a martingale representation is only a class of processes defined almost everywhere with respect to the measure induced by the bracket of the corresponding local martingale integrator. Accordingly, by uniqueness in the above $(\mathbb{G},\mathbb{Q})$ local martingale representation property, we mean that, for any $\mathbb{G}$ predictable, M integrable process Ψ^X and $\mathbb{G}$ predictable, M^e integrable processes Δ^e $(e\in E)$, we have:

$$\Psi^X\centerdot M+\sum_{e\in E}\Delta^e\centerdot\mathsf{M}^e=0\Longleftrightarrow\begin{cases}\Psi^X=0,\ \ d[M,M]_s\text{-}a.e.\\ \Delta^e=0,\ \ q^e_s\gamma_s\boldsymbol{\lambda}(ds)\text{-}a.e.,\ \ e\in E,\end{cases} \tag{4.3}$$

with $d[M,M]_s$-*a.e.* in the multivariate sense (see e.g. Jacod and Shiryaev (2003)).

Remark 4.1 These assumptions fit the concrete models of Crépey and Song (2015) (see also Sect. 4.1). In particular, the fact that M doesn't jump at θ means that the invariant time θ satisfies the avoidance property, discussed at the theoretical level in Crépey and Song (2014, Sect. 4.1) and verified in the concrete models of Sect. 4.1 (see also Sect. 4.1). In the case where it's only g that depends on u (but not G), there is no need to suppose a marked stopping time setup, which is only required, in conjunction with the condition (J) below, for analyzing the jump of X at default when G depends on u.

Given the terminal function G in (4.1) and a $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R}^{1+d+|E|})$ measurable coefficient $g_t(\omega,x,u)$, we consider the true BSDE form of the counterparty risk equation (2.1) given as

$$\begin{cases}\left(|g_\cdot(X_-,\Psi^X,\Delta)|\centerdot\boldsymbol{\lambda}\right)_\vartheta<\infty,\\ M^X:=\left(X+g_\cdot(X_-,\Psi^X,\Delta)\centerdot\boldsymbol{\lambda}\right)^\vartheta=X_0+\Psi^X\centerdot M+\sum_{e\in E}\Delta^e\centerdot\mathsf{M}^e\text{ on }\mathbb{R}_+,\\ X_\vartheta=\mathbb{1}_{\{\theta<T\}}\Gamma_\theta(\epsilon,X_{\theta-},\Psi^X_\theta,\Delta_\theta),\end{cases} \tag{4.4}$$

to be solved for $(X,\Psi^X,\Delta=(\Delta^e)_{e\in E})$, where X is a $(\mathbb{G},\mathbb{Q})$ semimartingale on $\mathbb{R}_+$ and Ψ^X (resp. Δ^e, $e\in E$) is a $\mathbb{G}$ predictable, M (resp. M^e) integrable process on $\mathbb{R}_+$. As we will see in the remark 4.2, the following jump consistency condition is a prerequisite for the well-posedness of the true BSDE (4.4).

Condition (J) There exists a $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}\otimes\mathcal{B}(\mathbb{R}^{1+d})$ measurable function $\Delta^\star_t(\omega,e,x,\psi)$ such that, ω almost surely: for each $t\geq 0$ and (x,ψ) in $\mathbb{R}^{1+d}$, the vector $(\delta_e)_{e\in E}$ given as $\delta_e=\mathbb{1}_{q^e_t\gamma_t>0}\Delta^\star_t(\omega,e,x,\psi)$, $e\in E$, is the unique solution to the system of equations

$$\delta_e=\mathbb{1}_{q^e_t\gamma_t>0}\mathbb{1}_{\{t<T\}}\left(\Gamma_t(e,x,\psi,(\delta_{e'})_{e'\in E})-x\right),\ \ e\in E. \tag{4.5}$$

The feasibility of this condition is illustrated on an example in Sect. 4.1.

First we reduce (4.4) to the following true BSDE analog of (3.8) for a $(\mathbb{G},\mathbb{Q})$ semimartingale Z on $\mathbb{R}_+$ and a $\mathbb{G}$ predictable, M integrable process Ψ^Z on $\mathbb{R}_+$:

$$\begin{cases} \Big((|g|+\gamma|\widehat{\mathsf{G}}|)_{\cdot}(Z_-,\Psi^Z,(\Delta^\star_{\cdot}(e,Z_-,\Psi^Z))_{e\in E})\boldsymbol{\cdot}\boldsymbol{\lambda}\Big)_{\vartheta}<\infty,\\[2ex] M^Z := \Big(Z+g_{\cdot}(Z_-,\Psi^Z,(\Delta^\star_{\cdot}(e,Z_-,\Psi^Z))_{e\in E})\boldsymbol{\cdot}\boldsymbol{\lambda} \\ \qquad +\sum_{e\in E}\big(\Gamma_{\cdot}(e,Z_-,\Psi^Z,(\Delta^\star_{\cdot}(e',Z_-,\Psi^Z))_{e'\in E})-Z_-\big)\,q^e\gamma\boldsymbol{\cdot}\boldsymbol{\lambda}\Big)^{\vartheta-} \\ \qquad = Z_0+\Psi^Z\boldsymbol{\cdot} M \text{ on } \mathbb{R}_+,\\[2ex] Z_{T-}\mathbb{1}_{\{T\le\theta\}}=0. \end{cases} \tag{4.6}$$

The following result is the true BSDE analog of Theorem 2.1.

Theorem 4.1 *Given a terminal condition G as of (4.1) for a marked stopping time θ satisfying the condition (J), assume a $(\mathbb{G},\mathbb{Q})$ local martingale unique representation property for some process M jointly with the $\mathsf{M}^e=\mathbb{1}_{\{\theta=\theta^e\}}\mathbb{1}_{[\theta,+\infty)}-q^e\gamma\boldsymbol{\cdot}\boldsymbol{\lambda}, e\in E$. If the true BSDE (4.4) has a solution (X,Ψ^X,Δ), then*

$$\Big(\gamma|\widehat{\mathsf{G}}|_{\cdot}(X_-,\Psi^X,\Delta)\boldsymbol{\cdot}\boldsymbol{\lambda}\Big)_T<\infty \tag{4.7}$$

and, for any $\mathbb{G}$ semimartingale on $\mathbb{R}_+$ such that $JZ=JX$ (e.g. $Z=X^{\vartheta-}$), the process (Z,Ψ^X) solves the true BSDE (4.6). Conversely, if (Z,Ψ^Z) is a solution to the true BSDE (4.6), then the process (X,Ψ^Z,Δ) such that

$$\Delta^e=\Delta^\star_{\cdot}(e,Z_-,\Psi^Z)\ (e\in E)\ ,\ \ X=JZ+H\mathbb{1}_{\{\theta<T\}}\Gamma_\theta(\epsilon,Z_{\theta-},\Psi^Z_\theta,\Delta_\theta), \tag{4.8}$$

solves the true BSDE (4.4). In both cases, we have (cf. (3.9))

$$\begin{aligned} M^X &= M^Z+\mathbb{1}_{[0,T)}(\Gamma_{\cdot}(\epsilon,X_-,\Psi^X,\Delta)-X_-)\boldsymbol{\cdot} H-\mathbb{1}_{[0,T)}\sum_{e\in E}(\Gamma_{\cdot}(e,X_-,\Psi^X,\Delta)-X_-)q^e\gamma\boldsymbol{\cdot}\boldsymbol{\lambda}\\ &= M^Z+\mathbb{1}_{[0,T)}(\Gamma_{\cdot}(\epsilon,Z_-,\Psi^Z,\Delta)-Z_-)\boldsymbol{\cdot} H-\mathbb{1}_{[0,T)}\sum_{e\in E}(\Gamma_{\cdot}(e,Z_-,\Psi^Z,\Delta)-Z_-)q^e\gamma\boldsymbol{\cdot}\boldsymbol{\lambda}\ \blacksquare \end{aligned} \tag{4.9}$$

Proof. By application of the direct part in Theorem 2.1 (cf. also (3.8)), if the true BSDE (4.4) has a solution (X,Ψ^X,Δ), then (4.7) holds and any $\mathbb{G}$ semimartingale Z on $\mathbb{R}_+$ such that $JZ=JX$ satisfies

$$\begin{cases} \Big((|g|+\gamma|\widehat{\mathsf{G}}|)_{\cdot}(Z_-,\Psi^X,\Delta)\boldsymbol{\cdot}\boldsymbol{\lambda}\Big)_{\vartheta}<\infty,\\[2ex] \Big(Z+g_{\cdot}(Z_-,\Psi^X,\Delta)\boldsymbol{\cdot}\boldsymbol{\lambda}+\sum_{e\in E}\big(\Gamma_{\cdot}(e,Z_-,\Psi^X,\Delta)-Z_-\big)q^e\gamma\boldsymbol{\cdot}\boldsymbol{\lambda}\Big)^{\vartheta-}\\ \qquad \text{defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } \mathbb{R}_+,\\[2ex] Z_{T-}\mathbb{1}_{\{T\le\theta\}}=0 \end{cases} \tag{4.10}$$

and (cf. (3.9) and (4.1)-(4.2)):

$$\begin{aligned} M^X &= M^Z + \mathbb{1}_{[0,T)}\big(\,\Gamma_\cdot(\epsilon, Z_-, \Psi^X, \Delta) - Z_-\big) \centerdot H \\ &\qquad - \mathbb{1}_{[0,T)}\big(\sum_{e\in E} q^e \Gamma_\cdot(e, Z_-, \Psi^X, \Delta) - Z_-\big)\gamma \centerdot \boldsymbol{\lambda} \\ &= M^Z + \mathbb{1}_{[0,T)}\big(\sum_{e\in E} \Gamma_\cdot(e, Z_-, \Psi^X, \Delta) - Z_-\big) \centerdot \mathsf{M}^e \\ &= X_0 + \Psi^X \centerdot M + \sum_{e\in E} \Delta^e \centerdot \mathsf{M}^e, \end{aligned} \tag{4.11}$$

by the martingale representation in (4.4). Hence, (4.3) implies that $M^Z = X_0 + \Psi^X \centerdot M$ and

$$\Delta^e = \mathbb{1}_{[0,T)}\left(\sum_{e\in E} \Gamma_\cdot(e, Z_-, \Psi^X, \Delta) - Z_-\right), \;\; q^e_s\gamma_s\boldsymbol{\lambda}(ds)\text{-}a.e., \;\; e \in E. \tag{4.12}$$

By the uniqueness in the condition (J), it follows that

$$\Delta^e = \Delta^\star_\cdot(e, X_-, \Psi^X, \Delta), \;\; q^e_s\gamma_s\boldsymbol{\lambda}(ds)\text{-}a.e., \;\; e \in E, \tag{4.13}$$

which, together with (4.10), shows that (Z, Ψ^X) solves the BSDE (4.6).

Likewise, by application of the converse part in Theorem 2.1, if the true BSDE (4.6) has a solution (Z, Ψ^Z), then, letting $\Delta^e = \Delta^\star_\cdot(e, Z_-, \Psi^Z)$, $e \in E$, the process

$$X = JZ + H\mathbb{1}_{\{\theta<T\}}\Gamma_\theta(\epsilon, Z_{\theta-}, \Psi^Z_\theta, \Delta_\theta)$$

satisfies

$$\begin{cases} \big(|g_\cdot(X_-, \Psi^Z, \Delta)| \centerdot \boldsymbol{\lambda}\big)_\vartheta < \infty, \\ \\ \big(X + g_\cdot(X_-, \Psi^Z, \Delta) \centerdot \boldsymbol{\lambda}\big)^\vartheta = X_0 + \Psi^Z \centerdot M \\ +\mathbb{1}_{[0,T)}\big(\Gamma_\cdot(\epsilon, Z_-, \Psi^Z, \Delta) - Z_-\big) \centerdot H - \mathbb{1}_{[0,T)} \sum_{e\in E}\big(\Gamma_\cdot(e, Z_-, \Psi^Z, \Delta) - Z_-\big) q^e\gamma \centerdot \boldsymbol{\lambda} \text{ on } \mathbb{R}_+, \\ \\ X_\vartheta = \mathbb{1}_{\{\theta<T\}}\Gamma_\theta(\epsilon, Z_{\theta-}, \Psi^Z_\theta, \Delta_\theta), \end{cases}$$

where the martingale representation follows from (3.9), the martingale representation in (4.6) and (4.1)-(4.2). In addition, the jump consistency condition (J) yields that

$$\begin{aligned} &\mathbb{1}_{[0,T)}\big(\Gamma_\cdot(\epsilon, Z_-, \Psi^Z, \Delta) - Z_-\big) \centerdot H - \mathbb{1}_{[0,T)} \sum_{e\in E}\big(\Gamma_\cdot(e, Z_-, \Psi^Z, \Delta) - Z_-\big) q^e\gamma \centerdot \boldsymbol{\lambda} \\ &\quad = \Delta^\epsilon \centerdot H - \sum_{e\in E} \Delta^e q^e \gamma \centerdot \boldsymbol{\lambda} = \sum_{e\in E} \Delta^e \centerdot \mathsf{M}^e. \end{aligned}$$

Hence, the process (X, Ψ^Z, Δ) solves the true BSDE (4.4).

Last, (4.9) follows from Corollary 2.1 and the form (3.9) of (2.12). ∎

Remark 4.2 In view of the first part of the proof, for the well-posedness of (4.4) in (X, Ψ^X, Δ), a prerequisite is the well-posedness in Δ of (4.12), given tentative solution components X and Ψ^X. This is the motivation for the condition (J).

Let $(\mathbb{F},\mathbb{P})$ be a reduced stochastic basis satisfying the condition (A). By virtue of Lemma 3.1 1), we may and do assume $\mathcal{P}(\mathbb{F})\otimes\mathcal{E}\otimes\mathcal{B}(\mathbb{R}^{1+d+|E|})$ measurable and $\mathcal{P}(\mathbb{F})\otimes\mathcal{E}\otimes\mathcal{B}(\mathbb{R}^{1+d})$ measurable data $\Gamma_t(\omega,e,x,u)$ and $\Delta^\star_t(\omega,e,x,\psi)$, respectively. In addition, we assume the existence of a d-variate $(\mathbb{F},\mathbb{P})$ local martingale P endowed with the $(\mathbb{F},\mathbb{P})$ local martingale property and such that, consistent with the condition (A),

$$M^\theta = P^{\theta-}. \tag{4.14}$$

Writing

$$\begin{aligned}
k.(V_-,\Phi^V) &= \big(|f| + (\mathsf{F}+\mathsf{F}')\,\varphi\big)_.(V_-,\Phi^V,(\Delta^\star_.(e,V_-,\Phi^V))_{e\in E})\\
h.(V_-,\Phi^V) &= \big(f + (\mathsf{F}-\mathsf{F}')\,\varphi\big)_.(\,V_-,\Phi^V,(\Delta^\star_.(e,V_-,\Phi^V))_{e\in E}) - V_-\varphi,
\end{aligned}$$

the true BSDE analog to (3.15) is written as

$$\begin{cases}
\big(k.(V_-,\Phi^V)\cdot\boldsymbol{\lambda}\big)_T < \infty \text{ on } \{\mathsf{S}_- > 0\}\,,\\[1ex]
\big(V + h.(V_-,\Phi^V)\cdot\boldsymbol{\lambda}\big)^T = V_0 + \Phi^V\cdot P \text{ on } \{\mathsf{S}_- > 0\}\,,\\[1ex]
V_T\mathsf{S}_{T-} = 0,
\end{cases} \tag{4.15}$$

where V is an $(\mathbb{F},\mathbb{P})$ semimartingale on $\{\mathsf{S}_- > 0\}$ and Φ^V is an $\mathbb{F}$ predictable, P integrable process on $\{\mathsf{S}_- > 0\}$.

Theorem 4.2 *Assuming the condition (A) as well as the relation* (4.14) *between the* $(\mathbb{G},\mathbb{Q})$ *local martingale* M *and a* $(\mathbb{F},\mathbb{P})$ *local martingale* P *endowed with the* $(\mathbb{F},\mathbb{P})$ *local martingale property, the true BSDEs (4.6) and (4.15) are equivalent. Specifically, if* (Z,Ψ^Z) *solves the true BSDE (4.6), then the pair process* $(V = U^{T-},\Phi^Z)$*, where* U *(resp.* Φ^Z*) is the* $\mathbb{F}$ *optional (resp. predictable) reduction of* $Z^{\theta-}$ *(resp.* Ψ^Z*), solves the true BSDE (4.15). Conversely, if* (V,Φ^V) *is a solution to the true BSDE (4.15), then* $(Z,\Psi^Z) = (V^{\theta-},\mathbb{1}_{[0,\theta]}\Phi^V)$ *solves the true BSDE (4.6). In both cases,* V *is the* $\mathbb{F}$ *optional reduction of* $Z^{\vartheta-}$ *with respect to the time* ϑ *and we have*

$$M^Z = \big(V + h.(V_-,\Phi^V)\cdot\boldsymbol{\lambda}\big)^{\vartheta-}. \tag{4.16}$$

Proof. By the direct parts in lemmas 3.3 through 3.5, if (Z,Ψ^Z) solves (4.6), then (V,Φ^Z) defined as in the direct part of the theorem satisfies

$$\begin{cases}
\Big(\big(|f| + (\mathsf{F}+\mathsf{F}')\,\varphi\big)_.(V_-,\Phi^Z,(\Delta^\star_.(e,V_-,\Phi^Z))_{e\in E})\cdot\boldsymbol{\lambda}\Big)_T < \infty \text{ on } \{\mathsf{S}_- > 0\}\,,\\[1ex]
P^V := \Big(V + \big(f + (\mathsf{F}-\mathsf{F}')\,\varphi\big)_.(\,V_-,\Phi^Z,(\Delta^\star_.(e,V_-,\Phi^Z))_{e\in E})\cdot\boldsymbol{\lambda} - V_-\varphi\cdot\boldsymbol{\lambda}\Big)^{T-}\\
\qquad = \Big(V + \big(f + (\mathsf{F}-\mathsf{F}')\,\varphi\big)_.(\,V_-,\Phi^Z,(\Delta^\star_.(e,V_-,\Phi^Z))_{e\in E})\cdot\boldsymbol{\lambda} - V_-\varphi\cdot\boldsymbol{\lambda}\Big)^{T}\\
\text{defines an } (\mathbb{F},\mathbb{P}) \text{ local martingale on } \{\mathsf{S}_- > 0\},\\[1ex]
V_{T-}\mathsf{S}_{T-} = V_T\mathsf{S}_{T-} = 0,
\end{cases} \tag{4.17}$$

and $(P^V)^{\theta-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale (via the condition (A)). Moreover, $Z^{\vartheta-} = V^{\vartheta-}$, hence the $(\mathbb{G},\mathbb{Q})$ canonical Doob-Meyer local martingale parts M^Z of $Z^{\vartheta-}$ and $(P^V)^{\theta-}$ of

$V^{\vartheta-}$ (by the above) coincide. In addition, by the martingale condition in (4.17) and the $(\mathbb{F},\mathbb{P})$ martingale representation property of P, we have

$$P^V = V_0 + \Phi \centerdot P \text{ on } \{\mathsf{S}_- > 0\}, \tag{4.18}$$

for some $\mathbb{F}$ predictable, P integrable process Φ on $\{\mathsf{S}_- > 0\}$. Hence (cf. (4.14) and the martingale representation in (4.6)),

$$V_0 + \mathbb{1}_{[0,\theta]}\Phi \centerdot M = V_0 + \Phi \centerdot M^\theta = V_0 + \Phi \centerdot P^{\theta-} = (P^V)^{\theta-} = M^Z = Z_0 + \Phi^Z \centerdot M,$$

thus

$$\mathbb{1}_{[0,\theta]}\Phi = \Phi^Z,\ \ d[M,M]_s\text{-}a.e.,\quad \text{hence } d[M^\theta,M^\theta]_s\text{-}a.e.,$$

hence (cf. (4.14))

$$\Phi = \Phi^Z,\ \ d[P^{\theta-},P^{\theta-}]_s\text{-}a.e..,\quad \text{i.e. } \mathbb{1}_{[0,\theta)}d[P,P]_s\text{-}a.e..$$

By optional projection, we obtain

$$\mathsf{S}\Phi = \mathsf{S}\Phi^Z,\ \ d[P,P]_s\text{-}a.e.,$$

under $\mathbb{Q}$ as under $\mathbb{P}$. In view of (4.17), (3.7) and of the first line in (4.3), (V,Φ^Z) solves

$$\begin{cases} \big(k_\cdot(V_-,\Phi^Z) \centerdot \boldsymbol{\lambda}\big)_T < \infty \text{ on } \{\mathsf{S}_- > 0\}, \\ \big(V + h_\cdot(V_-,\Phi^Z) \centerdot \boldsymbol{\lambda}\big)^{T-} = V_0 + \Phi^Z \centerdot P \text{ on } \{\mathsf{S}_- > 0\}, \\ V_{T-}\mathsf{S}_{T-} = V_T\mathsf{S}_{T-} = 0, \end{cases} \tag{4.19}$$

so that (V,Φ^Z) solves the true BSDE (4.15).

Conversely, by the converse parts in lemmas 3.3 through 3.5, if (V,Φ^V) solves (4.15), then, one the one hand, V cannot jump at T if $\mathsf{S}_{T-} > 0$, hence

$$\big(V + h_\cdot(V_-,\Phi^V) \centerdot \boldsymbol{\lambda}\big)^{\vartheta-} = \Big(\big(V + h_\cdot(V_-,\Phi^V) \centerdot \boldsymbol{\lambda}\big)^{T-}\Big)^{\theta-} = \Big(\big(V + h_\cdot(V_-,\Phi^V) \centerdot \boldsymbol{\lambda}\big)^{T}\Big)^{\theta-} \tag{4.20}$$

is a $(\mathbb{G},\mathbb{Q})$ local martingale (via the condition (A)) and, on the other hand, $Z = V^{\theta-}$ satisfies

$$\begin{cases} \Big((|g| + \gamma|\widehat{\mathsf{G}}|)_\cdot(Z_-,\Phi^V,(\Delta^\star_\cdot(e,Z_-,\Phi^V))_{e\in E}) \centerdot \boldsymbol{\lambda}\Big)_\vartheta < \infty, \\ N^Z := \Big(Z + g_\cdot(Z_-,\Phi^V,(\Delta^\star_\cdot(e,Z_-,\Phi^V))_{e\in E}) \centerdot \boldsymbol{\lambda} \\ \qquad + \sum_{e\in E}\big(\Gamma_\cdot(e,Z_-,\Phi^V,(\Delta^\star_\cdot(e',Z_-,\Phi^V))_{e'\in E}) - Z_-\big)\, q^e\gamma \centerdot \boldsymbol{\lambda}\Big)^{\vartheta-} \\ \text{defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } \mathbb{R}_+, \\ Z_{T-}\mathbb{1}_{\{T\le\theta\}} = 0. \end{cases} \tag{4.21}$$

Moreover, $Z^{\vartheta-} = V^{\vartheta-}$, hence the $(\mathbb{G},\mathbb{Q})$ canonical Doob-Meyer local martingale parts N^Z of $Z^{\vartheta-}$ (cf. the martingale condition in (4.21)) and

$$\big(V + h_\cdot(V_-,\Phi^V) \centerdot \boldsymbol{\lambda}\big)^{\vartheta-} = \Big(\big(V + h_\cdot(V_-,\Phi^V) \centerdot \boldsymbol{\lambda}\big)^{T}\Big)^{\theta-}$$

of $V^{\vartheta-}$ (by (4.20) and the surrounding comments) coincide, i.e., in view of the martingale representation in (4.15):

$$N^Z = ((V + h.(V_-, \Phi^V) \centerdot \boldsymbol{\lambda})^T)^{\theta-} = V_0 + (\Phi^V \centerdot P)^{\theta-} = V_0 + \Phi^V \centerdot M^\theta = V_0 + \mathbb{1}_{[0,\theta]} \Phi^V \centerdot M,$$

by (4.14). Thus, $(Z, \Psi^Z) = (V^{\vartheta-}, \mathbb{1}_{[0,\theta]}\Phi^V)$ solves (4.6). ∎

Theorem 4.3 *Under the assumptions of Theorems 4.1 and 4.2, the true BSDEs* (4.4) *and* (4.15) *are equivalent. Specifically, if (X, Ψ^X, Δ) solves the true BSDE (4.4), then the process $(V = U^{T-}, \Phi^X)$, where U (resp. Φ^X) is the $\mathbb{F}$ optional (resp. predictable) reduction of $X^{\theta-}$ (resp. Ψ^X), solves the true BSDE (4.15). Conversely, if (V, Φ^V) is a solution to the true BSDE (4.15), then (X, Ψ^X, Δ), where (cf.* (3.18)*)*

$$\Psi^X = \mathbb{1}_{[0,\theta]}\Phi^V,\ \Delta^e = \Delta^\star_\cdot(e, V_-, \Phi^V)\ (e \in E),\ X = JV + H\mathbb{1}_{\{\theta<T\}}\Gamma_\theta(\epsilon, V_{\theta-}, \Phi^V_\theta, \Delta_\theta), \quad (4.22)$$

solves the true BSDE (4.4). In both cases, V is the $\mathbb{F}$ optional reduction of $X^{\vartheta-}$ with respect to the time ϑ and we have (cf. (3.19)*):*

$$\begin{aligned} M^X &= V^{\vartheta-} + J_- h.(V_-, \Phi^X) \centerdot \boldsymbol{\lambda} \\ &\quad + \mathbb{1}_{[0,T)}(\Gamma.(\epsilon, V_-, \Phi^X, \Delta) - V_-) \centerdot H - \mathbb{1}_{[0,T)} \sum_{e \in E} (\Gamma.(e, V_-, \Phi^X, \Delta) - V_-) q^e \gamma \centerdot \boldsymbol{\lambda}. \end{aligned} \quad (4.23)$$

Proof. This follows by combining Theorems 4.2 and 4.3, using (4.16) to obtain (4.23) from (4.9). ∎

Comments similar to those made after Theorem 3.3 are applicable.

4.1 Example

This section illustrates the semimartingale approach of this paper on a Black-Scholes case considered from a Markov point of view in Crépey et al. (2014, Section 4.6 pages 106 through 113)[7], based on Burgard and Kjaer (2011a, 2011b). We refer to Sect. 2.1 for the broad financial background and to Crépey et al. (2014, Chapter 4) for all the financial details.

We consider a European option with payoff $\phi(S_T)$ on a Black-Scholes stock S sold by the bank to its counterparty at time 0. Both parties are defaultable but they cannot default simultaneously. The option position is hedged by the bank with the stock S and zero-recovery risky bonds B^c and B^b issued by the counterpartyand the bank, respectively. Repo markets (with zero repo bases) are assumed to exist for S, B^c and B^b. Assuming a constant risk free rate r, the gain process of a buy-and-hold position into the hedging assets, if securely funded through their repo markets, is written in differential form as

$$\begin{pmatrix} dS_t - rS_t dt \\ dB^c_t - rB^c_t dt \\ dB^b_t - rB^b_t dt \end{pmatrix}.$$

Consistent with martingale no-arbitrage requirements under the pricing measure $\mathbb{Q}$ (see the comments following Assumption 4.4.1 page 96 in Crépey, Bielecki, and Brigo (2014)[8]), we

[7]Or Section 5 in the journal version Crépey (2015, Part I).
[8]Or Assumption 4.1 in the journal version Crépey (2015, Part I).

assume the following model for (S, B^c, B^b):

$$\begin{cases} dS_t - rS_t dt = \sigma S_t dW_t =: dM_t \\ dB^c_t - rB^c_t dt = B^c_{t-}\left(d\mathsf{J}^c_t + \gamma_c dt\right) = -\widetilde{B}^c_t d\mathsf{M}^c_t \\ dB^b_t - rB^b_t dt = B^b_{t-}\left(d\mathsf{J}^b_t + \gamma_b dt\right) = -\widetilde{B}^b_t d\mathsf{M}^b_t, \end{cases} \tag{4.24}$$

where W_t is a $\mathbb{Q}$ Brownian motion, where $\mathsf{J}^c_t = \mathbb{1}_{t<\theta_c}$, $\mathsf{J}^b_t = \mathbb{1}_{t<\theta_b}$ are the survival indicator processes of the counterparty and the bank, with constant $\mathbb{Q}$ default intensities γ_c and γ_b, and where $\widetilde{B}^e(t) = B^e_0 e^{(r+\gamma_e)t}$, $e \in E = \{c, b\}$. The full model filtration $\mathbb{G}$ is given as the Brownian filtration $\mathbb{F}$ progressively enlarged by the default times θ_c and θ_b. Note that, by independence of W, J^c and J^b, we have $\mathsf{S}_t = e^{-(\gamma_b+\gamma_c)t} > 0$ and the condition (A) (that includes (B)) obviously holds with $\mathbb{P} = \mathbb{Q}$ for the $\mathbb{G}$ stopping time $\theta = \theta_b \wedge \theta_c$ with intensity $\gamma = \gamma^c + \gamma^b$. We assume that the bank trades S and B^c on their repo markets but that it externally funds its trading in B^b. In addition, we assume that the bank can get unsecured funding (borrow cash from an external lender) at an interest rate $(r + \bar{\lambda})$, for some positive borrowing basis $\bar{\lambda}$, but that it can only lend cash at the risk free rate r. Last, we assume no collateralization ($C = 0$ in the notation of Sect. 2.1). We denote by R_b and R_c constant recovery rates of the bank and the counterparty relative to each other and by $\bar{R}_b$ a constant recovery rate of the bank toward its external funder. In this case, following the methodology that is developed in Crépey et al. (2014, Chapters 4-5)[9], one can show that the counterparty risk replication equation comes in true BSDE form as (cf. (4.4) and (4.24)):

$$\begin{aligned} &X_\vartheta = \mathbb{1}_{\theta<T}\mathsf{G}(X_{\theta-}, \Psi^X_\theta, \Delta^c_\theta, \Delta^b_\theta) \text{ and, for } t \in [0, \vartheta], \\ &dX_t = g_t(X_t, \Psi^X_t, \Delta^c_t, \Delta^b_t)dt + \Psi^X_t dM_t + \left(\frac{-\Delta^c_t}{\widetilde{B}^c_t}\right)(-\widetilde{B}^c_t d\mathsf{M}^c_t) + \left(\frac{-\Delta^b_t}{\widetilde{B}^b_t}\right)(-\widetilde{B}^b_t d\mathsf{M}^b_t) \end{aligned} \tag{4.25}$$

(focusing on the interval of interest $[0, \vartheta]$ and omitting for brevity the integrability condition that is implicit on the gdt term), where, for x real and $u = (\psi, \delta_c, \delta_b)$ in $\mathbb{R}^3$:

$$g_t(x, u) = \bar{\lambda}\big(P_t - \delta_b - x\big)^+ - rx, \quad \mathsf{G}(x, u) = \Gamma_\theta(\epsilon, x, u), \tag{4.26}$$

with, for $t \in [0, T]$ and $e \in E = \{c, b\}$,

$$\Gamma_t(e, x, u) = \mathbb{1}_{e=c}(1 - R_c)P^+_t - \mathbb{1}_{e=b}\Big((1 - R_b)P^-_t + (1 - \bar{R}_b)(P_{t-} - \delta_b - x)^+\Big),$$

hence (cf. (4.2))

$$\gamma\widehat{\mathsf{G}}_t(x, u) = \gamma_c(1 - R_c)P^+_t - \gamma_b\Big((1 - R_b)P^-_t + (1 - \bar{R}_b)(P_{t-} - \delta_b - x)^+\Big). \tag{4.27}$$

Note that, in contrast with (2.3), g and G in (4.26) depend on $u = (\psi, \delta_c, \delta_b)$, through δ_b, reflecting the fact that the position of the bank in its own bond is externally funded (see Crépey et al. (2014, Example 4.4.3 page 97)[10]).

Assuming (4.25), an Itô computation yields, with $\mathsf{Z} = X^{\vartheta-}$ (cf. (2.11)):

$$d\mathsf{Z}_t = \Big(\bar{\lambda}\big(P_t - \Delta^b_t - \mathsf{Z}_t\big)^+ - r\mathsf{Z}_t\Big)dt + \Psi^X_t \sigma S_t\, dW_t - \Delta^c_t\gamma_c dt - \Delta^b_t\gamma_b dt \text{ on } (0, \vartheta), \tag{4.28}$$

[9] Or the journal version Crépey (2015).

[10] Or Example 4.1 in the journal version Crépey (2015, Part I).

along with the terminal condition $\Delta_\vartheta \mathsf{Z} = 0$ and the following jump consistency conditions:

$$\begin{aligned}\Delta^c_t &= (1-R_c)P^+_t - \mathsf{Z}_t \\ \Delta^b_t &= -(1-R_b)P^-_t - (1-\bar{R}_b)\left(P_t - \Delta^b_t - \mathsf{Z}_t\right)^+ - \mathsf{Z}_t,\end{aligned} \tag{4.29}$$

where the second line is equivalent to

$$0 = -P_t + (P_t - \Delta^b_t - \mathsf{Z}_t) - (1-R_b)P^-_t - (1-\bar{R}_b)\left(P_t - \Delta^b_t - \mathsf{Z}_t\right)^+,$$

i.e.

$$P^+_t - R_b P^-_t = \bar{R}_b\big(P_t - \Delta^b_t - \mathsf{Z}_t\big)^+ - \big(P_t - \Delta^b_t - \mathsf{Z}_t\big)^-, \tag{4.30}$$

hence (assuming $\bar{R}_b > 0$), taking the positive parts left and right,

$$\left(P_t - \Delta^b_t - \mathsf{Z}_t\right)^+ = \frac{1}{\bar{R}_b}P^+_t. \tag{4.31}$$

Thus, by the second line in (4.29),

$$\Delta^b_t \;\; = -(1-R_b)P^-_t - (\frac{1}{\bar{R}_b} - 1)P^+_t - \mathsf{Z}_t. \tag{4.32}$$

In other words, the condition (J) is satisfied with

$$\begin{aligned}\Delta^\star_t(c, x, \psi) &= (1-R_c)P^+_t - x, \\ \Delta^\star_t(b, x, \psi) &= -(1-R_b)P^-_t - (\frac{1}{\bar{R}_b} - 1)P^+_t - x\end{aligned} \tag{4.33}$$

(which in this case depend linearly on x and do not depend on ψ). Substituting the first line in (4.29), (4.32) and (4.31) into (4.28) yields

$$d\mathsf{Z}_t = \mathsf{g}_t(\mathsf{Z}_t)dt + \Psi^X_t \sigma S_t\, dW_t, \tag{4.34}$$

where, for any real number x,

$$\mathsf{g}_t(x) = \gamma_c(1-R_c)P^+_t - \gamma_b(1-R_b)P^-_t + \frac{\widetilde{\lambda}}{\bar{R}_b}P^+_t - (r+\gamma)x, \tag{4.35}$$

in which $\widetilde{\lambda} = \bar{\lambda} - (1-\bar{R}_b)\gamma_b$ represents the liquidity borrowing basis of the bank. Observe that in the context of this example, the risk-free price of the option P_t (price ignoring counterparty risk and excess funding costs) is given by its Black-Scholes price $P_t = v(t, S_t)$, where $v(t, S)$ is the unique classical solution to the following Black-Scholes PDE (assuming ϕ continuous with polynomial growth in S for ensuring well posedness of (4.36) in the class of classical solutions with polynomial growth in S):

$$\begin{cases} v(T, S) = \phi(S), \; S \in (0, \infty), \\ \left(\partial_t + \mathcal{A}^{bs}\right) v(t, S) - rv(t, S) = 0, \; t < T, S \in (0, \infty), \end{cases} \tag{4.36}$$

where $\mathcal{A}^{bs} = rS\partial_S + \frac{\sigma^2 S^2}{2}\partial^2_{S^2}$. In view of (4.34)-(4.35), one could be tempted to conclude that

$$\mathsf{Z}_t = z(t, S_t), \;\; \Psi^X_t = \partial_S z(t, S_t), \tag{4.37}$$

where $z(t,S)$ is the unique classical solution with polynomial growth in S to

$$\begin{cases} z(T,S)=0,\ S\in(0,\infty), \\ \left(\partial_t+\mathcal{A}^{bs}\right)z(t,S)+f(t,S)-(r+\gamma)z(t,S)=0,\ t<T, S\in(0,\infty), \end{cases} \tag{4.38}$$

with

$$f(t,S)=\gamma_c(1-R_c)v(t,S)^+-\gamma_b(1-R_b)v(t,S_t)^-+\frac{\widetilde{\lambda}}{\bar{R}_b}v(t,S)^+,$$

so that

$$z(t,S_t)=\mathbb{E}\big[\int_t^T e^{-(r+\gamma)(s-t)}f(s,S_s)ds\,|\,\mathcal{F}_t\big]. \tag{4.39}$$

By the direct part in Theorem 4.3, this is in fact true regarding not exactly $\mathsf{Z}=X^{\vartheta-}$ itself (obviously, thinking of the terminal no jump condition $\Delta_\vartheta\mathsf{Z}=0$), but its $\mathbb{F}$ optional reduction V (recalling $\mathsf{S}>0$), which must satisfy, jointly with some $\mathbb{F}$ predictable, SdW integrable process Φ^V (cf. (4.15)):

$$\begin{aligned} &V_T=0 \text{ and, for } t\in[0,T], \\ &dV_t=(f(t,S_t)-(r+\gamma)V_t)dt+\Phi^V_t\sigma S_t dW_t \end{aligned} \tag{4.40}$$

(in this case, a linear BSDE), i.e., under standard assumptions,

$$V_t=z(t,S_t),\ \ \Phi^V_t=\partial_S z(t,S_t) \tag{4.41}$$

(cf. (4.37), (4.39)). In addition, the converse part in Theorem 4.3 shows that, for (V,Φ^V) defined in this way, the process (X,Ψ^X,Δ) defined in terms of (V,Φ^V) by (4.22), with $\Delta^\star$ there given by (4.33), solves the true BSDE (4.25). These results are also consistent with the Markov analysis in Crépey et al. (2014, Section 4.6 pages 106 through 113)[11].

Remark 4.3 The case $\bar{R}_b=0$ can be dealt with similarly provided $P\le 0$, otherwise (4.30) reduces to

$$P_t=-\big(P_t-\Delta^b_t-\mathsf{Z}_t\big)^-,$$

which has no solution given that the signs of both sides differ (hence, replicability does not hold in this case).

5 Conclusion

To conclude this paper, we draw the practical consequences of our study for the motivating counterparty and credit risk problems.

Back to the counterparty risk setup of Sect. 2.1, assuming the condition (A) relative to a reduced stochastic basis $(\mathbb{F},\mathbb{P})$ of $(\mathbb{G},\mathbb{Q})$ (a mild assumption in view of the results of Crépey and Song (2014, 2015)), Theorem 3.3 says that in order to find the price X of counterparty risk and funding costs, i.e. to solve the original $(\mathbb{G},\mathbb{Q})$ BSDE (2.1) with data (2.3), it suffices to set

$$X=JW+H\mathbb{1}_{\{\theta<T\}}\mathsf{G}(W_{\vartheta-}), \tag{5.1}$$

[11] Or Section 5 in the journal version Crépey (2015, Part I).

where W solves the reduced $(\mathbb{F},\mathbb{P})$ BSDE (3.20) with data φ, f, F and F' given as the respective $\mathbb{F}$ predictable reductions (which exist under the condition (B)) of γ, g, $\widehat{\mathsf{G}^+}$ and $\widehat{\mathsf{G}^-}$ (cf. (2.3)). In fact, the BSDEs (2.1) and (3.20) are equivalent (at least for S positive; see Theorems 3.2 and 3.3). Moreover, Theorem 4.3 says that similar equivalence is also available for the true BSDE (4.4) with an additional dependence of the data on integrands in a martingale representation of the solution. Beyond its theoretical interest and its implications in terms of existence and uniqueness of solutions, the reformulation of the counterparty risk BSDEs (2.1) or (4.4) with implicit terminal condition at the random time ϑ as BSDEs with null terminal conditions at the fixed time T also represents a significant improvement from the point of view of numerical solutions (see e.g. Crépey and Song (2015)).

Moreover, this paper generalizes various recovery pricing formulas in the credit risk literature (see Sect. 1), in various respects: terminal time only assumed invariant (no need of the density hypothesis), implicit terminal condition depending on the solution right before that time, optional (nonpredictable) recovery process, general semimartingale setting, tractability of these results to model defaultable securities (with economically appealing decompositions such as the formula (3.19) for the martingale parts of the solutions to the BSDEs).

This work could be pursued in several directions. In particular, it would be interesting to study the equivalence between the full and the reduced BSDEs, not only at the most general level of this paper, where a solution is only required to give a meaning to the involved Lebesgue and stochastic integrals, but also at the more restricted level of, say, square integrable solutions, for which well posedness of the reduced BSDEs holds under classical assumptions (see the comments following Theorems 3.3 and 4.3). Also, a general study of the reduction of the true BSDEs could be conducted beyond the application driven setup of Sect. 4.

A Parameterized Conditioning

This section deals with conditioning issues in relation with single step martingales of the form (A.1) that appear via the terminal condition of the full BSDE (2.1).

We need to compute conditional expectations of the form $\mathbb{E}[\mathsf{G}(\xi)|\mathcal{G}_{\theta-}]$, for a measurable function $\mathsf{G}(\omega,x)$, a $\mathbb{G}$ stopping time θ and a $\mathcal{G}_{\theta-}$ measurable random variable ξ. The intuition suggests that, under the conditioning, the random variable ξ can be treated as a constant, so that the computation can be performed in two steps: first for a constant x instead of ξ, then by substituting ξ for x, i.e.

$$\text{“}\mathbb{1}_{\{\theta<\infty\}}\mathbb{E}[\mathsf{G}(\xi)|\mathcal{G}_{\theta-}] = \mathbb{1}_{\{\theta<\infty\}}\mathbb{E}[\mathsf{G}(x)|\mathcal{G}_{\theta-}]_{x=\xi}\text{”}.$$

However, this is not well defined because the conditional expectation $\mathbb{E}[\mathsf{G}(x)|\mathcal{G}_{\theta-}]$ is an equivalence class depending on the parameter x. A “bad” choice of the class (one for each x) in the first step may result in a nonmeasurable expression in the second step.

Example A.1 Let $\mathcal{B}$ denote the Borel σ-field over $[0,1]$, considered as a sub-σ-field, through the inverse of the first coordinate projection π, of the Borel σ-field over $\Omega=[0,1]^2$ equipped with the Lebesgue measure. Let $\mathsf{G}(\omega,x)$ be a nonnegative Borel function on $\Omega\times\mathbb{R}$. By Fubini's theorem, there exists a Borel function G' on $\Omega\times\mathbb{R}_+$ such that, for any x, $\omega\to\mathsf{G}'(\omega,x)$ is a version of $\mathbb{E}[\mathsf{G}(x)|\mathcal{B}](\omega)$ (“expectation with respect to v for u frozen” in

$\omega = (u, v) \in \Omega = [0, 1]^2$). Let

$$\mathsf{G}''(\omega, x) = \mathsf{G}'(\omega, x) + \mathbb{1}_{\mathcal{V}}(x)\mathbb{1}_{\{x\}}(\pi(\omega)),$$

where $\mathcal{V}$ is the Vitali set in $[0, 1]$ (assuming the axiom of choice). Since x is fixed here, $\mathsf{G}'(\cdot, x)$ and $\mathsf{G}''(\cdot, x)$ are almost surely equal, i.e. $\mathsf{G}''(\cdot, x)$ is a version of $\mathbb{E}[\mathsf{G}(x)|\mathcal{B}]$ (for fixed x). However, since the Vitali set is not Lebesgue measurable, the function

$$\mathsf{G}''(\omega, \pi(\omega)) = \mathsf{G}'(\omega, \omega) + \mathbb{1}_{V}(\pi(\omega))$$

is not Borel.

A "good" choice of $\mathbb{E}[\mathsf{G}(x)|\mathcal{G}_{\theta-}]$ is made in Stricker and Yor (1978), where, by the monotone class argument, it is proved that there exists a $\mathcal{G}_{\theta-} \otimes \mathcal{B}(\mathbb{R})$ measurable function $\widehat{\mathsf{G}}(\omega, x)$ such that

$$\mathbb{1}_{\{\theta<\infty\}}\mathbb{E}[\mathsf{G}(\xi)|\mathcal{G}_{\theta-}] = \mathbb{1}_{\{\theta<\infty\}}\widehat{\mathsf{G}}(\xi).$$

But the construction of the function $\widehat{\mathsf{G}}$ in Stricker and Yor (1978) depends on the given random variable ξ, whereas we need $\widehat{\mathsf{G}}$ in an equation (BSDE) for an unknown random variable ξ (i.e. we need a common function $\widehat{\mathsf{G}}$ that works for all ξ). This motivates the following.

Definition A.1 Let $(E, \mathcal{E})$ be a measurable space. For any nonnegative $\mathcal{A} \otimes \mathcal{E}$ measurable function $\mathsf{G}(\omega, x)$, we say that $\widehat{\mathsf{G}}$ exists in the wide sense if there exists a nonnegative $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ measurable function $\widehat{\mathsf{G}}_t(\omega, x)$ such that, for any E valued $\mathcal{G}_{\theta-}$ measurable random variable ξ,

$$\mathbb{1}_{\{\theta<\infty\}}\mathbb{E}[\mathsf{G}(\xi)|\mathcal{G}_{\theta-}] = \mathbb{1}_{\{\theta<\infty\}}\widehat{\mathsf{G}}_\theta(\xi).$$

For a general (not necessarily nonnegative) $\mathcal{A} \otimes \mathcal{E}$ measurable function $\mathsf{G}(x)$, we say that $\widehat{\mathsf{G}}$ exists (in the strict sense, as considered by default) if, for any $\mathcal{G}_{\theta-}$ measurable random variable ξ, $\mathsf{G}(\xi)$ is $\mathcal{G}_{\theta-}$ locally integrable[12] and there exists a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ measurable function $\widehat{\mathsf{G}}_t(\omega, x)$ such that, for any E valued $\mathcal{G}_{\theta-}$ measurable random variable ξ,

$$\mathbb{1}_{\{\theta<\infty\}}\mathbb{E}[\mathsf{G}(\xi)|\mathcal{G}_{\theta-}] = \mathbb{1}_{\{\theta<\infty\}}\widehat{\mathsf{G}}_\theta(\xi).$$

In both general or strict cases, we say then that $\widehat{\mathsf{G}}_t(\omega, x)$ is a version of $\widehat{\mathsf{G}}$.

Lemma A.1 *Let $(E, \mathcal{E})$ be a measurable space.*

1) *For any bounded $\mathcal{A} \otimes \mathcal{E}$ measurable function $\mathsf{G}(\omega, x)$, $\widehat{\mathsf{G}}$ exists. For any non negative $\mathcal{A} \otimes \mathcal{E}$ measurable function $\mathsf{G}(\omega, x)$, $\widehat{\mathsf{G}}$ exists in the wide sense.*

2) *The space of $\mathcal{A} \otimes \mathcal{E}$ measurable functions G such that $\widehat{\mathsf{G}}$ exists is a vector space; for any $a, b \in \mathbb{R}$ and $\mathsf{G}_1, \mathsf{G}_2$ in this space, $a\widehat{\mathsf{G}}_1 + b\widehat{\mathsf{G}}_2$ is a version of $\widehat{a\mathsf{G}_1 + b\mathsf{G}_2}$.*

3) *If $\widehat{\mathsf{G}}$ exists, then $\widehat{\mathsf{G}^+}, \widehat{\mathsf{G}^-}$ and $\widehat{|\mathsf{G}|}$ exist and any versions of $(\widehat{\mathsf{G}^+} + \widehat{\mathsf{G}^-})$ and $(\widehat{\mathsf{G}^+} - \widehat{\mathsf{G}^-})$ are respective versions of $\widehat{|\mathsf{G}|}$ and $\widehat{\mathsf{G}}$.*

[12] So that the notion of (generalized) conditional expectation can be applied to it; cf. Sect. 1.2 and He et al. (1992, Definition 1.15).

Proof. Let Φ denote the class of all bounded $\mathcal{A}\otimes\mathcal{E}$ measurable function G for which $\widehat{\mathsf{G}}$ exists. We check directly from the definition that the class Φ contains the constants and is stable by multiplication by constants and by finite sums. Let $(\mathsf{G}_n)_{n\in\mathbb{N}}$ be a nondecreasing uniformly bounded sequence of nonnegative functions in Φ. For any $n\in\mathbb{N}$, for any E valued $\mathcal{G}_{\theta-}$ measurable random variable ξ,

$$\mathbb{E}[\mathsf{G}_n(\xi)|\mathcal{G}_{\theta-}]\mathbb{1}_{\{\theta<\infty\}}=(\widehat{\mathsf{G}_n})_\theta(\xi)\mathbb{1}_{\{\theta<\infty\}}.$$

We can assume that the $\widehat{\mathsf{G}_n}$ are uniformly bounded. By the monotone convergence theorem,

$$\mathbb{E}[\sup_{n\in\mathbb{N}}\mathsf{G}_n(\xi)|\mathcal{G}_{\theta-}]\mathbb{1}_{\{\theta<\infty\}}=\sup_{n\in\mathbb{N}}\mathbb{E}[\mathsf{G}_n(\xi)|\mathcal{G}_{\theta-}]\mathbb{1}_{\{\theta<\infty\}}=(\sup_{n\in\mathbb{N}}\widehat{\mathsf{G}_n})_\theta(\xi)\mathbb{1}_{\{\theta<\infty\}}.$$

This formula shows that $\sup_{n\in\mathbb{N}}\mathsf{G}_n\in\Phi$ and

$$\widehat{\sup_{n\in\mathbb{N}}\mathsf{G}_n}=\sup_{n\in\mathbb{N}}\widehat{\mathsf{G}_n}.$$

Finally, let's consider $A\in\mathcal{A}, B\in\mathcal{E}$. Since the random variable $\mathbb{Q}[A|\mathcal{G}_{\theta-}]$ is $\mathcal{G}_{\theta-}$ measurable, there exists a $\mathbb{G}$ predictable process L such that[13]

$$L_\theta\mathbb{1}_{\{\theta<\infty\}}=\mathbb{Q}[A|\mathcal{G}_{\theta-}]\mathbb{1}_{\{\theta<\infty\}}.$$

We check directly that $L_t(\omega)\mathbb{1}_B(x)$ is a version of $\widehat{\mathbb{1}_A\mathbb{1}_B}$, which shows that $\mathbb{1}_A(\omega)\mathbb{1}_B(x)\in\Phi$. We can now apply the monotone class theorem[14] to say that Φ contains all the bounded $\mathcal{A}\otimes\mathcal{E}$ measurable functions. By taking suprema over sequences, the result is extended in the wide sense to any (non necessarily bounded) nonnegative $\mathcal{A}\otimes\mathcal{E}$ measurable functions. This proves 1). 2) is a direct consequence of the definition. Regarding 3), note that the existence of $\widehat{\mathsf{G}}$ implies the $\mathcal{G}_{\vartheta-}$ local integrability of $|\mathsf{G}|$. Therefore, $\widehat{\mathsf{G}^+},\widehat{\mathsf{G}^-}$ and $\widehat{|\mathsf{G}|}$ exist and for any of their versions $(\widehat{\mathsf{G}^+})_t(x),(\widehat{\mathsf{G}^-})_t(x)$ and $(\widehat{|\mathsf{G}|})_t(x)$, respective versions of $\widehat{|\mathsf{G}|}$ and $\widehat{\mathsf{G}}$ are given by $(\widehat{\mathsf{G}^+})_t(x)+(\widehat{\mathsf{G}^-})_t(x)$ and $(\widehat{\mathsf{G}^+})_t(x)-(\widehat{\mathsf{G}^-})_t(x)$. ∎

Lemma A.2 *Let G be a $\mathcal{A}\otimes\mathcal{B}(\mathbb{R}^k)$ measurable function such that $\widehat{\mathsf{G}}$, hence $\widehat{|\mathsf{G}|}$, exist. Let Υ be an $\mathbb{R}^k$ valued $\mathbb{G}$ predictable process. Suppose θ finite and $\int_0^\theta\widehat{|\mathsf{G}|}_s(\Upsilon_s)d\mathbf{v}_s<\infty$. Then, the process*

$$\mathsf{G}(\Upsilon_\theta)H-\widehat{\mathsf{G}}_{\cdot}(\Upsilon)\centerdot\mathbf{v} \tag{A.1}$$

is a $\mathbb{G}$ local martingale on $\mathbb{R}_+$.

Proof. First of all, one can check immediately from the definition that the integral $\int_0^t\widehat{|\mathsf{G}|}_s(\Upsilon_s)d\mathbf{v}_s$ is independent of the choice of a version of $\widehat{\mathsf{G}}$. Moreover, any versions of $\widehat{\mathsf{G}^+}+\widehat{\mathsf{G}^-}$ and $\widehat{\mathsf{G}^+}-\widehat{\mathsf{G}^-}$ are respective versions of $\widehat{|\mathsf{G}|}$ and $\widehat{\mathsf{G}}$, by Lemma A.1 3). Since the process $\int_0^t\widehat{|\mathsf{G}|}_s(\Upsilon_s)d\mathbf{v}_s<\infty$, $t\in\mathbb{R}_+$, is càdlàg and $\mathbb{G}$ predictable, there exists a nondecreasing sequence of $\mathbb{G}$ stopping times $(\tau_n)_{n\in\mathbb{N}}$ tending to infinity such that $\int_0^{\tau_n}\widehat{|\mathsf{G}|}_s(\Upsilon_s)d\mathbf{v}_s$ is bounded[15]. We have

$$\begin{aligned}\mathbb{E}[|\widehat{\mathsf{G}}_\vartheta(\Upsilon_\theta)|\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]&\le\mathbb{E}[|\mathsf{G}|(\Upsilon_\theta)\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]\\&=\mathbb{E}[\widehat{|\mathsf{G}|}_\vartheta(\Upsilon_\theta)\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]=\mathbb{E}[\int_0^{\tau_n}\widehat{|\mathsf{G}|}_s(\Upsilon_s)d\mathbf{v}_s]<\infty.\end{aligned}$$

[13] Cf. He et al. (1992, Corollary 3.22).
[14] Cf. He et al. (1992, Theorem 1.4).
[15] Cf. He et al. (1992, Theorem 7.7).

As a consequence, we can write

$$\begin{aligned}\mathbb{E}[\mathsf{G}(\Upsilon_\theta)\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] &= \mathbb{E}[\mathbb{E}[\mathsf{G}(\Upsilon_\theta)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]\\ &= \mathbb{E}[\mathbb{E}[\mathsf{G}^+(\Upsilon_\theta)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] - \mathbb{E}[\mathbb{E}[\mathsf{G}^-(\Upsilon_\theta)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{0<\vartheta\le\tau_n\}}]\\ &= \mathbb{E}[((\widehat{\mathsf{G}^+})_\vartheta(\Upsilon_\theta) - (\widehat{\mathsf{G}^-})_\vartheta(\Upsilon_\theta))\mathbb{1}_{\{0<\vartheta\le\tau_n\}}] = \mathbb{E}[\int_0^{\tau_n}\widehat{\mathsf{G}}_s(\Upsilon_s)d\mathbf{v}_s].\end{aligned}$$

Therefore, by He et al. (1992, Theorem 4.40)),

$$\Big(\mathsf{G}(\Upsilon_\theta)\mathbb{1}_{\{0<\vartheta\}}\mathbb{1}_{[\vartheta,\infty)} - \widehat{\mathsf{G}}_\cdot(\Upsilon)\centerdot\mathbf{v}\Big)^{\tau_n}$$

is a $\mathbb{G}$ uniformly integrable martingale, for each n. Hence, the process $\mathsf{G}(\Upsilon_\theta)\mathbb{1}_{\{0<\vartheta\}}\mathbb{1}_{[\vartheta,\infty)} - \widehat{\mathsf{G}}_\cdot(\Upsilon)\centerdot\mathbf{v}$, hence $\mathsf{G}(\Upsilon_\theta)\mathbb{1}_{[\vartheta,\infty)} - \widehat{\mathsf{G}}_\cdot(\Upsilon)\centerdot\mathbf{v}$, is a $\mathbb{G}$ local martingale on $\mathbb{R}_+$. ∎

A.1 Marked Stopping Times

The following result gives an explicit formula for $\widehat{\mathsf{G}}$ in the case, considered in Sect. 4 and in Crépey and Song (2015), of a marked stopping time.

We consider a stopping time $\theta(\omega)$ taking, for every ω, the value of one of the $\theta^e(\omega)$, where, for every e in a finite or at most countable space of marks E endowed with some σ-algebra $\mathcal{E}$, θ^e represents a totally inaccessible stopping time with intensity γ^e_t, such that $\mathbb{Q}[\theta^e \neq \theta^{e'}] = 1$ for $e' \neq e$. Writing $\epsilon(\omega) = \sum_{e\in E}\mathbb{1}_{\{\theta^e(\omega)=\theta(\omega)\}}e$, which is $\mathcal{G}_\theta$ measurable, let

$$\mathsf{G}(\omega, v) = \Gamma_{\theta(\omega)}(\omega, \epsilon(\omega), v), \tag{A.2}$$

for some nonnegative $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}\otimes\mathcal{B}(\mathbb{R}^k)$ measurable $\Gamma_t(\omega, e, v)$ (for some $k \in \mathbb{N}$).

Lemma A.3 *For any G of the form (A.2), a version of $\widehat{\mathsf{G}}$ is given by $\sum_{e\in E} q^e_t\Gamma_t(e, v)$, where, for every $e \in E$, q^e is a $[0, 1]$ valued $\mathbb{G}$ predictable process such that $q^e_\theta = \mathbb{Q}[\{\theta^e = \theta\}|\mathcal{G}_{\theta-}]$ on $\{\theta < \infty\}$.*

Proof. The existence of the q^e follows from Corollary 3.23 2) in He et al. (1992). As a consequence, we have on $\{\theta < \infty\}$, for any $\mathbb{R}^k$ valued $\mathcal{G}_{\theta-}$ measurable random variable v:

$$\mathbb{E}[\mathsf{G}(v)|\mathcal{G}_{\theta-}] = \mathbb{E}[\Gamma_\theta(\epsilon, v)|\mathcal{G}_{\theta-}] = \sum_{e\in E}\mathbb{Q}[\{\theta^e = \theta\}|\mathcal{G}_{\theta-}]\Gamma_\theta(e, v) = \sum_{e\in E} q^e_\theta\Gamma_\theta(e, v). \ ∎$$

Note that by class monotone, one can show that a representation of the form (A.2) for a $\mathcal{G}_\theta\otimes\mathcal{B}(\mathbb{R}^k)$ measurable function $\mathsf{G}(\omega, v)$ always exists in the case where

$$\mathcal{G}_\theta = \mathcal{G}_{\theta-}\vee\sigma(\epsilon). \tag{A.3}$$

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