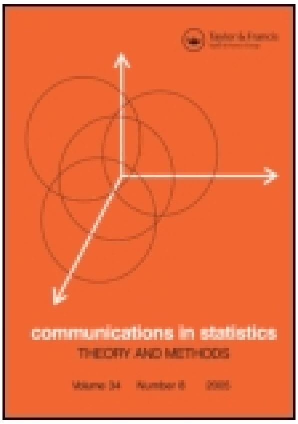

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A Bottom-Up Dynamic Model of Portfolio Credit Risk with Stochastic Intensities and Random Recoveries

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A Bottom-Up Dynamic Model of Portfolio Credit Risk with Stochastic Intensities and Random Recoveries

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In Bielecki et al. (2014a), the authors introduced a Markov copula model of portfolio credit risk where pricing and hedging can be done in a sound theoretical and practical way. Further theoretical backgrounds and practical details are developed in Bielecki et al. (2014b,c) where numerical illustrations assumed deterministic intensities and constant recoveries. In the present paper, we show how to incorporate stochastic default intensities and random recoveries in the bottom-up modeling framework of Bielecki et al. (2014a) while preserving numerical tractability. These two features are of primary importance for applications like CVA computations on credit derivatives (Assefa et al., 2011; Bielecki et al., 2012), as CVA is sensitive to the stochastic nature of credit spreads and random recoveries allow to achieve satisfactory calibration even for "badly behaved" data sets. This article is thus a complement to Bielecki et al. (2014a), Bielecki et al. (2014b) and Bielecki et al. (2014c).

Keywords Common shocks; Markov copula model; Portfolio credit risk; Random recoveries; Stochastic spreads.

Mathematics Subject Classification Primary 60J99, 91G40, 91B70; Secondary 33F05, 91G60.

1. Introduction

In Bielecki et al. (date 2014a) we introduced a common-shock Markov copula model of default times providing an effective joint calibration to single-name CDS and

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multi-name CDO tranches data. In this sense this model solves the portfolio credit risk top-down bottom-up puzzle (Bielecki et al., 2010). For earlier, partial progress in this direction, see Brigo et al. (2007a,b,2010); Elouerkhaioui, (2007); Brigo et al. (2010) and the introductory discussion in Bielecki et al. (2014b). The main model feature is the use of common jumps to default, triggered by common shocks, as a powerful dependence device also compatible with the Markov copula properties Bielecki et al. (2008), the latter being required to decouple the calibration of the individual (single-name) model parameters from the model dependence parameters.

The model presented in Bielecki et al. (2014a) is a fully dynamic model in which the critical issue of modeling counterparty risk embedded in credit derivatives, and consequently the issue of computation and hedging of CVA, can be consistently and practically addressed (Assefa et al., 2011; Bielecki and Crépey, 2013; Bielecki et al., 2012, 2013; Crépey et al., 2014). However, it was emphasized in a June 2011 Bank of International Settlements press release that "During the financial crisis of 2007–2009, roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults". In other words, the volatility of CVA matters as much as its level. Consequently (and also to be consistent with the optional nature of the CVA), for CVA computations on credit derivatives, practitioners strongly advocate the use of stochastic default intensities. Moreover, in case of some "badly behaved" data sets, a satisfying calibration accuracy can only be achieved by resorting to random recoveries.

In order to respond to these considerations, in the present paper, which is a follow-up to Bielecki et al. (2014a,b,c), we provide more background, implementation hints, as well as numerical illustration accounting for these two features which are important for applications: stochastic spreads and random recoveries. Section 2 reviews the model of default times which is used, including the specification of the stochastic intensities and recoveries. Regarding the default intensities, we resort to time-inhomogenous affine processes with time-dependent piecewise-constant mean-reversion level, resulting in analytical tractability and calibration flexibility (of the term-structure of CDS spreads in particular). For tractability reasons, random recoveries are taken to be independent between them, as well as independent from everything else in the model. Section 3 is about pricing in this setup. In particular, pricing of CDS, as well as of CDO tranches, ultimately boils down here to computations of Laplace transform for time-inhomogenous affine processes. Proposition 3.1 shows how this can be done explicitly, exploiting the piecewise-constant mean-reversion structure of the intensities. Again, effective joint calibration of this model to CDS and CDO data is an important achievement. Section 4 reviews in detail and illustrates the calibration methodology, regarding in particular the stochastic affine intensities and random recovery specifications which are used.

In the rest of the article we consider a risk neutral pricing model $(\Omega, \mathscr{F}, \mathbb{P})$, for a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\in[0,T]}$ which will be specified below, and where $T \geq 0$ is a fixed time horizon. We denote $\mathbb{N}_n = \{1, \ldots, n\}$ and we let $\mathscr{N}_n$ denote the set of all subsets of $\mathbb{N}_n$, where n represents the number of obligors in the underlying credit portfolio. We also let τ_i and $H_t^i = \mathbb{1}\tau_i \leq t$ denote the default time of name $i = 1, 2, \ldots, n$ and the corresponding indicator process.

2. Model of Default Times

We recall a common shocks portfolio credit risk model of Bielecki et al. (2014a,b,c). In order to describe the defaults we define a certain number m (typically small: a few units) of groups $I_j \subseteq \mathcal{N}_n$, of obligors who are likely to default simultaneously, for $j \in \mathbb{N}_m$. More precisely, the idea is that at every time t, there will be a positive probability that the survivors of the group of obligors I_j (obligors of group I_j still alive at time t) default simultaneously. Let $\mathcal{I} = \{I_1, \ldots, I_m\}$, $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$. Given non-negative constants a, c and non negative deterministic functions $b_Y(t)$ for $Y \in \mathcal{Y}$, let a *"shock intensity process"* X^Y be defined in the form of an extended CIR process as: X_0^Y a given constant, and for $t \in [0, T]$

$$dX_t^Y = a\left(b_Y(t) - X_t^Y\right)dt + c\sqrt{X_t^Y}dW_t^Y, \tag{1}$$

where the Brownian motions W^Y are independent.

Remark 2.1. We refer the reader to Bielecki et al. (2012) for a preliminary version of this model dedicated to valuation and hedging of counterparty risk on a CDS. The use of extended CIR processes as drivers of default intensities is motivated by the following arguments.

- The numerical results of Bielecki et al. (2012) illustrate that such extended CIR specifications of the intensities, with time dependent and piecewise constant functions $b_Y(\cdot)$,[1] in addition to being compatible with the underlying Markov copula structure of a portfolio credit risk model, are appropriate for dealing with counterparty credit risk. In particular, as shown in Sec. 8.4 of Bielecki et al. (2012), versions of the model are capable of generating a large range of implied volatilities for CDS spread options, broader and better behaved than with shifted CIR intensities (for results regarding the latter model we refer to Brigo and Alfonsi (2005), Brigo and El-Bachir (2010), or Brigo et al. (2011)).
- Compared to shifted CIR, the extended CIR (with piecewise constant parameter) specification allows for endogenous calibration of the term-structure of default probabilities whereas, with shifted CIR, one has to rely on arbitrary reconstruction methods. For instance, Brigo and Alfonsi (2005) uses a piecewise-linear specification of hazard rates to strip default probabilities from CDS spreads.
- The extended CIR model is very convenient when it turns to calibrate dependence parameters on CDO tranche spreads since the optimization constraints are linear (see Sec. 4 for more details).

Of course, extended CIR processes with piecewise constant coefficients can be seen as standard CIR processes on each time interval where the coefficients are constant. All the literature regarding simulation of standard CIR processes (in particular, how to cope with the numerical instabilities that may arise if the parameters do not satisfy a suitable Feller condition, e.g., by exact simulation based on chi-square distributions; (Glasserman, 2004, Fig. 3.5 p. 24)) can therefore be applied "piecewise" to such extended CIR processes.

For $\mathbf{k} = (k_1, \ldots, k_n) \in \{0, 1\}^n$, we introduce $\mathrm{supp}(\mathbf{k}) = \{i \in \mathbb{N}_n;\ k_i = 1\}$ and $\mathrm{supp}^c(\mathbf{k}) = \{i \in \mathbb{N}_n;\ k_i = 0\}$. Hence, $\mathrm{supp}(\mathbf{k})$ denotes the obligors who have

[1]Note that in this regard our CIR model is a special version of the *segmented square root model* of Schlögl and Schlögl (1997), where all three coefficients of the CIR diffusion are piecewise functions of time.

defaulted in the portfolio-state $\mathbf{k}$ and similarly $\operatorname{supp}^c(\mathbf{k})$ are the survived names in state $\mathbf{k}$. Given $\mathbf{X} = (X^Y)_{Y\in\mathcal{Y}}$, we aim for a model in which the predictable intensity of a jump of $\mathbf{H} = (H^i)_{i\in\mathbb{N}_n}$ from $\mathbf{H}_{t-} = \boldsymbol{\kappa}$ to $\mathbf{H}_t = \imath$, with $\operatorname{supp}(\boldsymbol{\kappa}) \subsetneq \operatorname{supp}(\imath)$ in $\{0, 1\}^n$, would be given by

$$\sum_{\{Y\in\mathcal{Y};\, \boldsymbol{\kappa}^Y=\imath\}} X_t^Y, \tag{2}$$

where $\boldsymbol{\kappa}^Y$ denotes the vector obtained from $\boldsymbol{\kappa} = (k_i)_{i\in\mathbb{N}_n}$ by replacing the components k_i, $i \in Y$, by numbers one. The intensity of a jump of $\mathbf{H}$ from $\boldsymbol{\kappa}$ to $\boldsymbol{\imath}$ at time t is thus equal to the sum of the intensities of the groups $Y \in \mathcal{Y}$ such that, if the default of the survivors in group Y occurred at time t, the state of $\mathbf{H}$ would move from $\boldsymbol{\kappa}$ to $\boldsymbol{\imath}$.

This is achieved by constructing $\mathbf{H}$ through an $\mathbf{X}$-related change of probability measure, starting from a continuous-time Markov chain with intensity one (see Bielecki et al., 2014b). As a result, the pair-process $(\mathbf{X}, \mathbf{H})$ is a Markov process with respect to the filtration $\mathcal{F}$ generated by the Brownian Motion $\mathbf{W}$ and the random measure counting the jumps of $\mathbf{H}$, with infinitesimal generator $\mathcal{A}$ of $(\mathbf{X}, \mathbf{H})$ acting on every function $u = u(t, \mathbf{x}, \boldsymbol{\kappa})$ with $t \in \mathbb{R}_+$, $\mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}$ and $\kappa = (k_i)_{i\in\mathbb{N}_n}$ as

$$\begin{aligned}\mathcal{A}_t u(t, \mathbf{x}, \boldsymbol{\kappa}) = \sum_{Y\in\mathcal{Y}} \Big(& a\,(b_Y(t) - x_Y)\,\partial_{x_Y} u(t, \mathbf{x}, \boldsymbol{\kappa}) + \frac{1}{2}c^2\; x_Y \partial^2_{x_Y^2} u(t, \mathbf{x}, \boldsymbol{\kappa}) \\ & + x_Y \delta_Y u(t, \mathbf{x}, \boldsymbol{\kappa})\Big),\end{aligned} \tag{3}$$

where we denote

$$\delta_Y u(t, \mathbf{x}, \boldsymbol{\kappa}) = u(t, \mathbf{x}, \boldsymbol{\kappa}^Y) - u(t, \mathbf{x}, \boldsymbol{\kappa}).$$

2.1. *Markov Copula Properties*

Note that the SDEs for processes the X^Y have the same coefficients except for $b_Y(t)$, to the effect that for $i \in \mathbb{N}_n$,

$$X^i := \sum_{\mathcal{Y}\ni Y\ni i} X^Y = X^{\{i\}} + \sum_{\mathcal{I}\ni I\ni i} X^I \tag{4}$$

is again an extended CIR process, with parameters a, c and

$$b_i(t) := \sum_{\mathcal{Y}\ni Y\ni i} b_Y(t) = b_{\{i\}}(t) + \sum_{\mathcal{I}\ni I\ni i} b_I(t), \tag{5}$$

driven by an $\mathcal{F}$-Brownian motion W^i such that

$$\sqrt{X_t^i}\,dW_t^i = \sum_{Y\ni i} \sqrt{X_t^Y}\,dW_t^Y, \quad dW_t^i = \sum_{Y\ni i} \frac{\sqrt{X_t^Y}}{\sqrt{\sum_{Y\ni i} X_t^Y}}\,dW_t^Y. \tag{6}$$

The fact that W^i defined by (6) is an $\mathcal{F}$-Brownian motion results from Paul Lévy's characterization of a Brownian motion as a continuous local martingale with bracket process equal to time t. One can then check, as is done in Bielecki et al. (2014b), that the so-called *Markov copula property* holds (see Bielecki et al., 2008), in the sense that for every $i \in \mathbb{N}_n$, (X^i, H^i) is an $\mathcal{F}$ – Markov process admitting the following generator, acting on functions $v_i = v_i(t, x_i, k_i)$ with $(x_i, k_i) \in \mathbb{R} \times \{0, 1\}$:

$$\mathcal{A}^i_t v_i(t, x_i, k_i) = a(b_i(t) - x_i)\partial_{x_i} v_i(t, x_i, k_i) + \frac{1}{2}c^2\ x_i \partial^2_{x_i^2} v_i(t, x_i, k_i) \qquad (7)$$
$$+x_i\big(v_i(t, x_i, 1) - v_i(t, x_i, k_i)\big).$$

Also, the $\mathcal{F}$ – intensity process of H^i is given by $(1 - H^i_t)X^i_t$. In other words, the process M^i defined by,

$$M^i_t = H^i_t - \int_0^t (1 - H^i_s)X^i_s ds, \qquad (8)$$

is an $\mathcal{F}$-martingale. Finally, the conditional survival probability function of name $i \in \mathbb{N}_n$ is given by, for every $t_i > t$,

$$\mathbb{P}(\tau_i > t_i \,|\, \mathcal{F}_t) = \mathbb{E}\left\{\exp\big(-\int_t^{t_i} X^i_s ds\big) \,|\, X^i_t\right\}. \qquad (9)$$

3. Pricing

Regarding the dynamics of CIR intensity processes, we assume in the sequel that the mean-reversion functions $b_Y(t)$ are piecewise-constant with respect to a time tenor $(T_k)_{k=1,\dots,M}$. So, for every $k = 1 \dots M$,

$$b_Y(t) = b_Y^{(k)},\ t \in [T_{k-1}, T_k), \qquad (10)$$

where $b_Y^{(k)}$ is a non negative constant and $T_0 = 0$. The time tenor (T_k) is a set of pillars corresponding to standard CDS maturities. In this framework, we are able to provide explicit expressions for survival probabilities of triggering events, which are the main building blocks in the calculation of CDS and CDO tranche spreads.

Remark 3.1. For comparison purposes, the (simpler) case of deterministic time-dependent intensities will also be considered in the numerical experiments of Sec. 5. In that case, the default intensities will be given as $X^Y_t := \lambda_Y(t)$ where λ_Y is a piecewise-constant function of time with respect to the same time tenor $(T_k)_{k=1,\dots,M}$ as for the mean-reversion function b_Y of the CIR intensity case. Then, for every $k = 1 \dots M$, there exists a non-negative constant $\lambda_Y^{(k)}$ such that

$$\lambda_Y(t) = \lambda_Y^{(k)},\ t \in [T_{k-1}, T_k). \qquad (11)$$

Survival probabilities (9) can be obtained explicitly in the case of piecewise-constant intensities as defined by (11). We will show now that similar analytical formulas can be obtained when the underlying intensities are driven by CIR processes with piecewise-constant mean-reversion parameters (see Proposition 3.1 and Remark 3.2).

3.1. *Survival Probabilities of Trigger-Events*

In this section, we provide an analytical expression for survival probabilities in an extended CIR intensity model with piecewise constant mean-reversion parameter. This result derives from Prop. 8.2 in Bielecki et al. (2012). However, contrary to the letter result which involves recursively defined parameters, Proposition 3.1 provides closed form expressions for survival probabilities and conditional density functions. This allows us to compute CDS and CDO tranche spreads almost as fast as in a (deterministic) piecewise constant intensity model.

Let X be an extended CIR process with dynamics

$$dX_t = a(b(t) - X_t)dt + c\sqrt{X_t}dW_t \tag{12}$$

where a and c are positive constants and $b(\cdot)$ is a non negative deterministic function. When the mean-reversion function b is constant, the following lemma is a standard result in the affine processes literature (see e.g. Duffie and Gârleanu (2001)). Note that (14), which is obtained from (13) with $y = 0$, by differentiating with respect to t, gives the conditional density of default time in a classical CIR intensity model.

Lemma 3.1. *Consider the process X in* (12). *If $b(\cdot)$ is constant on $[t_0, t]$, then for every $y \geq 0$,*

$$\mathbb{E}\left(e^{-\int_{t_0}^t X_s ds - yX_t} \big| X_{t_0}\right) = e^{-\phi(t-t_0;y)X_{t_0} - \xi(t-t_0;y)b}, \tag{13}$$

$$\mathbb{E}\left(X_t e^{-\int_{t_0}^t X_s ds} \big| X_{t_0}\right) = \left(\dot{\phi}(t-t_0;0)X_{t_0} + \dot{\xi}(t-t_0;0)b\right)e^{-\phi(t-t_0;0)X_{t_0} - \xi(t-t_0;0)b}, \tag{14}$$

where ϕ and ξ satisfy the following Riccati system of ODE:

$$\begin{cases} \dot{\phi}(s;y) = -a\phi(s;y) - \frac{c^2}{2}(\phi(s;y))^2 + 1 \ ; \ \phi(0;y) = y \\ \dot{\xi}(s;y) = a\phi(s;y) \qquad\qquad\qquad\qquad\quad ; \ \xi(0;y) = 0. \end{cases} \tag{15}$$

Note that the latter ODE can be solved explicitly, i.e.,

$$\phi(s;y) = \frac{1 + D(y)e^{-A(y)s}}{B + C(y)e^{-A(y)s}}, \tag{16}$$

$$\xi(s;y) = \frac{a}{B}\left\{\frac{C(y) - BD(y)}{A(y)C(y)} \log\frac{B + C(y)e^{-A(y)s}}{B + C(y)} + s\right\}. \tag{17}$$

where A, B, C and D are given by

$$B = \frac{1}{2}\left(a + \sqrt{a^2 + 2c^2}\right), \quad C(y) = (1 - By)\frac{a + c^2 y - \sqrt{a^2 + 2c^2}}{2ay + c^2 y - 2},$$

$$D(y) = (B + C(y))y - 1, \quad A(y) = \frac{-C(y)(2B - a) + D(y)(c^2 + aB)}{BD(y) - C(y)}.$$

In the following proposition, we extend Lemma 3.1 to the case of generalized CIR processes where the mean-reversion function $b(\cdot)$ is assumed to be piecewise constant. We denote $T_0 = 0$. The functions ϕ and ξ are those of Lemma 3.1. As a sanity-check observe that when $b_i = \cdots = b_j$, Proposition 3.1 gives the same results than the ones established in Lemma 3.1 for constant mean-reversion CIR processes. Also note that the partial derivative $\partial_s I$ of I with respect to s in (19) depends on the functions $\dot{\phi}$ and $\dot{\xi}$, which can be computed explicitly thanks to (15), (16), and (17).

Proposition 3.1. *Assume that $b(\cdot)$ is a piecewise constant function: $b(t) = b_k$ on $t \in [T_{k-1}, T_k]$ for $k = 1, ..., m$. For $t < s$, let $i \leq j$ such that $t \in [T_{i-1}, T_i)$ and $s \in (T_j, T_{j+1}]$. Then:*

(i) for any $y \geq 0$,

$$\mathbb{E}\Big(\exp(-\int_t^s X_u du - yX_s)|X_t\Big) = \exp\left(-I(t, s, X_t, y)\right) \tag{18}$$

where

$$\begin{aligned} I(t, s, x, y) := {} & x\phi(s-t; y) + b_i\left[\xi(s-t; y) - \xi(s-T_i; y)\right] \\ & + \sum_{k=i+1}^{j} b_k\left(\xi(s-T_{k-1}; y) - \xi(s-T_k; y)\right) \\ & + b_{j+1}\xi(s-T_j; y); \text{ and} \end{aligned}$$

(ii) one has

$$\mathbb{E}\Big(X_s \exp\big(-\int_t^s X_u du\big)|X_t\Big) = \partial_s I(t, s, X_t, 0)\mathbb{E}\Big(\exp\big(-\int_t^s X_u du\big)|X_t\Big). \tag{19}$$

Proof. Expression (19) can be obtained by differentiating (18) with respect to s and by letting $y = 0$. Let us prove (18). Note that the piecewise constant function $b(\cdot)$ is characterized by the time tenor $\mathbf{T} = (T_k)_{k=1,...,m}$ and the corresponding set of parameters $\mathbf{b} = (b_k)_{k=1,...,m}$. From Prop. 8.2 in Bielecki et al. (2012), we know that when $t < s$ such that $t \in [T_{i-1}, T_i)$ and $s \in (T_j, T_{j+1}]$ with $i \leq j$, the following relation holds:

$$\begin{aligned} \mathscr{E}(\mathbf{T}, \mathbf{b}, x, y) := {} & \mathbb{E}\Big(\exp(-\int_t^s X_u du - yX_s)|X_t = x\Big) \\ = {} & \exp\Big\{ -x\phi(T_i - t; y_i(s)) - b_i\xi(T_i - t; y_i(s)) \\ & - \sum_{k=i+1}^{j} b_k\xi(T_k - T_{k-1}; y_k(s)) - b_{j+1}\xi(s - T_j; y)\Big\} \end{aligned} \tag{20}$$

where the $y_k(s)$, $k = i, \ldots, j$, are defined by the following backward recursion

$$\begin{aligned} & y_j(s) := \phi(s - T_j; y), \\ & y_k(s) := \phi(T_{k+1} - T_k; y_{k+1}(s)), \qquad k < j. \end{aligned} \tag{21}$$

Note that in the particular case where $b_j = b_{j+1}$, expression (20) must be equal to the one obtained when $\boldsymbol{T}_j$ is removed from the set $\boldsymbol{T}$ and b_j is removed from the set $\boldsymbol{b}$. More precisely, if the adjusted set of parameters are defined by $\tilde{\boldsymbol{T}} := \boldsymbol{T} \setminus \{T_j\}$ and $\tilde{\boldsymbol{b}} := \boldsymbol{b} \setminus \{b_j\}$, then the following relation holds:

$$\mathscr{E}(\boldsymbol{T}, \boldsymbol{b}, x, y)|_{b_j=b_{j+1}} = \mathscr{E}(\tilde{\mathbf{T}}, \tilde{\mathbf{b}}, x, y). \tag{22}$$

By letting $x, b_i, \ldots, b_{j-1}$ be equal to zero in the previous relation, we obtain

$$\xi\left(T_j - T_{j-1}; y_j(s)\right) = \xi\left(s - T_{j-1}; y\right) - \xi\left(s - T_j; y\right). \tag{23}$$

We can then replace $\xi\left(T_j - T_{j-1}; y_j(s)\right)$ in (20) by the right-hand side of (23) and apply the previous argument in the particular case where $b_{j-1} = b_j = b_{j+1}$. We then obtain

$$\xi\left(T_{j-1} - T_{j-2}; y_{j-1}(s)\right) = \xi\left(s - T_{j-2}; y\right) - \xi\left(s - T_{j-1}; y\right). \tag{24}$$

This reasoning can be done backward from $k = j - 2$ to $k = i$ which shows that

$$\xi\left(T_k - T_{k-1}; y_k(s)\right) = \xi\left(s - T_{k-1}; y\right) - \xi\left(s - T_k; y\right) \tag{25}$$

and

$$\xi\left(T_i - t; y_i(s)\right) = \xi\left(s - t; y\right) - \xi\left(s - T_i; y\right). \tag{26}$$

Eventually, in the particular case where $b_i = \cdots = b_{j+1}$, expression (20) must be the same as the one given by application of Lemma 3.1, i.e.,

$$\mathscr{E}(\boldsymbol{T}, \boldsymbol{b}, x, y) = \exp\left(-x\phi(s-t; y) - b_i\xi(s-t; y)\right).$$

So, by letting b_i equal to 0, we obtain $\phi(T_i - t; y_i(s)) = \phi(s - t; y)$ which concludes the proof. □

Remark 3.2. Note that the expression of survival probabilities as computed in a deterministic piecewise-constant intensity set-up can be embedded in formulas (18). Indeed, if we assume that $X_t := \lambda(t)$ where $\lambda(\cdot)$ is a piecewise constant function, i.e., $\lambda(t) = \lambda_k$ on $t \in [T_{k-1}, T_k)$, then the function I in the right-hand side of Eq. (18) becomes

$$I_{s,y}(t, x) := (T_i - t)\,\lambda_i + \sum_{k=i+1}^{j-1} (T_k - T_{k-1})\,\lambda_k + (s - T_{j-1} + y)\lambda_j, \tag{27}$$

and Proposition 3.1 can be readily adapted to the deterministic intensity case by letting $\phi = 0$ and $\xi(s, y) = s + y$ in (18).

3.2. *CDS Pricing*

We assume in the sequel that recovery rates are independent of default times. Under this assumption, the CDS spread of a particular name can be expressed as deterministic functions of its survival probabilities and of its expected recovery. Let $t_1 < \cdots < t_p = T$ be the remaining premium payment dates where T stands for the maturity date. We assume for simplicity that the risk-free interest rate is constant and equal to r and we denote $\beta(t) = e^{-rt}$ the corresponding discount factor. In our numerical experiment, the current fair CDS spread of name i is approximated by the following expression:

$$S_i(T) = (1 - R_i^*)\frac{\sum_{j=1}^{p} \beta(t_j)\left(\mathbb{P}\left(\tau_i > t_{j-1}\right) - \mathbb{P}\left(\tau_i > t_j\right)\right)}{\sum_{j=1}^{p} \beta(t_j)(t_j - t_{j-1})\mathbb{P}\left(\tau_i > t_j\right)}, \tag{28}$$

where R_i^* denotes the expected recovery rate of name i. To derive the previous expression, we implicitly assume that, if a default occur at time $\tau_i < T$, the protection payment occurs at the premium payment date that immediately follows τ_i.

Recall that (cf. (9))

$$\mathbb{P}\left(\tau_i > t_j\right) = \mathbb{E}\left\{\exp\left(-\int_0^{t_j} X_s^i ds\right)\right\}, \tag{29}$$

where X^i is an extended CIR process with parameters a, c and piecewise-constant mean-reversion function $b_i(\cdot)$ in (5) (assuming piecewise-constant functions $b_Y(\cdot)$). Hence, provided the expected recovery is known, the CDS spread of name i can be efficiently calculated using part (i) of Proposition 3.1 with $y = 0$.

3.3. *CDO Tranche Pricing with Random Recoveries*

In this section, we outline how to modify the model to include stochastic recoveries. Let $\mathbf{L} = (L^i)_{1\le i\le n}$ represent the $[0,1]^n$-valued vector process of the loss given defaults in the pool of names. The process $\mathbf{L}$ is a multivariate process where $\mathbf{L}_0 \in \mathbf{0}$, and where each component L_t^i represents the fractional loss that name i may have suffered due to default until time t. Assuming unit notional for each name, the cumulative loss process for the entire portfolio is defined as $L_t := \sum_i (1 - R_i)H_t^i$ where the variables R_i are random and independent fractional recoveries with values in $[0, 1]$. The default times are defined as before, but at every time of jump of $\mathbf{H}$, an independent recovery draw is made for every newly defaulted name i, determining the recovery R_i of name i. In particular, the recovery rates resulting from a joint default are thus drawn independently for the affected names.

Note that independent recoveries do not break the dynamic properties developed in Bielecki et al. (2014b). However by introducing stochastic recoveries we can no longer use the exact convolution recursion procedures of Bielecki et al. (2014c) for pricing CDO tranches. Instead, we will here use an approximate procedure based on the exponential approximations of the so called hockey stick function, as presented in Iscoe et al. (2010, 2013) and originally developed by Beylkin and Monzon (2005). Here, we explain in detail how to use this method for computing the price of a CDO tranche in our Markov model when the individual losses are random.

The mathematical ideas underlying the method of exponential approximations were originally developed by Beylkin and Monzon (2005), and was later adopted by Iscoe et al. (2010, 2013) to price CDO tranches in a Gaussian copula model. While Iscoe et al. (2010, 2013) uses constant recoveries, we will in this paper adopt their techniques to random recoveries. Below we will outline the techniques given in Iscoe et al. (2010; 2013) and our presentation also introduces notation needed

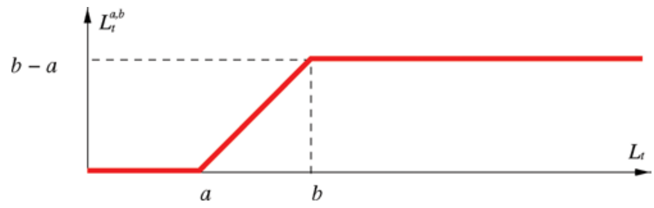

Figure 1. The tranche loss for $[a, b]$ as function of the total loss L_t.

later on. First, the so called tranche loss function $L_t^{a,b}$ for the tranche $[a, b]$ as a function of the portfolio credit loss L_t is given by

$$L_t^{a,b} = (L_t - a)^+ - (L_t - b)^+ \tag{30}$$

where $x^+ = \max(x, 0)$ (see Fig. 1).

As in Beylkin and Monzon (2005), we introduce the so-called *hockey stick function* $h(x)$ given by

$$h(x) = \begin{cases} 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x \end{cases} \tag{31}$$

(see Fig. 2).

Let $c > 0$ be scalar. By using (31) one can show that

$$\min(x, c) = c - ch\left(\frac{x}{c}\right) \tag{32}$$

and for any two scalars a and b it holds that

$$(x - a)^+ - (x - b)^+ = \min(x, b) - \min(x, a) \tag{33}$$

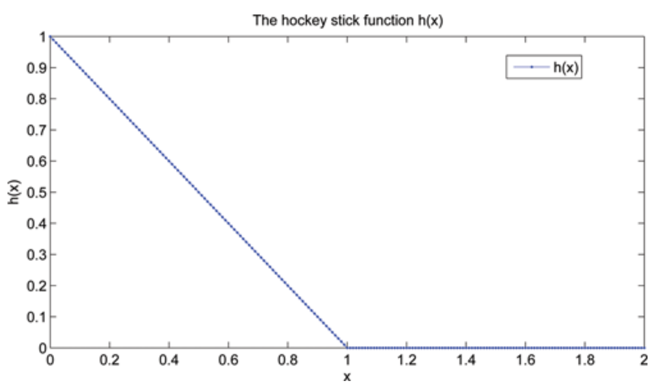

Figure 2. The hockey stick function $h(x)$ for $x \in [0, 2]$.

so (32) and (33) then yields

$$(x-a)^+ - (x-b)^+ = b\left(1 - h\left(\frac{x}{b}\right)\right) - a\left(1 - h\left(\frac{x}{a}\right)\right). \tag{34}$$

Hence, (30) and (34) implies that

$$L_t^{a,b} = b\left(1 - h\left(\frac{L_t}{b}\right)\right) - a\left(1 - h\left(\frac{L_t}{a}\right)\right). \tag{35}$$

This observation was done by Iscoe et al. (2010) which combined (35) with the results of Beylkin and Monzon (2005). More specifically, Beylkin and Monzon (2005) shows that for any fixed $\epsilon > 0$, the function $h(x)$ can be approximated by a function $h_{\exp}^{(q)}(x)$ on $[0, d]$ with $d = d(\epsilon)$ so that $|h(x) - h_{\exp}^{(q)}(x)| \leq \epsilon$ for all $x \in [0, d]$ where $q = q(\epsilon)$ is a positive integer and $h_{\exp}^{(q)}(x)$ is given by

$$h_{\exp}^{(q)}(x) = \sum_{\ell=1}^{q} \omega_\ell \exp\left(\gamma_\ell \frac{x}{d}\right). \tag{36}$$

where $(\omega_\ell)_{\ell=1}^q$ and $(\gamma_\ell)_{\ell=1}^q$ are complex numbers obtained as roots of polynomials whose coefficients can be computed numerically in a straightforward way. Figure 3 visualizes the approximation $h_{\exp}^{(q)}(x)$ of $h(x)$ on $x \in [0, 10]$ for $q = 2, 5, 10$ and

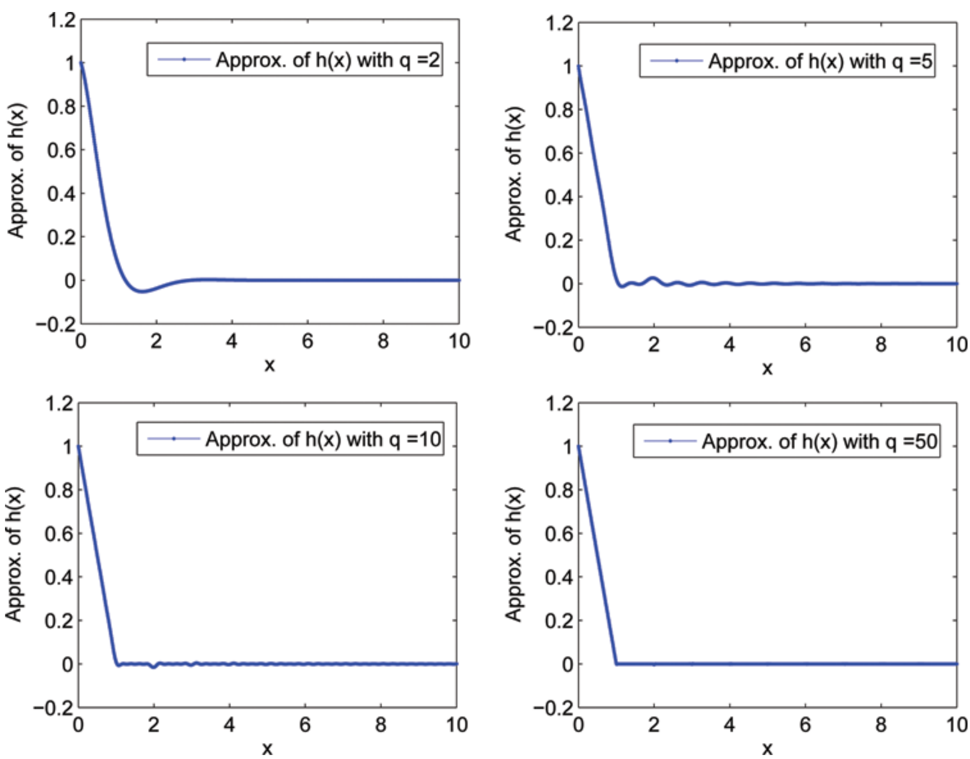

Figure 3. The function $h_{\exp}^{(q)}(x)$ as approximation of $h(x)$ for $x \in [0, 10]$ with $q = 2, 5, 10$ and $q = 50$.

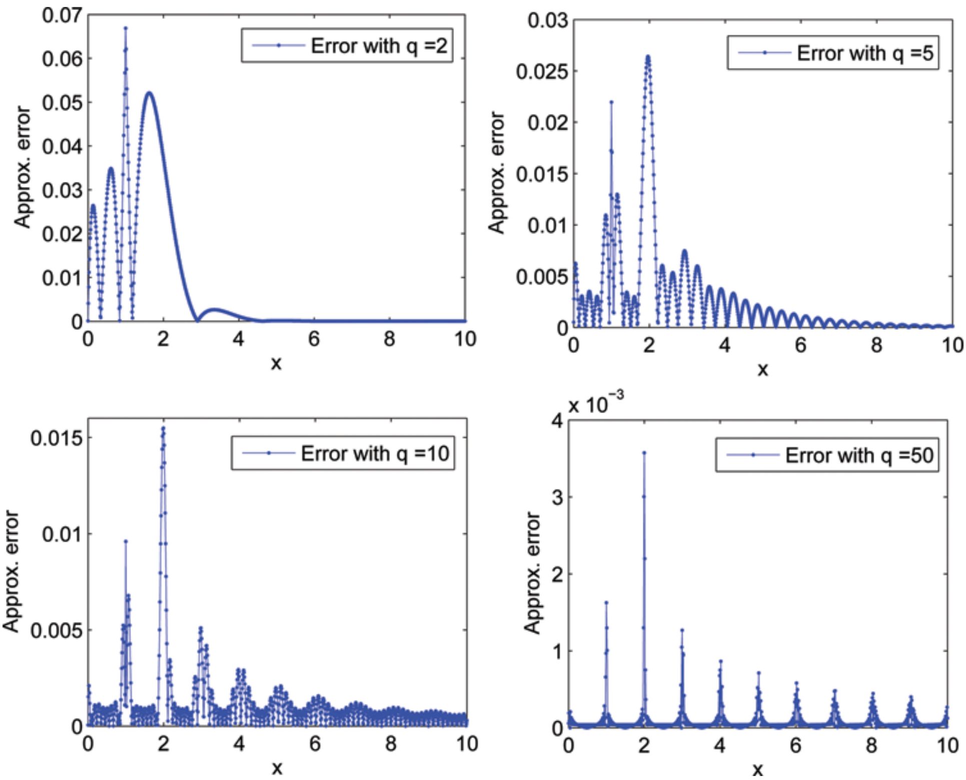

Figure 4. The approximation error $|h(x) - h^{(q)}_{\exp}(x)|$ for $x \in [0, 10]$ with $q = 2, 5, 10$ and $q = 50$.

$q = 50$. As can be seen in Fig. 3, the approximation is fairly good already for small values values of q. In Iscoe et al. (2010) the authors choose the algorithm for computing $(\omega_\ell)_{\ell=1}^q$ and $(\gamma_\ell)_{\ell=1}^q$ so that $d(\epsilon) = 2$, and then they show that the approximation accuracy ϵ satisfies $\frac{1}{4(q+1)} \leq \epsilon$ where q are the number terms in (36). Thus, q can be chosen first, implying an accuracy ϵ so that $\frac{1}{4(q+1)} \leq \epsilon$. In practice, the error $|h(x) - h^{(q)}_{\exp}(x)|$ will for almost all $x \in [0, \infty)$ be much smaller than the lower bound $\frac{1}{4(q+1)}$ for ϵ, as can bee seen in Figure 4. More specifically, in Iscoe et al. (2010, 2013) the authors show that $\mathrm{Re}(\gamma_\ell) < 0$ for all ℓ (see also in Figure 5) which implies that $h^{(q)}_{\exp}(x) \to 0$ as $x \to \infty$ and, as pointed out by Iscoe et al. (2010), since $h(x) = 0$ for $x \geq 1$ this guarantees that $h^{(q)}_{\exp}(x) \to h(x)$ when $x \to \infty$. In the rest of this paper we will, just as in Iscoe et al. (2010, 2013) and Bielecki et al. (2014a), use $d = 2$ in the approximation $h^{(q)}_{\exp}(x)$ given by (36).

Since $(\omega_\ell)_{\ell=1}^q$ are roots to a certain polynomial, then if $\zeta \in (\omega_\ell)_{\ell=1}^q$ it will also hold that $\overline{\zeta} \in (\omega_\ell)_{\ell=1}^q$. The same also holds for the complex numbers $(\gamma_\ell)_{\ell=1}^q$. Thus, it will hold that $\mathrm{Im}\left(\sum_{\ell=1}^q \gamma_\ell \exp(\gamma_\ell x/d)\right) = 0$ and Fig. 5 displays the coefficients $(\omega_\ell)_{\ell=1}^q$ and $(\gamma_\ell)_{\ell=1}^q$ in the case $q = 50$.

It is well known that in order to price synthetic CDO tranches, one needs to compute the quantity $\mathbb{E}L_t^{a,b}$ for $t > 0$, see e.g. in Herbertsson (2008). So by replacing $h(x)$ with $h^{(q)}_{\exp}(x)$ in (35) with $d = 2$ and using (36) then implies that we can approximate $L_t^{a,b}$ as follows

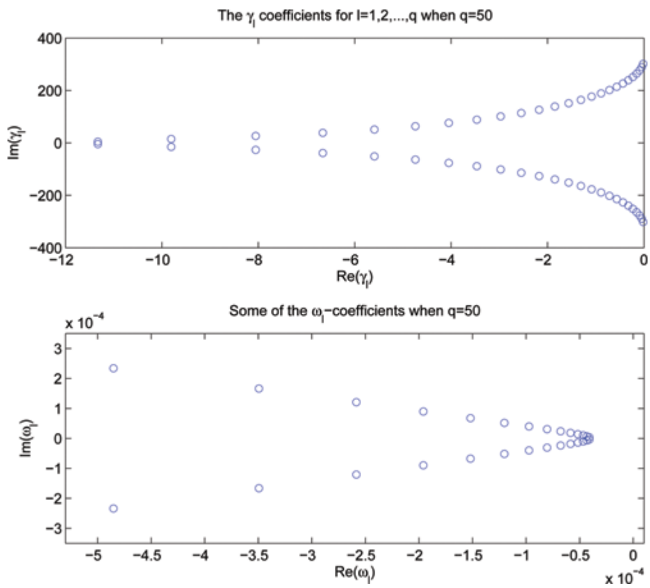

Figure 5. The coefficients $(\gamma_\ell)_{\ell=1}^q$ (top) and some of the coefficients $(\omega_\ell)_{\ell=1}^q$ (bottom) when $q = 50$.

$$L_t^{a,b} \approx b - a - b \sum_{\ell=1}^{q} \omega_\ell \exp\left(\gamma_\ell \frac{L_t}{2b}\right) + a \sum_{\ell=1}^{q} \omega_\ell \exp\left(\gamma_\ell \frac{L_t}{2a}\right) \tag{37}$$

and, consequently,

$$\mathbb{E}L_t^{a,b} \approx b - a - b \sum_{\ell=1}^{q} \omega_\ell \mathbb{E}\exp\left(\gamma_\ell \frac{L_t}{2b}\right) + a \sum_{\ell=1}^{q} \omega_\ell \mathbb{E}\exp\left(\gamma_\ell \frac{L_t}{2a}\right). \tag{38}$$

Thus, in view of (38), the pricing of a CDO tranche of maturity T, boils down to computation of expectations of the form

$$\mathbb{E}e^{\gamma_\ell \frac{L_t}{2c}} \tag{39}$$

for $\ell = 1, 2, \ldots, q$ and different attachment points c and time horizons $0 \leq t \leq T$.

Remark 3.3. One can extend the present developments to conditional expectations given $\mathcal{F}_s$ for any $0 < s < t$. The case $s = 0$ is used in the calibration (our focus in this paper), while the case $s > 0$ is needed for pricing the credit valuation adjustment (CVA) on a CDO tranche in a counterparty risky environment, a topical issue since the 2007-09 credit crisis (see Crépey and Rahal, in press).

Since the algorithm for computing $\mathbb{E}e^{\gamma_\ell \frac{L_t}{2c}}$ is the same for each $\ell = 1, 2, \ldots, q$ and any attachment point c, we will below for notational convenience simply write $\mathbb{E}e^{\gamma L_t}$ instead of $\mathbb{E}e^{\gamma_\ell \frac{L_t}{2c}}$

We now use the common shock model representation developed in Sec. 3 of Bielecki et al. (2014c), with the same notation that was introduced there except that θ there is t here, and t there is simply 0 here, as we focus here on expectations and not conditional expectations; moreover we now use a "$\hat{.}$" notation for the common shocks model representation at the starting time 0 below, instead of a "(t)" which is used for common shocks model representation with a varying forward starting time t in Bielecki et al. (2014a).

We thus introduce a common shocks copula model of default times $\hat{\tau}_Y$ defined by, for every $Y \in \mathcal{Y}$,

$$\hat{\tau}_Y = \inf\{t > 0; \int_0^t X_s^Y ds > E_Y\},$$

where the random variables E_Y are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\hat{\tau}_i = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \hat{\tau}_Y, \tag{40}$$

which defines the default time of obligor i in the common shocks copula model. We also introduce the indicator processes $\hat{H}_t^Y = \mathbb{1}\hat{\tau}_Y \le t$ and $\hat{H}_t^i = \mathbb{1}\hat{\tau}_i \le t$, for every triggering-event Y and obligor i. One then has much like in Proposition 2.10(ii) of Bielecki et al. (2014a) that

$$\mathbb{E}e^{\gamma L_t} = \mathbb{E}e^{\gamma \hat{L}_t} \tag{41}$$

where $\hat{L}_t := \sum_i (1 - R_i)\hat{H}_t^i$.

We henceforth assume a nested structure of the sets I_j given by

$$I_1 \subset \cdots \subset I_m. \tag{42}$$

This structure implies that if all obligors in group I_k have defaulted, then all obligors in group $I_1, \ldots, I_{k-1}$ have also defaulted. As detailed in Bielecki et al. (2014a), the nested structure (42) yields a particularly tractable expression for the portfolio loss distribution. This nested structure also makes sense financially with regards to the hierarchical structure of risks which is reflected in standard CDO tranches. Denoting conventionally $I_0 = \emptyset$ and $\hat{H}_t^{I_0}(0) = 1$, then the event-sets

$$\hat{\Omega}_t^j := \{\hat{H}_t^{I_j} = 1, \hat{H}_t^{I_{j+1}} = 0, \ldots, \hat{H}_t^{I_m} = 0\}^0 \le j \le m$$

form a partition of Ω with

$$\mathbb{P}(\hat{\Omega}_t^j) = \left(1 - \mathbb{E}e^{-\int_0^t X_s^{I_j} ds}\right) \prod_{j+1 \le l \le m} \mathbb{E}e^{-\int_0^t X_s^{I_l} ds},$$

where the expectations are explicitly given by Proposition 3.1(i). One then has in (41) that

$$\mathbb{E}e^{\gamma \hat{L}_t} = \sum_{0 \le j \le m} \mathbb{E}\left(e^{\gamma \hat{L}_t} \mid \hat{\Omega}_t^j\right) \mathbb{P}(\hat{\Omega}_t^j) \tag{43}$$

in which by conditional independence of the $\hat{H}^i_t$ given every $\hat{\Omega}^j_t$

$$\mathbb{E}\big(e^{\gamma \hat{L}_t} \mid \hat{\Omega}^j_t\big) = \mathbb{E}\big(e^{\gamma \sum_i (1-R_i)\hat{H}^i_t} \mid \hat{\Omega}^j_t\big) = \textstyle\prod_{i\in Z} \mathbb{E}\big(e^{\gamma(1-R_i)\hat{H}^i_t} \mid \hat{\Omega}^j_t\big).$$

Now observe that by independence of R_i

$$\mathbb{E}\big(e^{\gamma(1-R_i)\hat{H}^i_t} \mid \hat{\Omega}^j_t\big) = \begin{cases} \mathbb{E}e^{\gamma(1-R_i)}, & i \in I_j \\ \mathbb{E}e^{\gamma(1-R_i)\hat{H}^{\{i\}}_t}, & \text{else} \end{cases} \tag{44}$$

with

$$\mathbb{E}e^{\gamma(1-R_i)\hat{H}^{\{i\}}_t} = 1 - \hat{p}^{i,j}_t \left(1 - \mathbb{E}e^{\gamma(1-R_i)}\right), \tag{45}$$

where

$$\hat{p}^{i,j}_t = \begin{cases} 1, & i \in I_j, \\ 1 - \mathbb{E}e^{-\int_0^t X^{\{i\}}_s ds} & \text{else} \end{cases}$$

in which the expectation is explicitly given by Proposition 3.1(i) for CIR intensities or Remark 3.2 for deterministic intensities. Hence, the above formulas together with (41) will determine the quantity (39) which in turn is needed to compute the expected tranche loss given by (38). Furthermore, from the above equations we see that what is left to compute is the quantity $\mathbb{E}e^{\gamma(1-R_i)}$ and in Sec. 5 we will give an explicitly example of the recovery rate R_i (and the quantity $\mathbb{E}e^{\gamma(1-R_i)}$) which will be used in in Sec. 5.2 with the above hockey-stick method when calibrating the Markov copula against market data on CDO tranches. As will be seen in Sec. 5.2, using random recoveries will for some data sets render much better calibration results compared with the case of using constant recoveries.

4. Calibration with Stochastic Intensities and Constant Recovery

In this section, we discuss the calibration methodology used for fitting the stochastic intensity Markov copula model against CDO tranches on CDX.NA.IG series. We use here extended CIR intensities with piecewise-constant mean-reversion coefficients (as described previously) and we assume that recovery rates are constant. In Sec. 5, we will investigate the "dual" model specification where intensities are deterministic and recoveries are stochastic.

Recall that, given non negative constants a and c, the intensity process of any group $Y \in \mathcal{Y}$ is defined by

$$dX^Y_t = a\left(b_Y(t) - X^Y_t\right) dt + c\sqrt{X^Y_t}\, dW^Y_t \tag{46}$$

where X^Y_0 is a given constant and $b_Y(t)$ is a piecewise constant function such that, for every $k = 1 \dots M$, $b_Y(t) = b^{(k)}_Y$, $t \in [T_{k-1}, T_k)$ with $T_0 = 0$. In this paper we will use a time tenor consisting of two maturities $T_1 = 3y$ and $T_2 = 5y$. Moreover, in order to reduce the number of parameters at hands, we consider that, for every group $Y \in \mathcal{Y}$, the starting point of the corresponding intensity process is given by its first-pillar mean-reversion parameter, i.e., $X^Y_0 = b^{(1)}_Y$. Note that in that case, given a and c, the intensity dynamics of any group $Y \in \mathcal{Y}$ is completely characterized by

$b_Y^{(k)}$, $k = 1, 2$. In particular, thanks to (4), the survival probability of name i up to T_2 is characterized by $(b_i^{(k)})_{k=1,2}$ where

$$b_i^{(k)} = b_{\{i\}}^{(k)} + \sum_{\mathcal{I} \ni I \ni i} b_I^{(k)}. \tag{47}$$

The calibration is done in two steps. The first step consists in boostrapping $(b_i^{(k)})_{k=1,2}$ on the single-name CDS curve associated with obligor i, for any $i = 1, \ldots, n$. The CDS curve of name i is composed of two market spreads: $S_i^*(T_1)$ corresponding to maturity T_1 and $S_i^*(T_2)$ corresponding to maturity T_2. We first remark from (28) and Proposition 3.1 that the model spread of CDS i with maturity T_1 only depends on $b_i^{(1)}$ whereas the model spread of CDS i with maturity T_2 depends on $b_i^{(1)}$ and $b_i^{(2)}$. As soon as a and c are fixed, we can then find $b_1^{(1)}$ as the solution of the non-linear (univariate) equation $S_i(T_1) = S_i^*(T_1)$, plugged this solution into the expression of $S_i(T_2)$ and then find $b_1^{(2)}$ as the solution of the non-linear (univariate) equation $S_i(T_2) = S_i^*(T_2)$. Figure 5, 6 and 7, respectively, show the 3y- and the 5y-implied mean-reversion coefficients bootstrapped from the 125 CDS curves of the CDX.NA.IG index constituents as of December 17, 2007. We compare three different specifications of the underlying individual intensities : piecewise-constant deterministic intensities (standard bootstrap procedure), CIR intensities with $a = 3$ and $c = 0.05$ and CIR intensities with $a = 3$ and $c = 2$. We can see that the volatility parameter c has little impact on implied coefficients whereas individual intensities may be relatively volatile even for small volatility parameter as illustrated by Fig. 8.

Remark 4.1. We checked that, for $a = 3$ and $c = 0.05$, the Feller's condition holds for all names after calibration of the mean-reversion levels $b^{(k)}$'s on CDS spreads. This eases Monte Carlo path generation considerably compared to a situation where the Feller's condition would be violated.

The second step is to calibrate group parameters $(b_{I_j}^{(k)})_{k=1,2}$, $j = 1, \ldots, m$ so that the model CDO tranche spreads coincide with the corresponding market spreads. The hockey-stick method described in Sec. 3.3 can be used to compute model CDO tranche spreads.

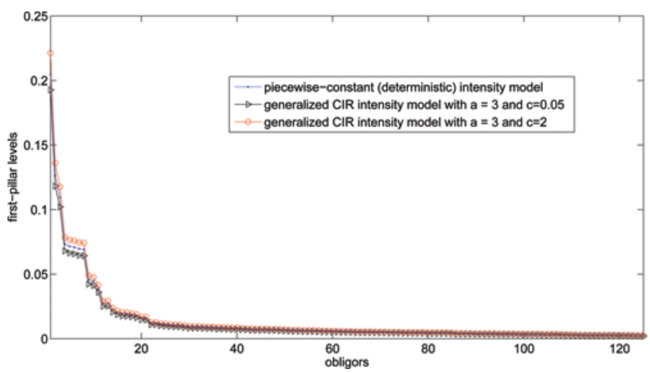

Figure 6. Three-year mean-reversion coefficients $b_i^{(1)}$, $i = 1, \ldots, 125$ bootstrapped from CDX.NA.IG December 17, 2007 single-name CDS curves and sorted in decreasing order.

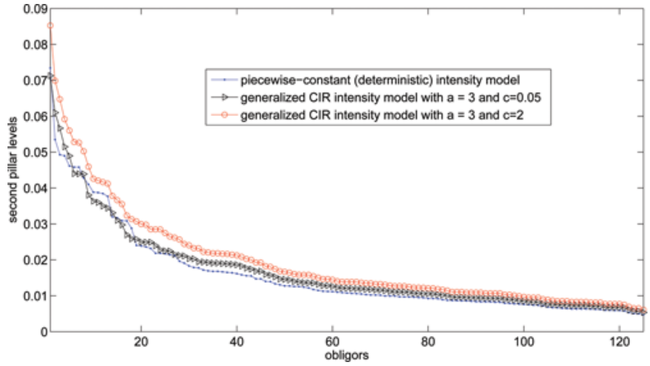

Figure 7. Five-year mean-reversion coefficients $b_i^{(2)}$, $i = 1, \ldots, 125$ bootstrapped from CDX.NA.IG December 17, 2007 single-name CDS curves and sorted in decreasing order.

Moreover, in view of (47), we impose that, for all $k = 1, 2$ and $i = 1, \ldots, n$, the group parameters are such that

$$\sum_{\mathcal{J} \ni I \ni i} b_I^{(k)} \leq b_i^{(k)} \tag{48}$$

for all $i = 1, \ldots, 125$. The previous constraints guarantee that the long-term averages $b_{\{i\}}^{(k)}$ of single-group intensities are all positive. This in turn implies by construction that the starting points of single-group intensities $X_0^{\{i\}}$ are all positive. Given the nested structure of the groups I_j-s specified in (42), the following constrains must hold for all $l = 1, \ldots, m$ and $k = 1, 2$:

$$\sum_{j=l}^{m} b_{I_j}^{(k)} \leq \min_{i \in I_l \setminus I_{l-1}} b_i^{(k)}. \tag{49}$$

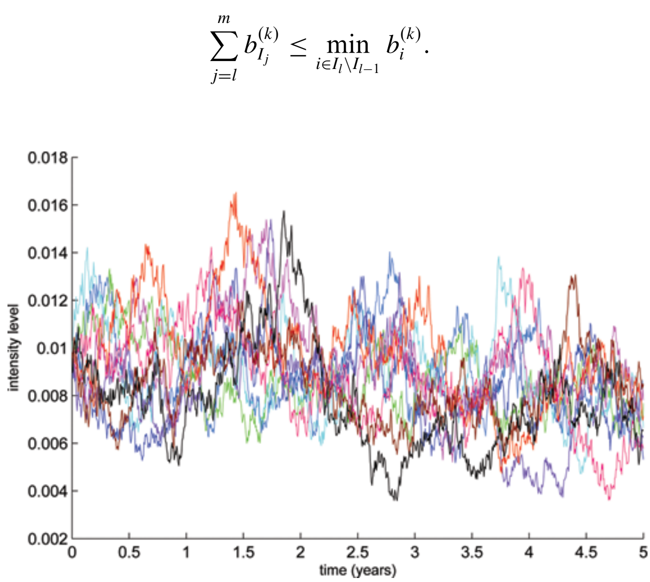

Figure 8. Sample paths of generalized CIR intensities with $a = 3$ and $c = 0.05$ where mean-reversion parameters are implied from AIG CDS curve at December 17, 2007. The first and the second pillar coefficients are (resp.) equal to $b^{(1)} = 0.096$ and $b^{(2)} = 0.075$.

Next, the group parameters $\boldsymbol{b} = (b_{I_j}^{(k)})_{j,k} = \{b_{I_j}^{(k)} : j = 1, \dots, m \text{ and } k = 1, 2\}$ are then calibrated so that the five-year model spread $S_{a_l,b_l}(\mathbf{b}) =: S_l(\boldsymbol{\lambda})$ will coincide with the corresponding market spread S_l^* for each tranche l. To be more specific, the parameters $\boldsymbol{b} = (b_{I_j}^{(k)})_{j,k}$ are obtained according to

$$\boldsymbol{b} = \underset{\hat{\boldsymbol{b}}}{\operatorname{argmin}} \sum_l \left(\frac{S_l(\hat{\boldsymbol{b}}) - S_l^*}{S_l^*} \right)^2 \tag{50}$$

under the constraints that all elements in $\boldsymbol{b}$ are non negative and that $\boldsymbol{b}$ satisfies the inequalities (49). In $S_l(\hat{\boldsymbol{b}})$ we have emphasized that the model spread for tranche l is a function of $\boldsymbol{b} = (b_{I_j}^{(k)})_{j,k}$ but we suppressed the dependence in other parameters like interest rate, payment frequency or b_i, $i = 1, \dots, n$. In the calibration we used an interest rate of 3%, the payments in the premium leg were quarterly and the integral in the default leg was discretized on a quarterly mesh. We use a constant recovery of 40%.

As can be seen in Table 1, we obtain a correct fit for CDX 2007-12-17 even in the case where no name is removed from the calibration constraints. Here, we use 5 groups $I_1, I_2, \dots, I_5$ where $I_j = \{1, \dots, i_j\}$ for $i_j = 8, 19, 27, 102, 125$. However, for the two cases, we label the obligors by decreasing level of riskiness. We use the average over 3-year and 5-year CDS spreads as a measure of riskiness. Consequently, obligor 1 has the highest average CDS spread while company 125 has the lowest average CDS spread. We use Matlab in our numerical calculations and the related objective function is minimized under the suitable constraints by using the built in optimization routine `fmincon` (e.g. in this setup, minimizing the criterion (50) under the constraints given by equations on the form (49)).

For iTraxx Europe 2008-03-31, the calibration results are not improved with respect to the piecewise-constant intensity model and constant recovery.

As a matter of comparison, we plot in Fig. 9 the loss distribution functions obtained from fitted parameters of the generalized CIR intensity model with $a = 3$ and $c = 0.5$ and from the fitted parameters of the piecewise-constant (deterministic) intensity model (see Sec. 5.2 for more details). Note that the grouping is not the same in the two calibrated models. For the deterministic intensity model, we used 5 groups $I_1, I_2, \dots, I_5$ where $I_j = \{1, \dots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$ when calibrating the joint default intensities. Moreover, the obligors in the set I_5 I_4

Table 1
CDX.NA.IG Series 9, December 17, 2007. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The [0, 3] spread is quoted in %. All maturities are for five years.

CDX 2007-12-17: Calibration with constant recovery					
Tranche	[0,3]	[3,7]	[7,10]	[10,15]	[15,30]
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	50.37	258.01	124.68	61.32	41.91
Absolute error in bp	2.301	4.016	0.684	0.327	0.912
Relative error in %	4.787	1.581	0.552	0.536	2.225

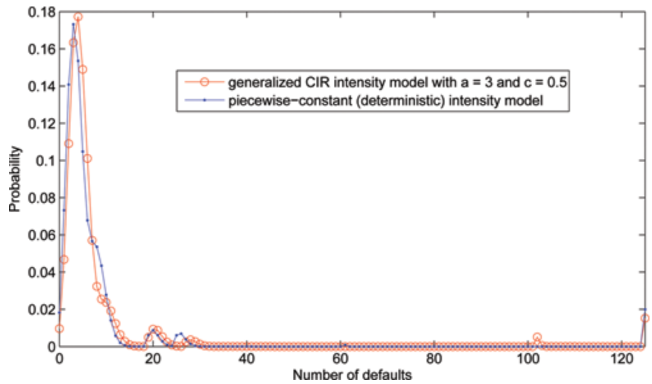

Figure 9. Comparison of 5-year implied loss distributions ($\mathbb{P}(\sum_i H_t^i = k)$, $k = 0, \ldots, 125$) from CDX.NA.IG December 17, 2007 calibration of the generalized CIR intensity model and the piecewise constant intensity model.

consisting of the 64 safest companies are assumed to never default individually, and the corresponding CDSs are excluded from the calibrations constraints. This specification renders a perfect fit. For the CIR intensity model, we also use 5 groups but with $i_j = 8, 19, 27, 102, 125$ and, contrary to the deterministic intensity model, we do not remove any name from the calibration constraints. This specification renders a very good fit.

5. Calibration with Deterministic Intensities and Random Recoveries

In this section we discuss the second calibration methodology used when fitting the Markov copula model against CDO tranches on the iTraxx Europe and CDX.NA.IG series in Sec. 5.2. This method relies on piecewise constant default intensities and random recoveries. Recall that compared with constant recoveries, using random recoveries requires a more sophisticated method in order to compute the expected tranche losses, as was explained in Sec. 3.3.

The piecewise-constant intensity model used in this section is the one presented in Remark 3.1 (see also numerical applications in Bielecki et al., 2014c). Remark 3.2 can be used to compute survival probabilities in this setting.

The calibration methodology and constraints connected to the piecewise constant default intensities are the same as for the mean-reversion coefficients in the CIR intensity case of Section 4: one only needs to replace b by λ in formulas (47), (48), and (49). Therefore we will in this Section only discuss the distribution for the individual stochastic recoveries R_i as well as accompanying constraints used in the calibration. This distribution will determine the quantity $\mathbb{E}\left(e^{\gamma(1-R_i)}\right)$ in (45) which is needed to compute the expected tranche losses.

5.1. *Random recoveries specification and calibration methodology*

We assume that the individual recoveries $\{R_i\}$ are i.i.d and have a binomial mixture distribution of the following form:

$$R_i \sim \frac{1}{K}\mathrm{Bin}\left(K, R^*(p_0 + (1-\Theta)p_1)\right) \text{ where } \Theta \in \{0,1\} \text{ and } \mathbb{P}[\Theta = 1] = q, \quad (51)$$

where R^*, q, p_0 and p_1 are positive constants and K is an integer (in this paper and in Bielecki et al., (2014a) we let $K = 10$). As a result, the distribution function for the recovery rate is given by

$$\mathbb{P}, \left(R_i = \frac{k}{K}\right) = \sum_{\xi=0}^{1} \mu(\xi) \binom{K}{k} p(\xi)^k (1 - p(\xi))^{K-k} \quad \text{where} \quad p(\xi) = R^* \left(p_0 + (1-\xi) p_1\right) \quad (52)$$

where $\xi \in \{0, 1\}$ and $\mu(1) = q$, $\mu(0) = 1 - q$.

In view of (51) and (52) we can give an explicit expression for the quantity $\mathbb{E}e^{\gamma(1-R_i)}$ specified in Sec. 3.3, as follows:

$$\mathbb{E}e^{\gamma(1-R_i)} = \sum_{\xi=0}^{1} \sum_{k=0}^{K} e^{\gamma\left(1-\frac{k}{K}\right)} \binom{K}{k} p(\xi)^k (1 - p(\xi))^{K-k}. \quad (53)$$

Recall that $\mathbb{E}e^{\gamma(1-R_i)}$ together with the corresponding computations in Sec. 3.3 and Eq. (41) will determine the quantity $\mathbb{E}e^{\gamma_\ell \frac{L_t}{2c}}$ in (39) which in turn is needed to compute the expected tranche loss given by Eq. (38).

Let R^* be a constant representing the average recovery for each obligor in the portfolio. We now impose the constraint $\mathbb{E}R = R^*$ which is necessary in order to have a calibration of the single-name CDSs that is separate from the calibration of the common-shock parameters. The condition $\mathbb{E}R = R^*$ leads to constraints on the parameters p_0, q and p_1 that must be added to the constraints for the common shock intensities used in the calibration of the CDO tranches (recall that the calibration is a constrained minimization problem for these parameters). Below we derive these constraints for p_0, q, and p_1. First, note that

$$\mathbb{E}R = \frac{R^*}{K} K \mathbb{E}\left[p_0 + (1-\xi)p_1\right] = R^* p_0 + (1-q)p_1$$

so the condition $\mathbb{E}R = R^*$ implies $p_0 + (1-q)p_1 = 1$ which yields

$$p_1 = \frac{1 - p_0}{1 - q}. \quad (54)$$

Thus, p_1 can be seen as a function of q and p_0. Next, in view of (51) we have for any scalar $\xi \in \{0, 1\}$ that

$$\mathbb{P}\left[R = \frac{k}{K} \,\middle|\, \Theta = \xi\right] = \binom{K}{k} p(\xi)^k (1 - p(\xi))^{K-k} \quad (55)$$

where $p(\xi)$ is defined as in (52). Since $p(\xi)$ is a probability for $\xi \in \{0, 1\}$ it must hold that

$$p(1) = R^* p_0 \in (0, 1) \quad \text{and} \quad p(0) = R^*(p_0 + p_1) \in (0, 1)$$

that is,

$$0 < R^* p_0 < 1 \quad \text{and} \quad 0 < R^*(p_0 + p_1) < 1. \quad (56)$$

We can always assume that $p_0 > 0$ and $0 < R^* < 1$ so the first condition in (56) then implies

$$p_0 < \frac{1}{R^*}. \tag{57}$$

Furthermore, by inserting (54) into the second condition in (56) we retrieve the following constraint:

$$0 < R^* \frac{1 - p_0 q}{1 - q} < 1. \tag{58}$$

Since $q \in (0, 1)$ and consequently $1 - q > 0$, then (58) implies $1 - p_0 q > 0$, that is

$$q < \frac{1}{p_0}. \tag{59}$$

However, we note that this is a "soft" condition since (57) implies that $p_0 < \frac{1}{R^*}$ and if $p_0 < 1$ then (59) is superfluous since we already know that $0 < q < 1$. Next, (58) also tells us that $R^*(1 - p_0 q) < (1 - q)$ which after some computation yields

$$q < \frac{1 - R^*}{1 - R^* p_0}. \tag{60}$$

Finally, it must obviously hold that $q < 1$ since $\mathbb{P}[\Theta = 1] = q$. Thus, combining this with (59) and (60) gives us the following final constraint for the parameter q,

$$q < \min\left(1, \frac{1}{p_0}, \frac{1 - R^*}{1 - R^* p_0}\right). \tag{61}$$

Consequently, using the same notation as in Sec. 4 and replacing the group parameters identifier $\boldsymbol{b}$ by $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k} = \{\lambda_{I_j}^{(k)} : j = 1, \dots, m \text{ and } k = 1, 2\}$, the parameters $\boldsymbol{\theta} = (\boldsymbol{\lambda}, q)$ are obtained according to

$$\boldsymbol{\theta} = \operatorname*{argmin}_{\hat{\theta}} \sum_l \left(\frac{S_l(\hat{\theta}) - S_l^*}{S_l^*} \right)^2, \tag{62}$$

where $\boldsymbol{\lambda}$ must satisfies the same constraints as $\boldsymbol{b}$ in Sec. 4 and q must obey (61). The rest of the notation in (62) are defined as in Sec. 4. In our calibrations the parameters p_0 and R^* will be treated as exogenously given parameters where we set $R^* = 40\%$ while p_0 can be any positive scalar satisfying $p_0 < \frac{1}{R^*}$. The scalar p_0 will give us some freedom to fine-tune our calibrations. In Sec. 5.2 we use the above setting with stochastic recoveries when calibrating this model against two different CDO data-sets.

Finally, note that if the i.i.d recoveries R_i would follow other distributions than (51) we simply modify $\mathbb{E}e^{\gamma(1-R_i)}$ in (45) in Sec. 3.3 but the rest of the computations are the same. Of course, changing (51) will also imply that the constraints in (61) will no longer be relevant.

5.2. *Calibration Results*

In all the numerical calibrations below we use an interest rate of 3%, the payments in the premium leg are quarterly and the integral in the default leg is discretized on a quarterly mesh. Constant or average recoveries (as relevant) are set equal to 40%.

In this section, we calibrate our model against CDO tranches on the iTraxx Europe and CDX.NA.IG series with maturity of five years. We use the random recoveries and the calibration methodology as described in Sec. 5.1. Hence, the 125 single-name CDSs constituting the entities in these series are bootstrapped from their market spreads for $T_1 = 3$ and $T_2 = 5$ using piecewise constant individual default intensities on the time intervals $[0, 3]$ and $[3, 5]$. Figure 10 displays the 3 and 5-year market CDS spreads for the 125 obligors used in the single-name bootstrapping, for the two portfolios CDX.NA.IG sampled on December 17, 2007 and the iTraxx Europe series sampled on March 31, 2008. The CDS spreads are sorted in decreasing order.

When calibrating the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the CDX.NA.IG Series 9, December 17, 2007 we used 5 groups $I_1, I_2, \ldots, I_5$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$. Recall that we label the obligors by decreasing level of riskiness. We use the average over 3-year and 5-year CDS spreads as a measure of riskiness. Consequently, obligor 1 has the highest average CDS spread while company 125 has the lowest average CDS spread. Moreover, the obligors in the set $I_5 \setminus I_4$ consisting of the 64 safest companies are assumed to never default individually, and the corresponding CDSs are excluded from the calibration, which in turn relaxes the constraints for $\boldsymbol{\lambda}$ Hence, the obligors in $I_5 \setminus I_4$ can only bankrupt due to a simultaneous default of the companies in the group $I_5 = \{1, \ldots, 125\}$, i.e., in an Armageddon event. With this structure the calibration against the December 17, 2007 data-set is very good as can be seen in Table 2. By using stochastic recoveries specified as in (51) and (52) we get a perfect fit of the same data-set. The calibrated common shock intensities $\boldsymbol{\lambda}$ for the 5 groups in the December 17, 2007 data-set, both for constant and stochastic recoveries, are displayed in the left subplot in Fig. 11. Note that the shock intensities $\lambda_{I_j}^{(1)}$ for the first pillar (i.e. on the

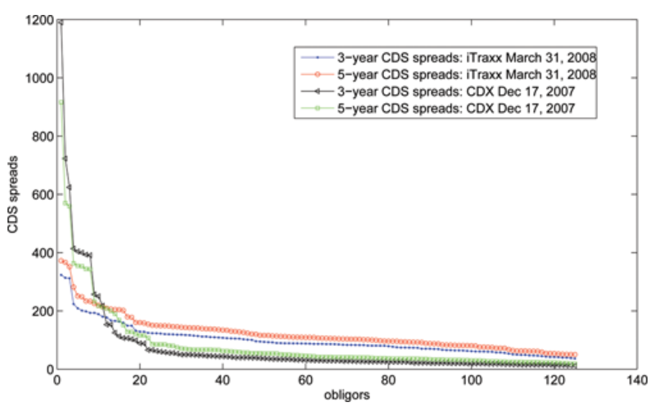

Figure 10. The 3- and 5-year market CDS spreads for the 125 obligors used in the single-name bootstrapping, for the two portfolios CDX.NA.IG sampled on December 17, 2007 and the iTraxx Europe series sampled on March 31, 2008. The CDS spreads are sorted in decreasing order.

Table 2

CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The [0, 3] spread is quoted in %. All maturities are for five years.

CDX 2007-12-17: Calibration with constant recovery

Tranche	[0, 3]	[3, 7]	[7, 10]	[10, 15]	[15, 30]
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	38.94
Absolute error in bp	0.010	0.000	0.000	0.000	2.061
Relative error in %	0.0001	0.000	0.000	0.000	5.027

CDX 2007-12-17: Calibration with stochastic recovery

Tranche	[0, 3]	[3, 7]	[7, 10]	[10, 15]	[15, 30]
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	41.00
Absolute error in bp	0.000	0.000	0.000	0.000	0.000
Relative error in %	0.000	0.000	0.000	0.000	0.000

iTraxx Europe 2008-03-31: Calibration with constant recovery

Tranche	[0, 3]	[3, 6]	[6, 9]	[9, 12]	[12, 22]
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	41.68	429.7	309.4	215.1	103.7
Absolute error in bp	153.1	49.81	0.0441	0.0331	5.711
Relative error in %	3.812	10.39	0.0142	0.0154	5.218

iTraxx Europe 2008-03-31: Calibration with stochastic recovery

Tranche	[0, 3]	[3, 6]	[6, 9]	[9, 12]	[12, 22]
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	40.54	463.6	307.8	215.7	108.3
Absolute error in bp	39.69	15.90	1.676	0.5905	1.153
Relative error in %	0.9886	3.316	0.5414	0.2745	1.053

interval [0, 3]) follows the same trends both in the constant and stochastic recovery case, while the shock intensities $\lambda_{I_j}^{(2)}$ for the second pillar (i.e. on the interval [3, 5]) has less common trend.

The calibration of the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the data sampled at March 31, 2008 is more demanding. This time we use 18 groups $I_1, I_2, \ldots, I_{18}$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 1, 2, \ldots, 11, 13, 14, 15, 19, 25, 79, 125$. In order to improve the fit, as in the 2007-case, we relax the constraints for $\boldsymbol{\lambda}$ by excluding from the calibration the CDSs corresponding to the obligors in $I_{18} \setminus I_{17}$. Hence, we assume that the obligors in $I_{18} \setminus I_{17}$ never default individually, but can only bankrupt due to an simultaneous default of all companies in the group $I_{18} = \{1, \ldots, 125\}$. In this setting, the calibration of the 2008 data-set with constant recoveries yields an acceptable fit except for the [3, 6] tranche, as can be seen in Table 2. However, by including stochastic recoveries (51), (52) the fit is substantially improved as seen in Table 2. Furthermore, in both recovery versions, the more groups added the better the fit, which explain why we use as many as 18 groups.

The calibrated common shock intensities $\boldsymbol{\lambda}$ for the 18 groups in the March 2008 data-set, both for constant and stochastic recoveries, are displayed in the right subplot in Figure 11. In this subplot we note that for the 13 first groups $I_1, \ldots, I_{13}$, the common shock intensities $\lambda_{I_j}^{(1)}$ for the first pillar are identical in the constant and stochastic recovery case, and then diverge quite a lot on the last five groups $I_{14}, \ldots, I_{18}$, except for group I_{16}. Similarly, in the same subplot we also see that for the 11 first groups $I_1, \ldots, I_{11}$, the shock intensities $\lambda_{I_j}^{(2)}$ for the second pillar are identical in the constant and stochastic recovery case, and then differ quite a lot on the last seven groups, except for group I_{13}.

The optimal parameters q and p_0 used in the stochastic recovery model was given by $q = 0.4405$ and $p_0 = 0.4$ for the 2007 data set and $q = 0.6002$ and $p_0 = 0.4$ for the 2008 case. Fig. 12 displays the recovery distribution with calibrated parameters q for the two different data sets CDX.NA.IG series sampled at 2007-12-07 and iTraxx Europe sampled at 2008-03-31. Here $\mathbb{E}[R] = R^* = 0.4$ and $p_0 = 0.4$ in both cases. As seen in Fig. 12, the implied probability for a recovery of 0%, 10% and 20% was consistenly higher in the 2008 sample compared with the 2007 data set (in March 2008 Bear Stearns was bailed out leading to around three times higher credit spreads than in December 2007, both in Europe and North America). Recall that a recovery of 0% means that everything is lost at a default.

Let us finally discuss the choice of the groupings $I_1 \subset I_2 \subset \cdots \subset I_m$ in our calibrations. First, for the CDX.NA.IG Series 9, December 17, 2007 data set, we used $m = 5$ groups with as always $i_m = n$. For $j = 1, 2,$ and 4 the choice of i_j

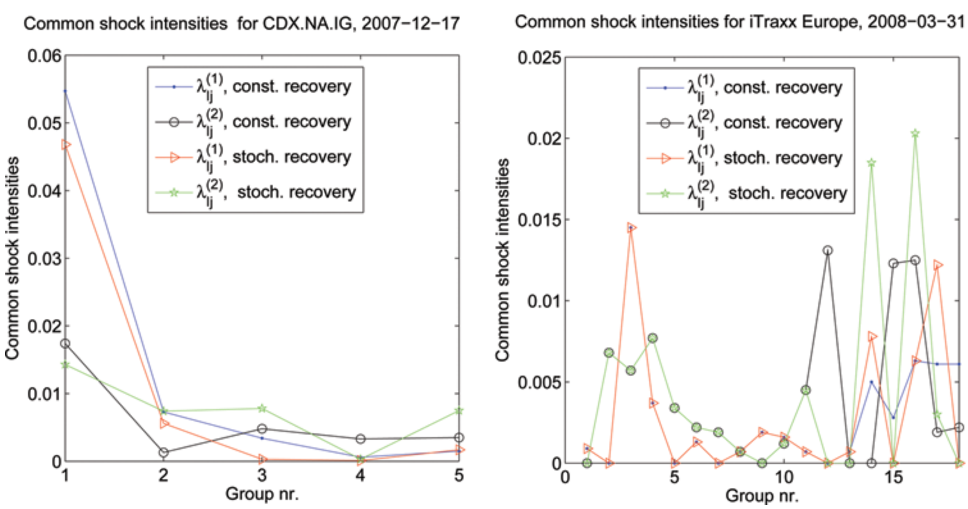

Figure 11. The calibrated common shock intensities $(\lambda_{I_j}^{(k)})_{j,k}$ both in the constant and stochastic recovery case for the two portfolios CDX.NA.IG sampled on December 17, 2007 (left) and the iTraxx Europe series sampled on March 31, 2008 (right).

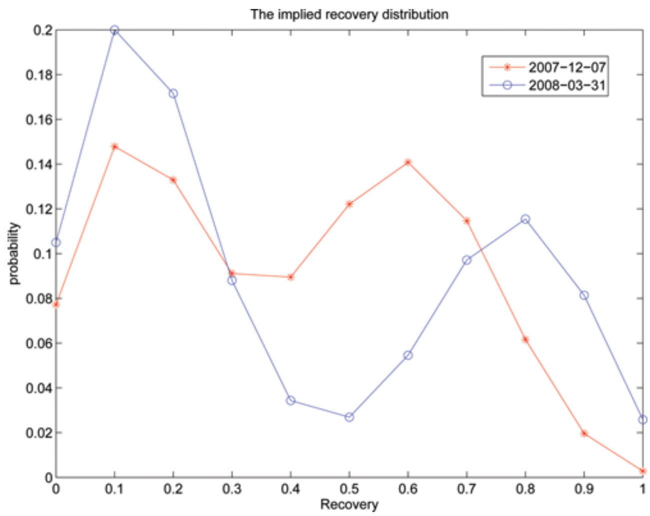

Figure 12. The implied recovery distribution with calibrated parameters q in Bielecki et al. (2014a), for the two different data sets CDX.NA.IG series sampled at 2007-12-07 and iTraxx Europe sampled at 2008-03-31, where $\mathbb{E}[R] = R^* = 0.4$ in both cases.

corresponds to the number of defaults needed for the loss process with constant recovery of 40% to reach the j-th attachment points. Hence, $i_j \cdot \frac{1-R}{n}$ with $R = 40\%$ and $n = 125$ then approximates the attachment points 3%, 10%, 30% which explains the choice $i_1 = 6, i_2 = 19, i_4 = 61$. The choice of $i_3 = 25$ implies a loss of 12% and gave a better fit than choosing i_3 to exactly match 15%. Finally, no group was chosen to match the attachment point of 7% since this made the calibration worse off for all groupings we tried. With the above grouping structure we got almost perfect fits in the constant recovery case, and perfect fit with stochastic recovery, as was seen in Table 2. Unfortunately, using the same technique on the market CDO data from the iTraxx Europe series sampled on March 31, 2008 was not enough to achieve good calibrations. Instead more groups had to be added and we tried different groupings which led to the optimal choice rendering the calibration in Table 2. To this end, it is of interest to study the sensitivity of the calibrations with respect to the choice of the groupings on the form $I_1 \subset I_2 \subset \cdots \subset I_m$ where $I_j = \{1, \ldots, i_j\}$ for $i_j \in \{1, 2, \ldots, m\}$ and $i_1 < \cdots < i_m = 125$ on the March 31, 2008, data set. Three such groupings are displayed in Table 3 and the corresponding calibration results on the 2008 data set is showed in Table 4. From Table 4 we see that in the case with constant recovery the relative calibration error in percent of the market spread decreased monotonically for the first three thranches as the number of groups increased. Furthermore, in the case with stochastic recovery the relative calibration error decreased monotonically for all five tranches as the number of groups increased in each grouping. The rest of the parameters in the calibration where the same as in the calibration in Table 2.

Finally, we remark that the two optimal groupings used in Table 2 in the two different data sets CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008 differ quite a lot. However, the CDX.NA.IG Series is composed by North American obligors while the iTraxx Europe Series is formed by European companies. Thus, there is no model risk or inconsistency created by

Table 3
Three different groupings (denoted A, B and C) consisting of $m = 7, 9, 13$ groups having the structure $I_1 \subset I_2 \subset \cdots \subset I_m$ where $I_j = \{1, \ldots, i_j\}$ for $i_j \in \{1, 2, \ldots, m\}$ and $i_1 < \cdots < i_m = 125$.

Three different groupings													
i_j	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	i_{12}	i_{13}
Grouping A	6	14	15	19	25	79	125						
Grouping B	2	4	6	14	15	19	25	79	125				
Grouping C	2	4	6	8	9	10	11	14	15	19	25	79	125

Table 4
The relative calibration error in percent of the market spread, for the three different groupings A, B, and C in Table 3, when calibrated against CDO tranche on iTraxx Europe Series 9, March 31, 2008 (see also in Table 2)

Relative calibration error in % (constant recovery)					
Tranche	[0, 3]	[3, 6]	[6, 9]	[9, 12]	[12, 22]
Error for grouping A	6.875	18.33	0.0606	0.0235	4.8411
Error for grouping B	6.622	16.05	0.0499	0.0206	5.5676
Error for grouping C	4.107	11.76	0.0458	0.0319	3.3076
Relative calibration error in % (stochastic recovery)					
Tranche	[0, 3]	[3, 6]	[6, 9]	[9, 12]	[12, 22]
Error for grouping A	3.929	9.174	2.902	1.053	2.109
Error for grouping B	2.962	7.381	2.807	1.002	1.982
Error for grouping C	1.439	4.402	0.5094	0.2907	1.235

using different groupings for these two different data sets, coming from two disjoint markets. If on the other hand the same series is calibrated and assessed (e.g., for hedging) at different time points in a short time span, it is of course desirable to use the same grouping in order to avoid model risk.

Conclusions and Perspectives

In this paper we make a focus on two practically important features of the Markov copula portfolio credit risk model of Bielecki et al. (2014a,b,c): random recoveries and stochastic intensities. Regarding random recoveries it would be interesting to find ways to add some dependence features without breaking the model tractability (in the current specifications one is only able to work with independent recoveries). As for stochastic intensities it would nice to find a good way of fixing the parameters a and c, maybe based on historical observation of the dynamics of CDS spreads, rather than quite arbitrarily in this paper, as these dynamic parameters have little impact on CDS and CDO spreads. Also note that other specification of the intensities could be used, in particular Lévy Hull-White intensities driven by subordinators (for the sake of non-negativity, cf. Example 3.6 in Crépey et al., 2012). Finally, it would be interesting to apply these alternative specifications and to compare them in the context of CVA computations on portfolios of CDS and/or CDOs.

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