Expected Credit Loss vs. Credit Value Adjustment: A Comparative Analysis

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I. Introduction

It is admitted that 2/3 of the losses that occured during the 2007-2008 subprime and financial crisis are due to the deterioration of credit counterparties rather than defaults or price variations on other assets. Both regulators and accountants now want banks to better recognize credit deterioration in capital charge and P&L.

On trading books, the price of counterparty risk is called Credit Value Adjustment (CVA). It is equal to the difference between the price of a derivative contract negociated with a riskless counterparty and the price of the same instrument negociated with a real (credit risky) counterparty. CVA was already taken into account in the pricing of books of derivatives before the crisis. This is why banks who were in the business of Negative Base Trading of ABS suffered large losses when the credit rating of their swap counterparties that were insuring them dropped from AAA to CCC or to default in a few weeks period of time only during the crisis. However, market risk related to CVA variations was not, at that time, embedded in the market VaR. In the Basel 3 regulation, the regulators ask banks to compute a dedicated capital charge for CVA risk.

Similarly, the crisis revealed that the provisioning process within banks was much too late when the creditworthiness of banking book exposures was deteriorating. The current IAS 39 (International Accounting Standards) accounting rules prescribe an "incurred loss" model for loan impairment: this norm considers that a loan is impaired when, on the basis of objective evidence, it is partly or wholly uncollectible, so that its carrying amount is greater than its estimated recoverable amount. Objective evidence in this context includes significant financial difficulty of the issuer, actual breach of contract or a high probability of bankruptcy. However, under the current standard, impairment losses are recognized only upon the occurrence of a credit event of the issuer, even if this event was already expected to happen. This generates procyclical effects that were pointed out to be responsible for accelerating the financial crisis, because of delayed recognition of credit losses. In order to circumvent this, the Spanish regulator has asked banks to set statistical provisions as soon as in 2000 (see Jimenez (2006)). IASB (2014) released the final version of the IFRS 9 new accounting standards aiming at overcoming the concerns that arose during the financial crisis because of the former IAS 39 incurred loss model. The new requirement on performing loan books is to recognize loss allowances or provisions before a default event occurs, based on the measurement of an Expected Credit Loss (ECL hereafter).

It appears that the definitions and mathematics of ECL and CVA are similar to each other because ECL and CVA measure analogous quantities, the first one on banking books and the second one on trading books. However, they are not sensitive to the same risk drivers. The CVA measures the expected credit loss, as seen by the market, on a derivative instrument. The main drivers of this expected loss are market risk factors that drive the exposure amount to counterparty risk and the credit spread of the counterparty. By contrast, the risk drivers of the IFRS 9 provisions are not market implied but rather fundamental since they aim at measuring the forthcoming credit losses and their sensitivity to the macroeconomic context. As the risk drivers between these two measures are different, the modelling challenges they generate are not the same. In a nutshell, the challenges to measure CVA are linked to the correct measure of the future exposure on derivatives contracts, whereas the main challenges on ECL estimation are to measure accurately the significant risk deterioration and default probability term structures for any tenor.

In this article, we review the notions of ECL (Section 2) and CVA (Section 3). Section 4 concludes by a comparative analysis of the two measures. Section 5 provides a review of the mathematical and estimation tools involved.

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Let $(\Omega, \mathbb{G})$ stand for a filtered space, which is used throughout this paper as the space containing random events that underlie modeling the stochastic evolution of a financial market. In particular, all our *processes* are $\mathbb{G}$ -adapted. The filtration will be given as $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{T}}$ with $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}_+$ in the respective ECL and CVA case, consistent with the respective banking/statistical and trading/pricing tradition for these notions. Expectations under the statistical probability measure $\mathbb{P}$ and a risk-neutral pricing measure $\mathbb{Q}$ (assumed chosen by the market) are denoted by $\mathbb{E}^{\mathbb{P}}$ and $\mathbb{E}^{\mathbb{Q}}$, respectively.

Here is a recapitulative list of acronyms, by order of apparition of their main use in the paper.

CSA	Credit Support Annex
CVA	Credit Valuation Adjustment
DVA	Debit Valuation Adjustment
FVA	Liquidity Funding Valuation Adjustment
XVA	Total Valuation Adjustment.
ECL	Expected Credit Loss
IFRS	International Financial Reporting Standards
IAS(B)	International Accounting Standards (Board)
EIR	Effective Interest Rate
LGD	Loss Given Default
EAD	Exposure At Default
PIT	Point-In-Time
IRB	Internal Rating Based
PD	Probability of Default
TTC	Through-The-Cycle
EPE	Expected Positive Exposure
EEPE	Effective Expected Positive Exposure
RWA	Risk Weighted Assets
EE	Expected Exposure
PFE	Potential Future Exposure
MPFE	Maximum Potential Future Exposure
HR	Hit Rate
CAP	Cumulative Accuracy Profile
AR	Accuracy Ratio

II. ECL AS A MEASURE OF THE IFRS 9 PROVISION

In the context of ECL computations, time is modeled as discrete and a finite time horizon $T < \infty$ represents the maturity of a credit exposure of the bank. As stated in IASB (2014) (the final version of the IFRS 9 new accounting standards), the Expected Credit Loss (ECL) aims at measuring collective provisions on non defaulted instruments that are measured at amortised cost or fair value through Other Comprehensive Income (OCI). In what follows, we will refer to the "instrument" for any financial instrument that is in the scope of the IFRS 9 norm; it can be a loan, a bond or an other debt instrument or asset on the balance sheet, as well as a liability such as an undrawn commitment or a guarantee issued by the bank. The IASB standard defines the ECL as the difference between all contractual discounted cash-flows that are due on the instrument and all discounted cash-flows that the bank expects to receive (i.e. all cash shortfalls). Both the interest amounts and the discount factor are computed based on the Effective Interest Rate (EIR). The EIR is calculated at initial recognition of the financial asset. It is the rate that exactly discounts estimated cash-flows through the expected life of the instrument to the gross carrying amount. It includes transaction costs or fees, as well as future lifetime expected credit losses for originated credit-impaired financial assets. ECL measurement in then based on all terms of the contract (including all the options such as prepayment, extensions,...) over the life of the instrument, as well as cash-flows coming for instance from the sale of the loan collateral or other credit enhancement mechanism, if any.

The loss allowance to non defaulted instruments is done in two stages, depending on the observed credit risk deterioration since origination of the instrument:

- Stage 1: if the credit risk on a financial instrument has not increased significantly since initial recognition, the loss allowance is equal to the 12 months ECL.
- Stage 2: if the credit risk on a financial instrument has increased significantly since initial recognition, the loss allowance is equal to the lifetime ECL.

As a result, stage 1 loans are of better credit quality than stage 2 loans. Paragraphs 5.5.10 and 5.5.11 of IASB (2014) provide details about the transfer criteria to be used from stage 1 to stage 2. For instance, an entity may assume that the credit risk on instruments has not increased significantly if the instrument is determined to have low credit risk. Conversely, as soon as contractual payments on an instrument are more than 30 days past due, the norm requires transfering it automatically to stage 2. However, beyond these two special cases, the norm is "principle-based" so that it does not detail how to determine the instruments that should be in stage 1 or in stage 2: implementing the new standards is subject to interpretation of the text and to some subjective choices in terms of credit risk quantification. We mention that the the defaulted instruments correspond to the stage 3 instruments, for which the provision is specific (and no longer collective) and is equal to the full lifetime ECL.

ECL is an estimate of the total credit loss over the life of the instrument. The measurement of ECL needs to take into account:

- unbiased and probability weighted amounts in order to take into account a full range of possible outcomes. The bank needs to specify the amount and timing of cash-flows as well as the estimated probability of each outcome.
- time value of money has to be taken into account by discounting cash-flows at a rate that approximates the Effective Interest Rate of the instrument.
- risk parameters estimated from historical data and adjusted to reflect the effects of current conditions and forecasts of future economic conditions.

Based on the definition stated earlier, the standard defines the ECL as the weighted-average of future credit losses on the instrument, where the weights must reflect the probabilities of occurence of a default. Let's start from the initial definition. For a given instrument, the ECL is equal to the difference between the sum of discounted contractual and expected cash-shortfalls:

$$ECL = \sum_{t\geq 0} CF_t^C.DF(t) - \mathbb{E}^{\mathbb{P}}\left[\sum_{t\geq 0} CF_t^R.DF(t)\right]s \quad (1)$$

where the horizon T is equal to one year for stage 1 instruments and to the maturity date for stage 2 instruments; the second sum is done for all values of t larger than 0 because the recovery process in case of a default may be longer than the contractual maturity of the instrument. As stated by the norm, the discount rate is the EIR and the discount factor associated with horizon t is $DF(t) = 1/(1+EIR)^t$. $(CF_t^C)_{t\geq 0}$ and $(CF_t^R)_{t\geq 0}$ are respectively the sequences of contractual and real (random) cash-flows paid by the instrument, including both principal and interest payments. In the case of a loan, the quantity CF_t^R embeds prepayments, additional drawings (for instance for revolving facilities) or an occurrence of a default ; indeed, in case of default, the cash-shortfalls are the proceeds linked to the recovery or liquidation process and no longer with the withdrawal of the loan from the client. We can decompose the real cash-flows into two terms, depending on whether a default has already occured at time t or not. We obtain:

$$CF_t^R = CF_t^0.1_{\{\tau>t\}} + R_t.1_{\{\tau\leq t\}} \quad (2)$$

where τ is the default date on the instrument, R_t is the recovery cash shortfall at date t and CF_t^0 is the cash-flow occuring at date t in the non default scenario. The sequences $(CF_t^0)_{t\geq 0}$ and $(R_t)_{t\geq 0}$ represent the cash-flows before and after the default event occurs. We can then rearrange the ECL formula in order to make the expected credit loss appear explicitly:

$$ECL = \mathbb{E}^{\mathbb{P}}\left[\sum_{t\geq 0}\left(CF_t^C - CF_t^0\right).DF(t)\right] \quad (3)$$
$$+\mathbb{E}^{\mathbb{P}}\left[\sum_{t\geq 0}\left(CF_t^0 - R_t\right).1_{\{\tau\leq t\}}.DF(t)\right]$$

The cash-flows CF_t^C and CF_t^R both include principal and interest payments. The first term of equation (3) is equal to 0 from the following proposition:

Proposition 2.1. For any sequence of cash-flows $CF(t)$ that includes principal and interest payments, the sum of the discounted cash-flows is independent on the dates and amounts of the cash-flows. In particular, we have $\sum_{t\geq 0} CF_t^0 DF(t) = \sum_{t\geq 0} CF_t^C DF($

Proof. Let's call N_t the principal amount withdrawn at date t. We have:

$$\sum_{t_i\leq T} CF_{t_i}\cdot DF(t_i)$$
$$= \sum_{t_i\leq T}\left(\begin{array}{c} N_{t_i}\cdot DF(t_i) \\ + \dfrac{DF(t_{i-1}) - DF(t_i)}{DF(t_i)}\left(\sum_{t_j > t_{i-1}} N_{t_j}\right)\cdot DF(t_i)\end{array}\right)$$

After rearranging this relationship, we get:

$$\sum_{t_i\leq T} CF_{t_i}\cdot DF(t_i) = \sum_{t_i\leq T} N_{t_i}$$

This relationship means that the sum of all the discounted cash-flows are equal to the total nominal exposure, whatever the amounts and the dates of payment of the principal cash-flows. Then whatever the sequence of cash-flows for a given nominal amount, the sum of discounted cash-flows remains unchanged. ■

Equation (3) then writes:

$$ECL = \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{t\geq\tau} CF_t^0 \frac{DF(t)}{DF(\tau)} - \sum_{t\geq\tau} R_t \frac{DF(t)}{DF(\tau)}\right) DF(\tau)\right] \quad (4)$$

We set $EXP(\tau) = \sum_{t\geq\tau} CF_t^0 \frac{DF(t)}{DF(\tau)}$, which is the total credit exposure on the instrument at the date of the default event. We note that this definition differs from the Basel 2 Exposure At Default (EAD) definition. We then get for the ECL:

$$ECL = \mathbb{E}^{\mathbb{P}}\left[EXP(\tau)\left(1 - \frac{\sum_{t\geq\tau} R_t \frac{DF(t)}{DF(\tau)}}{EXP(\tau)}\right) DF(\tau)\right] \quad (5)$$

We introduce the loss rate, or Loss Given Default $LGD = 1 - \dfrac{\sum_{t\geq\tau} R_t \frac{DF(t)}{DF(\tau)}}{EXP(\tau)}$. Through this definition, we see that the loss rate depends on the default date τ and on the recovery cash-flow sequence $\left(R_t\right)_{t\geq 0}$ which is random as well. However, it is usually assumed that the loss rate is a constant. We get the formula for the expected credit loss:

$$ECL = \int_0^\infty DF(t)\cdot EXP(t)\cdot LGD\cdot \mathbb{P}[\tau\in(t,t+dt)]. \quad (6)$$

If we discretize this relationship, we get:

$$ECL \sim \sum_{t_i} EXP(t_i)\cdot LGD\cdot DF(t_i)\cdot \mathbb{P}[\tau\in(t_i,t_{i+1})] \quad (7)$$

where $\mathbb{P}[\tau\in(t_i,t_{i+1})]$ can be estimated by the techniques reviewed in Sect. 5.1-5.2 (whereas for loans the estimation of the exposure in (7) is straightforward).

Depending on whether the instrument is in stage 1 or in stage 2, the expected credit loss is measured either over a one year time horizon or over the full lifetime of the instrument. This generates a jump in the provision amount when the instrument is transfered from stage 1 to 2: as the credit risk associated with the instrument has increased significantly, the default probabilities are much higher compared to their values at origination, and additionally, the ECL is measured over the full lifetime instead of one year only.

The philosophy of the IFRS 9 is similar to a pricing approach, motivating the full lifetime horizon retained for instruments in stage 2. Thus, the choice of the one year horizon for instruments in stage 1 looks arbitrary, but it is the same time horizon as the one used to measure regulatory capital charge in the advanced approaches. There is however a major difference between the two, in particular from the way that credit risk parameters (PDs and loss rates) are measured. In the IFRS 9 framework, parameters should reflect current economic conditions (the so called Point-In-Time -PIT- parameters) whereas

Basel PDs for instance are supposed to be average parameters over the credit cycle (Through-The-Cycle or TTC parameters) and LGDs have downturn values. Despite, IRB banks are supposed to use their existing Basel models at least as a starting point for measuring IFRS 9 provisions. Basel Committee on Banking Supervision (2015) yields guidelines for the evolution of risk management frameworks to comply with the ECL measurement. The interplay betweeen the Basel and the IFRS 9 frameworks is out of the scope of this article, but it will obviously lead to an upgrade of risk management practices within banks.

The new aspect that the IFRS 9 norm introduces is about including forecasts into the metrics, PDs and LGDs in particular. Historical information is a starting point, and adjustments are required based on forward-looking information. From a modelling point of view, this raises several issues. First of all, the period over which the historical data have to be considered is an issue in order to reflect both past events and current conditions. Second, it should be assessed whether the historical data capture TTC information or PIT information. Third, the information captured through historical data has to be assessed too and adjusted to include forecasts. Finally, the estimates of ECLs shall be recalibrated regularly, and backtested as well. Backtesting raises an important theoretical issue too because the ECL embeds both estimation from historical data and forward-looking forecasts: these are the two legs in the quatification which need to be assessed separately.

III. Credit Value Adjustment (CVA)

III.1. CASH FLOWS

In the context of CVA computations, time is modeled continuously and a finite horizon $T < \infty$ represents the maturity of a credit support annex (CSA) regarding the "instrument", in the sense of a generic netted portfolio of OTC derivatives between a bank and its counterparty. The perspective of the bank is taken in the sequel. The CSA is a legal agreement between the two parties, which frames the liquidation process in case of default of one or the other party. In principle, a CSA also prescribes a collateralization scheme in order to mitigate counterparty risk, but since there is no collateral involved in ECL computations, we assume no collateralisation henceforth. By "netted portfolio" above, we mean that in case of default of a party, the debt of each party to the other will be valued by the liquidator on the basis of the cumulative value of all the ingredients of the portfolio, rather than as the cumulative debt over the different ingredients in the nonnetted case. This setup can be considered as general since, for any partition of a portfolio into netted sub-portfolios, the results may be applied separately to every netting set. The results at the portfolio level are then simply derived as the sum of the results of the netting sets.

Given our purpose of emphasizing the analogy between ECL and CVA computations, we assume the bank default-free and only consider the default time τ of the counterparty. We assume that τ cannot occur at fixed times, an assumption that is for instance satisfied in all the intensity models of credit risk. In particular, the scenario $\{\tau = T\}$ has zero probability and is immaterial in any expectation (hence price and hedge), so that we can ignore it for simplicity, e.g. we write $\{\tau < T\}$ interchangeably with $\{\tau \le T\}$. We denote $\bar{\tau} = \tau \wedge T$, which represents the effective time horizon of our problem, since there are no cash flows after $\bar{\tau}$.

We represent by a finite variation process D the promised (or clean) cumulative dividend process of the instrument, with jump process denoted by Δ, i.e. $\Delta_t = D_t - D_{t-}$. All cash flows are considered from the bank's point of view in the sense that $\Delta_t = 1$ means $+1$ to the bank. A promised dividend is only effectively paid at time t if the counterparty is still alive at time t, resulting in the effective dividend process $1_{\{t<\tau\}} dD_t$. A close-out cash flow χ closes the bank's position at time τ (if $\tau < T$). This close-out cash flow is based on the (algebraic) debt ε of the counterparty to the bank at the counterparty default time τ, namely

$$\varepsilon = Q_\tau + \Delta_\tau, \tag{8}$$

where Q is a CSA close-out valuation process and where $\Delta_\tau = D_\tau - D_{\tau-}$ denotes the jump of D at τ, representing any dividend unpaid to the bank by the counterparty defaulted at time τ. In the absence of collateral, the close-out cash flow χ is defined, for $\tau < \infty$, as

$$\chi = R\varepsilon^{+} - \varepsilon^{-}, \tag{9}$$

where R denotes the recovery rate of the counterparty toward the bank, assumed constant in this article for simplicity. In words, if, at the counterparty default time, the counterparty is net debtor to the bank (case where $\varepsilon > 0$), then, as the counterparty is in default, it's only a fraction R of the debt that is recovered by the bank. If the bank is net debtor to the counterparty (case where $\varepsilon < 0$), then the bank settles its debt in totality (otherwise it would itself be in default, which we excluded by assumption).

III.2.PRICES

We assume that there is only one funding asset, the so-called savings account, growing at the risk-free rate r_t. The savings account is thus the inverse of the risk-free discount factor $\beta_t = e^{-\int_0^t r_s ds}$. In the ensuing classical risk neutral valuation framework we can proceed with presenting the following definitions, which are consistent with the standard theory of arbitrage (cf. Delbaen and Schachermayer (2005)). Recall that D stands for the cumulative promised (or clean) dividend process of the instrument on $[0,T]$, ignoring counterparty risk. We denote by $\mathbb{E}_t^{\mathbb{Q}}$ the conditional expectation under the risk-neutral measure $\mathbb{Q}$ given $\mathcal{G}_t$.

Definition 3.1. (i) The (counterparty-)clean price[1] P_t of the instrument is given, for $t \in [0,T]$, by

$$\beta_t P_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \beta_s dD_s\right]. \tag{10}$$

The clean cumulative value process of the portfolio is given by

$$\widehat{P}_t = P_t + p_t, \qquad (11)$$

where p_t represents the discounted cumulative clean dividend up to time t, so

$$\beta_t p_t = \int_0^t \beta_s dD_s. \qquad (12)$$

(ii) The (counterparty-)risky price Π_t of the instrument is given, for $t \in [0,\bar{\tau}]$, by

$$\beta_t \Pi_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\bar{\tau}} \beta_s 1_{s<\tau} dD_s + \beta_\tau 1_{\tau<T} \chi\right]. \qquad (13)$$

The risky cumulative value process of the portfolio is given, for $t \le \bar{\tau}$, by

$$\widehat{\Pi}_t = \Pi_t + \pi_t, \qquad (14)$$

where

$$\beta_t \pi_t = \int_0^t \beta_s 1_{s<\tau} dD_s. \qquad (15)$$

In the sequel we assume the most standard CSA close-out valuation process $Q = P$.

III.3. CREDIT VALUATION ADJUSTMENT

As for the ECL in (1), we will begin with the CVA defined as a counterparty risk valuation adjustment process. Then, in Proposition 3.4, which is the CVA analog of the ECL identity (4), we will demonstrate that the CVA is a price process for a so-called contingent credit default swap (CCDS), which pays the so-called counterparty risk exposure at default.

Definition 3.2. The CVA process, denoted as CVA, is given, for $t \in [0,\bar{\tau}]$, as:

$$CVA_t = P_t - \Pi_t = P_t - \Pi_t + 1_{t \geq \tau, t<T} \Delta_\tau. \qquad (16)$$

The left equality appears as a natural definition for what a (cumulative) counterparty risk valuation adjustment should be. The right equality follows since, by Definition 3.1, we have on $[0,\bar{\tau}]$:

$$p_t - \pi_t = 1_{t \geq \tau, t<T} \Delta_\tau.$$

Remark 3.3. The term $1_{t \geq \tau, t<T} \Delta_\tau$ is needed in the right hand side of (16) so that we get a *cumulative* CVA. In most cases, there will be no promised cash flows at τ, so that $\Delta_\tau = 0$, then of course this term vanishes. The reader may ask why we are interested in a cumulative CVA. The reason is that the discounted cumulative CVA is a $\mathbb{Q}$ martingale (assuming integrability), which makes it convenient mathematically for valuation and hedging purposes. Note however that on the time interval $[0,\tau)$, which only matters for the bank in practice (as the bank only manages its CVA before τ), we have $CVA_t = P_t - \Pi_t$ (whether or not $\Delta_\tau = 0$).

In the rest of this section we will discuss three alternative representations of the CVA.

Exposure at Default. We define the (counterparty risk) exposure at default as the $\mathcal{G}_\tau$-measurable random variable

$$\xi := P_\tau + \Delta_\tau - \chi = \epsilon - (R\epsilon^+ - \epsilon^-) = (1-R)\epsilon^+, \qquad (17)$$

where the second equality holds by the definition (9) of χ and the third one follows by the definition (8) of ε, accounting for our specification $Q = P$ for the CSA close-out valuation process.

Proposition 3.4. For $t \in [0,\bar{\tau}]$, we have:

$$\beta_t CVA_t = \mathbb{E}_t^{\mathbb{Q}}[\beta_\tau 1_{\tau<T} \xi]. \qquad (18)$$

Proof. This follows from the martingale property of the process βCVA, which is apparent on the left-hand-side identity in (16) (assuming integrability), whereas we have by the right-hand side identity:

$$\begin{aligned} CVA_{\bar{\tau}} &= P_{\bar{\tau}} - \Pi_{\bar{\tau}} + 1_{\tau<T} \Delta_\tau \qquad (19) \\ &= 1_{\tau<T}(P_\tau + \Delta_\tau - \chi) = 1_{\tau<T} \xi. \end{aligned}$$

As first observed in Brigo and Pallavicini (2008), we obtain an interpretation of the CVA as the price of a contingent credit default swap (CCDS), which, as visible in the definition (17) of the payoff ξ, is an option on the (algebraic) debt ε of the counterparty toward the bank at time τ (if $\tau < T$).

A major issue in regard to counterparty credit risk is the so-called wrong-way risk. From the perspective of the bank, this occurs when the exposure is adversely correlated with the credit quality of the counterparty (risk that the value of the instrument is particularly high at the counterparty's default). In simultaneous default models of portfolio credit risk, an extreme form of wrong-way risk ("instantaneous default contagion") is represented by a term $\Delta_\tau \neq 0$ in ε (see (8) and Remark 3.3).

Expected Positive Exposure. In the case where β is deterministic, the following representation of CVA at time 0 follows from (18):

$$\begin{aligned} CVA_0 &= \mathbb{E}^{\mathbb{Q}}\left(\beta_\tau (1-R) 1_{\tau<T} \epsilon^+\right) \qquad (20) \\ &= \int_0^T \beta_s \mathbb{E}^{\mathbb{Q}}\left((1-R) 1_{\tau \in ds} \epsilon^+\right) \\ &= \int_0^T \beta_s \mathbb{E}^{\mathbb{Q}}\left((1-R)\epsilon^+ \mid \tau = s\right) \mathbb{Q}(\tau \in ds) \\ &= \int_0^T \beta_s EPE(s) \mathbb{Q}(\tau \in ds), \end{aligned}$$

where the expected positive exposure EPE is the function of time defined, for $t \in [0,T]$, by

$$EPE(t) = \mathbb{E}^{\mathbb{Q}}\left[(1-R)\varepsilon^+ \mid \tau = t\right]. \qquad (21)$$

Note that the EPE is frequently computed under the assumption that $R = 0$. A loss-given-default factor is then introduced at the stage of CVA computation based on this "zero recovery EPE".

Remark 3.5. The CVA representation formula (20) can be compared with the ECL representation formula (4), except for the nature of the probability measures that are used in the expectations. For CVA computations, the pricing measure $\mathbb{Q}$ prevails. However, for regulatory capital computations, the historical measure $\mathbb{P}$ is typically used. In particular, this is the case for measuring the capital charge linked to the variations of the CVA, the so-called CVA-VaR, which is the 99% $\mathbb{P}$ percentile of CVA variations over 10 business days (where the involved CVAs are computed under $\mathbb{Q}$ as usual). In EEPE (Effective Expected Positive Exposure) internal model approaches, RWA measurements related to counterparty credit risk are also done with respect to the historical probability measure. Specifically, in this context, the following quantities are defined:

- Expected Exposure (EE): $\mathbb{P}$ expected value of the positive part of the future market value of the instrument;
- Expected Positive Exposure (regulatory EPE): time average of the Expected Exposure profile, i.e.

$$\frac{1}{T}\int_0^T EE(t)dt$$

(note that the $\mathbb{P}$ analog of the CVA related EPE (21) is EE and not regulatory EPE);

- Effective Expected Positive Exposure: time average of the running maximum Expected Exposure profile, i.e.

$$\frac{1}{T}\int_0^T \max_{s \leq t} EE(s)dt.$$

Other indicators are used by banks in their internal processes of deal acceptance or risk limit definition for front offices. In particular, the Potential Future Exposure (PFE) is a percentile under the historical measure of the positive part of the market value and the MPFE is the maximum of the PFE over the time horizon of the instrument.

Example 3.6. Figure 3.6 shows the mean and the 2.5% and 97.5% quantiles as a function of time for the price process a 3 months versus 6 months basis swap in the multicurve interest rate two-factor log-normal model of Crepey, Macrina, Nguyen and Skovmand (2015). A basis swap exchanges two streams of floating payments based on a nominal cash amount N or, more generally, a floating leg against another floating leg plus a fixed leg. In the classical single-curve interest rate setup, the value of a basis swap (without fixed leg) is zero throughout its life. Since the onset of the financial crisis in 2007, markets quote positive basis swap spreads that have to be added to the smaller tenor leg. Hence, basis swaps have to be handled through multicurve interest rate models. Our $\mathbb{Q}$ measure in this example is calibrated to the EONIA 3m and 6m tenor initial term structures and to 3m tenor swaption data of January 4, 2011; Our $\mathbb{P}$-measure is fudged from our $\mathbb{Q}$-measure based on a view that the LIBOR rate $L(10.75y; 10.75y, 11y)$ will be either 2% with $\mathbb{P}$ probability $p = 0.7$ or 5% with $\mathbb{P}$ probability $1 - p = 0.3$. The dashed lines in the left and right panels indicate the maximum in time of the 97.5% quantile of the price process of the basis swap under the respective $\mathbb{Q}$ and $\mathbb{P}$ measures. In particular, the dashed line on the right panel corresponds to the MPFE. These graphs show that the impact of the probability measure on extreme percentiles of the price process, hence the impact of the choice of the probability measure $\mathbb{Q}$ and $\mathbb{P}$ in counterparty risk computations, can be very large.

Intensity-Based CVA Formula. The formula (20) looks nice, but it is only really practical when τ is independent of ε so that

$$EPE(t) \begin{aligned} &= \mathbb{E}^{\mathbb{Q}}\left[(1-R)\varepsilon^+ \mid \tau = t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[(1-R)(P_t + \Delta_t)^+\right]. \end{aligned}$$

In order to obtain a formula similar to (20), but under a weaker *conditional* independence (called immersion) assumption between the credit risk of the counterparty and the underlying exposure, we suppose further that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where the filtration $\mathbb{H}$ is generated by the indicator process of τ and where $\mathbb{F}$ is some reference filtration. Assuming $Z_t := \mathbb{Q}(\tau > t | \mathcal{F}_t)$ nonincreasing, positive and time-differentiable, an application of Lemma 5.4 below (case where $f = 0$ in the lemma, where the case of a nonzero f could be used to extend our results to the presence of funding costs, i.e. FVA computations) yields the following intensity-based CVA formula:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \gamma_t (1 - R) \alpha_t P_t^+ dt\right], \tag{22}$$

where γ and α are the respective hazard default intensity and credit-risk-adjusted-discount-factor formally defined in terms of Z in (40).

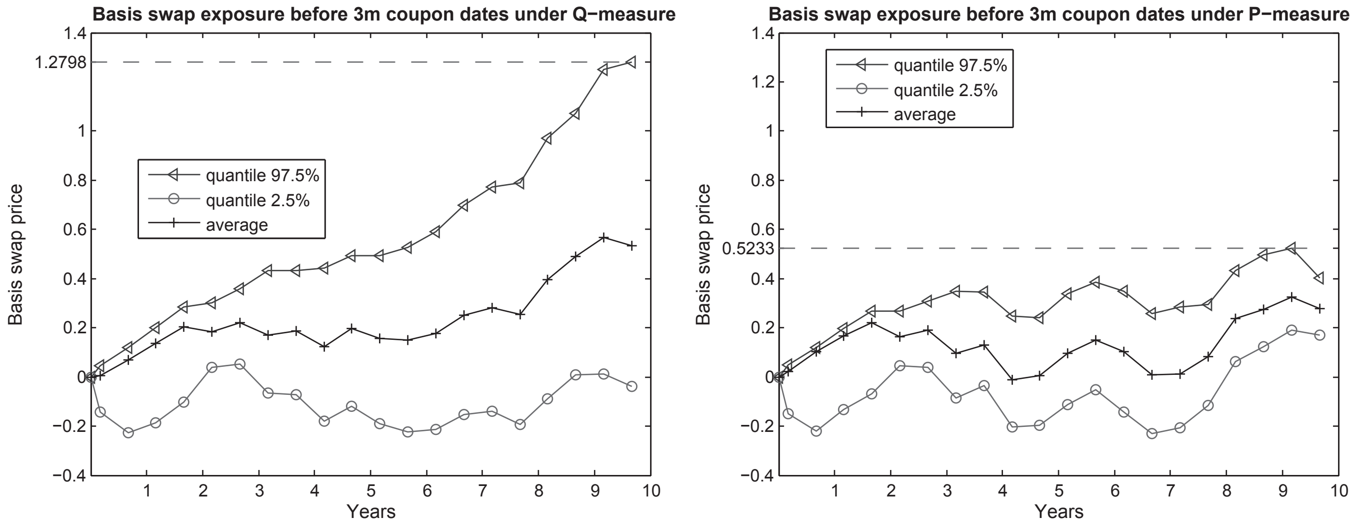
The reason why Δ does not show up in the formula (22) is the simplistic information structure $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, which implies that $\Delta_\tau = 0$ (cf. Lemma (5.3)(ii) below). To go beyond the conditional independence (basic immersion) setup for this formula, one can work in a larger filtration $\mathbb{G}$, where the additional information in $\mathbb{G}$ is used for modeling different wrong-way risk scenarios. An alternative is a similar formula valid in general (beyond a basic immersion setup) under a changed probability measure, where the measure change reflects the lack of immersion in the model (see Crépey and Song (2015)).

IV. Comparison

We see from (1) and (16) that ECL and CVA have the same definitions, i.e. the difference between riskless and risky probability weighted discounted cash shortfalls. We have shown in (6) and (18) that these cash-shortfall were equal to the present value of future losses on the intruments. In a risk neutral world, as it is the case for CVA, this is equal to a price, associated with the fact that counterparties are credit risky. In a historical world, this is equal to the ECL provision that measures the average risk on the instruments. To conclude this paper we underline the similarities and differences between ECL and CVA from the point of view of calibration, modeling, management and impact on the bank.

Both measures embed forward looking parameters and, in particular, forward-looking default probabilities. The CVA is a probability weighted average of all the possible outcomes in terms of default risk of the counterparty. The weighting of the outcomes is computed from market implied default probabilities of the counterparty. In essence, these probabilities are Point-In-Time parameters

Figure 1. Mean and quantiles of the price process right before the 6m coupon times for a 3 months versus 6 months basis swap in the multicurve interest rate two-factor lognormal model of Crepey, Macrina, Nguyen and Skovmand (2015). Both legs of the basis swap are worth 27.96 at time 0. (*Left*) Exposure under the $\mathbb{Q}$-measure. (*Right*) Exposure under the $\mathbb{P}$-measure



(reflect current economic conditions), because market implied parameters are measured from spot market prices, and are forward-looking, because they embed the view of the market related to the default likelihood of the counterparty at different horizons in the future. Regarding the ECL, PDs are PIT forward-looking as well. However, they are not calibrated from market spreads, but estimated from historical data and adjusted to take into account the curent conditions and the available information about the foreseeable future. Some players are pushing for the use of market implied parameters of the corporate perimeter, but this seems disconnected from the goal of measuring provisions on the loan book.

Regarding loss rates (or LGDs), we observe the same difference between CVA and ECL. For CVA, the LGD used in the pricing is linked to the recovery rate used for the pricing of credit derivatives, which is a market parameter. By contrast, for ECL, the loss rate is estimated from the bank's loss data base or from the bank's provision rates database, which are not linked to market parameters.

One of the main aspect of counterparty risk is the notion of wrong way risk that encompasses the potential positive correlation between the increase of the exposure and the increase of default likelihood of the counterparty. A similar effect occurs on the loan book, in particular for revolving facilities or term loans with a drawing period. For this type of exposure, it is well known that the increase of the default likelihood is often associated with an increase of the exposure because the client draws her line of credit when financial difficulties appear. The regulator is of course aware of this type of wrong way risk and requires the bank to measure a Credit Conversion Factor (CCF) for off-balance sheet exposures that are likely to be drawn before the client goes to default. Prepayments are another component of the dynamics of the exposure on the loan book, and the difference in the prepayment behavior of the clients depending on their credit quality has to be assessed in IFRS 9 models as well, because it may contribute to generate a wrong way risk.

Finally, we have focused in this paper on unilateral counterparty risk and CVA, only accounting for the default risk of the bank's counterparty. However, a change in the bank's own credit spread may impact the P&L of the bank. When the credit spread of the bank increases, the present value of all the future cash-flows that the bank is committed to pay decreases, resulting in a gain in P&L. The correction on the PV of all the outgoing cash-flows due to the positive market spread of the bank, i.e. the symmetrical accounting companion of the CVA, is called Debit Valuation Adjustment (DVA). Such an effect does not exist on the ECL because the IFRS 9 norm requires to compute purely unilateral ECLs, even for issued guarantees or other off-balance sheet commitments that are considered as liabilities. ■

1 In the market terminology, it is frequently called the mark-to-market (MtM).

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Appendix A – Review of the Mathematical Toolbox

A.1. Credit Rating and Scoring models

One of the main differences between CVA and ECL (provisions) is that they are not based on the same risk measures because they are not sensitive to the same risk drivers. Regarding credit risk, CVA is measured from the risk parameters as assessed by the market, namely credit spreads, which is possible because derivative transactions are often done with non retail counterparties (corporate, financial institutions, sovereigns) whose credit risk is quoted on the CDS markets. Conversely, provisions are assigned to any type of client of the bank, and most of them are not quoted on the CDS markets, such as Small and Medium Enterprises (SMEs) or retail banking clients. It is widely accepted that the provision has to be estimated based on credit ratings or credit scores instead of any other risk measure (either market implied or pool based).

A.1.1. Ratings

A credit rating is an opinion on the client's or counterparty's creditworthiness, i.e. its abillity or willingness to pay back the loan it has been granted. Credit ratings are provided by rating agencies (Standard and Poor's, Moody's, FitchRatings for instance) which focus on evaluating credit risk. There are also regional or "niche" rating agencies that specialize in a geographical area or industry. Banks provide credit ratings as well, based on their own internal models.

Ratings embed an evaluation of the current and historical information as well as the potential impact of the foreseeable future conditions or events, such as the evolution of the business cycle. However, the forward-looking feature of the rating is neither a prediction of a forthcoming default nor an exact quantification of the probability of default: the rating is a relative opinion about creditworthiness, a ranking of obligors based on their credit risk. A rating grade or a risk class gathers clients having the same level of risk.

A.1.2. Scoring

Scoring and rating models have been used in the field of credit granting and credit risk management for long now. Initially, scoring models were used to assess the creditworthiness on potential clients of the bank as a decision tool for granting loans. In the 1980s, banks developped behavioral scores, which assessed the credit risk of existing clients, and were useful for risk management and marketing purposes. When the Basel 2 regulation came into force in 2007, behavioral scores became the cornerstone of the Internal Rating Based (IRB) approach for measuring capital allocation under that accord.

Rating and scoring have the same goal, i.e. assessing the risk of insolvency of a debt issuer or instrunment. The main difference between the two is that the rating process is based on a financial audit of the client, whereas the scoring process is more automatic. Scoring models take into account certain characteristics of the customer and of the loan (for instance, for a given borrower, real estate loans and consumer loans don't have the same risk drivers), which are integrated into a single number, the score. This process is mainly backward-looking and contains limited or no subjectivity.

Scoring models link the behavior (repayments, arrears, bank account information,...) or the features of clients to the defaults that the bank suffers over a one year period. Regression analysis, discriminant analysis, neural networks, and many other machine learning techniques are available for scoring. Scoring is a classification method where the inputs are the explicative factors, $F_1, ..., F_n$, and the output is a partition of the clients into two groups, the "goods" (G) and the "bads" (B). We refer to Thomas2000 and references therein for a survey on credit scoring. Many classification problems, in particular those for which the conditional distribution given "goods" and "bads" of the explicative factors is multivariate normal, reduce to a linear rule (see for a review Lachenbruch1975, Choi1986, Hand1981). In this case, the set of explicative factors for the goods is defined by:

$$A_G = \{(F_1, ..., F_n) | w_1 F_1 + w_2 F_2 + ... w_n F_n > c\}. \tag{23}$$

If we introduce the score function $s(F_1, ... F_n) = w_1 F_1 + w_2 F_2 + ... w_n F_n$, we reduce the classification problem from a problem with n dimensions to a problem with one dimension only. The value of the cut-off parameter c is the result of an optimization problem because a classification error generates losses: classifying a "good" as a bad and rejecting it generates a loss of profit, and classifying a "bad" as a good and accepting it generates a credit loss. The cut-off value is the one that minimizes the expected loss on the estimation sample. It can be easily shown that the optimal cut-value does not depend directly on the misclassification costs themselves but rather on the ratio of the misclassification costs (see Hand (2009)).

Computing the weights in the score function is a classical optimization problem addressed for instance in Fisher (1936). Discriminant analysis has been applied by Altman in the field of corporate bankruptcies (see Altman (1968, 1981, 2010). Following Fisher's method, the estimation of the score function is equivalent to maximizing the variance between the groups and minimizing the variance within individual groups.

For single borrowers, the default probability cannot be observed directly. For groups of borrowers, however, observed default rates are a proxy of the average default probability. We can link the individual default probability to the score:

$$p_i = L(s(F_1^i, ..., F_n^i)) \tag{24}$$

where $F_1^i, ..., F_n^i$ are the realization of the factors for client number i. The use of a logistic link function $L(\cdot)$ was first introduced by Wiginton (1980) in the context of credit scoring and then became very popular. The resulting logit model is defined by

$$p_i = \frac{1}{1 + e^{\square\square + s(F_1^i, ..., F_n^i)}} \tag{25}$$

The coefficient γ is estimated so that the average default probability on the sample is equal to the observed (or target) default rate of the portfolio. Another link function, the normal cumulative distribution function $N(\cdot)$, was suggested by Grablowsky (1981) in the framework of regression analysis. This is the probit model. The difference between logit and probit models is often negligible because the shapes of the associated link function are very close to each other.

A.1.3. Rating and scoring models performance

As mentioned above, a scoring model consists in ranking the loans of a portfolio. Some loans are assigned a low score and are going to default. Others with low scores are not going to default. The better the performance of the model, the better the ranking it generates compared to the observed defaults. We consider a homogeneous portfolio of loans, which means that the loans have the same risk drivers. These loans are granted to the same type of

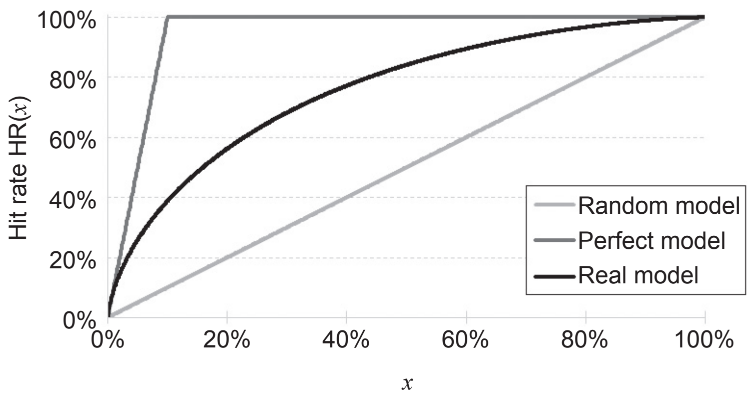
Appendix A (continued)

clients, in the same geographic area and belong to the same asset class (for instance prime residential mortgages in the UK originated by entity X of the bank). We call p the one year unconditional probability of default within the loan portfolio. We consider a rating model that produces a continuous score over the set of debtors in the portfolio. The higher the score related to a loan, the lower its probability of default. We rank the debtors respective to their creditworthiness, starting with those that have lowest score and going to those with the highest score.

Let's consider the fraction x of the debtors having the lowest scores. Among all the defaulters of the portfolio, we call "Hit Rate" $HR(x)$ is the proportion of defaulters that have been predicted correctly regarding a threshold on the score value equal to x.

The Cumulative Accuracy Profile curve (CAP curve; see figure 1) is obtained by plotting the function $HR(x)$ when x ranges from 0% to 100%. A perfect scoring model will assign the lowest scores to the defaulters. In this case, the CAP curve is increasing linearly for $x \leq p$ and remains equal to one for $x \geq p$. For a scoring model without any discriminatory power (random model), the fraction of all debtors with the lowest rating scores contains a fraction of defaulters equal to x, and the CAP curve is the diagonal of the unit square. Real rating systems lie somewhere between these two extremes. The quality of a rating system is measured by the accuracy ratio AR defined as the ratio of the area between the CAP curve of the real scoring model and the CAP curve of the random model, and the area between the CAP curve of the perfect scoring model and the CAP curve of the random model. This ratio ranges from 0% for the random model to 100% for the perfect model (see Engelmann et al., 2003).

Figure 2. CAP curve: Hit Rate as a function of *x*



We mention that the Accuracy ratio AR is criticized for being flawed, particularly when expressed in terms of misclassification costs, and that a more objective measure exists (see Hand (2009)). When reformulated in terms of misclassification cost, a perfect model corresponds to an expected misclassification loss equal to 0 whereas a random model corresponds to an expected misclassification loss being equal to its maximum value L_{Max}. For a realistic model, the expected misclassification loss L is between these two extreme values. Hand (2009) proposes to measure the model performance with the H-measure, defined as $H = 1 - L / L_{Max}$.

In the context of the IFRS 9 norm, Brunel (2015) uses these analytics and shows how the discriminatory power of the scoring model drives the size of the stage 2 portfolio and the IFRS 9 provision amount itself.

A.2. PD estimation methods

The parameters used to measure the ECL must be sensitive to the provisions' risk drivers. The IFRS 9 standard allows one to compute the parameters based on a statistical approach for instruments gathered into homogeneous portfolios of assets that share the same risk drivers, called segments. The estimation process leads to a set of identical parameters for all the instruments within a segment. Segmentation is a key feature of the provision model.

A.2.1. Overview of the approaches

PD estimation is one of the cornerstones of the quantification of IFRS 9 provision within a given segment. The requirement of the norm is quite heavy because PD estimations have to include the relevant information about the past, the current position in the credit cycle (Point-in-Time, "PIT") and any relevant information about the foreseeable future. PDs must reflect default risk over all time periods.

It is clear that market spreads natively encompass these requirements because market prices are forward-looking in essence. However, market spreads exist for large corporates only and don't exist for the majority of bank's clients or counterparties that don't have any listed debt instrument or credit derivative quoted on the market. This is the case for retail clients or small corporates for instance. Additionally, the estimation of PDs from market data would overestimate the ECL because market spreads are biased by liquidity premia.

It is now widely accepted that the IFRS 9 framework should be based upon the regulatory framework. This doesn't mean that the parameters are the same, in particular regarding PDs since regulatory PDs is expected to be the average of the default rate over an economic cycle (Through-the-Cycle, "TTC"), whereas IFRS 9 PDs must be PIT. As PDs are not directly observable and are not calibrated from market data, they are estimated in a portfolio approach, by considering homogeneous groups of loans or counterparties, the segments. This is similar to the the estimation of the Basel 2 parameters.

The mainstream approach is to consider risk classes or internal ratings as a starting point of the IFRS 9 framework for PD estimation. Two options are possible for PD estimation:

- option 1: estimation from observed rating migrations,
- option2: estimation from observed default rates within each risk class or rating level.

Appendix A (continued)

When choosing between option 1 and options 2, we have to distinguish between two types of portfolios, high default portfolios on one hand and low default portfolios on the other hand. High default portfolios are portfolios that contains a sufficient number of loans so that statistical analysis can be performed with a good degree of accuracy on default and rating migrations. Retail portfolios, small corporate portfolios, large corporate portfolios are all considered as high default portfolios, and both option 1 and options 2 are relevant because of the large quantity of available data. In the case when maturation effects are important (for some retail portfolios for instance), Markovianity or time homogeneity may be broken, and it is more relevant to estimate directly the term structures of PDs from observed default rates. Conversely, very large corporate portfolios, banks or sovereign portfolios generate very few default events, and they are called low default portfolios. In this case, estimation from observed default rates is not a relevant option because of the scarcity of data and estimation from observed rating migrations often leads to much robust results (see for instance Fuertes (2007)).

A.2.2. The PD term structure from the observed migrations

In this paragraph we describe the main lines of the method based on observed rating migrations, which is suitable for non retail portfolios or retail portfoliois based on the roll-rates. The most common approach to estimate migration matrices from observed rating migrations is the cohort method (see Schuermann (2005)), but is not very robust in practice, in particular for low default portfolios. Another approach, the continuous approach, leads to much robust results and is widely recognized to be the best estimation approach: "It is crucial to base the estimation of transition rates on continuously observed histories to get efficient estimates of transition rates. This point is particularly important when estimating rare events" (see Lando (2002)).

We assume that the data set includes all the observed rating migrationsbetween dates 0 and T. In the continuous approach, the waiting times between rating migrations (called "durations") follow an exponential distribution whose parameters are the elements of a generating matrix representing migration rates that we seek to estimate. In this paragraph, we follow the model described as in Brunel and Roger (2014). At time t, and for $i,j=\{1,\cdots,K\}$, the elements of the generating matrix $\lambda_t=\{\lambda_{i,j}(t)\}$ are the instantaneous migration rates from rating i to rating j, conditional to being in rating i at date t. The likelihood of the transition from rating i to rating j after a time interval t_m is equal to the probability that an issuer rated i remains in this class for a time period equal to t_m multiplied by the probability to jump to rating j at time t_m. This writes:

$$L(i,j,t_m)=e^{\int_0^{t_m}\lambda_{ii}(t)dt}\lambda_{ij}(t) \tag{26}$$

Similarly, the likelihood that an issuer with rating i stays in this rating class up to time t_m is equal to:

$$L(i,i,t_m)=e^{\int_0^{t_m}\lambda_{ii}(t)dt} \tag{27}$$

If we observe N_{ij} migrations from rating i to rating j over the full historical data set, each having occured by time t_m ($m=1\cdots N_{ij}$), and a number N_{ic} of censored durations from rating i at dates t_m ($m=1\cdots N_{ic}$), then we can write the total likelihood as the product of individual likelihoods of all issuers that passed in the rating class i in the data set. The relationship $\lambda_{ii}(t)=-\sum_{j\neq i}\lambda_{ij}(t)$ leads to:

$$L_i=\left[\prod_{j\neq i}\prod_{m=1}^{N_{ij}}\lambda_{ij}(t)e^{-\int_0^{t_m}\sum_{j\neq i}\lambda_{ij}(t)dt}\right]\prod_{m=1}^{N_{ic}}e^{-\int_0^{t_m}\sum_{j\neq i}\lambda_{ij}(t)dt} \tag{28}$$

Under the assumption of time homogeneity $\lambda_{ij}(t)=\lambda_{ij},\forall i,j=1\cdots K$, the estimator of the instantaneous migration rate from rating i to rating j is obtained by maximizing the above likelihood function with respect to each of the λ_{ij}:

$$\hat{\lambda}_{ij}=\frac{N_{ij}(T)}{\sum_{m=1}^{N_i}t_m} \tag{29}$$

where $N_{ij}(T)$ is the number of observed migrations from rating i to rating j between dates 0 and T, and $N_i=\sum_j N_{ij}(T)+N_{ic}$ is the number of migrations starting from rating i between dates 0 et T.

This approach leads to a one year TTC migration matrix $\exp(\widehat{\Lambda})$, and PD curves by raising the matrix to the power of time horizon. This is the starting point from which one can estimate PIT forward-looking PD curves as required by the IFRS 9 norm. PIT migration rates are time dependent because of variations of the macroeconomic conditions and they can be made forward-looking by taking into account some forecasting of the macroeconomic factors into the estimation process. The credit cycle is represented by a continuous time process $(F_t)_{t\geq 0}$ where the date $t=0$ is the starting point of the historical data. We assume, as in Vasicek's model (Vasicek, 2007), that a rating migration or a default for obligor number l between date t and date t +1 year is driven by a standard normal random variable defined by

$$R_{l,t}=\sqrt{\rho}F_t+\sqrt{1-\rho}\,\varepsilon_{l,t} \tag{30}$$

where F_t and $\varepsilon_{l,t}$ are independent standard normal variables. The transition from rating i to a rating equal or lower than j for obligor l occurs when $R_{l,t}$ is lower than a threshold $s_{i,j}$ which is linked to the TTC transition probability from rating i to a rating equal or lower than j. If we denote $p_{i,j}$ this probability, we have:

$$s_{i,j}=N^{-1}\left(p_{i,j}\right), \tag{31}$$

Appendix A (continued)

where N denotes the standard normal cumulative distribution function. Conditional to the value of the systemic factor F_t, the average migration rate from rating i to rating j over one year is then equal to:

$$M_{i,j}^{F_t,\rho} = N\left(\frac{N^{-1}\left(p_{i,j-1}\right) - \sqrt{\rho}F_t}{\sqrt{1-\rho}}\right) - N\left(\frac{N^{-1}\left(p_{i,j}\right) - \sqrt{\rho}F_t}{\sqrt{1-\rho}}\right). \tag{32}$$

The systemic factors $\left(F_t\right)_{t\geq 0}$ and the correlation parameter ρ are estimated by maximizing the likelihood of observed migrations:

$$L(\rho, F_0, F_1, ..., F_T) = \prod_{i=1}^{D-1}\prod_{j=1}^{D}\prod_{t=0}^{T-1}\left(M_{i,j}^{F_t,\rho}\right)^{n_{i,j,t}} \tag{33}$$

where D is the number of rating grades including default and $n_{i,j,t}$ is the number of observed migration from rating i to rating j between time t and time $t+1$. A forward-looking measure of PDs requires to have a view on the values of the systemic factor for dates $t > T$. This can be obtained by assuming a functional dependence between this macroeconomic factor and some macroeconomic variables (GDP growth, unemployment rate, interest rates...) on which economists may express some forecasts.

The term structure of PDs is obtained by multiplying the resulting transition matrices. The estimated correlation parameter, called $\widehat{\rho}$, is assumed to be constant over the estimation period and over the future horizons. The dynamics of the systemic factor is assumed to be linked to the dynamics of macroeconomic factors via an econometric relationship which can be written in a discrete time framework as:

$$F_{t+1} = a.F_t + \sum_i \beta_i.Y_t^i + \varepsilon_t \tag{34}$$

where the Y_t^i are the macroeconomic factors at time t (for instance GDP growth, unemployement rate variation...) and $(\varepsilon_t)_{t\geq 0}$ is a white noise process. Let's assume that we are at time t. When the economists provide forecasts for macroeconomic factors over the next few years, *i.e.* they provide the values for Y_t^i, Y_{t+2}^i, ... Y_{t+k}^i, then eq. (34) provide forecasts for the systemic factor at all future dates up to date $t+k$. The PIT forward-looking PDs at horizon k are given by the last column of the cumulative transition matrix over the time period going from date T to date $T+k$, equal to:

$$M(T, T+k) = M^{F_T,\widehat{\rho}} \times M^{F_{T+1},\widehat{\rho}} \times \cdots \times M^{F_{T+k-1},\widehat{\rho}} \tag{35}$$

This approach describes the general formalism to estimate TTC parameters and adjust them to get PIT forward-looking parameters. It can be extended to the estimation of PD curves directly from obseverd defaults, but this would require a global review of all the approaches to estimate PD curves which is out of the scope of this section devoted to the mathematical toolbox.

A.3. Reduced-Form Credit Risk Modeling

This section, based on Crépey, Bielecki, and Brigo (2014, Sect. 13.7), gives mathematical tools underlying the so-called reduced-form intensity credit risk modeling approach that grounds the intensity-based CVA formula (22). Given a $[0,T]\cup\{+\infty\}$-valued stopping time τ without atom on $[0,T]$, let $J_t = 1_{\{\tau>t\}}$ denote the related survival indicator and let $\bar{\tau} = \tau \wedge T$. We assume further that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where the filtration $\mathbb{H}$ is generated by the process J and where $\mathbb{F}$ is some reference filtration. The Azéma supermartingale associated with τ is the $[0,1]$ valued process Z defined, for $t \in [0,T]$, by

$$Z_t = \mathbb{E}^{\mathbb{Q}}(J_t|\mathcal{F}_t). \tag{36}$$

Assuming a positive $Z_t =: e^{-\Gamma_t}$, where Γ is called the hazard process, we have the following "key lemma" of single-name credit risk (see e.g. page 143 of Bielecki and Rutkowski (2001)).

Lemma A.1. If ξ is an integrable random variable, then

$$J_t\mathbb{E}^{\mathbb{Q}}[\xi \mid \mathcal{G}_t] = J_t\frac{\mathbb{E}^{\mathbb{Q}}(\xi J_t|\mathcal{F}_t)}{\mathbb{Q}(\tau > t|\mathcal{F}_t)} = J_t e^{\Gamma_t}\mathbb{E}^{\mathbb{Q}}(\xi J_t|\mathcal{F}_t). \tag{37}$$

For ξ of the form $J_s\chi$, for some $\mathcal{F}_s$-measurable χ with $s \geq t$, we have:

$$\mathbb{E}^{\mathbb{Q}}[J_s\chi \mid \mathcal{G}_t] = J_t e^{\Gamma_t}\mathbb{E}^{\mathbb{Q}}(\chi J_s|\mathcal{F}_t) = J_t\mathbb{E}^{\mathbb{Q}}(\chi e^{-(\Gamma_s-\Gamma_t)}|\mathcal{F}_t). \tag{38}$$

Proof. The left-hand side in (37) (where the right-hand side is only notational) results from the fact that, on $\{\tau > t\}$, the σ-field $\mathcal{G}_t$ is generated by $\mathcal{F}_t$ and the random variable $\{\tau > t\}$. In (38), the left-hand side follows by an application of (37) to $\xi = J_s\chi$; the right-hand side then results from the tower law by taking an inner conditional expectation with respect to $\mathcal{F}_s$. ■

In particular, (38) with $\chi = e^{\Gamma_s}$ proves that the process $X_t = J_t e^{\Gamma_t}$ is a $\mathbb{G}$-martingale, since for $s \geq t$:

$$\mathbb{E}^{\mathbb{Q}}[J_s e^{\Gamma_s} \mid \mathcal{G}_t] = J_t\mathbb{E}^{\mathbb{Q}}(e^{\Gamma_s}e^{-(\Gamma_s-\Gamma_t)} \mid \mathcal{F}_t] = J_t e^{\Gamma_t}.$$

Appendix A (continued)

Lemma A.2. For any $\mathbb{G}$ -adapted, respectively $\mathbb{G}$ -predictable, process Y, there exists a unique $\mathbb{F}$ -adapted, respectively $\mathbb{F}$ -predictable, process $\widetilde{Y}$ such that $JY = J\widetilde{Y}$, respectively $J_{-}Y = J_{-}\widetilde{Y}$.

Proof. In view of (37), we can take, in the adapted case, $\widetilde{Y}_t = e^{\Gamma_t}\mathbb{E}^{\mathbb{Q}}(Y_t J_t|\mathcal{F}_t)$. For the predictable case see §75, page 186 in Dellacherie et al. (1992) and Proposition 9.12 in Nikeghbali (2006). ■

Further assuming the process Z continuous and nonincreasing, letting $M_t = -(J_t + \Gamma_{t\wedge\tau})$, we have that $dX_t = -X_{t-}dM_t$ and therefore $dM_t = -e^{\Gamma_t}dX_t$, so that M also is a $\mathbb{G}$ -martingale. Moreover:

Lemma A.3. Under the above assumptions: **(i)** An $\mathbb{F}$ -martingale stopped at τ is a $\mathbb{G}$ -martingale.
(ii) An $\mathbb{F}$ -adapted càdlàg process cannot jump at τ.

Proof. **(i)** Since τ has a positive, continuous and nonincreasing Azéma supermartingale, it is known from Elliot et al. (2000) that an $\mathbb{F}$ -martingale stopped at τ is a $\mathbb{G}$ -martingale.

(ii) As Z is continuous, τ avoids $\mathbb{F}$ -stopping times, i.e. $\mathbb{Q}(\tau = \sigma) = 0$ for any $\mathbb{F}$ -stopping time σ (see for instance Nikeghbali (2006)). Moreover, by Theorem 4.1, page 120 in He et al. (1992), there exists a sequence of $\mathbb{F}$ -stopping times exhausting the jump times of an $\mathbb{F}$ -adapted càdlàg process. ■

Letting $\beta_t = e^{-\int_0^t r_s ds}$ denote the discount factor at some $\mathbb{F}$ -progressively measurable risk-free rate r, we model the cumulative discounted future cash flows of a defaultable claim in the form of the $\mathcal{G}_{\bar{\tau}}$ -measurable random variable π^t defined, at any $t \in [0, T]$, by

$$\beta_t \pi^t = \int_t^{\bar{\tau}} \beta_s f_s ds + \beta_\tau (1_{\{t<\tau<T\}} R_\tau + 1_{\tau>T}\xi), \tag{39}$$

for some $\mathbb{F}$ -progressively measurable dividend rate process f, some $\mathbb{F}$ -predictable recovery process R and some $\mathcal{F}_T$-measurable payment at maturity (random variable) ξ. Note that the assumption that the data r_t, f, R and ξ are in $\mathbb{F}$ is not restrictive in view of Lemma 5.2.
Now, assuming Z_t time-differentiable, we define the hazard intensity γ and the credit-risk-adjusted-discount-factor α as the $\mathbb{F}$ -adapted processes defined, for $t \in \mathbb{R}_+$, by

$$\gamma_t = -\frac{d \ln Z_t}{dt} = \frac{d\Gamma_t}{dt}, \alpha_t = \beta_t \exp(-\int_0^t \gamma_s ds) = \exp(-\int_0^t (r_s + \gamma_s) ds). \tag{40}$$

The next result shows that the computation of conditional expectations of cash flows π^t with respect to $\mathcal{G}_t$ can be reduced to the computation of conditional expectations of "$\mathbb{F}$ -equivalent" cash flows $\tilde{\pi}^t$ with respect to $\mathcal{F}_t$.

Lemma A.4. We have

$$\mathbb{E}(\pi^t|\mathcal{G}_t) = J_t \mathbb{E}(\tilde{\pi}^t|\mathcal{F}_t),$$

where $\tilde{\pi}^t$ is given, with $g = f + \gamma R$, by

$$\alpha_t \tilde{\pi}^t = \int_t^T \alpha_s g_s ds + \alpha_T \xi. \tag{41}$$

Proof. Since $M_t = -(J_t + \Gamma_{t\wedge\tau})$ is a $\mathbb{G}$ -martingale,

$$\mathbb{E}^{\mathbb{Q}}[1_{\{t<\tau<T\}}\beta_\tau R_\tau \mid \mathcal{G}_t] = -\mathbb{E}^{\mathbb{Q}}[\int_t^T \beta_s R_s dJ_s \mid \mathcal{G}_t] = \int_t^T \mathbb{E}^{\mathbb{Q}}[\beta_s R_s J_s \gamma_s \mid \mathcal{G}_t] ds.$$

The proof is concluded by repeated applications of (38). ■

Hence, the valuation of defaultable claims can be handled in essentially the same way as default-free claims, provided the default-free discount factor process β is replaced by a credit risk adjusted discount factor α and a fictitious dividend continuously paid at rate γ is introduced to account for recovery on the claim upon default (note that a "default-free" discount factor β can itself be interpreted in terms of a default risk with "intensity" r_t).