Hedging Valuation Adjustment and Model Risk

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Abstract

We revisit Burnett (2021); Burnett and Williams (2021)'s notion of hedging valuation adjustment (HVA) in the direction of model risk. The resulting HVA can be seen as the bridge between a global fair valuation model and the local models used by the different desks of the bank. Moreover, model risk and dynamic hedging frictions indeed deserve a reserve, but a risk-adjusted one, so not only an HVA, but also a contribution to the KVA of the bank. We also argue that the industry-standard XVA metrics are jeopardized by cash flows risk, which is in fact of the same mathematical nature than the one regarding pricing models, although at the higher level of aggregation characteristic of XVA metrics.

1 Introduction

The 2008 global financial crisis triggered a shift from trade-specific pricing to netting-set CVA analytics. For tractability reasons, the market models used by banks for their CVA analytics are simpler than the ones that they use for individual deals. Given this coexistence of models, it is no surprise if FRTB emphasized the issue of model risk.

In the context of structured products, Albanese, Crépey, and Iabichino (2021) introduced the notion of Darwinian model risk, whereby the trader of a bank prefers to a reference fair valuation model an alternative pricing model, which renders a trade more competitive (more attractive for clients) in valuation terms. The trader thus closes the deal, at some valuation loss, but the latter is more than compensated by gains on the hedging side of the position. However these overall positive gains on the product and its hedge are only a short to medium term view. In the long run, large losses are

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incurred by the bank when market conditions reveal the unsoundness of the trader's pricing and hedging model.

Typically[1] model risk is accounted for by setting aside as a reserve the difference between the valuation computed with good models minus the valuation given by bad models, which indeed corresponds to the first layer of defense against model risk in this paper. However, under current market practice, no model risk provision is linked to erroneous hedges. Model risk reserves that are predicated on differences in valuations have no effect on risk capital and risk exposures, which instead are a function of hedge ratios. Regulators have attempted to remedy this shortcoming. We already mentioned FRTB, which insists that models used for risk management must be the exact same used to keep books and records. There is also Volker's prohibition of proprietary trading. But both requirements can be skirted by using low quality models.

Contribution We propose to revise model risk reserves by adding an add-on sensitive to hedge ratios. Toward this aim, we put Darwinian model risk in the XVA perspective of (Albanese, Crépey, Hoskinson, and Saadeddine, 2021). This leads us to propose a reserve for model risk and dynamic hedging frictions, in two parts: an HVA component, encapsulating Burnett (2021); Burnett and Williams (2021) within a broader model risk perspective, restoring the correct prices that should have been used by the trader in the first place. This HVA can be seen as the bridge between a global fair valuation model and the local models used by the different desks of the bank. But the reserve should also be risk-adjusted, via a KVA component, i.e. a related contribution to the KVA of the bank.

Related literature Detering and Packham (2016) already insisted on "capital buffers necessary to sufficiently protect trading book positions against unexpected losses from model risk". However this was mostly envisioned at the level of individual deals. In this paper we also provide the global picture after aggregation throughout all deals and inclusion of hedging nonlinear imperfections at the hedging sets level, before capital at risk implications are eventually assessed at the balance-sheet level. We refer the reader to Detering and Packham (2016) for a discussion of the literature, including (Karoui, Jeanblanc-picqué, and Shreve, 1998; Cont, 2006; Elices and Giménez, 2013), and for model risk regulatory guidelines, until 2014. More recent references include Barrieu and Scandolo (2015), who "introduce three quantitative measures of model risk when choosing a particular reference model within a given class", and Farkas, Fringuellotti, and Tunaru (2020), who "propose a general method to account for model risk in capital requirements calculus related to market risk".

On the XVA side, Bichuch, Capponi, and Sturm (2020) deal with XVAs that are conservative, or "robust" in the sense of superhedging with respect to the uncertainty of the credit spread of a client. Regarding our baseline cost-of-capital XVA approach, they write in their introduction: "Despite the merits of this approach, in particular not having to rely on replication arguments for the value adjustments, it makes two critical assumptions. First, it assumes that the counterparty-free payoffs of the contract are perfectly replicated, rather than designing the replication strategy from first principles

[1]see e.g. (European Parliament, 2016, L 21/54, point (2)).

(and ignoring potential interaction of risk factors). Second, and most importantly, they assume that the historical and risk-neutral probability measure coincide. This, of course, exposes the calculation of the valuation adjustments to a substantial amount of model risk, which can be accounted for by the techniques proposed in this paper." The present paper provides an element of answer to their first point[2]. As for their second point, we no longer assume (as in early stages of our cost-of-capital XVA theory) that the historical and risk-neutral probability measure coincide: see Definition 2.1. Even then, there is of course still an XVA model risk and uncertainty issue. But to address the latter, we prefer to a worst-case approach the Bayesian-robust one alluded to in Remark 2.6, which we think is more scalable[3] and does better justice to considerations of model realism[4].

More fundamentally, at the economic level, a key difference between the XVA approaches underlying (Bichuch, Capponi, and Sturm, 2020) and our cost-of-capital XVA approach, even before model risk is introduced, is that they view XVAs as replication prices, whereas we say CVA and (especially) FVA cannot be replicated, hence the ensuing pnl deserves capital at risk, the KVA cost of which plays a key role[5] in the overall structure. This divergence then also impacts the treatment of model risk. Regarding for instance model risk on the CVA, an important point of this paper is thus that, even if the correct CVA value is restored by boosting the default probability of the client (if too small in the first place), then the CVA hedge is still wrong and this should be reflected in the KVA. Regarding the FVA (which we say cannot seriously be hedged), our emphasis is on cash flows risk, even prior to model risk. Cash flows risk is easily integrable to our HVA setup, but seems less amenable to a robust approach. Note that the robust FVA embedded in Bichuch, Capponi, and Sturm (2020) is still implicitly[6] a simplified specification additive across counterparties, like (from this viewpoint) the FCA and FBA discussed in Section 4.1.

As Bichuch, Capponi, and Sturm (2020), Silotto, Scaringi, and Bianchetti (2021) is also about XVA valuation uncertainty linked to the existence of a range of different parameterizations and it also ignores the intrinsic non-replicability of counterparty credit risk. The latter is manifested by their emphasis on the DVA metric, whereas, accounting for the counterparty credit risk incompleteness, the DVA cannot be monetized by the bank shareholders and should therefore not be considered in financial derivatives entry prices (Albanese et al., 2021, Section 3.5). The contribution of (Silotto, Scaringi, and Bianchetti, 2021) is more on the modeling and implementation sides, with the proposal of a generalized G2++ model to multi-curve framework also allowing for the time dependency of volatility parameters for both risk factor dynamics and pricing formulas for swaps and swaptions underlying uncollateralized CVA/DVA formulas, extended further to variation and initial margin. The model risk AVA formula (Silotto

[2]see before Remark 2.8.

[3]can more realistically be applied to model parameterizations at the level of the derivative portfolio of a bank.

[4]when a robust approach is typically unnecessarily over-conservative, hence unusable in practice, let aside the hopeless computational issues at large scale.

[5]often even predominant, see e.g. Figure 3.1.

[6]as their XVA setup involves a single client, which, unless the portfolio XVA numbers can be retrieved by addition across counterparties, is at odds with the reality of a banking portfolio involving thousands of them.

et al., 2021, (5.10)) is expressed as the difference between the XVA (CVA or DVA, in their case) obtained in their baseline framework optimizing the trade-off between accuracy and performance and a 10th percentile XVA value[7] across a whole collection of XVA frameworks. This is a model risk AVA in the direct line of the regulation European Parliament (2013, 2016). Our prospective[8] AVA formula (2.19) obviously takes more freedom with the latter.

Finally, Singh and Zhang (2019b,a) study XVA uncertainty in the robust[9] Wasserstein sense, computing worst-case values in an uncertainty set of probability measures given by a Wasserstein ball around a reference measure. However, their approach, based on infinite dimensional Lagrangian duality results, is only presented in a discrete time setting and, more importantly, for a finitely supported reference probability measure. In addition, as discussed in (Singh and Zhang, 2019b, Section 2.2.4), they disregard no arbitrage drift conditions.

Outline of the paper Section 2 casts the HVA and the related risk adjustment in a global valuation framework à la Albanese, Crépey, Hoskinson, and Saadeddine (2021), also encompassing mark-to-market valuation (MtM), CVA, FVA, and KVA. Sections 3 and 4 illustrate the MtM and XVA sides of the HVA topic. Sections A, B and C recapitulate the baseline cost-of-capital XVA approach of Albanese, Crépey, Hoskinson, and Saadeddine (2021) on which this paper is rooted, establish the pricing analytics in the Merton jump-to-ruin model used in Section 3, and derive the transaction costs component of the HVA in a delta-hedged jump-diffusion setup.

Standing notation All processes are adapted to the filtration $\mathfrak{F} = (\mathfrak{F}_t)$ of a reference stochastic basis. The risk-free asset chosen as a numéraire everywhere. We denote by $\mathcal{N}$, the standard normal cumulative distribution function; $\overline{T}$, a bound on the final maturity of the bank portfolio[10], also including the time (assumed bounded) of liquidating defaulted positions; $\boldsymbol{\delta}_\vartheta$, a Dirac measure at a stopping time ϑ; X^ϑ, a process X stopped at time ϑ; Q_t and q_t, a reference fair valuation as per Definition 2.2 vs. a trader time-t price of a financial claim (cumulative cash flow stream) of interest; $X_{(0)} = X - X_0$, for any process X.

We use bold letters for cash flows and straight letters for prices (uppercase and lowercase for fair and trader valuations, respectively).

2 The Global Valuation Framework

In the incomplete market setup intrinsic to the XVA issue (Albanese et al., 2021, Section 3.5), our reference probability measure $\mathbb{R}$ (like "regulatory") is the hybrid of pricing and physical probability measures advocated in Albanese, Crépey, Hoskinson, and Saadeddine (2021, Remark 2.3):

[7] ensuring that one can exit the position at corresponding price with a degree of certainty equal to or larger than 90%.

[8] also a model risk AVA, but in a different sense as developed in the above.

[9] again, deserving the same related comments as above.

[10] assessed on a run-off basis as relevant for XVA computations (Albanese et al., 2021, Section 4.2).

Definition 2.1. *Let there be given a σ-field $\mathcal{A}$, on which the physical probability measure $\mathbb{P}$ is defined, and a financial sub-σ-field $\mathcal{B}$ of $\mathcal{A}$, on which a risk-neutral measure $\mathbb{Q}$, equivalent to the restriction to $\mathcal{B}$ of the physical probability measure, is defined. Our probability measure $\mathbb{R}$ in the paper is the uniquely defined probability measure on $\mathcal{A}$, provided by Artzner, Eisele, and Schmidt (2020, Proposition 2.1), such that (i) $\mathbb{R}$ coincides with $\mathbb{Q}$ on $\mathcal{B}$ and (ii) $\mathbb{R}$ and $\mathbb{P}$ coincide conditionally on $\mathcal{B}$.*

More precisely, we work throughout the paper under the *bank survival probability measure* associated with $\mathbb{R}$[11], with related time-t expectation, value-at-risk[12], and expected shortfall [13] denoted by $\mathbb{E}_t$, $\mathbb{V}a\mathbb{R}_t$ and $\mathbb{ES}_t$ (and for $t = 0$, we drop all indices t).

All (cumulative) cash flows are finite variation processes (starting from 0) and all prices are special semimartingales in a càdlàg version.

Definition 2.2. *Given an optional, integrable process $\mathcal{Y}$ stopped at $\overline{T}$[14], its value process $Y = va(\mathcal{Y})$ is the optional projection of $(\mathcal{Y}_{\overline{T}} - \mathcal{Y})$, i.e.*

$$Y_t = \mathbb{E}_t(\mathcal{Y}_{\overline{T}} - \mathcal{Y}_t),\ t \leq \overline{T}, \tag{2.1}$$

and Y vanishes on $[\overline{T}, +\infty)$.

In particular, $(\mathcal{Y} + Y)$ is a martingale on $[0, \overline{T}]$.

The global valuation approach of Albanese, Crépey, Hoskinson, and Saadeddine (2021) can then be summarized as follows. $\mathrm{MtM} = va(\mathcal{M})$, $\mathrm{CVA} = va(\mathcal{C})$ and $\mathrm{FVA} = va(\mathcal{F})$ are the value processes (2.1) of the cash flows impacting the eponymous desks of the bank, namely the cash flow processes $\mathcal{M}$ to the trading desks, $\mathcal{C}$ from the CVA desk and $\mathcal{F}$ from the FVA desk[15]. The ensuing loss process of the bank is then given by[16]

$$\mathcal{L} = -(\mathcal{M} + \mathrm{MtM}_{(0)}) + \mathcal{C} + \mathcal{F} + \mathcal{H} + \mathrm{CA}_{(0)}, \tag{2.2}$$

where

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} \tag{2.3}$$

and the zero-valued martingale $\mathcal{H}$ represents the dynamic hedging losses of the bank. ***In the extension of the theory provided by the present paper, the (no longer martingale) process $\mathcal{H}$ additionally accounts for model risk and dynamic hedging frictions, and* (2.3) *is generalized as***

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{HVA}\text{, } \textit{where } \mathrm{HVA} = va(\mathcal{H}). \tag{2.4}$$

[11] see Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 4) and Crépey and Song (2017, Section 4.2) and see also (Crépey et al., 2020, Section 5) for a practically equivalent reduction of filtration viewpoint.

[12] lower quantile at some given confidence level $\alpha \in (\frac{1}{2}, 1)$

[13] of a loss ℓ: expectation of ℓ given ℓ exceeds the corresponding value-at-risk of ℓ, cf. (A.1).

[14] cumulative cash flow stream stopped at the final maturity of the portfolio in the financial interpretation.

[15] An amount paid means effectively paid if positive, received if negative. A similar convention applies to the notions of loss and gain or cost and benefit.

[16] recall the notation $X_{(0)} = X - X_0$.

The theory then proceeds as in Albanese, Crépey, Hoskinson, and Saadeddine (2021). Via compensation by the CA process[17], the loss process $\mathcal{L}$ is a martingale. As detailed by Definitions A.1 and A.2, the KVA process is then designed so as to turn the dividend process

$$-(\mathcal{L} + \mathrm{KVA}_{(0)})$$

of the bank shareholders into a submartingale with drift coefficient rSCR, for some nonnegative constant hurdle rate r (e.g. 10%), where the shareholder capital at risk process (SCR) is sized dynamically based on the economic capital (EC) of the bank, i.e. the conditional expected shortfall of the increment of $\mathcal{L}$ over the next year.

2.1 HVA for Mark-to-Market

We index by "$\cdot$" the client originating deals of the bank, which may be hedged statically with other banks[18] and/or dynamically through exchanges. At the time of explosion (or "model switch") of the trader's strategy introduced in Section 1, the valuation of the deal and its hedge in the trader's pnl respectively pass from local to fair valuation. This justifies the following expression for the raw pnl of the trader, where "raw" is in reference to the fact that this pnl still ignores the to-be-defined HVA liability, as well as extra contributions[19] that can only be addressed at the portfolio level:

Definition 2.3. *The raw pnl of the client originating deal "$\cdot$" of the bank, with maturity $T^{\cdot}$, is given by*

$$pnl^{\cdot} = \big(\mathcal{Q}^{\cdot} + J^{s,\cdot} q^{\cdot} + (1 - J^{s,\cdot})Q^{\cdot}\big)^{\tau^{\cdot}}_{(0)} - \big(\mathcal{P}^{\cdot} + J^{s,\cdot} p^{\cdot} + (1 - J^{s,\cdot})P^{\cdot}\big)^{\tau^{\cdot}}_{(0)} - h^{\cdot}, \tag{2.5}$$

where $\mathcal{Q}^{\cdot}$ represents the cash flows contractually promised to the bank through the deal, $Q^{\cdot} = va(\mathcal{Q}^{\cdot})$[20], $\mathcal{P}^{\cdot}$ and $P^{\cdot}$ are the analogous quantities regarding a static hedge component of the deal, $\tau^{\cdot} = \tau_d^{\cdot} \wedge \tau_e^{\cdot}$ in which $0 < \tau_e^{\cdot} \leq T^{\cdot}$ is a deactivation time[21] and $\tau_d^{\cdot} > 0$ is the time of default of the client of the deal, $q^{\cdot}$ and $p^{\cdot}$ are the prices of the product and of its static hedge in the local model of the trader, $h^{\cdot}$ is the dynamic hedging loss of the trader, and $J^{s,\cdot} = \mathbb{1}_{[\![0,\tau_s^{\cdot}[\![}$, where $\tau_s^{\cdot} > 0$ is the time of model switch.

Remark 2.1. *The bank may consider liquidating the deal at $\tau_s^{\cdot}$ when earlier than $\tau^{\cdot}$. To render this case, one just needs to redefine $\tau^{\cdot}$ as $\tau_d^{\cdot} \wedge \tau_e^{\cdot} \wedge \tau_s^{\cdot}$ (instead of $\tau_d^{\cdot} \wedge \tau_e^{\cdot}$).*

Remark 2.2. *We could also consider American claims with exercise times possibly $< T^{\cdot}$ under the control of the bank and/or client, in which case $\tau_e^{\cdot}$ in $\tau^{\cdot} = \tau_d^{\cdot} \wedge \tau_e^{\cdot}$ (or $\tau_d^{\cdot} \wedge \tau_e^{\cdot} \wedge \tau_s^{\cdot}$ as above) should be understood as the corresponding exercise time. Further adjustments are then required to deal with possibly unoptimal stopping by the bank[22]. Callability by the bank is actually the source of the model risk in Albanese, Crépey, and Iabichino (2021). In this paper American early exercise features are ignored to alleviate the setup.*

[17] cf. the sentence following Definition 2.2.

[18] directly or via CCPs (Albanese, Armenti, and Crépey, 2020).

[19] to be introduced later.

[20] cf. (2.1).

[21] possibly $< T^{\cdot}$ in the case of products with knock-out features.

[22] unoptimal stopping by the client can be conservatively ignored in the modeling.

The process $h^{\cdot}$ is meant for standard dynamic hedging cash flows ignoring nonlinear frictions such as transaction costs, which can only be assessed at the portfolio level and will therefore be added later[23]. The raw pnl ignores likewise the to-be-defined HVA liability, starting with:

Definition 2.4. $\mathrm{HVA}^{\cdot} = -va(pnl^{\cdot})$.

Lemma 2.1. *We have*

$$\mathrm{HVA}^{\cdot} = va(\mathcal{H}^{\cdot}), \tag{2.6}$$

where

$$\begin{aligned}\mathcal{H}^{\cdot} &= (\mathcal{Q}^{\cdot} + Q^{\cdot})^{\tau^{\cdot}}_{(0)} - (\mathcal{P}^{\cdot} + P^{\cdot})^{\tau^{\cdot}}_{(0)} - pnl^{\cdot} &(2.7)\\ &= \big(J^{s,\cdot}\big(Q^{\cdot} - q^{\cdot} - (P^{\cdot} - p^{\cdot})\big)\big)^{\tau^{\cdot}}_{(0)} + h^{\cdot}. &(2.8)\end{aligned}$$

Proof. In view of (2.5) and (2.7), we have (2.8). Moreover, by the observation following Definition 2.2, $(\mathcal{Q}^{\cdot} + Q^{\cdot})^{\tau^{\cdot}}_{(0)}$ and $(\mathcal{P}^{\cdot} + P^{\cdot})^{\tau^{\cdot}}_{(0)}$ are zero-valued martingales. Hence (2.6) proceeds from Definition 2.4. ∎

Remark 2.3. *In the "continuously recalibrated case" $p^{\cdot} = P^{\cdot}$, the static side of the hedge does not impact $\mathcal{H}^{\cdot}$*[24] *nor* $\mathrm{HVA}^{\cdot}$*.*

Hereafter we postulate that, as natural in view of their financial interpretation:

Assumption 2.1. *All prices $Q^{\cdot}, q^{\cdot}, P^{\cdot}, p^{\cdot}$ share a nil terminal condition at time $T^{\cdot}$ and $h^{\cdot}$ is an $\mathbb{R}$ martingale stopped at $\tau^{\cdot}$.*

Proposition 2.1. *We have*

$$\begin{aligned}&\mathrm{HVA}^{\cdot} = \big(J^{s,\cdot}\big(P^{\cdot} - p^{\cdot} - (Q^{\cdot} - q^{\cdot})\big)\big)^{\tau^{\cdot}} - D^{\cdot},\\ &\mathcal{H}^{\cdot} + \mathrm{HVA}^{\cdot}_{(0)} = h^{\cdot} - D^{\cdot}_{(0)},\end{aligned} \tag{2.9}$$

where[25]

$$D^{\cdot}_t = \mathbb{E}_t\Big(\mathbb{1}_{\{\tau^{\cdot} < \tau^{\cdot}_s \wedge T^{\cdot}\}}\big(P^{\cdot} - p^{\cdot} - (Q^{\cdot} - q^{\cdot})\big)_{\tau^{\cdot}}\Big),\ t \geq 0. \tag{2.10}$$

In particular,

$$\mathrm{HVA}^{\cdot}_0 = \big(P^{\cdot} - p^{\cdot} - (Q^{\cdot} - q^{\cdot})\big)_0 - \mathbb{E}\Big(\mathbb{1}_{\{\tau^{\cdot} < \tau^{\cdot}_s \wedge T^{\cdot}\}}\big(P^{\cdot} - p^{\cdot} - (Q^{\cdot} - q^{\cdot})\big)_{\tau^{\cdot}}\Big). \tag{2.11}$$

Proof. Under Assumption 2.1, (2.6)–(2.8) yield (2.9)–(2.11). ∎

[23] see Section 2.3.
[24] cf. (2.8).
[25] we use D for "Darwinian".

Corollary 2.1. *In the pure static hedging case where* $h^{\cdot}=0$*,* (2.5)–(2.11) *reduce to*

$$\begin{aligned}
&pnl^{\cdot}=\big(\mathcal{Q}^{\cdot}+J^{s,\cdot}q^{\cdot}+(1-J^{s,\cdot})Q^{\cdot}\big)_{(0)}^{\tau^{\cdot}}-\big(\mathcal{P}^{\cdot}+J^{s,\cdot}p^{\cdot}+(1-J^{s,\cdot})P^{\cdot}\big)_{(0)}^{\tau^{\cdot}},\\
&\mathcal{H}^{\cdot}=\big(J^{s,\cdot}\big(Q^{\cdot}-q^{\cdot}-(P^{\cdot}-p^{\cdot})\big)\big)_{(0)}^{\tau^{\cdot}}\\
&\mathrm{HVA}^{\cdot}=\big(J^{s,\cdot}\big(P^{\cdot}-p^{\cdot}-(Q^{\cdot}-q^{\cdot})\big)\big)^{\tau^{\cdot}}-D^{\cdot},\\
&\qquad\mathrm{HVA}^{\cdot}_0=\big(P^{\cdot}-p^{\cdot}-(Q^{\cdot}-q^{\cdot})\big)_0-\mathbb{E}\Big(\mathbb{1}_{\{\tau^{\cdot}<\tau_s^{\cdot}\wedge T^{\cdot}\}}\big(P^{\cdot}-p^{\cdot}-(Q^{\cdot}-q^{\cdot})\big)_{\tau^{\cdot}}\Big),
\end{aligned}\tag{2.12}$$

whereas, in the pure dynamic hedging case where $P^{\cdot}=p^{\cdot}=0$*,*

$$\begin{aligned}
&pnl^{\cdot}=\big(\mathcal{Q}^{\cdot}+J^{s,\cdot}q^{\cdot}+(1-J^{s,\cdot})Q^{\cdot}\big)_{(0)}^{\tau^{\cdot}}-h^{\cdot},\\
&\mathcal{H}^{\cdot}=\big(J^{s,\cdot}(Q^{\cdot}-q^{\cdot})\big)_{(0)}^{\tau^{\cdot}}+h^{\cdot},\\
&\mathrm{HVA}^{\cdot}=(J^{s,\cdot}(q^{\cdot}-Q^{\cdot}))^{\tau^{\cdot}}-D^{\cdot}\ ,\ \textit{where}\ D^{\cdot}_t=\mathbb{E}_t\Big(\mathbb{1}_{\{\tau^{\cdot}<\tau_s^{\cdot}\wedge T^{\cdot}\}}(q^{\cdot}-Q^{\cdot})_{\tau^{\cdot}}\Big),\ t\geq 0,\\
&\qquad\mathrm{HVA}^{\cdot}_0=(q^{\cdot}-Q^{\cdot})_0-\mathbb{E}\big(\mathbb{1}_{\{\tau^{\cdot}<\tau_s^{\cdot}\wedge T^{\cdot}\}}(q^{\cdot}-Q^{\cdot})_{\tau^{\cdot}}\big).
\end{aligned}\tag{2.13}$$

Remark 2.4. *This* $\mathrm{HVA}^{\cdot}$ *corresponds to the current market practice for handling model risk, in the form of a reserve put aside at initial time. In fact, rather than paying* $q^{\cdot}_0$ *to the client*[26] *while the client would provide* $\mathrm{HVA}^{\cdot}_0$ *as reserve capital to the bank, the trader pays* $Q^{\cdot}_0$ *to the client and puts by himself* $\mathrm{HVA}^{\cdot}_0$ *in the reserve capital account, which is equivalent (at least if* $D^{\cdot}_0=0$ *and* $P^{\cdot}=p^{\cdot}$*, as then* $\mathrm{HVA}^{\cdot}_0=q^{\cdot}_0-Q^{\cdot}_0$*).*

Remark 2.5. *As will be illustrated in later sections*[27]*, in line with the notion of Darwinian model risk, the net* $(pnl^{\cdot}-\mathrm{HVA}^{\cdot}_{(0)})$ *is a martingale, typically of the "gamma negative" type, i.e. the trader makes systematic profits in the short-to-medium term followed by a large loss when a related extreme event occurs. This is at least the case unless risk-adjusted model risk provisions are used*[28]*.*

2.2 HVA for CVA and FVA

Accounting methods are also models in the sense of SR-11-7[29] because they produce numbers, are based on assumptions, and have an impact on strategies. If they are misaligned with economics they cause a misalignement of interests between executives and shareholders. Hence, model risk is a concept that does not apply only to pricing models, but should be extended to accounting principles for dealer banks, including the specification of their CVA and FVA metrics (as these are liabilities to the bank[30]).

In the case of the FVA, which is not marketed and can hardly be hedged[31], the model risk game is in fact simpler than for client deals and their hedges (but at the higher level of aggregation characteristic of XVA metrics). It is just the temptation for

[26] as implied by (2.5).
[27] see Remark 3.2.
[28] cf. Section 2.4
[29] cf. https://www.federalreserve.gov/supervisionreg/srletters/sr1107.htm.
[30] see (Albanese et al., 2021, Figure 1).
[31] for the bank, hedging its FVA would mean trading its own default, which is unpractical (Albanese et al., 2021, Section 3.5).

the bank to underestimate its FVA for being more competitive with clients, at the long-term risk of seeing the FVA spiking and triggering huge losses to the bank (as happened for instance during the Covid-19 Q1 2020 financial crisis, see Section 4.1). The CVA, which can only partially be hedged (only its market risk and only to some extent, fault of liquid enough CDS markets), is from this point of view in an intermediate situation between FVA and MtM. Accordingly:

Assumption 2.2. *The CVA is purely dynamically hedged and the FVA is unhedged.*

Denoting by $\mathcal{C}$ and $\mathcal{F}$ the counterparty default and risky funding cash flows[32], with fair vs. trader prices CVA/FVA vs. cva/fva, developments similar to (2.5)–(2.13) with analogous notation[33], detailed in Table 2.1, yield under Assumption 2.2:

$$\begin{aligned}
-\,pnl^{cva} &= \big(\mathcal{C} + J^{s,cva}\mathrm{cva} + (1-J^{s,cva})\mathrm{CVA}\big)_{(0)} - h^{\mathrm{cva}},\\
\mathcal{H}^{cva} &= \big(J^{s,cva}(\mathrm{CVA}-\mathrm{cva})\big)_{(0)} + h^{\mathrm{cva}},\\
\mathrm{HVA}^{cva} &= va(-\mathcal{H}^{cva}) = va(-pnl^{cva}) = J^{s,cva}\big(\mathrm{CVA}-\mathrm{cva}\big),\\
\mathrm{HVA}^{cva}_0 &= \big(\mathrm{CVA}-\mathrm{cva}\big)_0,\\
\mathcal{H}^{cva} - \mathrm{HVA}^{cva}_{(0)} &= h^{cva},
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
-\,pnl^{fva} &= \big(\mathcal{F} + J^{s,fva}\mathrm{fva} + (1-J^{s,fva})\mathrm{FVA}\big)_{(0)},\\
\mathcal{H}^{fva} &= \big(J^{s,fva}(\mathrm{FVA}-\mathrm{fva})\big)_{(0)},\\
\mathrm{HVA}^{fva} &= va(-\mathcal{H}^{fva}) = va(-pnl^{fva}) = J^{s,fva}\big(\mathrm{FVA}-\mathrm{fva}\big),\\
\mathrm{HVA}^{fva}_0 &= \big(\mathrm{FVA}-\mathrm{fva}\big)_0,\\
\mathcal{H}^{fva} - \mathrm{HVA}^{fva}_{(0)} &= 0.
\end{aligned} \tag{2.15}$$

To build a bad CVA model satisfying the Darwinian principle for medium-to-long term sustainability, one can for instance neglect credit vegas. It is well known that credit spread volatilities spike up whenever spreads jump up. If one uses a credit model where hazard rates for default are either deterministic or follow a diffusion process, credit hedge ratios are under-stated and the CVA is under-valued, giving rise to more competitive pricing and unavoidable blow up.

Another route to build a bad CVA model is to ignore wrong way risk (Li and Mercurio, 2015; Crépey and Song, 2014). In particular, upon the occurence of default of a large and systemically important entity, a waterfall of consequences and market disruptions ensue. Accordingly, defaults by major counterparties should be modeled as binary occurrences, not as a probabilities of default. By modeling defaults only through probability distributions and not as binary events, one reduces the CVA and benefits out of selling uncovered puts on the default of those names.

[32]see Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section A).

[33]as local XVA pricing models typically underestimate CVA and FVA whereas local mark-to-market models overestimate MtM, our sign convention for the XVA related HVA cash flows is opposite to the one regarding the MtM component, so that we arrive at a final HVA formula as per the last line in (2.16), with all terms nonnegative there.

Regarding funding valuation adjustments, as detailed in Section 4, a bad FVA model widely used in banks for its simplicity is the symmetric FVA, i.e. fva = FCA − FBA as per (4.1), while the fair valuation model should value an asymmetric FVA. The stochastic models to evaluate fva and FVA may or not be the same, there is in any case a cash flow risk. In practice, the situation is even worse than that as banks also use a too conservative model for accounting purposes, i.e. FCA as per (4.2), instead of what should be the asymmetric FVA again, then compensating for the lack of modeling of the rehypothecation option by slashing the funding rate by a factor 3 to 5. This is commonly achieved by pulling data from peers and adopting a similar funding spread. Although in tranquil times this method yields a number roughly in the right FVA ballpark, it leads to unavoidable blow up at times of stress (see Section 4.1).

Remark 2.6. *As for KVA computations, to enhance its competitiveness in the short term, a bank might be tempted to use a model understating the risk and economic capital of the bank. A sound practice in this regard is to combine different, equally valid (realistic and co-calibrated) models for simulating the set of trajectories underlying the economic capital and KVA computations (Albanese et al., 2022, Section 4.3). Such a Bayesian KVA approach typically fattens the tails of the simulated distributions and avoids under-stated risk estimates.*

2.3 HVA for Dynamic Hedging Frictions

The above processes h are meant for standard dynamic hedging cash flows ignoring nonlinear frictions such as transaction costs, i.e. ignoring HVA originating cash flows à la Burnett (2021); Burnett and Williams (2021). Indeed, as these are nonlinear, they can only be addressed at the level of each book of contracts or exposures that are hedged together, or hedging sets "$\star$". Let $proc^{mtm} = \sum_{\cdot} proc^{\cdot}$, for each process $proc = pnl, h, \mathcal{H}, \mathrm{HVA}, D$, and let[34]

$$
\begin{aligned}
&\mathcal{M} + \mathrm{MtM}_{(0)} = \sum_{\cdot} \big((\mathcal{Q}^{\cdot} + Q^{\cdot})^{\tau^{\cdot}}_{(0)} - (\mathcal{P}^{\cdot} + P^{\cdot})^{\tau^{\cdot}}_{(0)} \big), \\
&\mathcal{H} = \mathcal{H}^{mtm} - \mathcal{H}^{cva} - \mathcal{H}^{fva} + f, \\
&\mathrm{HVA} = \mathrm{HVA}^{mtm} + \mathrm{HVA}^{cva} + \mathrm{HVA}^{fva} + \mathrm{HVA}^{f},
\end{aligned}
\tag{2.16}
$$

where $f = \sum_{\star} f^{\star}$ is the sum of the hedging friction on each hedging set, valued by $\mathrm{HVA}^f = va(f)$. Accounting for raw pnls, hedging frictions, and HVA compensators for all, we obtain the overall trading loss of the bank

$$
\mathcal{L} = -pnl^{mtm} + \mathrm{HVA}^{mtm}_{(0)} - pnl^{cva} + \mathrm{HVA}^{cva}_{(0)} - pnl^{fva} + \mathrm{HVA}^{fva}_{(0)} + f + \mathrm{HVA}^{f}_{(0)}. \tag{2.17}
$$

Lemma 2.2. ***(i)*** *The process $\mathcal{L}$ as per* (2.17) *is a martingale of the form* (2.2)-(2.4)*, with the different terms specified as in* (2.16).

[34] the first line in (2.16) is not a definition but the statement established as Crépey (2022, Part I, Eqn. (3.5)).

(ii) *It holds:*

$$\begin{aligned}\mathcal{H} + \mathrm{HVA}_{(0)} &= h^{mtm} - D^{mtm}_{(0)} - h^{cva} + f + \mathrm{HVA}^{f}_{(0)},\\ \mathcal{L} &= -\sum_{\cdot} \big((\mathcal{Q}^{\cdot} + Q^{\cdot})^{\tau^{\cdot}}_{(0)} - (\mathcal{P}^{\cdot} + P^{\cdot})^{\tau^{\cdot}}_{(0)} - h^{\cdot} + D^{\cdot}_{(0)}\big)\\ &\qquad + \mathcal{C} + \mathrm{CVA}_{(0)} - h^{cva} + \mathcal{F} + \mathrm{FVA}_{(0)} + f + \mathrm{HVA}^{f}_{(0)}.\end{aligned} \tag{2.18}$$

Proof. **(i)** follows from the first line in (2.7) and of its credit and funding analogs

$$-pnl^{cva} + \mathcal{H}^{cva} = \mathcal{C} + \mathrm{CVA}_{(0)} \text{ and } -pnl^{fva} + \mathcal{H}^{fva} = \mathcal{F} + \mathrm{FVA}_{(0)}$$

that stem from (2.14) and (2.15).
(ii) The first line in (2.18) proceeds from (2.16) by the last lines in (2.9), (2.14) and (2.15). Substituting the first line in (2.18) for $\mathcal{H} + \mathrm{HVA}_{(0)}$ in the identity $\mathcal{L} = -(\mathcal{M} + \mathrm{MtM}_{(0)}) + \mathcal{C} + \mathcal{F} + \mathrm{CVA}_{(0)} + \mathrm{FVA}_{(0)} + \mathcal{H} + \mathrm{HVA}_{(0)}$ that stems from (2.2)-(2.4) yields

$$\mathcal{L} = -(\mathcal{M} + \mathrm{MtM}_{(0)}) + h^{mtm} - D^{mtm}_{(0)} + \mathcal{C} + \mathrm{CVA}_{(0)} - h^{cva} + \mathcal{F} + \mathrm{FVA}_{(0)} + f + \mathrm{HVA}^{f}_{(0)},$$

where by (2.16)

$$-(\mathcal{M} + \mathrm{MtM}_{(0)}) + h^{mtm} - D^{mtm}_{(0)} = -\sum_{\cdot} \big((\mathcal{Q}^{\cdot} + Q^{\cdot})^{\tau^{\cdot}}_{(0)} - (\mathcal{P}^{\cdot} + P^{\cdot})^{\tau^{\cdot}}_{(0)} - h^{\cdot} + D^{\cdot}_{(0)}\big).$$

This yields the second line in (2.18). ∎

Table 2.1 recapitulates the HVA related data of the global valuation problem of the bank. From an organizational viewpoint, the computation of the $\mathrm{HVA}^{\cdot}$ (summing up to HVA^{mtm}), HVA^{cva} and HVA^{fva} components could be delegated to each related trader (under regulatory control). The HVA^{f} component(s) calculations would typically require a dedicated (regulated) HVA desk, as such computations typically need a mix of data from the different trading and/or CVA/FVA desks of the bank.

2.4 KVA Adjustment for the HVA Risks

After compensation by the HVA, the price is right[35], but the hedge is still wrong (is not the hedging strategy corresponding to the corrected price). Under a cost-of-capital valuation approach, the reserve for model risk and dynamic hedging frictions does not reduce to HVA terms summed over all the pnl centers of the banks. This reserve is also risk-adjusted, via the impact of model risk and dynamic hedging frictions on $\mathcal{L}$ as per (2.2)-(2.17)-(2.18). Going by Definitions A.1 and A.2, the ensuing volatile swings of $\mathcal{L}$ are reflected in the economic capital and in the KVA of the bank.

If there was no model risk, i.e. if all the pnl centers of the bank were relying on the fair valuation model for all their purposes, then all the $\mathcal{H}$ processes in the above would reduce to related components h^*,[36] all $\mathbb{R}$ martingales, and one would fall back on an

[35] cf. Remark 2.4.
[36] cf. the last lines in (2.9), (2.14) and (2.15).

$\cdot$	a generic client deal of the bank
$\mathcal{Q}^{\cdot}, \mathcal{P}^{\cdot}$	cash flows promised to the bank on the deal and its static hedge components
$Q^{\cdot}, P^{\cdot}$	corresponding fair value processes (2.1)
$q^{\cdot}, p^{\cdot}$	trader's local model price for the deal "·" and its static hedge component (if any)
$\tau^{\cdot}_{e/s/d}$	related deactivation (barrier) time, model switch time, default time
$h^{\cdot}$	dynamic (martingale) hedging cash flows related to the deal "·"
$\mathcal{C}, \mathcal{P}^{cva} = 0, \mathrm{CVA}, P^{cva} = 0,$ $\mathrm{cva}, p^{cva} = 0$ $\tau^{cva}_{e/d} = +\infty, \tau^{cva}_{s}, h^{cva}$	similar data regarding the CVA of the bank
$\mathcal{F}, \mathcal{P}^{fva} = 0, \mathrm{FVA}, P^{fva} = 0,$ $\mathrm{fva}, p^{fva} = 0$ $\tau^{fva}_{e/d} = +\infty, \tau^{fva}_{s}, h^{fva} = 0$	similar data regarding the FVA of the bank
f	dynamic hedging friction costs at the aggregated bank level

Table 2.1: HVA related data.

$\mathrm{HVA}^* = \mathrm{HVA}^{f,*}$ à la Burnett (2021); Burnett and Williams (2021)[37], along with the related risk adjustment. Note that we wrote h^*, HVA^* and $\mathrm{HVA}^{f,*}$ above, not h, HVA and HVA^f as before, to emphasize that using the fair valuation model for all purposes by the bank would also imply different and presumably much better dynamic (as well as static) hedges, triggering much less volatile swings of $\mathcal{L}$ than the ones implied by local models, hence in turn much lower economic capital and KVA.

An additional valuation adjustment (AVA, or model risk component thereof, cf. European Parliament (2013, 2016)[38] could thus be defined as the difference between HVA + KVA as above and a baseline $\mathrm{HVA}^{f,*} + \mathrm{KVA}^*$ corresponding to a loss process (to be compared with $\mathcal{L}$ in (2.18))

$$\begin{aligned} \mathcal{L}^* = -\sum_{\cdot} \big(- (\mathcal{Q}^{\cdot} + Q^{\cdot})^{\tau^{\cdot}}_{(0)} - (\mathcal{P}^{*,\cdot} + P^{*,\cdot})^{\tau^{\cdot}}_{(0)} - h^{*,\cdot} \big) \\ + \mathcal{C} + \mathrm{CVA}_{(0)} - h^{*,\mathrm{cva}} + \mathcal{F} + \mathrm{FVA}_{(0)} + f^* + \mathrm{HVA}^{f,*}, \end{aligned} \tag{2.19}$$

so

$$\mathrm{AVA} = \mathrm{HVA} + \mathrm{KVA} - (\mathrm{HVA}^{f,*} + \mathrm{KVA}^*). \tag{2.20}$$

This AVA depends on the detailed specification of the baseline XVA setup, including the choice of the corresponding hedges. In fact, as a dealer bank should not do proprietary trading, the reference MtM hedging case is when the sum in the first line simply vanishes in (2.19). Conversely, a bank cannot really hedge its CVA, hence a

[37] detailed in Section C.

[38] see also https://www.eba.europa.eu/regulation-and-policy/market-risk/draft-regulatory-technical-standards-on-prudent-valuation.

reference CVA hedging case could be $h^{*,\mathrm{cva}} = 0$, yielding to the following minimalist specification of (2.19):

$$\mathcal{L}^* = \mathcal{C} + \mathrm{CVA}_{(0)} + \mathcal{F} + \mathrm{FVA}_{(0)} + f^* + \mathrm{HVA}^{f,*}, \tag{2.21}$$

which could be taken as a reference for defining KVA* and in turn the AVA via (2.20).

Remark 2.7. *For a bank there is no economic necessity of computing a baseline* KVA**, nor of identifying the corresponding* AVA*. All that matters is that the bank passes to its clients the total add-on* CA + KVA*, with* CA = CVA + FVA + HVA + KVA*. However, after the introduction of the* HVA *and its risk adjustment as a contribution to the* KVA*, the use of bad quality local models should imply a positive* AVA *in* (2.20)*. Computing their AVAs could usefully incite banks to consider using higher quality models for all purposes, pricing and accounting numbers computations, leading to diminished rebates and provisions under the baseline* $*$ *model.*

Note that, under the reference specification (2.21) for the loss process $\mathcal{L}^*$ in the baseline XVA approach (2.2)-(2.3), market risk is assumed fully hedged and it does therefore not contribute to the economic capital or to the KVA of the bank. Once Darwinian model risk is included into the analysis, instead, one can see a very significant amount of market risk (and corresponding contributions to the economic capital and KVA of the bank) due to the fact that, even after HVA^{mtm} has been added to restore the correct MtM values, the price (MtM) has become right but the hedge is still wrong. Hence, Darwinian model risk is the way market risk reintroduces itself into the KVA. By contrast, the FVA side of the HVA does not trigger significant additional KVA, because the FVA is not (or can only very partially be) hedged in the first place[39]. So adding HVA^{fva} to fva restores the right funding pnl $\mathcal{F} + \mathrm{FVA}_{(0)}$[40]. The CVA side of the HVA is in between[41].

Remark 2.8. *We saw in Remark 2.3 that, in the continuously recalibrated case, the static side of the hedge does not impact* $\mathcal{H}^{\cdot}$ *nor* HVA$^{\cdot}$*. But it does modify the raw pnl*$^{\cdot}$ *of the trader* (2.5)*, hence the trading loss of the bank* (2.17)*, and therefore the related contribution to the KVA.*

3 MtM Example

In the following example, a trader is short an extreme (default) event but pretends he does not see it, only hedging market risk. Hence the hedged position is still short the default event, which can be seen as an extreme case of "gamma negative" type position[42].

Remark 3.1. *This example was devised for the sake of analytical tractability. But the Darwinian model risk mechanism at hand here is essentially the same as the one*

[39] cf. the beginning of Section 2.2.
[40] cf. (2.21).
[41] cf. Assumption 2.2.
[42] see Remark 2.5.

affecting huge amounts of structured derivative products, including range accruals in the fixed-income world, autocallables and cliquets on equities, or power-reversal dual currency options and target redemption forwards on foreign-exchange: cf. https://www.risk.net/derivatives/6556166/remembering-the-range-accrual-bloodbath (11 April 2019, last accessed on 17 May 2022), illustrated in Albanese, Crépey, and Iabichino (2021) by a case study regarding a callable range accrual hedged by digital swaption streams. In their case, Darwinian model risk enters the picture through the callability of the asset[43]*: a trader long an extreme (corridor exit) event ends-up being short in the event, hence "gamma negative", by shorting excessive vanilla option (digital swaption) positions as his hedge, as these are computed with a model overestimating the probability of the extreme event and therefore the optimal call time by the bank. If the extreme event happens, the mis-hedged position blows up. Risk magazine thus reported that Q4 of 2019, a $70bn notional of range accrual had to be unwound at very large losses by the industry.*

We denote by

$$Q^{mo}(\text{t, market risk parameters[; model parameters, whenever relevant]}),$$

or simply Q_t^{mo} when the parameters are clear enough from the context, the value process (2.1) of a given product (or cumulative cash flow stream) in the fair valuation model mo, and we use analogous conventions with P instead of Q regarding a static hedge component of the claim.

We consider financial derivatives of maturity $T = \overline{T}$ on a stock S, with dividend yields on S and interest rates in the economy set to 0 for notational simplicity. The role of the fair valuation model is played by the jump-to-ruin (jr) model

$$dS_t = \lambda S_t dt + \sigma S_t dW_t - S_{t-} dN_t = \sigma S_t dW_t - S_{t-} dM_t, \tag{3.1}$$

where W is a standard Brownian motion, $\sigma > 0$ is a constant volatility parameter and N is a Poisson process with intensity $\lambda > 0$ and compensated martingale $M = N - \lambda t$, with W and N independent[44]. Hence the stock S jumps to 0 at the first jump time τ_s of the driving Poisson process N.

The role of a local model is played by a Black-Scholes model bs, with volatility Σ continuously recalibrated to the jump-to-ruin price P^{jr} of a European vanilla put with maturity T and strike K, which can be used by the trader as an hedging asset along with S and the risk-free constant asset. The claim of interest consists in an exotic variation on this put including an additional barrier clause. At each time t the option is valued and hedged by the trader in the Black-Scholes model with implied volatility Σ_t[45] of the vanilla put. At least this holds for $t < \tau_s \wedge T$. Note that $P^{jr} = K$ on $[\![\tau_s, T[\![$[46]. So, at time τ_s (if $< T$), the implied volatility of the vanilla put ceases to be well-defined[47], the falsity of the local model is revealed and the position must be unwound by the trader at exit prices dictated by the fair valuation (jr) model.

[43] see Remark 2.2.

[44] as in fact always the case for a Brownian motion and a Poisson process with respect to a common stochastic basis, see e.g. He, Wang, and Yan (1992, Theorem 11.43 page 316).

[45] see Definition B.1.

[46] by application of the formula (B.9) for $S = 0$ and $-d_{\pm} = +\infty$.

[47] cf. Remark B.1.

3.1 Static Hedging of a Vulnerable Put

We first consider the limiting case in the above where the option bought (at $t = 0$) and hedged by the trader is a put option of maturity T and strike K with a deactivating barrier at the level $S = 0$.

In the jump-to-ruin model, this barrier is attainable and absorbing and the above option is equivalent to the vulnerable put of Proposition B.3, so that, by (B.12), $Q^{jr} = 0$ from time τ_s onward. But the trader values the vulnerable put at its bs price with implied volatility Σ_t of the vanilla put. As the barrier is immaterial in bs, for $t < \tau_s \wedge T$,

$$\begin{aligned} q_t &= Q^{bs}(t, S_t; \Sigma_t) = P^{bs}(t, S_t; \Sigma_t) = P^{jr}(t, S_t) \\ &= Q^{jr}(t, S_t) + K(1 - e^{-\lambda(T-t)}) > Q^{jr}(t, S_t) = Q_t, \end{aligned} \tag{3.2}$$

where the first equality in the second line follows from (B.13). The inequality in (3.2) is in line with the first Darwinian principle of Section 1. We assume the portfolio of the bank restricted to the vulnerable put and its hedge, as well as a risk-free bank, client and hedge counterparties. In the notation of Section 2 summarized in Table 2.1, we thus have:

$$\begin{aligned} &\mathcal{Q}^{\cdot} = \mathbb{1}_{\{\tau_s > T\}}(K - S_T)^+ \mathbb{1}_{[T,+\infty)},\ Q^{\cdot} = Q^{jr},\ \mathcal{P}^{\cdot} = (K - S_T)^+ \mathbb{1}_{[T,+\infty)},\ q^{\cdot} = p^{\cdot} = P^{\cdot} = P^{jr} \\ &\tau^{\cdot} = +\infty,\ D^{\cdot} = 0 \end{aligned} \tag{3.3}$$

(cf. (2.10)).

In view of (3.3), (B.13), (2.12) and (2.17) (cf. also (2.16)), the HVA equations reduce to

$$\begin{aligned} pnl^{\cdot} &= -\mathbb{1}_{\{\tau_s \le T\}} K \mathbb{1}_{[T,+\infty)} + (1 - J^s)(Q^{jr} - P^{jr}) = -\mathbb{1}_{\{\tau_s \le T\}} K \mathbb{1}_{[T,+\infty)} - (1 - J^s) K \mathbb{1}_{[0,T)} \\ &= -\mathbb{1}_{\{\tau_s \le T\}} K \mathbb{1}_{[T,+\infty)} \mathbb{1}_{[\tau_s,+\infty)} - (1 - J^s) K \mathbb{1}_{[0,T)} \mathbb{1}_{\{\tau_s \le T\}} = -(1 - J^s) K \mathbb{1}_{\{\tau_s \le T\}} \\ \mathcal{H} &= \mathcal{H}^{\cdot} = J^s \big(Q^{jr} - P^{jr}\big) - \big(Q^{jr} - P^{jr}\big)_0 = -J^s K(1 - e^{-\lambda(T-\cdot)}) + K(1 - e^{-\lambda T}) \\ \mathrm{HVA} &= \mathrm{HVA}^{\cdot} = J^s \big(P^{jr} - Q^{jr}\big) = J^s K (1 - e^{-\lambda(T-\cdot)}),\ \mathrm{HVA}_0 = K(1 - e^{-\lambda T}), \\ \mathcal{L} &= -pnl^{\cdot} + \mathrm{HVA}^{\cdot}_{(0)}. \end{aligned} \tag{3.4}$$

Remark 3.2. *The raw pnl process in* (3.4) *and the corresponding pnl net of* $\mathrm{HVA}^{\cdot}_{(0)}$[48] *satisfy (starting from 0)*

$$\begin{aligned} dpnl^{\cdot}_t &= -K \mathbb{1}_{\{t \le T\}} \boldsymbol{\delta}_{\tau_s}(dt) = \mathbb{1}_{\{t \le \tau_s \wedge T\}} \big(-\lambda K dt - (K dN_t - \lambda K dt)\big) \\ dpnl^{\cdot}_t - d\mathrm{HVA}^{\cdot}_t &= K \mathbb{1}_{\{t \le \tau_s \wedge T\}} e^{-\lambda(T-t)} \big(\lambda dt - \boldsymbol{\delta}_{\tau_s}(dt)\big). \end{aligned} \tag{3.5}$$

Consistently with the Darwinian model risk pattern[49]*, a seemingly positive drift*

$$\mathbb{1}_{\{t \le \tau_s \wedge T\}} \lambda K e^{-\lambda(T-t)} dt$$

in the second line is only the compensator of the loss

$$(-\mathbb{1}_{\{t \le \tau_s \wedge T\}} K e^{-\lambda(T-t)} dN_t)$$

that hits the bank in case the extreme event materializes[50].

[48] cf. Remark 2.5.

[49] see Section 1.

[50] cf. Remark 2.5.

Proposition 3.1. *Denoting* $\Theta = (T + \frac{\ln(\alpha)}{\lambda})^+ \leq T$*, where* α *is the confidence level at which economic capital is calculated*[51]*, and by* r *the hurdle rate of the bank*[52]*, we have* $\mathrm{EC} = J^s \widetilde{\mathrm{EC}}$ *and* $\mathrm{KVA} = J^s \widetilde{\mathrm{KVA}}$*, where*

$$
\begin{aligned}
\widetilde{\mathrm{EC}} &= \mathbb{1}_{\lambda > -\ln(\alpha)} \mathbb{1}_{[0,\Theta)} K e^{-\lambda(T-\cdot)},\\
\widetilde{\mathrm{KVA}} &= \mathbb{1}_{\lambda > -\ln(\alpha)} K e^{-\lambda(T-\cdot)} \mathbb{1}_{[0,\Theta)} (1 - e^{-r(\Theta-\cdot)}),\\
\mathrm{KVA}_0 &= \mathbb{1}_{\lambda > -\ln(\alpha)} K e^{-\lambda T} \mathbb{1}_{\Theta > 0} (1 - e^{-r\Theta}).
\end{aligned}
\tag{3.6}
$$

Proof. For $t < t' \leq T$, the last line in (3.4) yields

$$
\begin{aligned}
\mathcal{L}_{t'} - \mathcal{L}_t &= (-pnl^{\cdot} + \mathrm{HVA}^{\cdot})_{t'} - (-pnl^{\cdot} + \mathrm{HVA}^{\cdot})_t\\
&= \mathbb{1}_{\{t' \geq \tau_s > t\}} K + \mathbb{1}_{\{t' < \tau_s\}} K(1 - e^{-\lambda(T-t')}) - \mathbb{1}_{\{t < \tau_s\}} K(1 - e^{-\lambda(T-t)})\\
&= \mathbb{1}_{\{t < \tau_s\}} \Big(\mathbb{1}_{\{t' \geq \tau_s\}} (K - K(1 - e^{-\lambda(T-t')})) + K(1 - e^{-\lambda(T-t')}) - K(1 - e^{-\lambda(T-t)}) \Big)\\
&= \mathbb{1}_{\{t < \tau_s\}} B^t_{t'}, \text{ where } B^t_{t'} = \mathbb{1}_{\{t' \geq \tau_s\}} K e^{-\lambda(T-t')} + K(e^{-\lambda(T-t)} - e^{-\lambda(T-t')}).
\end{aligned}
$$

On $\{t < \tau_s\}$, the Bernoulli random variable $\mathbb{1}_{\{t' \geq \tau_s\}}$ satisfies $\mathbb{E}_t \left[\mathbb{1}_{\{t' \geq \tau_s\}} = 0 \right] = e^{-\lambda(t'-t)}$ and, for any confidence level $\alpha > e^{-\lambda(t'-t)}$, i.e. such that $t' - t > \frac{-\ln(\alpha)}{\lambda}$, $\mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t)$ is the largest of the two possible values of $(\mathcal{L}_{t'} - \mathcal{L}_t)$, so that the latter never exceeds $\mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t)$. As a consequence, for $t' - t > \frac{-\ln(\alpha)}{\lambda}$, we have[53]:

$$
\begin{aligned}
&\mathbb{ES}_t(\mathcal{L}_{t'} - \mathcal{L}_t) = \mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t) =\\
&\qquad \mathbb{1}_{\{t < \tau_s\}} \big(K e^{-\lambda(T-t')} + K(e^{\lambda(T-t)} - e^{-\lambda(T-t')} \big) = \mathbb{1}_{\{t < \tau_s\}} K e^{-\lambda(T-t)}.
\end{aligned}
$$

For $t' - t \leq \frac{-\ln(\alpha)}{\lambda}$, we have

$$
(\mathcal{L}_{t'} - \mathcal{L}_t) \mathbb{1}_{\{\mathcal{L}_{t'} - \mathcal{L}_t \geq \mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t)\}} = \mathcal{L}_{t'} - \mathcal{L}_t,
$$

which is a time-t conditionally centered random variable as the increment of the martingale $\mathcal{L}$. Hence[54]

$$
0 = \mathbb{E}_t(\mathcal{L}_{t'} - \mathcal{L}_t) = \mathbb{E}_t\big((\mathcal{L}_{t'} - \mathcal{L}_t) \mathbb{1}_{\{\mathcal{L}_{t'} - \mathcal{L}_t \geq \mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t)\}} \big) = \mathbb{ES}_t(\mathcal{L}_{t'} - \mathcal{L}_t).
$$

Setting $t' = (t+1) \wedge T$ so that $t' - t > \frac{-\ln(\alpha)}{\lambda} \Leftrightarrow t < \Theta$, we obtain by Definition A.1:

$$
\mathrm{EC}_t = \mathbb{ES}_t(\mathcal{L}_{t'} - \mathcal{L}_t) = \mathbb{1}_{\{t < \tau_s\}} \mathbb{1}_{\lambda > -\ln(\alpha)} \mathbb{1}_{t < \Theta} K e^{-\lambda(T-t)},
$$

which is the first line in (3.6).

Assuming $\lambda > -\ln(\alpha)$ (otherwise $\mathrm{EC} = \mathrm{KVA} = 0$), let us define the process

$$
\mathrm{KVA}^\dagger_t := r\mathbb{E}_t \int_t^T e^{-r(u-t)} \mathrm{EC}_u du = r\mathbb{E}_t \int_t^T \big(\mathrm{EC}_s - \mathrm{KVA}^\dagger_s \big) ds,\ t \leq T. \tag{3.7}
$$

[51] see Definition A.1.
[52] cf. Definition A.3.
[53] cf. (A.1).
[54] cf. (A.1).

We have

$$
\begin{aligned}
\mathrm{KVA}^{\dagger}_t &= rK\mathbb{E}_t \mathbb{1}_{\{t<\Theta\}} \int_t^{\Theta} e^{-r(u-t)} \mathbb{1}_{\{u<\tau_s\}} e^{-\lambda(T-u)} du \\
&= rKe^{-\lambda(T-\Theta)} \mathbb{1}_{\{t<\Theta\}} \mathbb{1}_{\{t<\tau_s\}} \int_t^{\Theta} e^{-r(u-t)} e^{-\lambda(u-t)} e^{-\lambda(\Theta-u)} du \\
&= \mathbb{1}_{\{t<\tau_s\}} rKe^{-\lambda(T-\Theta)} e^{-\lambda(\Theta-t)} \mathbb{1}_{\{t<\Theta\}} \int_t^{\Theta} e^{-r(u-t)} du \\
&= \mathbb{1}_{\{t<\tau_s\}} Ke^{-\lambda(T-t)} \mathbb{1}_{t<\Theta} (1 - e^{-r(\Theta-t)}) \leq \mathbb{1}_{\{t<\tau_s\}} K \mathbb{1}_{t<\Theta} e^{-\lambda(T-t)} = \mathrm{EC}_t.
\end{aligned}
\tag{3.8}
$$

Back to the right-hand side in (3.7), the process $\mathrm{KVA}^{\dagger}$ therefore satisfies

$$
\mathrm{KVA}^{\dagger}_t = \mathbb{E}_t \int_t^T r\big(\mathrm{EC}_s - \mathrm{KVA}^{\dagger}_s\big) ds = \mathbb{E}_t \int_t^T r\big(\mathrm{EC}_s - \mathrm{KVA}^{\dagger}_s\big)^+ ds, \; t \leq T, \tag{3.9}
$$

which is the KVA equation (A.3). As EC and $\mathrm{KVA}^{\dagger}$ are bounded processes, hence $\mathrm{KVA}^{\dagger}$ is the unique bounded (or even square integrable) solution to this equation[55], i.e. $\mathrm{KVA}^{\dagger} = \mathrm{KVA}$. The first identity in the last line of (3.8) then yields the second line in (3.6). ∎

Numerical Results For $\lambda = 1\%$, $T = 10y$ and $r = 10\%$, (3.5) and (3.6) yield as $\alpha \downarrow e^{-0.01} \approx 99\%$:

$$
\begin{aligned}
&\mathrm{HVA}_0 = K(1 - e^{-0.1}) \approx 0.095K, \; \mathrm{KVA}_0 \downarrow Ke^{-0.1}(1 - e^{-1+0.1}) \approx 0.54K \\
&\frac{\mathrm{KVA}_0}{\mathrm{HVA}_0} \downarrow \frac{(1 - e^{-0.9})}{(e^{0.1} - 1)} \approx 5.64.
\end{aligned}
\tag{3.10}
$$

In the present case where $f = 0$ and a pure frictions HVA^f à la Burnett (2021); Burnett and Williams (2021) vanishes, we see from the top panels of Figure 3.1 that the Darwinian model risk HVA alone can be extreme. As visible on the bottom panels of Figure 3.1, the corresponding KVA adjustment can be even several times larger. The latter holds for $\alpha > e^{-\lambda}$. For $\alpha \leq e^{-\lambda}$, instead, there is no tail risk at the envisioned confidence level, hence $\mathrm{EC} = \mathrm{KVA} = 0$.

For a baseline setup (cf. Section 2.4) corresponding to dynamic, assumed frictionless, replication of the vulnerable put by the stock and the vanilla put in the jr model as per Proposition B.4, the AVA (2.20) reduces to HVA + KVA—as $\mathrm{HVA}^{f,*} + \mathrm{KVA}^* = 0$.

3.2 Delta Hedging of a Vulnerable Put

Instead of the previous static hedge, we now assume a dynamic delta hedging scheme, whereby the trader uses, on top of the Black-Scholes implied price[56]

$$
q_t = Q^{bs}(t, S_t; \Sigma_t) = P^{jr}(t, S_t) > Q^{jr}_t,
$$

[55] see the sentence following Definition A.2.
[56] cf. (3.2).

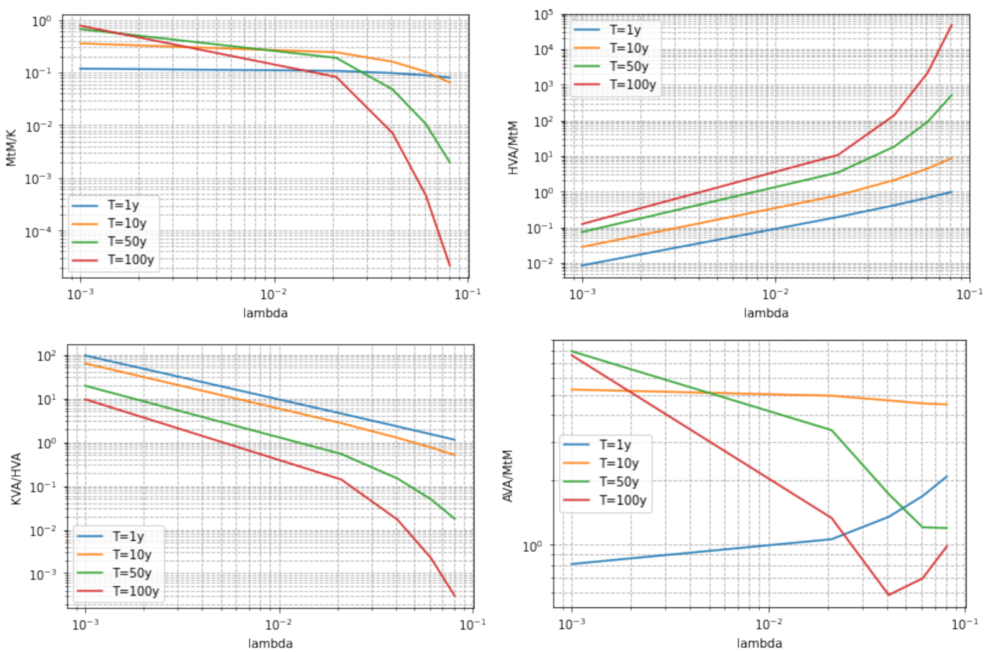

Figure 3.1: At-the-money $S_0 = K$, denoting $\mathrm{MtM}_0 = Q_0^{jr}$ and assuming $\alpha \downarrow e^{-\lambda}$ everywhere in the bottom panels (where the limiting value of the confidence level α that underlies the KVA therefore depends of the abscissa λ): *(Top left)* $\frac{\mathrm{MtM}_0}{K}$; *(Top right)* $\frac{\mathrm{HVA}_0}{\mathrm{MtM}_0}$; *(Bottom left)* $\frac{\mathrm{KVA}_0}{\mathrm{HVA}_0}$; *(Bottom right)* $\frac{\mathrm{AVA}_0}{\mathrm{MtM}_0}$.

the delta $\delta_t = \Delta^{bs}_{t-}$ in S, with

$$\Delta^{bs}_t := \partial_S Q^{bs}(t, S_t; \Sigma_t) = \partial_S P^{bs}(t, S_t; \Sigma_t) < 0$$

before τ_s. Hence $h^{\cdot} = \int_0^{\cdot} \delta_t dS_t$ (a zero-valued martingale).

Using (3.3)[57], (B.12)-(B.13), (2.13) and (2.17), the HVA$^{\cdot}$ equations reduce to

$$\begin{aligned}
&pnl^{\cdot} = \mathbb{1}_{\{\tau_s > T\}}(K - S_T)^+ \mathbb{1}_{[T,+\infty)} + J^s P^{jr} - P^{jr}_0 - h^{\cdot},\\
&\mathcal{H}^{\cdot} = J^s(Q^{jr} - P^{jr}) - (Q^{jr} - P^{jr})_0 + h^{\cdot} =\\
&\qquad - J^s K(1 - e^{-\lambda(T-\cdot)})\mathbb{1}_{[0,T)} + K(1 - e^{-\lambda T}) + h^{\cdot},\\
&\mathrm{HVA}^{\cdot} = J^s(P^{jr} - Q^{jr}) = J^s K(1 - e^{-\lambda(T-\cdot)}),\ \mathrm{HVA}^{\cdot}_0 = K(1 - e^{-\lambda T}).
\end{aligned} \tag{3.11}$$

Remark 3.3. *The raw $pnl^{\cdot}$ in* (3.11)*, satisfies, for $t < \tau_s$,*

$$dpnl^{\cdot}_t = \boldsymbol{\delta}_T(dt)(K - S_T)^+ + dP^{jr}_t - \delta_t dS_t,$$

whereas at τ_s (if $\leq T$) the bank incurs a loss

$$\begin{aligned}
pnl^{\cdot}_{\tau_s} - pnl^{\cdot}_{\tau_s-} &= -P^{jr}_{\tau_s-} + h^{\cdot}_{\tau_s-} - h^{\cdot}_{\tau_s} = -P^{bs}_{\tau_s-} + \Delta^{bs}_{\tau_s-}(S_{\tau_s-} - S_{\tau_s})\\
&= -P^{bs}_{\tau_s-} + \Delta^{bs}_{\tau_s-} S_{\tau_s-} = -K\mathcal{N}(-d_-(\tau_s, S_{\tau_s-}; 0, \Sigma_{\tau_s-})) < 0
\end{aligned} \tag{3.12}$$

(cf. the Black-Scholes formula for puts and (B.3)*), consistent with the blow-up pattern of Darwinian model risk described in Section 1.*

Remark 3.4. *While the static hedge of Section 3.1 is perfect before τ_s and the continuous-time delta hedge is not (due to the continuous recalibration of the pricing model), one observes a smaller loss at $\tau_s < T$ in the delta hedge case:*

$$P^{jr}_{\tau_s-} - \Delta^{bs}_{\tau_s-} S_{\tau_s-} = K\mathcal{N}(-d_-(\tau_s, S_{\tau_s-}; 0, \Sigma_{\tau_s-})) \leq K = P^{bs}_{\tau_s},$$

cf. (3.12), (3.5)*, and see Figure 3.2.*

Hence[58]

$$\mathcal{L} = -pnl^{\cdot} + \mathrm{HVA}^{\cdot}_{(0)} + f + \mathrm{HVA}^{f}_{(0)}, \tag{3.13}$$

where, for $t < t' \leq T$,

$$\begin{aligned}
&(-pnl^{\cdot} + \mathrm{HVA}^{\cdot}_{(0)})_{t'} - (-pnl^{\cdot} + \mathrm{HVA}^{\cdot}_{(0)})_t\\
&\quad = -\big((K - S_T)^+ \mathbb{1}_{[T,+\infty)} \mathbb{1}_{\{\tau_s > T\}} + J^s Q^{jr}\big)_{t'} +\\
&\qquad\quad \big((K - S_T)^+ \mathbb{1}_{[T,+\infty)} \mathbb{1}_{\{\tau_s > T\}} + J^s Q^{jr}\big)_t + (h_{t'} - h_t)\\
&\quad = -(K - S_T)^+ \mathbb{1}_{t < T \leq t'} \mathbb{1}_{\{\tau_s > T\}} - \big(J^s Q^{jr}\big)_{t'} + \big(J^s Q^{jr}\big)_t + (h_{t'} - h_t).
\end{aligned} \tag{3.14}$$

In view of (C.6) and Theorem C.1, we set for some constant $\mathrm{k} \geq 0$[59]:

$$\begin{aligned}
&df_t = \mathbb{1}_{\{S_t > 0\}} \frac{\mathrm{k}}{\sqrt{2\pi}} \Sigma_t S_t \Gamma^{bs}_t dt, \text{ with}\\
&\Gamma^{bs}_t = \partial^2_{S^2} q(t, S_t) = \partial^2_{S^2} P^{bs}(t, S_t; \Sigma_t) \geq 0.
\end{aligned} \tag{3.15}$$

[57] but with $\mathcal{P}^{\cdot} = p^{\cdot} = P^{\cdot} = 0$ in the present purely dynamic hedging setup.
[58] cf. (2.17).
[59] assuming the position unwound at $\tau^{\cdot}_s$, cf. Remark 2.1.

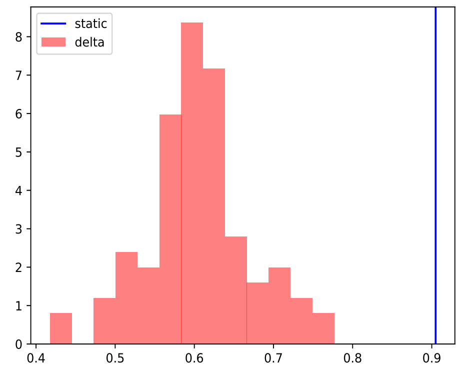

Figure 3.2: The red histogram is the density of $-pnl^{\cdot}_1 + \mathrm{HVA}^{\cdot}_1 - \mathrm{HVA}^{\cdot}_0$ conditional on model switch occuring, i.e. on $\{0 < \tau_s \leq 1\}$, for delta hedging without frictions. The vertical blue line in the right panel corresponds to the deterministic loss $-pnl^{\cdot}_1 + \mathrm{HVA}^{\cdot}_1 - \mathrm{HVA}^{\cdot}_0 = K + \mathrm{HVA}^{\cdot}_1 - \mathrm{HVA}^{\cdot}_0$ for static hedging, also conditional on $\{0 < \tau_s \leq 1\}$. Note that, in both cases, $\mathrm{HVA}^{\cdot}_1 - \mathrm{HVA}^{\cdot}_0 = 0 - K(1 - e^{-\lambda T}) \simeq 0.095$ on $\{0 < \tau_s \leq 1\}$.

Numerical Results The numerical parameters are the same as in Section 3.2 (but with delta hedging here instead of static hedging there), along with $S_0 = 1$ and $\sigma = 0.3$, and with $k_0 = 0.1$ in (3.15). Note that, in the present (Markovian) framework, each process $X = \mathrm{HVA}^f, \mathrm{EC}, \mathbb{V}a\mathbb{R}.(\mathcal{L}_{\cdot'} - \mathcal{L}_{\cdot})$ and KVA satisfies

$$X_t = \widetilde{X}(t, S_t) = J^s_t \widetilde{X}(t, \widetilde{S}_t),$$

where $\tilde{S}$ is the auxiliary Black-Scholes model (B.1) and $\widetilde{\mathrm{HVA}}^f(t,0) = \widetilde{\mathbb{V}a\mathbb{R}}(t,0) = \widetilde{\mathrm{EC}}(t,0) = \widetilde{\mathrm{KVA}}(t,0) = 0$, while, for all $(t, S) \in [0,T] \times (0,\infty)$, setting $t' = (t+1) \wedge T$,

$$\begin{aligned}
&\widetilde{\mathrm{HVA}}^f(t,S) = \mathbb{E}\left[f_T - f_t \,|\, S_t = S\right],\\
&\widetilde{\mathbb{V}a\mathbb{R}}(t,S) = \mathbb{V}a\mathbb{R}\left[\mathcal{L}_{t'} - \mathcal{L}_t \,|\, S_t = S\right],\\
&\widetilde{\mathrm{EC}}(t,S) = \mathbb{ES}\left[\mathcal{L}_{t'} - \mathcal{L}_t \,|\, S_t = S\right],\\
&\widetilde{\mathrm{KVA}}(t,S) = \mathbb{E}\left[h\int_t^T (\mathrm{EC}_u - \mathrm{KVA}_u)^+ \, du \,\middle|\, S_t = S\right].
\end{aligned} \tag{3.16}$$

We first perform a Monte-Carlo with $M = 50,000$ paths to estimate HVA^f_0 and, as a sanity check, $\mathrm{HVA}^{\cdot}_0$, already known from (3.11) and (3.10). We can see from Figure 3.3, where the horizontal red line corresponds to $\mathrm{HVA}^{\cdot}_0 = 1 - e^{-0.1}$, that $\mathrm{HVA}^{\cdot}_0$ dominates over HVA^f_0.

We then compute the HVA and KVA processes at all nodes of a forward simulated grid $(S^m_{t_k})^{0 \leq k \leq 10}_{1 \leq m \leq M}$ of S, backward in time by neural net regressions that are used for solving the corresponding equations numerically.

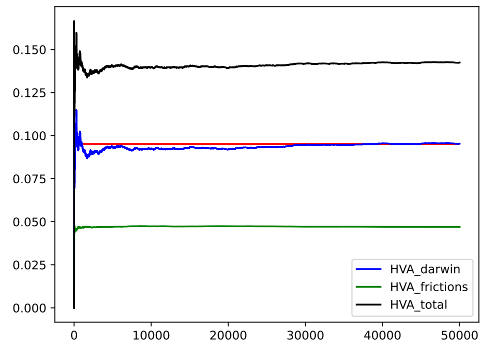

Figure 3.3: Monte-Carlo approximation of $\mathrm{HVA}_0^{\cdot}$ and HVA_0^f.

Namely, the function $\widetilde{\mathrm{HVA}}^f$ in (3.16) is such that $\widetilde{\mathrm{HVA}}^f(T,\cdot)=0$, $\widetilde{\mathrm{HVA}}^f(t,0)=0$ for all t and, for $u<t$ and $S\in(0,\infty)$,

$$\begin{aligned}\widetilde{\mathrm{HVA}}^f(u,S)&=\mathbb{E}\left[(f_t-f_u)+\widetilde{\mathrm{HVA}}^f(t,S_t)\,\middle|\,S_u=S\right]\\&=\mathbb{E}\left[(f_t-f_u)+\widetilde{\mathrm{HVA}}^f(t,S_t)\mathbb{1}_{\{S_t>0\}}\,\middle|\,S_u=S\right].\end{aligned}\tag{3.17}$$

Accordingly, we approximate on $(0,\infty)$ the functions $\widetilde{\mathrm{HVA}}^f(t_i,\cdot)$ for $t_i:=i\frac{T}{10}$, as follows. Set $\widehat{\mathrm{HVA}}^f(t_{10},\cdot)=0$ and assume that we have already trained neural networks $\widehat{\mathrm{HVA}}^f(t_k,\cdot)$, $i+1\le k<10$. Based on sampled data

$$(X,Y)=\left(\widetilde{S}^m_{t_i},(f_{t_{i+1}}-f_{t_i})^m+\widehat{\mathrm{HVA}}^f(t_{i+1},S^m_{t_{i+1}})\mathbb{1}_{\{S^m_{t_{i+1}}>0\}}\right)_{1\le m\le M},$$

where each $S^m_{t_{i+1}}$ is a obtained from (3.1) with initial condition $S^m_{t_i}=\widetilde{S}^m_{t_i}>0$ simulated from (B.1), in view of (3.17) and of the L_2 projection characterization of conditional expectation (in the square integrable case), we seek for $\widehat{\mathrm{HVA}}^f(t_i,\cdot)$ in

$$\mathrm{Argmin}_{\varphi\in\mathcal{NN}}\sum_{m=1}^{M}\left(\widehat{\mathrm{HVA}}^f(t_{i+1},S^m_{t_{i+1}})\mathbb{1}_{\{S^m_{t_{i+1}}>0\}}+\left(f_{t_{i+1}}-f_{t_i}\right)^m-\varphi(\widetilde{S}^m_{t_i})\right)^2,\tag{3.18}$$

where $\mathcal{NN}$ denotes the set of feedforward neural networks with three hidden layers of 10 neurons each and ReLU activation functions. We then obtain $\widehat{\mathrm{HVA}}^f(0,S_0)=0.04613$ from $f_{t_1}+\widehat{\mathrm{HVA}}^f(t_1,S_{t_1})$ as a sample mean. The corresponding standard deviation, 95% confidence interval and relative error at 95% are $\hat{\sigma}^f\simeq 6\times10^{-3}$, $[0.04601,0.04624]$ and $\frac{1.96\hat{\sigma}^f}{\widehat{\mathrm{HVA}}^f_0\sqrt{M}}\simeq 0.25\%$, where $\hat{\sigma}^f$ denotes the empirical standard deviation of $f_1+\widehat{\mathrm{HVA}}^f(t_1,S_{t_1})$.

Next we approximate $\widetilde{\mathrm{EC}}(t,\cdot)$ on $(0,\infty)$ by the 2-step scheme of (Barrera et al., 2022, Section 4.3), for each $t = t_i, 1 \le i < 10$. Recall $t' = (t+1)\wedge T$. We first train a neural network $\widehat{\mathbb{V}a\mathbb{R}}(t,\cdot)$ approximating $\widetilde{\mathbb{V}a\mathbb{R}}(t,\cdot)$ based on sampled data $(X,Y) = \Big(\widetilde{S}_t^m, (\mathcal{L}_{t'}-\mathcal{L}_t)^m\Big)_{1\le m\le M}$ and on the pinball-type loss[60] $(y-\varphi(x))^+ + (1-\alpha)\varphi(x)$, i.e. we seek for $\widehat{\mathbb{V}a\mathbb{R}}(t,\cdot)$ in

$$\mathrm{Argmin}_{\varphi\in\mathcal{NN}}\,\frac{1}{M}\sum_{m=1}^{M}\Big((\mathcal{L}_{t'}-\mathcal{L}_t)^m - \varphi(\widetilde{S}_t^m)\Big)^+ + (1-\alpha)\varphi(\widetilde{S}_t^m).$$

Note from (3.13) that, for $t = t_i$, sampling $\mathcal{L}_{t'}-\mathcal{L}_t$ uses the already trained neural network $\widehat{\mathrm{HVA}}^f(t_{i+1},\cdot)$. We also compute $\widehat{\mathbb{V}a\mathbb{R}}(0,S_0) = 0.0120$ as an empirical (unconditional) value-at-risk. The corresponding 95% confidence interval and relative error at 95% are $[0.0117, 0.0123]$ and $\frac{1.96}{\widehat{\mathbb{V}a\mathbb{R}}(0,S_0)\hat{f}(\widehat{\mathbb{V}a\mathbb{R}}(0,S_0))}\sqrt{\frac{\alpha(1-\alpha)}{M}} \simeq 2.3\%$, where $\hat{f}$ denotes the empirical density of $\mathcal{L}_{t_1}-\mathcal{L}_{t_0}$. For $t = 1yr$ (where the approximation should be the worst due to accumulated error on $\widetilde{\mathrm{HVA}}^f$ from dynamic programming), the Monte Carlo estimate of (Barrera et al., 2022, (4.10)) for the distance in p-values between the estimate $\widehat{\mathbb{V}a\mathbb{R}}(t,S_t)$ and the targeted (unknown) $\mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'}-\mathcal{L}_t)$ is less than $3.6\times 10^{-3} \le 1-\alpha = 10^{-2}$ with 95% probability.

We then train neural networks $\widehat{\mathrm{EC}}(t,\cdot)$ approximating $\widetilde{\mathrm{EC}}(t,\cdot)$ on $(0,\infty)$ at times $t = t_i$ based on sampled data $(X,Y) = \Big(\widetilde{S}_t^m, (\mathcal{L}_{t'}-\mathcal{L}_t)^m\Big)_{1\le m\le M}$ and on the loss

$$\left(\frac{1}{1-\alpha}(y-\widehat{\mathbb{V}a\mathbb{R}}(t,x))^+ + \widehat{\mathbb{V}a\mathbb{R}}(t,x) - \varphi(x)\right)^2,$$

i.e. we seek for $\widehat{\mathrm{EC}}(t,\cdot)$ in

$$\mathrm{Argmin}_{\varphi\in\mathcal{NN}}\,\frac{1}{M}\sum_{m=1}^{M}\left(\frac{1}{1-\alpha}\Big((\mathcal{L}_{t'}-\mathcal{L}_t)^m - \widehat{\mathbb{V}a\mathbb{R}}(S_t^m)\Big)^+ + \widehat{\mathbb{V}a\mathbb{R}}(S_t^m) - \varphi(x)\right)^2.$$

We also compute $\widehat{\mathrm{EC}}(0,S_0) = 0.493$ using the recursive algorithm of Costa and Gadat (2021, Eqn (4)). Using the central limit theorem for expected shortfalls derived in Costa and Gadat (2021, Theorem 1.3), a 95% confidence interval is $[0.451, 0.534]$ and the relative error at 95% is $\sqrt{\frac{b_M}{2}}\frac{1.96\hat{\sigma}^s}{(1-\alpha)\widehat{\mathrm{EC}}(0,S_0)} \simeq 0.08$, where $\hat{\sigma}^s$ denotes the empirical standard deviation of $(\mathcal{L}_1-\mathcal{L}_0)\mathbb{1}_{\{(\mathcal{L}_1-\mathcal{L}_0)>\widehat{\mathbb{V}a\mathbb{R}}(0,S_0)\}}$ and b_M is defined in (Costa and Gadat, 2021, Assumption H_{a_n,b_n}).

For $t = 1yr$, the Monte Carlo estimate of (Barrera et al., 2022, (4.8)) for the L_2-norm of the difference between the estimate $\widehat{\mathrm{EC}}(t,\widetilde{S}_t)$ and the targeted (unknown) $\widetilde{\mathrm{EC}}(t,\widetilde{S}_t)$ is smaller than 0.067 (itself significantly less then the orders of magnitude of EC visible on the left panels of Figure 3.4) with probability 95%.

[60] instead of the quadratic loss $(y-\varphi(x))^2$ in the previous conditional expectation case (3.18).

Last, we approximate $\widetilde{\mathrm{KVA}}(t,\cdot)$ at times $t = t_i$ on $(0,\infty)$, for i decreasing from 10 to 1, by neural networks $\widehat{\mathrm{KVA}}(t_i,\cdot)$, based on the following dynamic programming equation, for $0 \le i < 10$:

$$\begin{aligned}\mathrm{KVA}_{t_i} &= \mathbb{E}_{t_i}\left[\mathrm{KVA}_{t_{i+1}} + h\int_{t_i}^{t_{i+1}} (\mathrm{EC}_u - \mathrm{KVA}_u)^+ \, du\right] \\ &\approx \mathbb{E}_{t_i}\left[\mathrm{KVA}_{t_{i+1}} + h(t_{i+1}-t_i)\left(\mathrm{EC}_{t_{i+1}} - \mathrm{KVA}_{t_{i+1}}\right)^+\right].\end{aligned}$$

Starting from $\widehat{\mathrm{KVA}}(t_n,\cdot) = 0$ and having already trained the $\widehat{\mathrm{KVA}}(t_j,\cdot), j > i > 0$, we train $\widehat{\mathrm{KVA}}(t_i,\cdot)$ based on sampled data

$$\begin{aligned}(X,Y) = \bigg(\widetilde{S}^m_{t_i}, h(t_{i+1}-t_i)\Big(\widehat{\mathrm{EC}}(t_{i+1}, S^m_{t_{i+1}}) - \widehat{\mathrm{KVA}}(t_{i+1}, S^m_{t_{i+1}})\mathbb{1}_{\{S^m_{t_{i+1}}>0\}}\Big)^+ \\ + \widehat{\mathrm{KVA}}(t_{i+1}, S^m_{t_{i+1}})\mathbb{1}_{\{S^m_{t_{i+1}}>0\}}\bigg)_{1\le m\le M}\end{aligned}$$

and on the quadratic loss $(y-\varphi(x))^2$. We then compute $\widehat{\mathrm{KVA}}(0,S_0) = 0.407$ from $r(t_1-t_0)(\widehat{\mathrm{EC}}(t_1,S_{t_1}) - \widehat{\mathrm{KVA}}(t_1,S_{t_1})\mathbb{1}_{\{S_{t_1}>0\}}) + \widehat{\mathrm{KVA}}(t_1,S_{t_1})$ as a sample mean. The corresponding standard deviation, 95% confidence interval and relative error at 95% are $\hat{\sigma}^{kva} \simeq 6\times 10^{-2}$, $[0.4056, 0.4082]$ and $\frac{1.96\hat{\sigma}^{kva}}{\widehat{\mathrm{KVA}}_0\sqrt{M}} \simeq 0.0028$, where $\hat{\sigma}^{kva}$ denotes the empirical standard deviation of $r(t_1-t_0)(\widehat{\mathrm{EC}}(t_1,S_{t_1}) - \widehat{\mathrm{KVA}}(t_1,S_{t_1})\mathbb{1}_{\{S_{t_1}>0\}}) + \widehat{\mathrm{KVA}}(t_1,S_{t_1})$.

We plot on Figure 3.4 the processes $\widehat{\mathrm{EC}}(\cdot,\widetilde{S}_\cdot)$ and $\widehat{\mathrm{KVA}}(\cdot,\widetilde{S}_\cdot)$ represented by the term structures of their means (in green) and quantiles at level 10%, 90% (in blue) and 2.5% and 97.5% (in red), both with and without frictions f, as well as in the (deterministic) static hedging case (3.6). In particular, we obtain

$$\begin{aligned}&\widehat{\mathrm{HVA}}^{\cdot}_0 \simeq 0.095 \text{ and } \widehat{\mathrm{HVA}}^{f}_0 \simeq 0.046, \text{ hence } \widehat{\mathrm{HVA}}_0 \simeq 0.141, \\ &\widehat{\mathrm{KVA}}_0 \simeq 0.407, \ \frac{\widehat{\mathrm{KVA}}_0}{\widehat{\mathrm{HVA}}_0} \simeq 2.881.\end{aligned} \tag{3.19}$$

As could be expected from Remark 3.4 (see also Example 3.1 below), there is ultimately less risk (as assessed by economic capital and KVA, cf. (3.10) and Figure 3.4) with the delta hedge than with the static hedge.

In the frictionless case $f = 0$, we obtain by the same methodology

$$\begin{aligned}&\widehat{\mathrm{HVA}}_0 = \widehat{\mathrm{HVA}}^{\cdot}_0 \simeq 0.095, \\ &\widehat{\mathrm{KVA}}_0 \simeq 0.433, \ \frac{\widehat{\mathrm{KVA}}_0}{\widehat{\mathrm{HVA}}_0} \simeq 4.550.\end{aligned} \tag{3.20}$$

By comparison with (3.19) (see also Figure 3.4), the dynamic hedging frictions happen to be slightly risk-reducing, meaning that the components $-pnl^{\cdot} + \mathrm{HVA}^{\cdot}_{(0)}$ and $f + \mathrm{HVA}^{f}_{(0)}$ of (3.13) tend to be negatively correlated (for which we have no particular explanation).

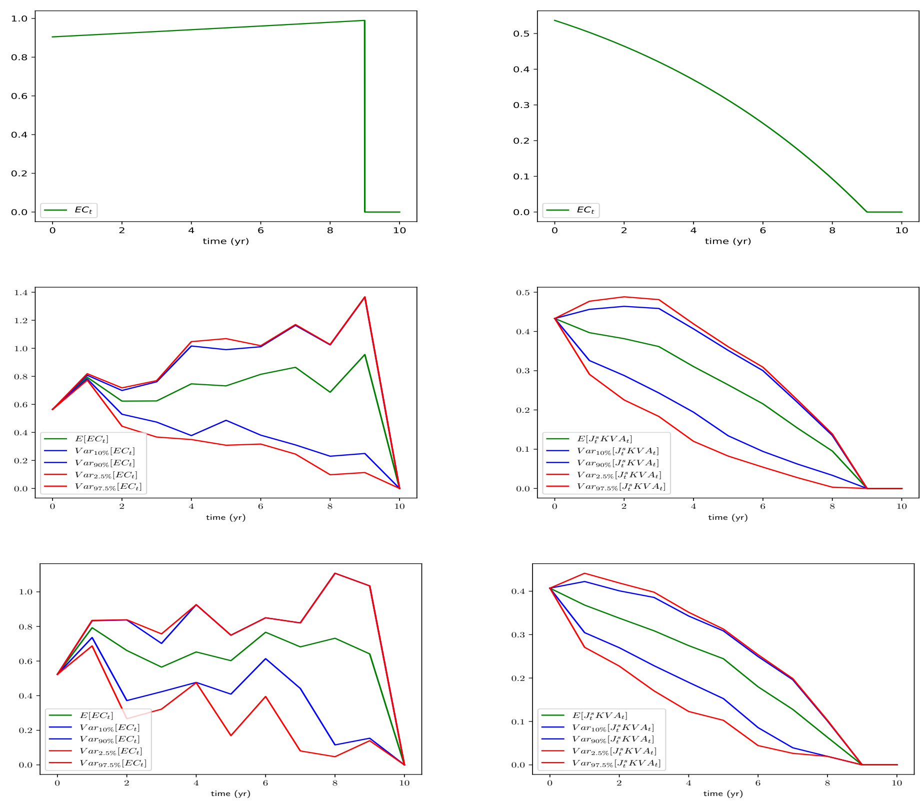

Figure 3.4: Plot of the deterministic maps $t \mapsto \widetilde{\mathrm{EC}}(t)$ [Top left] and $t \mapsto \widetilde{\mathrm{KVA}}(t)$ [Top right] corresponding to the static hedging case (3.6). Plots of mean (in green) and quantiles at levels 10% and 90% (in blue) and 2.5% and 97.5% (in red) of $\widehat{\mathrm{EC}}(t, \widetilde{S}_t)$ in the delta hedging case without friction [Middle left] and in the delta hedging case with frictions [Bottom left]. Plot of the deterministic map (3.6) for the static hedging case . Plots of mean (in green) and quantiles at levels 10% and 90% (in blue) and 2.5% and 97.5% (in red) of $\widehat{\mathrm{KVA}}(t, \widetilde{S}_t)$ in the delta hedging case without friction [Middle right] and in the delta hedging case with frictions [Bottom right].

3.3 Delta and Vega Hedging a More General Barrier Option

We finally consider a more general form of barrier option, delta-vega hedged by means of the underlying stock S and the European vanilla part of the option, i.e. the vanilla put $P = P^{jr}$ in the above. Denoting by $\Delta_t^{exo} = \partial_S Q^{bs}(t, S_t; \Sigma_t)$ and $\mathcal{V}_t^{exo} = \partial_\sigma Q^{bs}(t, S_t; \Sigma_t)$ the Black-Scholes implied delta and vega of the option and by $\Delta_t^{van} = \partial_S P^{bs}(t, S_t; \Sigma_t)$ (as before) and $\mathcal{V}_t^{van} = \partial_\sigma P^{bs}(t, S_t; \Sigma_t)$ the ones of the vanilla put, we now have (still assuming a default-free client but for a now nontrivial deactivating barrier crossing time $\tau^\cdot = \tau_e^\cdot$):

$$h_t^\cdot = \int_0^{t\wedge\tau_e^\cdot} (\zeta_s dS_s + \eta_s dP_s),\ t \leq T,$$

for càglàd[61] hedging ratios ζ and η in S and P such that

$$\eta_t \mathcal{V}_t^{van} = \mathcal{V}_t^{exo},\ \zeta_t + \eta_t \Delta_t^{van} = \Delta_t^{exo}. \tag{3.21}$$

This process $h^\cdot$ is again a zero-valued (assumed true) martingale and (2.13) and (3.3)[62] yield

$$\begin{aligned} pnl^\cdot &= \big(\mathcal{Q}^\cdot + J^s Q^{bs} + (1 - J^s)Q^{jr}\big)^{\tau_e^\cdot} - Q_0^{bs} - \int_0^{\cdot\wedge\tau_e^\cdot} (\zeta_t dS_t + \eta_t dP_t^{jr}),\\ \mathcal{H}^\cdot &= J^s(Q^{jr} - Q^{bs})^{\tau_e^\cdot} - (Q^{jr} - Q^{bs})_0 + \int_0^{\cdot\wedge\tau_e^\cdot} (\zeta_t dS_t + \eta_t dP_t^{jr}),\\ \mathrm{HVA}^\cdot &= \big(J^s(Q^{bs} - Q^{jr})\big)^{\tau_e^\cdot} - D^\cdot,\ \mathrm{HVA}_0^\cdot = (Q^{bs} - Q^{jr})_0 - D_0^\cdot, \end{aligned} \tag{3.22}$$

where, by (2.10),

$$D_t^\cdot = \mathbb{E}_t\big(\mathbb{1}_{\{\tau_e^\cdot < \tau_s^\cdot \wedge T\}} (Q^{bs} - Q^{jr})_{\tau_e^\cdot}\big),\ t \geq 0.$$

The above $\mathrm{HVA}^\cdot$ formula is in fact valid for any dynamic hedging scheme in S and P. In particular, vega hedging the option does not diminish $\mathrm{HVA}^\cdot$ (whereas it enhances f and HVA^f), nor necessarily the related KVA contribution: delta-vega hedging may be a better strategy before the model switch in terms of loss fluctuations, but this can be at the cost of a higher exposure at the model switch time τ_s.

Example 3.1. *When the option is the vulnerable put of Section 3.1 , then, in view of* (3.21)*, delta-vega hedging the option reduces to the static hedging strategy of Section 3.1, and we already saw after* (3.19) *that the ensuing risk is higher than the one triggered by the delta-hedging strategy.*

4 XVA Example

The 2013-2016 XVA debate revolved around the definition of suitable FVA (cost of funding) and KVA (cost of capital) metrics. The FVA number supposedly captures

[61] more precisely, left-limits of càdlàg processes, hence predictable and locally bounded processes (He et al., 1992, Theorem 7.7 1) page 192), so that stochastic integrals of such processes against local martingales are again local martingales (Protter, 2004, Theorem IV.29 page 173).

[62] but with $\mathcal{P}^\cdot = p^\cdot = P^\cdot = 0$, in the present purely dynamic hedging setup.

the present value of funding costs by projecting out into the future requirements and credit spreads. The KVA number[63] is a cost of capital metric designed as an overall proxy for shareholder return and for general guidance on a broad spectrum of strategic actions from hedging to executive compensation, credit limits, and dividend policy. Accordingly, FVA and KVA require computations at an even broader level than the netting-set CVA, i.e. at the funding-set level for the FVA and at the overall derivative portfolio (i.e. balance-sheet) level of the bank for the KVA[64].

Under current market practice, however, banks are calculating their funding costs by aggregation of client (or netting-set) specific numbers, which does not reflect the economics of collateral management. This arises from the desire of the banks to arrive at the FVA numbers by simply retrofitting their CVA calculators, which are based on distributed computing and are performed netting set by netting set, often with netting set specific approximations.

CVA and FVA related HVA model risk use cases as per Section 3 could be conducted using the path-wise XVA regression techniques of Abbas-Turki, Crépey, and Saadeddine (2022). In this section, instead, we analyze the limits and dangers of this strategy revealed by the Covid-19 financial crisis of Q1 2020.

4.1 FVA Proxies at the Test of the Covid-19

Quoting a statement on the website of the Bank of International Settlements[65]:

> The coronavirus (Covid-19) pandemic is a major disruptive event for the global economy. *It is revealing financial vulnerabilities and testing the post-financial crisis economic system.*

As we detail in what follows, the Covid-19 Q1 2020 financial crisis demonstrated the inappropriateness of using netting-set aggregation typical to CVA analytics for FVA and KVA computations.

Let γ_b and $x_c(\omega)$ respectively denote the bank credit spread and the (possibly negative) debt of client c to the bank in the scenario ω (skipped in the notation hereafter). Assuming for simplicity no collateral on the client portfolios of the bank, but perfect variation margining on their hedges (and no initial margins), the amount that needs be borrowed by the bank to be posted as collateral on its hedge is $(\sum_c x_c)^+$. The reason why the positive part sits above the sum is because variation margin is rehypothecable, i.e. fungible across netting sets (clients). In view of this, the economically correct cost of funding formula, which should be used both for decision taking and as a capital deduction by the bank (as the cost of its future funding expenses is a liability to the bank), is the asymmetric FVA $= \gamma_b \mathbb{E}[(\sum_c x_c)^+]$. Instead of this:

- for all their decision taking purposes, such as pricing and executives compensation, banks use

$$\gamma_b \mathbb{E} \sum_c x_c = \gamma_b \mathbb{E} \sum x_c^+ - \gamma_b \mathbb{E} \sum x_c^- = \text{FCA} - \text{FBA}; \tag{4.1}$$

[63] cf. (A.3).

[64] see the formulas in Albanese, Crépey, Hoskinson, and Saadeddine (2021, Appendix A) with proofs in Crépey (2022).

[65] see https://www.bis.org/topic/coronavirus.htm, last accessed on December 23, 2021.

- as a capital deduction, they use

$$\mathrm{FCA} = \gamma_b \mathbb{E} \sum x_c^+. \tag{4.2}$$

Indeed, regulators insist, rightly so, that only asymmetric, nonnegative numbers should be used for the purpose of calculating a capital deduction. They do not specify the aggregation level, which could be at the netting set or funding set level. They are indifferent, which from their point of view is understandable, as the smaller is the level of aggregation, the larger and more conservative is the size of capital deduction.

Now, in calm times:

- equity capital buffers are large enough to absorb the conservative capital deduction;
- banks' balance sheets are dominated by assets, i.e.

$$0 < \sum_c x_c = (\sum_c x_c)^+ \text{ holds in most scenarios,} \tag{4.3}$$

so

$$\gamma_b \mathbb{E} \sum_c x_c \approx \gamma_b \mathbb{E}[(\sum_c x_c)^+] \text{ , i.e. } \mathrm{FCA} - \mathrm{FBA} \approx \mathrm{FVA}. \tag{4.4}$$

The reason for (4.3) is that, if a corporate holds a bank payable, it typically has a desire to close it, receive cash, and restructure the trade with a par contract (the bank would agree to close the deal as a market maker, charging fees for the new trade). Because of this natural selection, in normal times, a bank is mostly in the receivables in its derivative business with corporates.

However, during the Covid-19 financial crisis:

- markdowns swinged bank balance sheets towards liabilities, invalidating (4.3) and the ensuing approximation (4.4);
- we saw an 8-fold credit spreads widening;
- increasing default rates put pressure on bank capital.

As a result:

- the FCA − FBA number (4.1) used for decision taking by banks went further and further from the correct FVA number, implying erroneous hedges and executive compensation;
- the FCA number (4.2) exploded and the corresponding capital reduction became needlessly punitive for banks, at the precise bad time where capital was becoming scarce for banks;
- the discrepancy between the (both wrong) FCA − FBA and FCA numbers increased, enhancing the corresponding misalignment of interest between the executives and the shareholders of the bank.

As FVA fluctuations contribute to economic capital[66], the cost of which is the KVA, wrong FVA computations then compromise the KVA computations. This is a perfect storm weather, through which only a mathematically and numerically rigorous treatment of accounting numbers, capital models and funding strategies can be of guidance. The intent of the cost-of-capital XVA approach of Crépey (2022), completed by the present paper, is precisely to define economically and mathematically correct accounting principles for dealer banks. Until banks adopt a correct accounting framework, they are exposed to major cash flows risk, like with FVA desks pricing through their fva metric a wrong $\mathcal{F}$, say fva $= \tilde{va}(\tilde{\mathcal{F}})$, with not only $\tilde{va} \neq va$ (wrong FVA pricing model), but $\tilde{\mathcal{F}} \neq \mathcal{F}$ (erroneous FVA originating cash flows in the first place), implying a very significant HVA^{fva}.[67]

The above even shows a schizophrenic use by the bank of two different (and both wrong) specifications (or "models") $\sum_c x_c$ and $\sum_c x_c^+$ of the same cash flow $(\sum_c x_c)^+$ for serving two different purposes within the bank: pricing and capital reduction. Instead, our high-level message in this work, in the line of Albanese, Crépey, and Iabichino (2021, 2022), is that the same good model should be used for every purposes within the bank.

Conclusion

The fact evidenced by Example 3.1 that vega hedging may actually increase Darwinian model risk is a striking illustration of the fact that (Darwinian, at least) model risk cannot be hedged. Model risk can only be provisioned against or, preferably, compressed, by improving the quality of the models that are used by traders. One way to incentivize the use of high-quality models by traders would be to forbid upticks, i.e. excluding time-0 departures from fair valuation unduly increasing the competitiveness of the bank. At least, an HVA reserve for model risk should also be risk-adjusted. But risk-adjusted HVA computations are also very demanding. In particular, beyond analytical toy examples such as the one of Section 3.1[68], HVA risk-adjusted KVA computations (and already HVA^f computations) require dynamic recalibration in a simulation setup, for assessing the hedging ratios used by the traders at future time points as well as the time of explosion of the trader's strategy (time of model switch) $\tau_s^{\cdot}$[69]

Even if dynamic recalibration is not necessarily out-of-scope with the help of the emerging machine learning calibration tools (Horvath, Muguruza, and Tomas, 2021), from the computational workload viewpoint too, the best practice would be that banks only rely on high-quality models, so that such computations are simply not needed. Morever, as the paper illustrates, a risk-adjusted reserve would be much greater than the uptick, by a factor 3 to 5 in our experiments[70], which could be even more if one accounted for liquidity impact[71]. In the end requiring risk-adjusted reserves for model risk would probably be tantamount to forbid upticks.

[66] cf. (A.1) and (2.2).

[67] cf. (2.15).

[68] and already in the case of Section 3.2.

[69] see Definition 2.3 and above.

[70] cf. (3.19), (3.20), and (3.10).

[71] cf. https://www.risk.net/derivatives/6556166/remembering-the-range-accrual-bloodbath effects, last accessed on 17 May 2022.

A Cost-of-Capital in a Nutshell

Definition A.1. *The economic capital (*EC*) of the bank is defined as the time-t conditional expected shortfall (*$\mathbb{ES}_t$*) of the random variable* $(\mathcal{L}_{t'} - \mathcal{L}_t)$ *at the confidence level* $\alpha \in (\frac{1}{2}, 1)$*, where* $\mathcal{L}$ *is the loss process* (2.2)-(2.4)[72] *of the bank and* $t' = (t+1) \wedge \overline{T}$*, i.e.*

$$\mathrm{EC}_t = \mathbb{ES}_t(\mathcal{L}_{t'} - \mathcal{L}_t) := \frac{\mathbb{E}_t\big((\mathcal{L}_{t'} - \mathcal{L}_t)\mathbb{1}_{\{\mathcal{L}_{t'} - \mathcal{L}_t \geq \mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t)\}}\big)}{\mathbb{E}_t \mathbb{1}_{\{\mathcal{L}_{t'} - \mathcal{L}_t \geq \mathbb{V}a\mathbb{R}_t(\mathcal{L}_{t'} - \mathcal{L}_t)\}}}. \tag{A.1}$$

Definition A.2. *We define the shareholder capital at risk (*SCR*), to be remunerated at a constant and nonnegative hurdle rate r, and the corresponding capital valuation adjustment (*KVA*) of the bank, as*

$$\mathrm{SCR} = \mathrm{EC} - \mathrm{KVA} = \max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA} = (\mathrm{EC} - \mathrm{KVA})^+, \tag{A.2}$$

where

$$\mathrm{KVA}_t = r\mathbb{E}_t \int_t^{\overline{T}} \big(\mathrm{EC}_s - \mathrm{KVA}_s\big)^+ ds,\ t \leq \overline{T}. \tag{A.3}$$

So KVA is the value process (2.1) of $\int_0^{\cdot} h\mathrm{SCR}_s ds$. By standard Lipschitz BSDE results[73], (A.3) defines a unique square integrable KVA process, assuming EC square integrable.

B Pricing Equations in the Jump-to-Ruin Model

In this section we provide pricing analytics in the *jr* model (3.1) for S. We also consider the auxiliary Black-Scholes model

$$d\tilde{S}_t = \lambda \tilde{S}_t dt + \sigma \tilde{S}_t dW_t, \tag{B.1}$$

starting from $\tilde{S}_0 = S_0$, where λ and σ (omitted in the notation for $d_\pm$ below when clear from the context) were introduced after (3.1). Hence

$$S_t = \mathbb{1}_{\{N_t = 0\}} \tilde{S}_t,\ t \geq 0. \tag{B.2}$$

Given the maturity $T = \overline{T}$ and strike $K > 0$ of an option, let, for every pricing time t and stock value S,

$$d_\pm(t, S; \lambda, \sigma) = \frac{\ln(\frac{S}{K}) + \lambda(T-t)}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}. \tag{B.3}$$

We first consider the pricing of a vanilla call option.

[72] or, equivalently, (2.17).

[73] valid in a general filtration (Kruse and Popier, 2016; Bouchard, Possamaï, Tan, and Zhou, 2018).

Proposition B.1. *The jr value process* (2.1) *of the call option with payoff* $(S_T - K)^+$ *at time* T *can be represented as*

$$C_t^{jr} = u(t, S_t)\mathbb{1}_{[0,T)},\ t \in [0, T],$$

where the pricing function $u = u(t, S) := \mathbb{E}\big((S_T - K)^+ \big| S_t = S\big)$ *is the unique classical solution*[74] *with linear growth in* S *to the PDE*

$$\begin{cases} u(T, S) = (S - K)^+,\ S \geq 0 \\ \partial_t u(t, S) + \lambda S \partial_S u(t, S) + \frac{\sigma^2 S^2}{2} \partial^2_{S^2} u(t, S) \\ \qquad\qquad -\lambda u(t, S) = 0,\ t < T,\ S \geq 0. \end{cases} \tag{B.4}$$

For $t < T$,

$$C_t^{jr} = S_t \mathcal{N}(d_+(t, S_t)) - K e^{-\lambda(T-t)} \mathcal{N}(d_-(t, S_t)). \tag{B.5}$$

Proof. We have $S_T = \mathbb{1}_{\{\tau_s > T\}} \tilde{S}_T = \mathbb{1}_{\{\tau_s > T\}} S_0 \exp\Big(\sigma W_T + (\lambda - \frac{\sigma^2}{2})T\Big)$. Since $(S_T - K)^+ = 0$ on $\tau_s \leq T$ and $S_T = \tilde{S}_T$ on $\tau_s > T$, it follows that, on $\{t < \tau_s\}$,

$$\begin{aligned} \mathbb{E}_t\left[(S_T - K)^+\right] &= \mathbb{E}_t\left[\mathbb{1}_{\{\tau_s > T\}}(S_T - K)^+\right] = \\ &= \mathbb{E}_t\left[\mathbb{1}_{\{\tau_s > T\}}(\tilde{S}_T - K)^+\right] = \mathbb{E}_t\left[e^{-\lambda(T-\tau)}(\tilde{S}_T - K)^+\right], \end{aligned} \tag{B.6}$$

by independence between W and N[75] in (2.1). One recognizes the probabilistic expression for the time-t price of the vanilla call option in the auxiliary Black-Scholes model (B.1), hence the proposition follows from standard Black-Scholes results. ∎

We now consider the pricing of a put option in the *jr* model, in two forms: either a vanilla put with payoff $(K - S_T)^+$, or a vulnerable put[76] with payoff $\mathbb{1}_{\{\tau_s > T\}}(K - S_T)^+$.

Proposition B.2. *The jr value process* (2.1) *of the vanilla put can be represented as*

$$P_t^{jr} = v(t, S_t)\mathbb{1}_{[0,T)},\ t \in [0, T], \tag{B.7}$$

where the vanilla put pricing function $v = v(t, S) := \mathbb{E}\big((K - S_T)^+ \big| S_t = S\big)$ *is the unique bounded classical solution to the PDE*

$$\begin{cases} v(T, S) = (K - S)^+,\ S \geq 0 \\ \partial_t v(t, S) + \lambda S \partial_S v(t, S) + \frac{\sigma^2 S^2}{2} \partial^2_{S^2} v(t, S) \\ \qquad\qquad -\lambda v(t, S) + \lambda K = 0,\ t < T,\ S \geq 0. \end{cases} \tag{B.8}$$

For $t < T$,

$$P_t^{jr} = K e^{-\lambda(T-t)} \mathcal{N}(-d_-(t, S_t)) - S_t \mathcal{N}(-d_+(t, S_t)) + K(1 - e^{-\lambda(T-t)}). \tag{B.9}$$

[74] of class $\mathcal{C}^{1,2}\big([0, T) \times [0, +\infty)\big) \cap \mathcal{C}^0\big([0, T] \times [0, +\infty)\big)$.

[75] independence always holds for a standard Brownian motion and a Poisson process on the same filtered probability space (He et al., 1992, Theorem 11.43).

[76] for a call option, vulnerable or not makes no difference in the *jr* model, where $S_T = (S_T - K)^+ = 0$ holds on $\{\tau_s \leq T\}$.

Proof. Taking expectation in the decomposition $S_T - K = (S_T - K)^+ - (S_T - K)^-$ yields the (model-free) call-put parity relationship

$$S_t - K = u(t, S_t) - v(t, S_t),\ t \le T, \tag{B.10}$$

hence $v = u - (S - K)$, from which the PDE characterization based on (B.8) for v results from the PDE characterization based on (B.4) for u. Moreover, we deduce from (B.5) that, for $t < T$,

$$\begin{aligned} P_t^{jr} &= C_t^{jr} - (S_t - K) = S_t\left(\mathcal{N}(d_+(t, S_t)) - 1\right) - K\left(e^{-\lambda(T-t)}\mathcal{N}(d_-(t, S_t)) - 1\right) \\ &= Ke^{-\lambda(T-t)}\mathcal{N}(-d_-(t, S_t)) - S_t\mathcal{N}(-d_+(t, S_t)) + K(1 - e^{-\lambda(T-t)}), \end{aligned}$$

which is (B.9). ∎

In accordance with (B.9):

Definition B.1. *For $t < \tau_s \wedge T$, given the observed spot price $S_t = S > 0$, the Black-Scholes implied volatility $\Sigma_t = \Sigma(t, S)$ of the vanilla put in the jr model is the unique solution Σ to*

$$\begin{aligned} &Ke^{-\lambda(T-t)}\mathcal{N}(-d_-(t, S; \lambda, \sigma)) - S\mathcal{N}(-d_+(t, S; \lambda, \sigma)) + K(1 - e^{-\lambda(T-t)}) \\ &\qquad = K\mathcal{N}(-d_-(t, S; 0, \Sigma_t)) - S\mathcal{N}(-d_+(t, S; 0, \Sigma_t)). \end{aligned} \tag{B.11}$$

We also set $\Sigma(t, 0) = 0$.

Remark B.1. *For $S = 0$, any $\Sigma \ge 0$ solves* (B.11)*: for any Σ, $d_\pm = -\infty$ as $\ln(\frac{0}{K}) = -\infty$, so $K\mathcal{N}(-d_-) - S\mathcal{N}(-d_+) = K - S = K$ (for $S = 0$).*

Proposition B.3. *The value process* (2.1) *of the vulnerable put is given by*

$$\begin{aligned} Q_t^{jr} &= \mathbb{1}_{t<\tau_s\wedge T}\big(P_t^{jr} - (1 - e^{-\lambda(T-t)})K\big) = \\ &\qquad \mathbb{1}_{t<\tau_s\wedge T}\Big(Ke^{-\lambda(T-t)}\mathcal{N}\big(-d_-(t, S_t)\big) - S_t\mathcal{N}\big(-d_+(t, S_t)\big)\Big). \end{aligned} \tag{B.12}$$

For $t < T$,

$$P_t^{jr} - Q_t^{jr} = \mathbb{1}_{t<\tau_s}K(1 - e^{-\lambda(T-t)}) + \mathbb{1}_{t\ge\tau_s}K. \tag{B.13}$$

Proof. We have

$$\mathbb{1}_{\{\tau_s>T\}}(S_T - K) = \mathbb{1}_{\{\tau_s>T\}}\big((S_T - K)^+ - (S_T - K)^-\big),$$

which in *jr* reduces to

$$S_T - \mathbb{1}_{\{\tau_s>T\}}K = (S_T - K)^+ - \mathbb{1}_{\{\tau_s>T\}}(S_T - K)^-.$$

By taking time-t conditional expectations, we have, on $\{t < \tau_s \wedge T\}$, that $S_t - Ke^{-\lambda(T-t)} = C_t^{jr} - Q_t^{jr}$, which yields

$$Q_t^{jr} = C_t^{jr} - S_t + Ke^{-\lambda(T-t)},$$

out of which (still on $\{t < \tau_s \wedge T\}$) the first identity in (B.12) follows from (B.10) and the second identity in turn follows from (B.9). Besides, on $\{t \ge \tau_s\}$, we have $Q^{jr} = 0$ and $P^{jr} = K$, whereas on $\{t \ge T\}$ we have $Q^{jr} = 0$, which completes the proof of (B.12) and (B.13). ∎

Proposition B.4. *Setting* $w(t,S) = v(t,S) - K(1 - e^{-\lambda(T-t)})$[77]*, the vulnerable put is replicable on* $[0, \tau_s \wedge T]$ *in the jr model (in the absence of model risk and hedging frictions), by the dynamic strategy* ζ *in* S *and* η *in the vanilla put*[78] *given by*

$$\zeta_t = -\frac{\mathcal{N}\big(-d_+(t,S_t)\big)}{1-\mathcal{N}\big(-d_-(t,S_t)\big)},\ \eta_t = -\frac{\mathcal{N}\big(-d_-(t,S_t)\big)}{1-\mathcal{N}\big(-d_-(t,S_t)\big)},\ t < \tau_s \wedge T, \tag{B.14}$$

and the number of constant riskless assets deduced from the budget condition $w(t, S_t)$ *on the strategy.*

Proof. The profit-and-loss associated with the hedging strategy ζ in S and η in the vanilla put[79], both assumed left-limits of càdlàg processes, evolves following (the position being assumed to be unwound at τ_s)

$$dpnl_t = \mathbb{1}_{\{t\le\tau_s\}}(dQ_t^{jr} - \zeta_t dS_t - \eta_t dP_t^{jr})$$

(with $pnl_0 = 0$). Itô formulas with (elementary) jump exploiting the results of Propositions B.2 and B.3 yield[80]

$$dpnl_t = \mathbb{1}_{\{t\le\tau_s\}}(\alpha_t dW_t + \beta_t dM_t),$$

where[81]

$$\alpha_t = \sigma S_t\Big(\partial_S w(t,S_{t-}) - \zeta_t - \eta_t\partial_S v(t,S_{t-})\Big),\ \beta_t = -w(t,S_{t-}) + \zeta_t S_{t-} + \eta_t\big(v(t,S_{t-}) - K\big).$$

Hence the replication condition $\alpha = \beta = 0$ reduces to the linear systems

$$\begin{aligned} &\partial_S w(t,S_{t-}) - \zeta_t - \eta_t\partial_S v(t,S_{t-}) = \\ &\qquad - w(t,S_{t-}) + \zeta_t S_{t-} + \eta_t\big(v(t,S_{t-}) - K\big) = 0 \end{aligned} \tag{B.15}$$

in the (ζ_t, η_t) (one system for each $t < \tau_s \wedge T$). Using (B.9) for the first line and (B.12) and (B.13) for the second line, one verifies that (B.14) solves (B.15). ∎

C The Limiting Friction Process for Proportional Transaction Costs

Our HVA encompasses nonlinear transaction costs à la Burnett (2021); Burnett and Williams (2021). In this section we derive a specification for the corresponding cumulative friction costs f[82], by passage to the continuous-time limit starting from a classical discrete-time specification. This is achieved in the setup of the following fair

[77] see Proposition B.2 and (B.13).
[78] both sought for as left-limits of càdlàg processes.
[79] and the quantity in the constant riskless asset deduced from the budget condition on the strategy.
[80] cf. (3.1).
[81] noting from the Itô isometry that $\int_0^{\cdot\wedge\tau_s}(\zeta_t\sigma S_t - \zeta_t\sigma S_{t-})dW_t = \int_0^{\cdot\wedge\tau_s}\big(\eta_t\sigma S_t\partial_S v(t,S_t) - \eta_t\sigma S_{t-}\partial_S v(t,S_{t-})\big)dW_t = 0$.
[82] cf. Section 2.3.

valuation model stated under the probability measure $\mathbb{R}$[83] with the risk-free asset as a numéraire, encompassing (3.1) for $\mathcal{X} := (X, J) = (\tilde{S}, N)$[84] and many more (including XVA) models as special cases[85]:

$$
\begin{aligned}
dX_t &= \mu(t, \mathcal{X}_t)dt + \sigma(t, \mathcal{X}_t)dW_t, \\
dJ_t &= \sum_{k=1}^{K} (k - J_{t-})d\nu_t^k, \\
\lambda_t^k &= \lambda_k(t, \mathcal{X}_{t-}),
\end{aligned}
\tag{C.1}
$$

where W is a multivariate Brownian motion and ν^k is a random measure counting the number of transitions of the "Markov chain like" component[86] J to the state k on $(0, \cdot]$, with compensated martingale $d\nu_t^k - \lambda_t^k dt$ of ν^k. We assume that the function-coefficients μ, σ, λ are continuous maps such that the above-model is well-posed, referring to Crépey (2013, Proposition 12.3.7) for a set of explicit assumptions ensuring it. In particular:

Assumption C.1. *1. The maps λ_k, $1 \le k \le K$, are bounded by a constant $\Lambda \ge 0$.*

2. The map $(t, x, k) \mapsto (\mu, \sigma)(t, x, k)$ is Lipschitz in $x \in \mathbb{R}^d$, uniformly in (t, k), and the map $(t, k) \mapsto (\mu, \sigma)(t, 0, k)$ is bounded.

Hence[87] there exists a constant $C_1 \ge 0$ such that

$$
\mathbb{E}\left[|X_t - X_s|^2\right]^{\frac{1}{2}} \le C_1 (t - s)^{\frac{1}{2}}. \tag{C.2}
$$

In addition, for all $1 \le l \le d$,

$$
C^l := \sup_{t \in [0,T]} \mathbb{E}\left[(X_t^l)^2\right]^{\frac{1}{2}} < +\infty. \tag{C.3}
$$

We denote, for any smooth map $\varphi = \varphi(t, x, k)$,

$$
\begin{aligned}
\mathcal{F}\varphi &:= \partial_t \varphi + \partial_x \varphi \mu + \frac{1}{2} \operatorname{tr}\left[\sigma\sigma^\top \partial_{x^2}^2 \varphi\right], \\
\mathcal{G}\varphi &:= \partial_t \varphi + \partial_x \varphi \mu + \frac{1}{2} \operatorname{tr}\left[\sigma\sigma^\top \partial_{x^2}^2 \varphi\right] + \sum_{k=1}^{K} \left(\varphi(\cdot, k) - \varphi\right) \lambda_k,
\end{aligned}
$$

where ∂_x is the row-gradient with respect to x, $\partial_{x^2}^2$ the Hessian matrix with respect to x and tr is the trace operator. We also abbreviate ∂_{x_l} into ∂_l, for all $1 \le l \le d$.

[83] or, as more precisely detailed after Definition 2.1, the bank survival probability measure associated with $\mathbb{R}$.

[84] cf. (B.2).

[85] subject to any relevant no arbitrage constraints regarding (C.1).

[86] but with transition probabilities modulated by X.

[87] see e.g. (Élie, 2006, (II.83) page 123).

C.1 Discrete Rebalancing

We assume that a trader values a hedging set as $q_t = q(t, \mathcal{X}_t)$, for a smooth map q, computed in the above model, and that the trader delta-hedges its position with respect to the d-dimensional risky asset X, discretely at the times of the uniform grid $(i\mathrm{h})_{0\le i\le n}$ with $\mathrm{h} = \frac{T}{n}$ for some $n \ge 1$, where T is the final maturity of this hedging set.

Remark C.1. *More generally, one could consider delta-hedging only some risky assets among all the underlyings X. The extension is straightforward, but we keep delta-hedging all the coordinates of X for notational simplicity.*

For all $t \in [0, T)$, we define

$$a_t = \left(a_t^l\right)_{1\le l\le d} \text{ with } a_t^l = \partial_l q(t, \mathcal{X}_t),\ 1 \le l \le d.$$

Assumption C.2. *The transaction cost to go from portfolio $a = \left(a^l\right)_{1\le l\le d}$ at time t to portfolio $a + \delta a = (a^l + \delta a^l)_{1\le l\le d}$ at time $t + \mathrm{h}$ is given by $X_{t+\mathrm{h}}^\top \delta a + \frac{1}{2} X_{t+\mathrm{h}}^\top \mathbf{k}(\delta a)^{\mathrm{abs}}\sqrt{\mathrm{h}}$, where $(\delta a)^{\mathrm{abs}} := (|\delta a_l|, 1 \le l \le d)$ and $\mathbf{k} := diag(\mathrm{k}_l, 1 \le l \le d)$ for some constants $\mathrm{k}_l \ge 0$, $1 \le l \le d$.*

The transaction costs are thus proportional to the risky assets price (measured in units of the risk-free asset price). In the context of proportional transaction costs, Assumption (C.2) is classical (Kabanov and Safarian, 2009, page 8).

Remark C.2. *Unless there is no Markov-chain-like component J involved in $\mathcal{X}$[88], the replication hedging ratios in such setups also involve finite differences (as opposed to partial derivatives only in the above): see e.g. Proposition B.4. However practitioners typically only use partial derivatives as their hedging ratios, motivating the present framework, which encompasses in particular the use-case of Section 3.2.*

The discrete-time hedging valuation adjustment for frictions ($\mathrm{HVA}^{\mathrm{h}}$) is then a process which aims at compensating the bank for these transaction costs.

Definition C.1. *The HVA for frictions associated to discrete hedging along the time-grid $(t_i^{\mathrm{h}} := i\mathrm{h})_{0\le i\le n}$ is defined as the (nonnegative) process $\mathrm{HVA}^{\mathrm{h}}$ such that $\mathrm{HVA}^{\mathrm{h}}_{t_n^{\mathrm{h}}} = 0$ and, for $0 \le i < n$,*

$$\begin{aligned} \mathrm{HVA}^{\mathrm{h}}_{t_i^{\mathrm{h}}} &= \mathbb{E}_{t_i^{\mathrm{h}}}\left[f^{\mathrm{h}}_{t_n^{\mathrm{h}}} - f^{\mathrm{h}}_{t_i^{\mathrm{h}}}\right] \\ &= \mathbb{E}_{t_i^{\mathrm{h}}}\left[f^{\mathrm{h}}_{t_{i+1}^{\mathrm{h}}} - f^{\mathrm{h}}_{t_i^{\mathrm{h}}} + \mathrm{HVA}^{\mathrm{h}}_{t_{i+1}^{\mathrm{h}}}\right] \\ &= \mathbb{E}_{t_i^{\mathrm{h}}}\left[\dot{f}^{\mathrm{h}}_{t_{i+1}^{\mathrm{h}}} + \mathrm{HVA}^{\mathrm{h}}_{t_{i+1}^{\mathrm{h}}}\right], \end{aligned} \tag{C.4}$$

where $f^{\mathrm{h}}_{t_i^{\mathrm{h}}} = \sum_{u=0}^{i} \dot{f}^{\mathrm{h}}_{u\mathrm{h}}$, with

$$\dot{f}^{\mathrm{h}}_{t_i^{\mathrm{h}}} = \frac{\sqrt{\mathrm{h}}}{2} X_{t_i^{\mathrm{h}}}^\top \mathbf{k}(\delta a_{t_i^{\mathrm{h}}})^{\mathrm{abs}}, \quad 0 < i < n, \quad \dot{f}^{\mathrm{h}}_0 = \dot{f}^{\mathrm{h}}_{t_n^{\mathrm{h}}} = 0,$$
$$\text{in which } (\delta a_{t_i^{\mathrm{h}}})^{\mathrm{abs}} = (|a^l_{t_i^{\mathrm{h}}} - a^l_{t_{i-1}^{\mathrm{h}}}|, 1 \le l \le d).$$

For $t \in [0, T]$, we set $\mathrm{HVA}^{\mathrm{h}}_{\lfloor \frac{t}{\mathrm{h}} \rfloor \mathrm{h}}$.

[88] i.e. for $K = 0$ in (C.1).

Remark C.3. *We neglect the transaction costs at time* $t = 0$*, given by (assuming* $d = 1$ *for simplicity)* $\sqrt{\mathrm{h}}\frac{\mathrm{k}}{2}X_0|a_0 - a_{0-}|$ *(where* a_{0-} *is the initial quantity of risky asset possessed before entering the deal), and at time* $t = T = t_n^{\mathrm{h}}$*, given by* $\sqrt{\mathrm{h}}\frac{\mathrm{k}}{2}X_T|a_{(n-1)\mathrm{h}}|$ *(to liquidate the hedging portfolio). In view of the numerical results of (Burnett and Williams, 2021, Figure 2, Table 2), in which their closeout HVA is negligible, we also neglect the liquidation costs occurring at the default time of a counterparty.*

Note that the last line in (C.4) yields a numerical scheme to recursively compute the discrete HVA process, backward in time starting from $\mathrm{HVA}^{\mathrm{h}}_{t_n^{\mathrm{h}}} = 0$.

C.2 Continuous-Time Rebalancing Limit

The next result specifies the HVA^f and the cumulative friction costs f[89] that arise in the above setup when the rebalancing frequency of the hedge goes to infinity, i.e. when $\mathrm{h} \to 0$.

For all $t \in [0, T]$, let

$$\mathrm{HVA}^f_t = \mathbb{E}_t\left[\int_t^T \dot{f}_s ds\right] \tag{C.5}$$

with $\dot{f} := \dot{f}(\cdot, \mathcal{X}.)$ and, for all $(t, x, k) \in [0, T] \times \mathbb{R}^d \times \{1, \cdots, K\}$,

$$\dot{f}(t, x, k) = \frac{1}{\sqrt{2\pi}} x^\top \mathbf{k}(\Gamma\sigma)^{\mathrm{abs}}(t, x, k), \tag{C.6}$$

where $(\Gamma\sigma)^{\mathrm{abs}} := (|\partial_x(\partial_l q)\sigma|, 1 \le l \le d)$. Note that the map HVA^f defined by $\mathrm{HVA}^f(t, x, k) := \mathbb{E}\left[\mathrm{HVA}^f_t \,\middle|\, \mathcal{X}_t = (x, k)\right]$ solves the PDE

$$\begin{aligned} &\mathrm{HVA}^f(T, \cdot) = 0 \text{ on } \mathbb{R} \times \{1, \ldots, K\}, \\ &(\partial_t + \mathcal{G})\mathrm{HVA}^f + \dot{f} = 0 \text{ on } [0, T) \times \mathbb{R} \times \{1, \ldots, K\}. \end{aligned} \tag{C.7}$$

We make the following technical hypothesis on the model coefficients:

Assumption C.3. *There exists* $0 < \alpha < \frac{1}{2}$ *such that, for all* $1 \le l \le d$ *and* $1 \le k \le K$*, the maps* $(t, x) \mapsto \partial_l q(t, x, k)$ *and* $(t, x) \mapsto (\partial_x(\partial_l q)\sigma(t, x, k)$ *is* α*-Hölder continuous in* t *and Lipschitz continuous in* x*. In addition, there exists* $C_2 > 0$ *such that, for any* $\varphi \in \{\partial_l q, (\partial_x(\partial_l q)\sigma \,|\, 1 \le l \le d, 1 \le k \le K\}$*,*

$$\sup_{(t,x,k,j)} |\varphi(t, x, k) - \varphi(t, x, j)| \le C_2 < \infty.$$

In addition, we assume that, for all $1 \le l \le d$*,*

$$\sup_{t \in [0,T]} \mathbb{E}\left[|(\partial_t + \mathcal{F})(\partial_l q)(t, \mathcal{X}_t)|^2\right]^{\frac{1}{2}} \le C_2 < \infty.$$

Remark C.4. *These assumptions, which are not minimal, are satisfied for regularized payoffs of vanilla European claims, in the jr setup* (3.1)*.*

[89] cf. Section 2.3.

Lemma C.1. *Under Assumptions C.1 and C.3, there exists $C_3 > 0$ such that, for all* $\mathrm{h} > 0$ *and* $\varphi \in \{\partial_l q, (\partial_x(\partial_l q)\sigma \,|\, 1 \leq l \leq d, 1 \leq k \leq K\}$,

$$\sup_{0<t-s<\mathrm{h}} \mathbb{E}\left[|\varphi(t,\mathcal{X}_t) - \varphi(s,\mathcal{X}_s)|^2\right]^{\frac{1}{2}} \leq C_3 \mathrm{h}^{\alpha}.$$

Proof. Since

$$1 - \prod_{k=1}^{K} \mathbb{1}_{\{\nu_t^k = \nu_s^k\}} \leq \sum_{k=1}^{K} (\nu_t^k - \nu_s^k),$$

we have, for some constant $C \geq 0$ varying from line to line,

$$\begin{aligned}
&\mathbb{E}\left[|\varphi(t,\mathcal{X}_t) - \varphi(s,\mathcal{X}_s)|^2\right]^{\frac{1}{2}} \\
&\leq \sqrt{2}\mathbb{E}\left[|\varphi(t,X_t,J_t) - \varphi(t,X_t,J_s)|^2\right]^{\frac{1}{2}} + \sqrt{2}\mathbb{E}\left[|\varphi(t,X_t,J_s) - \varphi(s,X_s,J_s)|^2\right]^{\frac{1}{2}} \\
&\leq C\left(\mathbb{E}\left[|\varphi(t,X_t,J_t) - \varphi(t,X_t,J_s)|^2 \sum_{k=1}^{K}(\nu_t^k - \nu_s^k)\right]^{\frac{1}{2}} + (t-s)^{\alpha} + (t-s)^{\frac{1}{2}}\right) \\
&\leq C\left(\sum_{k=1}^{K} \mathbb{E}\left[\int_s^t \lambda_r^k dr\right]^{\frac{1}{2}} + (t-s)^{\alpha}\right) \\
&\leq C\left(K\sqrt{\Lambda(t-s)} + (t-s)^{\alpha}\right) \leq C_3 \mathrm{h}^{\alpha},
\end{aligned}$$

where we used equation (C.2) and the bound on the maps λ^k. ∎

Theorem C.1. *Under Assumptions C.1, C.2 and C.3, we have, for all $t \in [0,T]$,*

$$\mathrm{HVA}_t^{\mathrm{h}} \rightarrow_{\mathrm{h}\rightarrow 0} \mathrm{HVA}_t^{f}.$$

Proof. We have, for $t = 0$ for notational simplicity,

$$\begin{aligned}
&\left|\mathrm{HVA}_0^{\mathrm{h}} - \mathrm{HVA}_0^{f}\right| = \left|\mathbb{E}\left[\sum_{i=1}^{n} \dot{f}^{\mathrm{h}}_{t_i^{\mathrm{h}}}\right] - \mathbb{E}\left[\int_0^T \dot{f}_t dt\right]\right| \\
&= \left|\frac{\sqrt{\mathrm{h}}}{2}\mathbb{E}\left[\sum_{i=1}^{n} X_{t_i^{\mathrm{h}}}^{\top}\mathbf{k}(\delta a_{t_i^{\mathrm{h}}})^{\mathrm{abs}}\right] - \frac{1}{\sqrt{2\pi}}\mathbb{E}\left[\sum_{i=1}^{n}\int_{t_{i-1}^{\mathrm{h}}}^{t_i^{\mathrm{h}}} X_t^{\top}\mathbf{k}(\Gamma\sigma)^{\mathrm{abs}}(t,\mathcal{X}_t)dt\right]\right| \\
&\leq \sum_{l=1}^{d} \mathrm{k}_l \left|\frac{\sqrt{\mathrm{h}}}{2}\mathbb{E}\left[\sum_{i=1}^{n} X_{t_i^{\mathrm{h}}}^{l}\left|a_{t_i^{\mathrm{h}}}^{l} - a_{t_{i-1}^{\mathrm{h}}}^{l}\right|\right] - \frac{1}{\sqrt{2\pi}}\mathbb{E}\left[\sum_{i=1}^{n}\int_{t_{i-1}^{\mathrm{h}}}^{t_i^{\mathrm{h}}} X_t^{l}|\partial_x(\partial_l q)\sigma|(t,\mathcal{X}_t)dt\right]\right| \\
&= \mathrm{h}\sum_{l=1}^{d}\sum_{i=1}^{n} \mathrm{k}_l \left|\mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}} X_{t_i^{\mathrm{h}}}^{l}\left|a_{t_i^{\mathrm{h}}}^{l} - a_{t_{i-1}^{\mathrm{h}}}^{l}\right| - \frac{1}{\mathrm{h}\sqrt{2\pi}}\int_{t_{i-1}^{\mathrm{h}}}^{t_i^{\mathrm{h}}} X_t^{l}|\partial_x(\partial_l q)\sigma|(t,\mathcal{X}_t)dt\right]\right| \\
&\leq T\sum_{l=1}^{d} \mathrm{k}_l \sup_{0\leq s<t\leq T, t-s=\mathrm{h}} \left|\mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}} X_t^{l}\left|a_t^{l} - a_s^{l}\right| - \frac{1}{\mathrm{h}\sqrt{2\pi}}\int_s^t X_u^{l}|\partial_x(\partial_l q)\sigma|(u,\mathcal{X}_u)du\right]\right|.
\end{aligned}$$

We fix $1 \le l \le d$ and we show that

$$\sup_{t-s=\mathrm{h}} \left| \mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}} X_t^l \left| a_t^l - a_s^l \right| - \frac{1}{\mathrm{h}\sqrt{2\pi}} \int_s^t X_u^l |\partial_x(\partial_l q)\sigma|(u, \mathcal{X}_u) du \right] \right| \to_{\mathrm{h}\to 0} 0. \tag{C.8}$$

In fact, for all $0 \le s < t \le T$ such that $t - s = \mathrm{h}$,

$$\begin{aligned}
&\left| \mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}} X_t^l \left| a_t^l - a_s^l \right| - \frac{1}{\mathrm{h}\sqrt{2\pi}} \int_s^t X_u^l |\partial_x(\partial_l q)\sigma|(u, \mathcal{X}_u) du \right] \right| \\
&\le \frac{1}{2\sqrt{\mathrm{h}}} \left| \mathbb{E}\left[(X_t^l - X_s^l) \left| a_t^l - a_s^l \right| \right] \right| \\
&+ \frac{1}{2\sqrt{\mathrm{h}}} \left| \mathbb{E}\left[X_s^l \left(\left| a_t^l - a_s^l \right| - |\partial_x(\partial_l q)\sigma(s, \mathcal{X}_s)(W_t - W_s)| \right) \right] \right| \\
&+ \left| \mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}} X_s^l \left| \partial_x(\partial_l q)\sigma(s, \mathcal{X}_s)(W_t - W_s) \right| - \frac{1}{\mathrm{h}\sqrt{2\pi}} \int_s^t X_u^l |\partial_x(\partial_l q)\sigma|(u, \mathcal{X}_u) du \right] \right|
\end{aligned} \tag{C.9}$$

Regarding the first term in the r.h.s. of (C.9), we have, by Assumption C.1 and Lemma C.1,

$$\begin{aligned}
&\frac{1}{2\sqrt{\mathrm{h}}} \left| \mathbb{E}\left[\left(X_t^l - X_s^l \right) \left| a_t^l - a_s^l \right| \right] \right| \\
&\le \frac{1}{2\sqrt{\mathrm{h}}} \mathbb{E}\left[\left| X_t^l - X_s^l \right|^2 \right]^{\frac{1}{2}} \mathbb{E}\left[\left| a_t^l - a_s^l \right|^2 \right]^{\frac{1}{2}} \\
&\le \frac{C_1}{2} \mathrm{h}^{-\frac{1}{2}+\frac{1}{2}+\alpha} = \frac{C_1}{2} \mathrm{h}^{\alpha}.
\end{aligned} \tag{C.10}$$

We now consider the second term in the r.h.s. of (C.9). With $\delta\partial_l q(t, x, j, k) := \partial_l q(t, x, k) - \partial_l q(t, x, j)$ and C^l defined in (C.3), recalling that $|\delta\partial_l q(t, x, j, k)| \le C_2$ by Assumption C.3, we compute by Itô's formula:

$$\begin{aligned}
&\frac{1}{2\sqrt{\mathrm{h}}} \left| \mathbb{E}\left[X_s^l \left(\left| a_t^l - a_s^l \right| - |\partial_x(\partial_l q)\sigma(s, \mathcal{X}_s)(W_t - W_s)| \right) \right] \right| \\
&\le \frac{1}{2\sqrt{\mathrm{h}}} \mathbb{E}\left[X_s^l \int_s^t |(\partial_t + \mathcal{F})\partial_l q(u, \mathcal{X}_u)| \, du \right] + \sum_{k=1}^{K} \frac{1}{2\sqrt{\mathrm{h}}} \mathbb{E}\left[X_s^l \left| \int_s^t \delta\partial_l q(u, \mathcal{X}_u, k) d\nu_u^k \right| \right] \\
&+ \frac{1}{2\sqrt{\mathrm{h}}} \mathbb{E}\left[X_s^l \left| \int_s^t (\partial_x(\partial_l q)\sigma(u, \mathcal{X}_u) - \partial_x(\partial_l q)\sigma(s, \mathcal{X}_s)) \, dW_u \right| \right] \\
&\le \frac{C^l}{2} \mathbb{E}\left[\int_s^t |(\partial_t + \mathcal{F})\partial_l q(u, \mathcal{X}_u)|^2 \, du \right]^{\frac{1}{2}} + \frac{C_2}{2\sqrt{h}} \sum_{k=1}^{K} \mathbb{E}\left[X_s^l (\nu_t^k - \nu_s^k) \right] \\
&+ \frac{C^l}{2\sqrt{\mathrm{h}}} \mathbb{E}\left[\int_s^t |\partial_x(\partial_l q)\sigma(u, \mathcal{X}_u) - \partial_x(\partial_l q)\sigma(s, \mathcal{X}_s)|^2 \, du \right]^{\frac{1}{2}} \\
&\le \frac{C^l C_2}{2} \sqrt{\mathrm{h}} + \frac{C^l C_2 \Lambda K \sqrt{\mathrm{h}}}{2} + \frac{C^l C_3}{2} \mathrm{h}^{\alpha} \le C \mathrm{h}^{\alpha},
\end{aligned} \tag{C.11}$$

by

$$\begin{aligned}\mathbb{E}\left[X_s^l(\nu_t^k-\nu_s^k)\right] &= \mathbb{E}\left[X_s^l\mathbb{E}_s\left[\nu_t^k-\nu_s^k\right]\right] = \mathbb{E}\left[X_s^l\int_s^t\lambda_u^k du\right]\\ &\le \mathbb{E}\left[\left(X_s^l\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(\int_s^t\lambda_u^k du\right)^2\right]^{\frac{1}{2}}\\ &\le C^l\Lambda\mathrm{h},\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\int_s^t\left|(\partial_t+\mathcal{F})\partial_l q(u,\mathcal{X}_u)\right|^2du\right]^{\frac{1}{2}} &= \left(\int_s^t\mathbb{E}\left[\left|(\partial_t+\mathcal{F})\partial_l q(u,\mathcal{X}_u)\right|^2\right]du\right)^{\frac{1}{2}}\\ &\le\sqrt{\mathrm{h}}\sup_{t\in[0,T]}\mathbb{E}\left[\left|(\partial_t+\mathcal{F})\partial_l q(u,\mathcal{X}_u)\right|^2\right]^{\frac{1}{2}}\\ &\le C_2\sqrt{\mathrm{h}},\end{aligned}$$

and Lemma C.1.

We finally deal with the last term in the r.h.s. of (C.9). As $\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\,(W_t-W_s)$ has, conditionally on $\mathfrak{F}_s$, the law $\mathcal{N}\left(0,\mathrm{h}\left|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\right|^2\right)$, we have

$$\frac{1}{2\sqrt{\mathrm{h}}}\mathbb{E}\left[X_s^l\left|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\,(W_t-W_s)\right|\right]=\frac{1}{\sqrt{2\pi}}\mathbb{E}\left[X_s^l\left|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\right|\right].$$

We then obtain

$$\begin{aligned}&\left|\mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}}X_s^l\left|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\,(W_t-W_s)\right|-\frac{1}{\mathrm{h}\sqrt{2\pi}}\int_s^t X_u^l|\partial_x(\partial_l q)\sigma|(u,\mathcal{X}_u)du\right]\right|\\ &=\frac{1}{\sqrt{2\pi}}\left|\mathbb{E}\left[X_s^l\left|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\right|-\frac{1}{\mathrm{h}}\int_s^t X_u^l|\partial_x(\partial_l q)\sigma|(u,\mathcal{X}_u)du\right]\right|\\ &\le\frac{1}{\sqrt{2\pi}}\left|\mathbb{E}\left[X_s^l\left|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)\right|-\frac{1}{\mathrm{h}}\int_s^t X_u^l|\partial_x(\partial_l q)\sigma|(u,X_u,J_s)du\right]\right|\\ &+\frac{1}{\mathrm{h}\sqrt{2\pi}}\left|\mathbb{E}\left[\int_s^t X_u^l\left(|\partial_x(\partial_l q)\sigma|(u,X_u,J_s)-|\partial_x(\partial_l q)\sigma|(u,X_u,J_u)\right)du\right]\right|\\ &\le\frac{1}{\sqrt{2\pi}}\left|\mathbb{E}\left[X_s^l|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)|-X_r^l|\partial_x(\partial_l q)\sigma(r,X_r,J_s)|\right]\right|\\ &+\frac{C^l}{\sqrt{\mathrm{h}2\pi}}\mathbb{E}\left[\int_s^t\left(\partial_x(\partial_l q)\sigma(u,X_u,J_s)-\partial_x(\partial_l q)\sigma(u,X_u,J_u)\right)^2du\sum_{k=1}^K(\nu_t^k-\nu_s^k)\right]^{\frac{1}{2}},\end{aligned}$$

where the (random) $r\in(s,t)$ in the first term is obtained via the mean value theorem. For this first term, we have, for a constant C changing from term to term,

$$\begin{aligned}&\left|\mathbb{E}\left[X_s^l|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)|-X_r^l|\partial_x(\partial_l q)\sigma(r,X_r,J_s)|\right]\right|\\ &\le\mathbb{E}\left[|X_s^l-X_r^l||\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)|\right]+\mathbb{E}\left[X_r^l|\partial_x(\partial_l q)\sigma(s,\mathcal{X}_s)|-\partial_x(\partial_l q)\sigma(r,X_r,J_s)|\right]\\ &\le C_1h^{\frac{1}{2}}\sup_{t\in[0,T]}\mathbb{E}\left[|\partial_x(\partial_l q)\sigma(t,\mathcal{X}_t)|^2\right]^{\frac{1}{2}}+\mathrm{C}\mathrm{h}^\alpha\mathbb{E}\left[X_r^l\right]+C\mathbb{E}\left[X_r^l|X_r-X_s|\right]\\ &\le Ch^\alpha,\end{aligned}\tag{C.12}$$

as

$$\sup_{t\in[0,T]} \mathbb{E}\left[|\partial_x(\partial_l q)\sigma(t,\mathcal{X}_t)|^2\right]^{\frac{1}{2}} \le CT^\alpha + C^l + C \max_{1\le k\le K} |\partial_x(\partial_l q)\sigma(0,0,k)| < \infty.$$

Eventually,

$$\begin{aligned}&\frac{C^l}{\sqrt{\mathrm{h}2\pi}}\mathbb{E}\left[\int_s^t \left(\partial_x(\partial_l q)\sigma(u,X_u,J_s) - \partial_x(\partial_l q)\sigma(u,X_u,J_u)\right)^2 du \sum_{k=1}^K(\nu_t^k - \nu_s^k)\right]^{\frac{1}{2}} \\ &\frac{C^l C_2}{\sqrt{2\pi}}\mathbb{E}\left[\sum_{k=1}^K \int_s^t \lambda_u^k du\right]^{\frac{1}{2}} \le \frac{C^l C_2}{\sqrt{2\pi}}\sqrt{K\Lambda \mathrm{h}}.\end{aligned} \tag{C.13}$$

Using (C.10)-(C.11)-(C.12)-(C.13), we obtain, for some constant $C \ge 0$,

$$\sup_{t-s=\mathrm{h}} \left|\mathbb{E}\left[\frac{1}{2\sqrt{\mathrm{h}}}X_t^l \left|a_t^l - a_s^l\right| - \frac{1}{\mathrm{h}\sqrt{2\pi}}\int_s^t X_u^l |\partial_x(\partial_l q)\sigma|(u,\mathcal{X}_u)du\right]\right| \le C\mathrm{h}^\alpha \to_{\mathrm{h}\to 0} 0,$$

which proves Theorem C.1. ∎

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