

CVA computation for counterparty risk assessment in credit portfolios

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Abstract

We first derive a general counterparty risk representation formula for the Credit Value Adjustment (CVA) of a netted and collateralized portfolio. This result is then specified to the case, most challenging from the modeling and numerical point of view, of counterparty credit risk. Our general results are essentially model free. Thus, although they are theoretically pleasing, they do not immediately lend themselves to any practical computations. We therefore subsequently introduce an underlying stochastic model, in order to put the general results to work. We thus propose a Markovian model of portfolio credit risk in which dependence between defaults and the wrong way risk are represented by the possibility of simultaneous defaults among the underlying credit names. Specifically, single-name marginals in our model are themselves Markov, so that they can be pre-calibrated in the first stage, much like in the standard (static) copula approach. One can then calibrate the few model dependence parameters in the second stage. The numerical results show a good agreement of the behavior of EPE (Expected Positive Exposure) and CVA in the model with stylized features.

Keywords: Counterparty Credit Risk, CVA, collateralization, Markov Copula, Joint Defaults, CDS.

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1 Introduction

Counter party risk is one of the most primitive forms of risk associated with contracts (financial and otherwise) between two, or more counterparties. The understanding and ability to measure and manage counterparty risk associated with the over-the-counter (OTC) financial contracts have been growing along with the growing volume of the literature on the subject. The brief discussion, given below, of some of the works studying counterparty risk, is by no means complete and comprehensive; it is rather meant to give a flavor of what has been done in this area so far.

In Canabarro and Duffie [7] an introduction to methods used to measure, mitigate and price counterparty risk is given. In addition, the authors provide a discussion of use of the Monte Carlo simulation to measure counterparty risk. Also, they discuss practical calculation of CVA in the case of OTC derivatives, by considering currency and interest rate swap agreements between two defaultable counterparties.

A discussion of counterparty risk and credit mitigation techniques at the portfolio level is given in De Prisco and Rosen [20]. In particular the authors provide a discussion of how Monte Carlo simulation and approximation methods, such as add-on approaches (as in Basel I for example), as well as some analytical approximations, can be used to calculate various statistics related to the measurement of counterparty credit risk. The paper provides a discussion of practical implementation of collateral modeling, and the calculation of expected exposure in credit derivatives portfolios with wrong way risk.

In [22], Zhu and Pykhtin provide a discussion of the simulation of credit exposure and collateral modeling in the presence of a call period. They also discuss calculation of expected exposure (EE) and credit value adjustment (CVA) assuming the wrong way risk.

Redon [21] gives two different analytical methods to calculate expected exposure so to account for the wrong way risk. The first analytical method accounts for wrong way risk due to currency depreciation in the case of country risk. This method gives expected exposure as a weighted average of the expected exposure when assuming there is a country crisis and of the expected exposure when assuming there is no country crisis. The second analytical method is based on the Merton-like model of default risk (cf. [5]) and the calculation of expected exposure assuming a Brownian motion model for the mark-to-market value of the portfolio subject to counterparty risk. These assumptions allow the author to account for wrong way risk by correlating the Brownian motions used to model default and mark-to-market. Moreover using this modeling framework it was possible to derive analytical expressions for expected exposure.

In [14], Gibson provides both an analytical and a Monte Carlo simulation method to calculate expected exposure and expected positive exposure to a margined and collateralized counterparty. The analytical method is based on a Gaussian model for the mark-to-market value of the portfolio exposed to counterparty risk, while the Monte Carlo simulation is based on a Gaussian random walk. Using the Gaussian model the author investigates the dependence of a collateralized exposure on: the initial mark-to-market, the threshold, the re-margining period, the grace period and the minimum transfer amount.

Brigo and Capponi [6] consider bilateral counterparty risk. They provide a general formula for the representation of CVA in the absence of joint defaults. Using a tri-variate Gaussian copula for modeling the joint default of the investor, the counterparty and the underlying CDS name, they show that in the absence of spread volatility the CVA decreases as the correlation between the investor and the counterparty is close to one. Hence they conclude that pure contagion models could be inappropriate to model CVA as they underestimate the risk in a high correlation environment. Moreover it is shown that simple add on approaches cannot be used effectively to capture the behavior of CVA.

Calculation of CVA in the case of bilateral credit risk on a portfolio consisting of over the counter derivatives is considered in Gregory [16]. Using a Gaussian copula model and allowing for simultaneous defaults, the author finds that the contribution of simultaneous default (which represents systematic risk) to the calculation of CVA is not significant. The paper concludes with a cautionary note regarding the use of bilateral CVA.

In [19]. Lipton and Sepp study counterparty credit risk in CDS contracts via a structural model based on the jump diffusion process. They develop original methods for computation of CVA in this context.

1.1 What is done in this paper

We consider in this paper several issues related to the valuation and mitigation of counterparty credit risk (CCR henceforth) on an OTC credit derivative contract, or, more generally, on a portfolio of OTC credit derivatives, written between two counterparties, relative to a pool of credit names. In particular, we study bilateral counterparty risk on a CDS, and unilateral counterparty risk on a portfolio of CDSs, in a suitable model of dependence for the underlying default times. In addition, we consider the issues of mitigation of counterparty credit risk by means of netting and/or collateralization (or margining).¹

Remark 1.1 We need to stress though that in order to simplify our presentation we give a highly stylized model for the collateral process. In particular we do not explicitly account for such aspects of the collateral formation as

- haircut provisions,
- margin period of risk,
- minimum transfer amounts,
- discrete tenor re-margining,
- various classes of assets used as collateral,
- collateral thresholds.

We shall incorporate these important considerations into our model in a future paper.

In this context we discuss the problem of representation and computation of the credit valuation adjustment (CVA). It is generally accepted that (cf. [7], page 128) the *CVA of an OTC derivatives portfolio with a given counterparty is the market value of the credit risk due to any failure to perform on agreements with that counterparty*. Thus, essentially, we have that (cf. Section 2.1)

$$CVA = P - \Pi,$$

where

P = market value of the portfolio not accounting for the counterparty risk,

and

Π = market value of the portfolio with accounting for the counterparty risk.

¹We refer to *ISDA Collateral Guidelines (2005)* for a detailed discussion setting industry standards for collateral formation.

On the other hand, CVA can be represented as (cf. (6))

$$CVA = \mathbb{E}(\text{discounted PFED}),$$

where \mathbb{E} represents a suitable mathematical expectation, and PFED is the potential future exposure at default. We characterize the PFED random variable in Lemma 2.1.

Remark 1.2 It needs to be stressed that typically, when assessing the counterparty risk, practitioners begin with modeling the flow of so called potential future exposure (PFE) and then they derive from the PFE various measures of the counterparty risk, such as the CVA (cf. [20]). In this regard we proceed in an opposite direction: we start with CVA defined as $P - \Pi$ and then deduce the relevant potential future exposure, which we dubbed PFED. It also needs to be noted that there are various ways in which financial institutions define PFE. Our definition of PFED is in line of the way the potential future exposure is understood in [20] or in [22].

In case of a deterministic discount factor we provide yet another representation of the CVA in terms of an appropriate integral of the so called expected positive exposures (EPE); see (14) and (15).

1.1.1 Outline of the Paper

Section 2 presents preparatory results about general counterparty risk. In Section 3 these results are specified to the case of counterparty credit risk. All the developments done in these first two sections are essentially model free. Thus, although they are theoretically pleasing, they do not immediately lend themselves to any practical computations. In Section 4 we propose an underlying stochastic model, that is able to put the previous results to work, also guaranteeing a satisfactory performance of the proposed methodology. In Section 5 numerical results are presented and discussed.

The full version of this paper will be included in the volume "Recent Advancements in the Theory and Practice of Credit Derivatives," edited by T.R. Bielecki, D. Brigo and F. Patras.

2 General Counterparty Risk

We consider two parties of a financial contract. We call them *the investor* and *the counterparty*. Furthermore, we denote by τ_{-1} and τ_0 the default times of the investor and of the counterparty, respectively. We shall study a unilateral counterparty risk from the perspective of the investor (i.e. $\tau_{-1} = \infty$ and $\tau_0 < \infty$), as well as the bilateral counterparty risk (i.e. $\tau_{-1} < \infty$ and $\tau_0 < \infty$).

We start by deriving a general (not specific to credit markets) representation formula for bilateral counterparty risk valuation adjustment for a fully netted and collateralized portfolio of contracts between the investor and his/her counterparty. This result can be considered as general since, for any partition of a portfolio into netted sub-portfolios, the results of this section may be applied separately to every sub-portfolio. The exposure at the portfolio level is then simply derived as the sum of the exposures of the sub-portfolios. This result is thus a generalization of that of Brigo and Capponi [6] to a possibly *collateralized* portfolio.

It needs to be emphasized that we do not exclude simultaneous defaults of the investor and his/her counterparty,² since our model in Section 4 precisely builds upon the possibility of simultaneous defaults as a way to account for defaults dependence. We do assume however that the default times

²We also allow for simultaneous defaults among the names in the underlying portfolio.

cannot occur at fixed times, as is for instance satisfied in all the intensity models of credit risk, such as the one introduced in Section 4.

For $i = -1$ or 0 , representing the two counterparties, let H^i and J^i stand for the default and non-default indicator processes of τ_i , so $H_t^i = \mathbf{1}_{\tau_i \leq t}, J_t^i = 1 - H_t^i$. We also denote $\tau = \tau_{-1} \wedge \tau_0$, with related default and non-default indicator processes denoted by H and J , respectively. In case where unilateral counterparty credit risk is considered, one simply sets $\tau_{-1} = +\infty$, so in this case $\tau = \tau_0$.

We fix the portfolio time horizon $T \in \mathbb{R}_+$, and we fix the underlying risk-neutral pricing model $(\Omega, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ such that τ_{-1} and τ_0 are \mathbb{F} -stopping times. One assumes that all the *processes* considered below are \mathbb{F} -adapted, all the *random variables* are \mathcal{F}_T -measurable, all the random times are $[0, T] \cup \{+\infty\}$ -valued, and all the semimartingales are càdlàg. We denote by \mathbb{E}_τ the conditional expectation under \mathbb{P} given \mathcal{F}_τ , for any stopping time τ . All the cash flows and prices (mark-to-market values of cash flows) are considered from the perspective of the investor. In accordance with the usual convention regarding *ex-dividend* valuation, \int_a^b is to be understood as $\int_{(a, b]}$ (so in particular $\int_a^b = 0$ whenever $a \geq b$).

Let D and \mathcal{D} represent the counterparty clean and counterparty risky cumulative dividend processes of the portfolio over the time horizon $[0, T]$, assumed to be finite variation processes. By *counterparty clean cumulative dividend process* we mean the cumulative dividend process that does not account for the counterparty risk, whereas by *counterparty risky cumulative dividend process* we mean the cumulative dividend process that does account for the counterparty risk.

Let β denote a finite variation and continuous *discount factor* process. The following definitions are consistent with the standard theory of arbitrage (cf. [10]).

Definition 2.1 (i) The *counterparty clean price process* (or *counterparty clean mark-to-market (MtM) process*) of the portfolio is given by $P_t = \mathbb{E}_t[p^t]$, where the random variable $\beta_t p^t$ represents the *cumulative discounted cash flows of the portfolio on the time interval $(t, T]$, not accounting for counterparty risk*, so, for $t \in [0, T]$,

$$\beta_t p^t = \int_t^T \beta_s dD_s . \quad (1)$$

(ii) The *counterparty risky MtM process* of the portfolio is given by $\Pi_t = \mathbb{E}_t[\pi^t]$, where the random variable π^t represents the *cumulative discounted cash flows of the portfolio adjusted for the counterparty risk* on the time interval $(t, T]$, so, for $t \in [0, T]$,

$$\beta_t \pi^t = \int_t^T \beta_s d\mathcal{D}_s . \quad (2)$$

Remark 2.2 Recall $\tau = \tau_{-1} \wedge \tau_0$, $J = 1 - H$. In the counterparty risky case there are no cash flows after $\tau \wedge T$, so the (\mathcal{F}_T -measurable) random variable π^t is in fact $\mathcal{F}_{\tau \wedge T}$ -measurable, and $\pi^t = \Pi_t = 0$ for $t \geq \tau \wedge T$.

We shall consider collateralized portfolios. In this regard we shall consider a *cumulative margin process* and we shall assume that no lump margin cash-flow can be asked for at time τ . Accordingly, given a finite variation cumulative margin process of the form $\gamma = J\mu + H\mu_{\tau-}$, so $\gamma_t = \mu_t \mathbf{1}_{t < \tau} + \mu_{\tau-} \mathbf{1}_{t \geq \tau}$ (note that γ does not jump at time τ), we define the *cumulative discounted margin process* by

$$\beta\Gamma = \int_{[0, \cdot]} \beta_t J_t d\gamma_t . \quad (3)$$

So, in particular, $\Gamma_0 = \gamma_0 - \gamma_{0-}$, and since J is killed at τ and γ does not jump at τ , one has for $\tau < \infty$,

$$\beta_\tau \Gamma_\tau = \int_{[0,\tau)} \beta_t d\gamma_t .$$

Comments 2.1 (i) Collateralization is a key modeling issue, particularly with regard to the development of centralized clearing houses for CDSs. Indeed, in the case of a centralized market, the counterparty risk is transferred from the market participants to the clearing house, and from the level of contracts or portfolios of contracts to the level of the margin calls. More precisely, the question becomes whether or not the clearing house will be able to face non recovered margin amounts called at times of defaults of markets participants. If the counterparty risk model of the clearing house is good enough so that the degree of confidence of the markets participants in the clearing house is sufficiently high, then the centralization of the market will have a beneficial effect. Otherwise the centralization may have little impact.

(ii) As postulated above, we consider just one collateral account, which may take either positive or negative value. Of course, a negative value from perspective of one of the two counterparties means a positive value from the perspective of the other counterparty, and vice versa.

(iii) The restriction that no margin can be asked for at time τ is of course motivated by the financial interpretation. In fact, to be more realistic in this regard, it would be important to introduce a notion of delay as in Zhu and Pykhtin [22].

(iv) Note that we are implicitly assuming in (3) that the margin account consists of cash which is put into a margin account and reinvested at the risk-free rate. According to the Basel Committee on Banking Supervision (cf. [2], page 31) other forms of collateral may be used, such as

- gold,
- debt securities rated by a recognized external credit assessment institution where these are either:
 - at least BB- when issued by sovereigns or PSEs that are treated as sovereigns by the national supervisor;
 - or at least BBB- when issued by other entities (including banks and securities firms);
 - or at least A-3/P-3 for short-term debt instruments,
- debt securities not rated by a recognized external credit assessment institution where these are:
 - issued by a bank;
 - and listed on a recognized exchange;
 - and classified as senior debt;
 - and all rated issues of the same seniority by the issuing bank that are rated at least BBB- or A-3/P-3 by a recognized external credit assessment institution; and the bank holding the securities as collateral has no information to suggest that the issue justifies a rating below BBB- or A-3/P-3 (as applicable) and;
 - the supervisor is sufficiently confident about the market liquidity of the security,
- equities (including convertible bonds) that are included in a main index.
- undertakings for Collective Investments in Transferable Securities (UCITS) and mutual funds.

However, according to the ISDA survey [15] it is mostly cash used for collateral, with some instances of sovereign bonds that are used as collateral. It is not clear whether credit derivatives have ever been or are used as collateral, although European Central Bank [13] indicates that CDS contracts can be used for this purpose. In a future work we shall investigate how well default sensitive instruments (such as CDS contracts) can mitigate, if used as a collateral, the value of default of the counterparty. (v) Here we take the margining process as given. In particular the issue of optimal collateralization (in some sense, cf. e.g. Aparicio and Cossin [1]) is not considered here, and is left for future research.

We assume for notational simplicity that γ and Γ are killed at T (so $\Gamma_t = \gamma_t = 0$ for $t \geq T$) and we define a random variable $\chi_{(\tau)}$ as

$$\chi_{(\tau)} = P_{(\tau)} + \Delta D_\tau - \Gamma_\tau \text{ if } \tau < T, \text{ and zero otherwise,} \quad (4)$$

in which, for $\tau < \infty$, $\Delta D_\tau = D_\tau - D_{\tau-}$ denotes the jump of D at τ ,³ and where the so called *legal value* $P_{(\tau)}$ is an \mathcal{F}_τ -measurable random variable representing an (ex-dividend) ‘fair value’ of the portfolio at time τ (in a sense to be specified later, cf. in particular Remark 2.4).

Let D^* denote the dividend process corresponding to the cash flows of D ‘stopped at $\tau-$ ’, that is

$$D^* = JD + HD_{\tau-} .$$

Assumption 2.1 The counterparty risky portfolio cumulative dividend process is given by

$$\begin{aligned} \mathcal{D} = & D^* + \Gamma_\tau H \\ & + (R_0 \chi_{(\tau)}^+ - \chi_{(\tau)}^-) [H, H^0] - (R_{-1} \chi_{(\tau)}^- - \chi_{(\tau)}^+) [H, H^{-1}] - \chi_{(\tau)} [[H, H^0], H^{-1}] , \end{aligned} \quad (5)$$

where the \mathcal{F}_{τ_0} – and $\mathcal{F}_{\tau_{-1}}$ – measurable random variables R_0 and R_{-1} , respectively, denote the recovery rates of the investor and of its counterparty upon default, and $[\cdot, \cdot]$ is the covariation process, i.e., in the case of pure jumps processes, it is the sum of products of common jumps.

Remark 2.3 Though the previous structural assumption on the cash flows makes perfect sense with respect to the financial interpretation of netting and margining, one may object that in case of the default of a counterparty the cash-flow (5) always has the same functional form regardless of what happens with the reference dividend process. For example, in case of modeling the counterparty risk relative to an underlying CDS contract, we postulate that the recovery structure in case of a default of a counterparty is the same regardless whether the counterparty defaults at the same time as the default time of the obligor referencing the CDS contract, or not. It might seem at first sight that this tacit assumption excludes *wrong way risk* [21] from the model, namely the risk that the value of the contract be particularly high from the perspective of the other party at the moment of default of the counterparty, which is a major issue regarding counterparty risk. In fact, the recovery *structure* in case of a default of a counterparty is the same, but the *ingredients* of the structure ($\chi_{(\tau)}$, specifically) may support wrong way risk, with for instance a significant chance that ΔD_τ in $\chi_{(\tau)}$ be more important than a ‘typical’ ΔD_t , as will be the case in our credit risk applications and models below.

2.1 CVA Representation Formula

As we shall now see, (cf. also, e.g., Brigo and Capponi [6]), one can represent the *Credit Valuation Adjustment* process $\text{CVA} := J(P - \Pi)$ in the form

$$\beta_t \text{CVA}_t = J_t \mathbb{E}_t [\beta_\tau \xi_{(\tau)}] , \quad (6)$$

³Recall that all processes are assumed to be càdlàg.

where $\xi_{(\tau)}$ is a suitable \mathcal{F}_τ -measurable random variable called *Potential Future Exposure at Default* (PFED).

Comments 2.2 (i) Since we consider the general case of the bilateral counterparty risk, the CVA process may take any real value. In case of the unilateral counterparty risk this process always takes either non-negative values or non-positive values, depending on which of the two parties entails the counterparty risk.

(ii) In general, the representation (6) does not uniquely define $\xi_{(\tau)}$. It is an open question under what assumptions on the model filtration \mathbb{F} the representation (6) uniquely defines $\xi_{(\tau)}$.

The following result is of interest:

Lemma 2.1 *A version of PFED is given by:*

$$\xi_{(\tau)} = P_\tau - P_{(\tau)} + (1 - R_0)\mathbf{1}_{\tau=\tau_0}\chi_{(\tau)}^+ - (1 - R_{-1})\mathbf{1}_{\tau=\tau_{-1}}\chi_{(\tau)}^- . \quad (7)$$

Proof. One has,

$$\begin{aligned} J_t \mathbb{E}_\tau \int_t^T \beta_s (dD_s - dD_s^*) &= J_t \mathbb{E}_\tau \int_{[\tau, T]} \beta_s dD_s \\ &= J_t \beta_\tau (P_\tau + \Delta D_\tau) = J_t \beta_\tau (P_\tau - P_{(\tau)} + \chi_{(\tau)} + \Gamma_\tau) . \end{aligned}$$

So, taking conditional expectation given \mathcal{F}_t ,

$$J_t \mathbb{E}_t \left\{ \int_t^T \beta_s (dD_s - dD_s^*) - \beta_\tau \Gamma_\tau \right\} = J_t \mathbb{E}_t \{ \beta_\tau (P_\tau - P_{(\tau)} + \chi_{(\tau)}) \} .$$

Thus, by Definition 2.1 and in view of (5),

$$\begin{aligned} J_t \beta_t (P_t - \Pi_t) &= J_t \mathbb{E}_t \{ \beta_\tau (P_\tau - P_{(\tau)} + \chi_{(\tau)}) \} \\ &\quad - J_t \mathbb{E}_t \left\{ \beta_\tau \left(\mathbf{1}_{\tau=\tau_0} (R_0 \chi_{(\tau)}^+ - \chi_{(\tau)}^-) - \mathbf{1}_{\tau=\tau_{-1}} (R_{-1} \chi_{(\tau)}^- - \chi_{(\tau)}^+) - \mathbf{1}_{\tau_0=\tau_{-1}} \chi_{(\tau)} \right) \right\} , \end{aligned}$$

which can be checked by inspection to coincide with $J_t \mathbb{E}_t [\beta_\tau \xi_{(\tau)}]$, with $\xi_{(\tau)}$ here defined by the RHS of (7). The result follows by definition (6) of a PFED. \square

In the rest of the paper we shall impose the following standing hypothesis,

Assumption 2.2 $P_{(\tau)} = P_\tau$.

Remark 2.4 This assumption appears to be considered as the current market standard. Arguably (see [8]), a more consistent choice, but also a more intensive one from the computational point of view, might be $P_{(\tau)} = \Pi_{\tau-}$. We observed in [8] that, in the simple reduced-form set-up of that paper, adopting either convention made little difference in practice. One should be aware that this might not be the case in different (like structural) set-ups however.

So, henceforth,

$$\xi_{(\tau)} = (1 - R_0)\mathbf{1}_{\tau=\tau_0}\chi_{(\tau)}^+ - (1 - R_{-1})\mathbf{1}_{\tau=\tau_{-1}}\chi_{(\tau)}^- , \quad (8)$$

with

$$\chi_{(\tau)} = P_\tau + \Delta D_\tau - \Gamma_\tau . \quad (9)$$

In case β is deterministic, the following alternative representation of the CVA at time 0 immediately follows from (6), (8):

$$\begin{aligned}
\beta_0 \text{CVA}_0 &= \mathbb{E}[\beta_\tau \xi_{(\tau)}] = \mathbb{E}\left(\beta_\tau(1 - R_0) \mathbf{1}_{\tau=\tau_0} \chi_{(\tau)}^+\right) - \mathbb{E}\left(\beta_\tau(1 - R_{-1}) \mathbf{1}_{\tau=\tau_{-1}} \chi_{(\tau)}^-\right) \\
&= \int_0^T \beta_s \mathbb{E}\left((1 - R_0) \mathbf{1}_{\tau=\tau_0, \tau_0 \in ds} \chi_{(\tau)}^+\right) - \int_0^T \beta_s \mathbb{E}\left((1 - R_{-1}) \mathbf{1}_{\tau=\tau_{-1}, \tau_{-1} \in ds} \chi_{(\tau)}^-\right) \\
&= \int_0^T \beta_s \mathbb{E}\left((1 - R_0) \chi_{(\tau)}^+ \mid \tau_0 = s, \tau_0 \leq \tau_{-1}\right) \mathbb{P}(\tau_0 \in ds, s \leq \tau_{-1}) \\
&\quad - \int_0^T \beta_s \mathbb{E}\left((1 - R_{-1}) \chi_{(\tau)}^+ \mid \tau_{-1} = s, \tau_{-1} \leq \tau_0\right) \mathbb{P}(\tau_{-1} \in ds, s \leq \tau_0) \\
&= \int_0^T \beta_s \text{EPE}_+(s) \mathbb{P}(\tau_0 \in ds, \tau_{-1} \geq s) - \int_0^T \beta_s \text{EPE}_-(s) \mathbb{P}(\tau_{-1} \in ds, \tau_0 \geq s)
\end{aligned} \tag{10}$$

where the *Expected Positive Exposures* EPE_\pm , also known as the *Asset Charge* and the *Liability Benefit*, respectively, are the functions of time defined by, for $t \in [0, T]$,

$$\text{EPE}_+(t) = \mathbb{E}\left[(1 - R_0) \chi_{(\tau)}^+ \mid \tau_0 = t \leq \tau_{-1}\right], \text{EPE}_-(t) = \mathbb{E}\left[(1 - R_{-1}) \chi_{(\tau)}^- \mid \tau_{-1} = t \leq \tau_0\right]. \tag{11}$$

Remark 2.5 Note that the following representation for CVA_0 may be better for computational purposes

$$\begin{aligned}
\beta_0 \text{CVA}_0 &= \int_0^T \int_s^T \beta_s \mathbb{E}\left((1 - R_0) \chi_{(\tau)}^+ \mid \tau_0 = s, \tau_{-1} = u\right) \mathbb{P}(\tau_0 \in ds, \tau_{-1} \in du) \\
&\quad - \int_0^T \int_s^T \beta_s \mathbb{E}\left((1 - R_{-1}) \chi_{(\tau)}^+ \mid \tau_{-1} = s, \tau_0 = u\right) \mathbb{P}(\tau_{-1} \in ds, \tau_0 \in du) \\
&= \int_0^T \int_s^T \beta_s \widetilde{\text{EPE}}_+(s, u) \mathbb{P}(\tau_0 \in ds, \tau_{-1} \in du) - \int_0^T \int_s^T \beta_s \widetilde{\text{EPE}}_-(u, s) \mathbb{P}(\tau_{-1} \in ds, \tau_0 \in du)
\end{aligned} \tag{12}$$

$$\widetilde{\text{EPE}}_+(t, r) = \mathbb{E}\left[(1 - R_0) \chi_{(\tau)}^+ \mid \tau_0 = t, \tau_{-1} = r\right], \widetilde{\text{EPE}}_-(r, t) = \mathbb{E}\left[(1 - R_{-1}) \chi_{(\tau)}^- \mid \tau_0 = r, \tau_{-1} = t\right]. \tag{13}$$

In the context of unilateral counterparty risk, i.e. $\tau_{-1} = \infty$, (10) of course reduces to:

$$\beta_0 \text{CVA}_0 = \int_0^T \beta_s \text{EPE}(s) \mathbb{P}(\tau \in ds), \tag{14}$$

where the *Expected Positive Exposure* EPE is the function of time defined by, for $t \in [0, T]$,

$$\text{EPE}(t) = \mathbb{E}[\xi_{(\tau)} \mid \tau = t] = \mathbb{E}\left[(1 - R_0) \chi_{(\tau)}^+ \mid \tau_0 = t\right]. \tag{15}$$

Remark 2.6 Frequently, in the case of unilateral counterparty risk, the EPE is computed under the assumption that $R_0 = 0$.

Remark 2.7 In a future paper we plan to study dynamics of the CVA process so to see how it reacts to varying credit quality of both counterparties, and, in case of counterparty credit risk, to the credit quality of the reference portfolio.

Comments 2.3 (i) As we already observed, the terminology used in the literature dealing with counterparty risk is quite fluid. For example, what we call *expected positive exposure*, and what we denote as $EPE(t)$, is frequently called *expected (conditional) exposure* and is denoted as $EE(t)$. Then, the expected positive exposure, over, say, the nominal lifetime of the portfolio, that is over the interval $[0, T]$, is defined as the average

$$\frac{1}{T} \int_0^T EE(t) dt.$$

(ii) CVA is one of the possible measures of the PFED.⁴ In the context of unilateral counterparty risk, alternative commonly used measurements of PFED are,

- The *effective EPE* (efEPE), computed at the time horizon one year ($1y$) as (to be compared with (14)),

$$\beta_0 \text{efEPE} = \mathbb{E}(\beta_\tau \xi_\tau | \tau < 1y) = \frac{\int_0^{1y} \beta_s \text{EPE}(s) \mathbb{P}(\tau \in ds)}{\mathbb{P}(\tau < 1y)}, \quad (16)$$

- The *exposure at default* profile $\text{EAD}(t)$ defined by, for t typically taken as the portfolio time horizon T , or, in case of credit limits with term structure (such as $1y$, $2y$, $5y\dots$), t varying in the term structure,

$$\beta_0 \text{EAD}(t) = \alpha \max_{s \in [0, t]} \mathbb{E}(\beta_\tau \xi_\tau | \tau < s) \quad (17)$$

for some ‘conservative’ factor $\alpha > 1$; or, alternatively to (17), no conservative factor α , but consideration of a high level quantile q_α (with, e.g., $\alpha = 95\%$) instead of the conditional expectation in (17), so,

$$\beta_0 \text{EAD}(t) = \max_{s \in [0, t]} q_\alpha (\beta_\tau \xi_\tau | \tau < s).$$

2.2 Collateralization Modeling

Modeling the margin process is very important too. In particular, for

$$\gamma_t = P_0 + \int_{(0, t]} J_s \beta_s^{-1} d(\beta_s P_s), \quad \Gamma_\tau = P_{\tau-}, \quad (18)$$

one gets,

$$\xi_{(\tau)} = \xi_{(\tau)}^1 = (1 - R_0) \mathbb{1}_{\tau=\tau_0} (\Delta(D + P)_\tau)^+ - (1 - R_{-1}) \mathbb{1}_{\tau=\tau_{-1}} (\Delta(D + P)_\tau)^-. \quad (19)$$

The choice $\Gamma_\tau = P_{\tau-}$ (cf. (18)) is, arguably, the extreme case of the collateral.

More generally, the good understanding of the principles of accumulation of the collateral is a crucial issue. Apart from the above extreme case, here is a possible suggestion in case of bilateral risk, when both counter-parties are contractually obliged to post the collateral, whenever called for.

We assume that adjustments of the account are done according to a discrete tenor, say $0 < t_1 < t_2 < \dots < t_n < T$. Here is the description of the mechanics of the collateral account, where $\Delta_{i,i+1}$ stands for $\Gamma_{t_{i+1}} - \frac{\beta_{t_i}}{\beta_{t_{i+1}}} \Gamma_{t_i}$:

⁴As already remarked above, in the literature the term *potential future exposure (PFE)* is used, rather than PFED, as well as the related term *PFE profile* (PFE as a function of future time). Accordingly, the terms: *measures of PFE* (cf. [11], page 6), and *risk measures of PFE profile* (cf. [20], page 11) are used.

- If $\Gamma_{t_i} > 0$ and if $\Delta_{i,i+1} > 0$, then counterparty ‘-1’ needs to post, at time t_{i+1} , the collateral amount equal to $\Delta_{i,i+1}$;
- If $\Gamma_{t_i} > 0$ and if $\Delta_{i,i+1} \leq 0$, then counterparty ‘-1’ receives back, at time t_{i+1} , the amount equal to $-\Delta_{i,i+1}$ from the collateral account in case $\Gamma_{t_{i+1}} > 0$; however, if $\Gamma_{t_{i+1}} \leq 0$ then counterparty ‘-1’ receives back, at time t_{i+1} , the amount equal to $\frac{\beta_{t_i}}{\beta_{t_{i+1}}} \Gamma_{t_i}$ from the collateral account, and counterparty ‘0’ posts, at time t_{i+1} , the collateral in the amount of $-\Gamma_{t_{i+1}} \leq 0$;
- If $\Gamma_{t_i} \leq 0$ and if $\Delta_{i,i+1} < 0$, then counterparty ‘0’ needs to post, at time t_{i+1} , the collateral amount equal to $-\Delta_{i,i+1}$;
- If $\Gamma_{t_i} \leq 0$ and if $\Delta_{i,i+1} \geq 0$, then counterparty ‘0’ receives back, at time t_{i+1} , the amount equal to $\Delta_{i,i+1}$ from the collateral account in case $\Gamma_{t_{i+1}} < 0$; however, if $\Gamma_{t_{i+1}} \geq 0$ then counterparty ‘0’ receives back, at time t_{i+1} , the amount equal to $-\frac{\beta_{t_i}}{\beta_{t_{i+1}}} \Gamma_{t_i}$ from the collateral account, and counterparty ‘-1’ posts, at time t_{i+1} , the collateral in the amount of $\Gamma_{t_{i+1}} \leq 0$.

Now, we come to the issue of how to compute the quantities Γ_{t_i} . Let V_t be the generic symbol for ‘the value’ of the contract’s exposure at time t due to bilateral counterparty risk. For example:

- $V_t = CVA_t^0$, where CVA_t^0 is the uncollateralized CVA, that is, CVA computed setting $\Gamma \equiv 0$,
- $V_t = \Pi_t^0$, where Π^0 is the uncollateralized risky MtM process,
- $V_t = P_t$, the intrinsic MtM process,
- V_t = the intrinsic MtM process adjusted for margin period of risk, for minimum transfer amount, for thresholds, for haircuts, etc.

Remark 2.8 At first sight it might seem reasonable to consider that the collateral cash flow should be somehow decided based on the fluctuations of the discounted CVA process, rather than based on the fluctuations of the MtM process. However, in view of (8)–(9) (under our standing assumption 2.2), the choice $V_t = P_t$ is probably the most appropriate.

The general idea for accumulating the collateral should be that the collateral amount changes in the same direction as the process V . For example, we might postulate that

$$\Gamma_{t_{i+1}} = \frac{\beta_{t_i}}{\beta_{t_{i+1}}} \Gamma_{t_i} + f(\beta_{t_i}, \beta_{t_{i+1}}, \Gamma_{t_i}, V_{t_i}, V_{t_{i+1}}), \quad (20)$$

where f is a function satisfying two general conditions:

$$\begin{aligned} f(\beta', \beta'', c, a, b) &\geq 0 \text{ if } \frac{\beta'}{\beta''} a < b, \quad f(\beta', \beta'', c, a, b) \leq 0 \text{ if } \frac{\beta'}{\beta''} a > b, \\ f(\beta', \beta'', c, a, b) &= 0 \text{ iff } \left| \frac{\beta'}{\beta''} a - b \right| \leq \epsilon \end{aligned}$$

for some $\epsilon > 0$.

Comments 2.4 (i) The above formulae need to be appropriately adjusted in case of the unilateral counterparty risk exposure. In particular, one will then need to take either positive or negative parts of V_t above, depending on the point of view.

(ii) In Zhu and Pykhtin [22], who consider the case of unilateral counterparty risk only, the collateral is computed using the following specifications,

- $V_t = \Pi_t$,
- $f(\beta', \beta'', c, a, b) = \max(a - H, 0) - \frac{\beta'}{\beta''} c$,

where, H represents a threshold, so that

$$\Gamma_{t_{i+1}} = \max(\Pi_{t_i} - H, 0).$$

3 Counterparty Credit Risk

We now specify the general setup presented above to the situation of the counterparty credit risk.

Towards this end we postulate that the contracts comprising the portfolio between the investor and his/her counterparty reference defaultable credit names. Next, we denote by τ_i , for $i = 1, \dots, n$, the default times of n credit names underlying the portfolio's contracts. We set $\mathbb{N}_n = \{-1, 0, 1, \dots, n\}$, $\mathbb{N}_n^* = \{1, \dots, n\}$. For $i \in \mathbb{N}_n$, we let H^i and J^i stand for the default and non-default indicator processes of τ_i (assumed to be an \mathbb{F} -stopping time), so $H_t^i = \mathbf{1}_{\tau_i \leq t}, J_t^i = 1 - H_t^i$. We assume that the default times τ_i 's cannot occur at fixed times. We do not however exclude simultaneous jumps of the credit names.

3.1 General Credit Risky Portfolio

We shall consider in this section counterparty credit risk associated with portfolios of contracts between two counter-parties and referencing *single underlyings*. Let thus the portfolio constituents cumulative dividend processes be given by, for $i = 1, \dots, n$,

$$D^i = \int_{[0, \cdot]} J_u^i dC_u^i + \int_{[0, \cdot]} \delta_u^i dH_u^i = J^i C^i + H^i (C_{\tau_i-}^i + \delta_{\tau_i}^i),$$

for a bounded variation *coupon process* $C^i = (C_t^i)_{t \in [0, T]}$, and a *recovery process* $\delta^i = (\delta_t^i)_{t \in [0, T]}$. We assume for simplicity that the C^i 's do not jump at the τ_j 's (otherwise this would induce further terms in some of the identities below).

The counterparty clean portfolio cumulative dividend process is thus $D = \sum_{i \in \mathbb{N}_n^*} D^i$. So, by linearity, the portfolio MtM process is given as $P = \sum_{i \in \mathbb{N}_n^*} P^i$, where for $i = 1, \dots, n$,

$$P_t^i = \mathbb{E}_t p_i^t \text{ with } \beta_t p_i^t = \int_t^T \beta_s dD_s^i. \quad (21)$$

Note that

$$\Delta D_\tau = \sum_{i \in \mathbb{N}_n^*} \mathbf{1}_{\tau_i=\tau} \delta_\tau^i. \quad (22)$$

So in particular, for the collateralization policy $\Gamma_\tau = P_{\tau-}$ (cf. (18)), we have that (19) yields

$$\xi_{(\tau)} = \xi_{(\tau)}^1 = (1 - R_0) \mathbf{1}_{\tau=\tau_0} \left(\sum_{i \in \mathbb{N}_n^*} \mathbf{1}_{\tau_i=\tau} \delta_\tau^i + \Delta P_\tau \right)^+ - (1 - R_{-1}) \mathbf{1}_{\tau=\tau_{-1}} \left(\sum_{i \in \mathbb{N}_n^*} \mathbf{1}_{\tau_i=\tau} \delta_\tau^i + \Delta P_\tau \right)^-, \quad (23)$$

which typically reduces to (assuming the P^i 's do not jump at τ unless $\tau = \tau_i$):

$$\xi_{(\tau)}^1 = (1 - R_0) \mathbf{1}_{\tau=\tau_0} \left(\sum_{i \in \mathbb{N}_n^*} \mathbf{1}_{\tau_i=\tau < T_i} (\delta_\tau^i - P_{\tau-}^i) \right)^+ - (1 - R_{-1}) \mathbf{1}_{\tau=\tau_{-1}} \left(\sum_{i \in \mathbb{N}_n^*} \mathbf{1}_{\tau_i=\tau < T_i} (\delta_\tau^i - P_{\tau-}^i) \right)^-. \quad (24)$$

We thus see that the choice $\Gamma_\tau = P_{\tau-}$ corresponds to an extreme case of the collateral: that is, the collected collateral balances the pre-default MtM at default of the counterparty, i.e. it balances $P_{\tau-}$, although it does not balance the i -th value of default if the counterparty defaults at the default time of the i -th reference obligor.

Remark 3.1 In case of the unilateral counterparty risk exposure, the ‘extreme collateralization policy’ corresponds to the choice $\Gamma_\tau = P_{\tau-}^+$ (see Comment 2.4(i)). The unilateral counterparty risk analog of formula (23) is thus, with $R = R_0$,

$$\xi_{(\tau)} = \xi_{(\tau)}^1 = (1 - R) \left(\sum_{i \in \mathbb{N}_n^*} \mathbb{1}_{\tau_i=\tau} \delta_\tau^i + P_\tau - P_{\tau-}^+ \right)^+. \quad (25)$$

We assume henceforth for simplicity that the face value of all the credit derivatives under consideration is equal to monetary unit and that all spreads are paid continuously in time.

3.2 Bilateral CDS CCR (Market Maker’s Perspective)

The first practical issue we want to consider is that of bilateral CCR on a CDS, which is important from the marking-to-market perspective (determination of a spread accounting for bilateral CCR).

Remark 3.2 We consider the pre-Big-Bang (cf. [18]) covenants regarding the cash flows of the CDS contract. That is, we do not include the up-front payment in the cash flows. The developments below can however be easily adapted to the post-Big-Bang universe of CDS contracts.

We consider a counterparty risky *payer CDS* on name 1 (CDS protection on name 1 bought by the investor, or credit name -1 , from its counterparty, represented by the credit name 0). Denoting by T the maturity, κ the contractual spread and $R_1 \in [0, 1]$ the recovery, this corresponds to the special case of Section 3.1 in which $n = 1$ (so we delete unnecessary indices in the notation), and

$$C_t = -\kappa(t \wedge T), \quad \delta_t = (1 - R_1)\mathbb{1}_{t < T}. \quad (26)$$

The following result follows by application of (8), (9), (22) and (24).

Proposition 3.1 *For a counterparty risky payer CDS, one has,*

$$\begin{aligned} \xi_{(\tau)} = \xi_{(\tau)}(\text{pay}; -1, 0, 1) &= (1 - R_0)\mathbb{1}_{\tau=\tau_0} \left(P_\tau + \mathbb{1}_{\tau_1=\tau < T}(1 - R_1) - \Gamma_\tau \right)^+ \\ &\quad - (1 - R_{-1})\mathbb{1}_{\tau=\tau_{-1}} \left(P_\tau + \mathbb{1}_{\tau_1=\tau < T}(1 - R_1) - \Gamma_\tau \right)^-. \end{aligned} \quad (27)$$

So, in case of no collateralization ($\Gamma = 0$),

$$\xi_{(\tau)} = \xi_{(\tau)}^0 = (1 - R_0)\mathbb{1}_{\tau=\tau_0} \left(P_\tau^+ + \mathbb{1}_{\tau_1=\tau < T}(1 - R_1) \right) - (1 - R_{-1})\mathbb{1}_{\tau=\tau_{-1}} P_\tau^-, \quad (28)$$

and in the case of extreme collateralization ($\Gamma_\tau = P_{\tau-}$, cf. (24)),

$$\xi_{(\tau)}^1 = (1 - R_0)\mathbb{1}_{\tau=\tau_0=\tau_1 < T}(1 - R_1 - P_{\tau-})^+ - (1 - R_{-1})\mathbb{1}_{\tau=\tau_{-1}=\tau_1 \leq T}(1 - R_1 - P_{\tau-})^-. \quad (29)$$

Remark 3.3 To derive (28) we used that $P_\tau = \mathbb{1}_{\tau < \tau_1} P_\tau$, and therefore

$$\left(P_\tau + \mathbb{1}_{\tau_1=\tau < T}(1 - R_1) \right)^+ = P_\tau^+ + \mathbb{1}_{\tau_1=\tau < T}(1 - R_1).$$

The symmetric concept of the receiver counterparty risky CDS complements the concept of the payer counterparty risky CDS. In case of a receiver counterparty risky CDS (CDS protection on name 1 sold by the investor, or credit name -1 , to its counterparty, represented by the credit name 0), one of course has, symmetrically to (26),

$$C_t = \kappa(t \wedge T), \quad \delta_t = -(1 - R_1) \mathbf{1}_{t < T}, \quad (30)$$

and one gets likewise,

$$\begin{aligned} \xi_{(\tau)} &= \xi_{(\tau)}(rec; -1, 0, 1) = -\xi_{(\tau)}(pay; 0, -1, 1) \\ \xi_{(\tau)}^1 &= \xi_{(\tau)}^1(rec; -1, 0, 1) = -\xi_{(\tau)}^1(pay; 0, -1, 1). \end{aligned} \quad (31)$$

3.2.1 Decomposition of the Fair Spread

Note that the decomposition of the (fair) MtM at time $t = 0$ for a CDS subject to the credit counterparty risk can be written as

$$0 = \Pi_0(\hat{\kappa}) = P_0(\hat{\kappa}) - CVA_0(\hat{\kappa}) = P_0(\hat{\kappa}) - \mathbb{E}_0(\beta_\tau \xi_\tau(\hat{\kappa})), \quad (32)$$

where we denoted by $\hat{\kappa}$ the CCR adjusted fair spread at initiation of the CDS, and we explicitly recorded dependence on $\hat{\kappa}$ of the relevant quantities. Of course, by the CCR adjusted fair spread we mean the constant $\hat{\kappa}$ that solves equation (32).

Likewise, the clean fair spread, say $\bar{\kappa}$, solves

$$0 = P_0(\bar{\kappa}). \quad (33)$$

In general, it is not possible to provide a closed form formula for the difference

$$\epsilon = \hat{\kappa} - \bar{\kappa}, \quad (34)$$

and thus, we do not have, in general, an explicit formula for the spread decomposition

$$\hat{\kappa} = \bar{\kappa} + \epsilon. \quad (35)$$

3.3 Unilateral Portfolio CCR (Bank's Perspective)

Here, we consider a bank that holds a portfolio of credit contracts referencing various credit names. This portfolio is subject to a counterparty credit risk with regard to a single counterparty. A bank typically disregards its own counterparty risk when assessing the counterparty risk of a portfolio with another party. Thus, we are led to considering unilateral counterparty risk from the perspective of the bank.

Remark 3.4 A further issue not dealt with in this paper is the case where a bank faces CCR with regard to several counter-parties.

So in this section we have that $\tau_{-1} = +\infty$ and $\tau = \tau_0$ is the default time of the investor's (bank's) counterparty, with recovery R_0 simply denoted as R .

More specifically, we now consider on each firm i , with $i = 1, \dots, n$, a payer (default protection bought by the investor from the counterparty) or receiver (default protection sold by the investor to the counterparty) CDS with maturity T_i , contractual spread κ_i and recovery R_i . Let P_i conventionally denote here the counterparty clean price of a *payer CDS* on the i -th firm, for every $i = 1, \dots, n$, so that the counterparty clean portfolio value is $P = (\sum_i \text{pay} - \sum_i \text{rec})P_i$.

One thus gets by application of (8), (9), (22) and (25),

Proposition 3.2 *For a portfolio of CDS with unilateral CCR, one has,*

$$\xi_{(\tau)} = (1 - R) \left(P_\tau + \left(\sum_{i \text{ pay}} - \sum_{i \text{ rec}} \right) \mathbb{1}_{\tau_i = \tau < T_i} (1 - R_i) - \Gamma_\tau \right)^+, \quad (36)$$

with $\Gamma = 0$ in the no collateralization case and $\Gamma_\tau = P_\tau^+$ in the unilateral extreme collateralization case.

4 Multivariate Markovian Default Model

In this section we propose an underlying stochastic model, that will be able to put the previous results to work. Towards this end we define a Markovian bottom-up model of multivariate default times with factor process $X = (X^{-1}, \dots, X^n) = (X^i)_{i \in \mathbb{N}_n}$ (recall $\mathbb{N}_n = \{-1, 0, \dots, n\}$), which will have the following key features (see [4, 3]):

- (i) The pair (X, H) is Markov in its natural filtration \mathbb{F} ,
- (ii) Each pair (X^i, H^i) is a Markov process,
- (iii) At every instant, either each alive obligor can default individually, or all the surviving names whose indices are in the set I_l , where the I_l 's are a few pre-specified subsets of the pool, can default simultaneously.

Remark 4.1 Property (ii) grants quick valuation of single-name credit derivatives and independent calibration of each model marginal (X^i, H^i) , whereas (iii) will allow us to account for dependence between defaults. Given property (i), various computational methods are, in principle, also available for basket derivatives. Since the Markovian dimension of the model will be of the order of, in general, 2^n , pricing of basket instruments by deterministic numerical schemes for the related Kolmogorov equations is precluded by the curse of dimensionality as soon as n exceeds a few units. However, in the special case of ‘static’ basket instruments with payoff of the form $\phi(N_T)$ where $N_t = \sum_{i=1}^n H_t^i$, efficient convolution recursion pricing schemes are also available (see [3]). In general, a practical alternative is to use Monte Carlo simulation methods (see Section 4.2).

We thus define a certain number of groups $I_l \subseteq \mathbb{N}_n$, of obligors who are likely to default simultaneously. Let $\mathcal{I} = (I_l)$. We define the generator of process $(X, H) = (X^i, H^i)_{i \in \mathbb{N}_n}$ as, for $u = u(t, \chi, \epsilon)$ with $\chi = (x_{-1}, \dots, x_n) \in \mathbb{R}^{n+2}$, $\epsilon = (e_{-1}, \dots, e_n) \in \{0, 1\}^{n+2}$:

$$\begin{aligned} \mathcal{A}_t u(t, \epsilon, \chi) = & \sum_{i \in \mathbb{N}_n} \left(b_i(t, x_i) \partial_{x_i} u(t, \chi, \epsilon) + \frac{1}{2} \sigma_i^2(t, x_i) \partial_{x_i}^2 u(t, \chi, \epsilon) \right) \\ & + \sum_{i, j \in \mathbb{N}_n; i < j} \varrho_{i,j}(t) \sigma_i(t, x_i) \sigma_j(t, x_j) \partial_{x_i, x_j}^2 u(t, \chi, \epsilon) \\ & + \sum_{i \in \mathbb{N}_n} \left(\eta_i(t, x_i) - \sum_{I \in \mathcal{I}; I \ni i} \lambda_I(t, \chi) \right) (u(t, \chi, \epsilon^i) - u(t, \chi, \epsilon)) \\ & + \sum_{I \in \mathcal{I}} \lambda_I(t, \chi) (u(t, \chi, \epsilon^I) - u(t, \chi, \epsilon)), \end{aligned} \quad (37)$$

where, for $i, j \in \mathbb{N}_n$ and $I \in \mathcal{I}$:

- b_i , σ_i^2 , $\varrho_{i,j}(t)$ and η_i denote suitable *drift*, *variance*, *correlation* and *pre-default intensity* function-coefficients,
- ϵ^i , resp. ϵ^I , denotes the vectors obtained from ϵ by replacing the component e_i , resp. the components e_j for $j \in I$, by number one,

- the non-negative bounded functions $\lambda_I(t, \chi)$ are chosen so that the following holds, for every $i, t, \chi = (x_{-1}, \dots, x_n)$:

$$\sum_{I \in \mathcal{I}; I \ni i} \lambda_I(t, \chi) \leq \eta_i(t, x_i). \quad (38)$$

For instance, one can set, for every $I \in \mathcal{I}$ and $t, \chi = (x_{-1}, \dots, x_n)$,

$$\lambda_I(t, \chi) = \alpha_I \inf_{i \in I} \eta_i(t, x_i) \quad (39)$$

for some non-negative *model dependence parameters* α_I such that $\sum_{I \in \mathcal{I}} \alpha_I \leq 1$.

Remark 4.2 It is possible to consider the situation of a *common factor process* $X^i = X$ by letting $b_i = b$, $\sigma_i = \sigma$, $\varrho = 1$ and $X_0^i = x$. On the opposite the choice $\varrho = 0$ will correspond to independent factor processes X^i 's. In the latter case dependency between defaults is then only represented in the model by the possibility of simultaneous defaults.

One can then verify (see [4]) that for $i \in \mathbb{N}_n$, the pair (X^i, H^i) is a jointly Markov process admitting the following generator, for $u = u(t, x_i, e_i)$ with $(x_i, e_i) \in \mathbb{R} \times \{0, 1\}$:

$$\mathcal{A}_t^i u(t, x_i, e_i) = b_i(t, x_i) \partial_{x_i} u(t, x_i, e_i) + \frac{1}{2} \sigma_i^2(t, x_i) \partial_{x_i}^2 u(t, x_i, e_i) + \eta_i(t, x_i) (u(t, x_i, 1) - u(t, x_i, e_i)). \quad (40)$$

4.1 Model Calibration

In the remainder of the paper we shall apply the Markovian model introduced above to valuation of the counterparty risk in the portfolio of CDS contracts, where the recovery rate associated with the i -th contract is R_i . It is implicit in the specification (37) (see also (40)) for the model generator that one shall work with constant recoveries R_i 's, for simplicity. In the case of affine coefficients, and assuming a constant interest rate r and a constant recovery rate R_i , the time- $t = 0$ fair spread for the i -th CDS contract, say κ_0^i , is then given by an explicit formula $\kappa_0^i = \kappa^i(0, X_0^i)$ in terms of r, R_i, T_i and the parameters of the coefficients b_i , σ_i^2 and η_i (See, e.g. Duffie and Garleanu [12], Crépey et al. [9]). Given pre-determined values of r and R_i , the parameters of the coefficients can then be calibrated so to match the spread curve of the related CDS.

Remark 4.3 More generally, assuming random recovery R_i with a suitably parameterized distribution, one could calibrate this distribution jointly with calibration of b_i , σ_i^2 and η_i using the corresponding CDS spread curve (cf. [3]).

The few residual model dependence parameters α_I with $I \in \mathcal{I}$ can then be calibrated to target prices of basket instruments, which can for instance be done by simulation.

Remark 4.4 We do not engage in any calibration exercise of the above Markovian model in this paper, as we only intend to illustrate qualitative features of our model. Calibration of the kind of Markov copula models that we use here is investigated in [3].

4.2 Model Simulation

Given that \mathcal{I} only consists of a few (no more than three or four, say) pre-specified subsets of the pool, the simulation of a set of random times $(\tau_i)_{i \in \mathbb{N}_n}$ is quite fast. Given a previously simulated

trajectory of \mathbf{X} , one essentially needs to simulate IID exponential random variables \mathcal{E}_ι , for $\iota \in \mathcal{I} \cup \mathbb{N}_n$. Then one computes, for every $I \in \mathcal{I}$,

$$\hat{\tau}_I = \inf\{t > 0; \int_0^t \lambda_I(s, \mathbf{X}_s) ds \geq \mathcal{E}_I\}$$

and for every $i \in \mathbb{N}_n$,

$$\hat{\tau}_i = \inf\{t > 0; \int_0^t \lambda_i(s, \mathbf{X}_s) ds \geq \mathcal{E}_i\}$$

with for $t, \chi = (x_{-1}, \dots, x_n)$,

$$\lambda_i(t, \chi) = \eta_i(t, x_i) - \sum_{I \in \mathcal{I}; I \ni i} \lambda_I(t, \chi).$$

One sets finally, for every $i \in \mathbb{N}_n$,

$$\tau_i = \hat{\tau}_i \wedge \left(\bigwedge_{I \in \mathcal{I}; I \ni i} \hat{\tau}_I \right).$$

5 Numerical Results

We present here some numerical results illustrating applications of the above theory. The computations are carried out by running 2×10^6 crude Monte Carlo simulations using Matlab R2009a on a laptop with an Intel Core 2 Duo, 2.66 GHz CPU. In the tables below the number between parentheses in each cell denotes the empirical standard error of the related Monte Carlo estimate (empirical average on its left).⁵

All recovery rates are fixed at 0.4. The intensities of default $\eta_i(t, X_t^i)$, $i \in \mathbb{N}_n$, are assumed to be of the affine form

$$\eta_i(t, X_t^i) = a_i + X_t^i, \quad (41)$$

where a_i is a constant, and where X^i is a homogenous Cox-Ingersoll-Ross (CIR) process generated by

$$dX_t^i = \zeta_i(\mu_i - X_t^i) dt + \sigma_i \sqrt{X_t^i} dW_i. \quad (42)$$

In this shorter version of the paper we only present numerical examples involving unilateral CCR. The full version of the paper presents numerical examples involving bilateral CCR.

We thus consider the issue of valuation of the unilateral counterparty credit risk on a portfolio of CDSs. We assume that we have a risk free investor which is trading a portfolio of payer and receiver CDSs with a defaultable counterparty. In the fully netted case, the PFEDs corresponding to a portfolio with no collateralization and to a portfolio with extreme collateralization are given by formula (36) for $\Gamma = 0$ or $\Gamma_\tau = P_{\tau-}^+$, respectively (see Proposition 3.2).

We conducted numerical experiments for the above two cases, and, for comparison we also considered the non-netted (and non-collateralized) case. By summation over the CDSs composing the portfolio,

⁵The random number generation was sufficiently rich to justify the large number of generated paths: The uniform random numbers were generated using Matlab's rand function that is based on the modified *subtract with borrow* generator, which has a period of 2^{1492} . The normally distributed random numbers were generated using Matlab randn function, which is an implementation of Marsaglia's shift-register generator summed with linear congruential generator, which has a period of 2^{64} .

one has in the latter case that the corresponding PFED is given by

$$\xi_{(\tau)} = (1 - R) \left(\sum_{i \text{ pay}} \left(P_\tau^i + \mathbf{1}_{\tau=\tau_i < T_i} (1 - R_i) \right)^+ + \sum_{i \text{ rec}} P_\tau^{i,-} \right), \quad (43)$$

where P^i denotes the counterparty clean price of a payer CDS on name i in the portfolio (see Section 3.3).

Here we consider an underlying portfolio of 100 CDS contracts. The corresponding market spreads (in basis points) are presented in listing (44) below,

$$\begin{aligned}
& (\kappa_1, \dots, \kappa_{100}) = \\
= & (405.936, 225.937, 620.786, 195.083, 37.97, 32.17, 1743.673, 348.411, 399.788, 297.902, 3013.286, \\
& 359.909, 327.962, 2085.618, 145.519, 234.948, 212.135, 120.000, 124.000, 304.845, 225.904, 39.78, \\
& 28.320, 229.577, 291.100, 349.071, 132.982, 889.620, 28.110, 25.110, 311.131, 210.919, 368.858, \\
& 480.993, 359.483, 200.581, 164.500, 127.000, 456.170, 130.027, 229.912, 343.500, 361.515, 300.346, \\
& 583.736, 342.688, 133.451, 141.984, 2440.458, 579.000, 306.745, 324.709, 647.019, 433.597, 201.960, \\
& 192.860, 243.031, 296.210, 333.747, 295.873, 374.750, 270.432, 436.182, 430.537, 127.000, 145.043, \\
& 52.270, 29.070, 123.000, 45.880, 29.930, 31.560, 190.000, 361.990, 36.980, 31.440, 33.880, \\
& 16.570, 39.000, 73.560, 70.470, 24.840, 21.270, 47.880, 53.450, 31.190, 499.517, 1092.000, \\
& 26.770, 49.810, 60.400, 180.000, 27.960, 130.876, 123.000, 47.110, 190.000, 38.170, 42.730, \\
& 200.000)
\end{aligned} \tag{44}$$

Now, we divide the portfolio into two categories: the first category composed of $p = 70$ payer CDS, and the second category composed of $r = 30$ receiver CDS. The choice of the names for the payer CDS and receiver CDS is made by taking the first 70 names from (44) and declaring them to represent payer contracts, and declaring the remaining 30 names to represent the receiver contracts.⁶

Next, setting $n = 100$, for $l \in L := \{20, 70, 101\}$, we define I_l as the set, possibly including the counterparty indexed by 0, containing the indices of the l riskiest obligors, as measured by the spread of the corresponding five year market CDS quote, and we set $\mathcal{I} = (I_l)_{l \in L}$, where each I_l stands for a specific subset of \mathbb{N}_n^* . More specifically, we consider a *nested* grouping of reference names of the CDS contracts in the portfolio, as well as the counterparty, which can default together. Thus, we have

$$I_{20} \subseteq I_{70} \subseteq I_{101} = \mathbb{N}_n^* = \{0, 1, 2, \dots, 100\}.$$

These groups were created using the market CDS spreads given in (44). The riskiness of the obligors was assessed based on the list obtained by sorting (44) in descending order.

The obligors in I_{20} are assumed to have high credit risk while the obligors in $I_{70} \setminus I_{20}$ are assumed to have middle credit risk and the obligors in $I_{101} \setminus I_{70}$ are assumed to have low credit risk.

The default intensities of the obligors are given as shifted CIR processes (cf. (41)–(42)). We assume that all obligors having high credit risk have the same credit risk parameters; hence I_{20} is a homogenous group with the CIR parameters given in the bottom row of Table 1. Similarly all the obligors with middle credit risk, that is obligors belonging to $I_{70} \setminus I_{20}$, have the same CIR parameters that are in the middle row of Table 1. We also assume that the obligors with low credit risk, that is

Credit Risk Level	ζ	μ	σ	X_0
low	0.9	0.001	0.01	0.001
middle	0.80	0.02	0.1	0.02
high	0.50	0.05	0.2	0.05

Table 1: Parameter for CIR process for CDS Portfolio

obligors belonging to $I_{101} \setminus I_{70}$, have the same CIR parameters that are in the upper row of Table 1.

The default intensities of the reference names and of the counterparty are assumed to be of the form $a_i + X^i$ where X^i is the CIR process common for the l -th group in case $i \in I_l$. This setup corresponds to taking a common factor process for each homogenous group (See Remark 4.2). The constants a_i of the reference names are calibrated so to match spreads listed in (44), and they are given below in listing (45):

$$\begin{aligned}
(a_1, \dots, a_{100}) = & \\
= (0.0194, 0.0178, 0.0552, 0.0126, 0.0053, 0.0044, 0.2424, 0.0382, 0.0184, 0.0297, 0.4540, \\
& 0.0401, 0.0348, 0.2993, 0.0044, 0.0193, 0.0155, 0.0001, 0.0008, 0.0309, 0.0177, 0.0056, \\
& 0.0037, 0.0184, 0.0286, 0.0383, 0.0023, 0.1000, 0.0037, 0.0032, 0.0320, 0.0153, 0.0132, \\
& 0.0319, 0.0400, 0.0135, 0.0075, 0.0013, 0.0278, 0.0018, 0.0184, 0.0373, 0.0404, 0.0302, \\
& 0.0490, 0.0372, 0.0023, 0.0038, 0.3585, 0.0482, 0.0312, 0.0342, 0.0596, 0.0240, 0.0138, \\
& 0.0122, 0.0206, 0.0295, 0.0357, 0.0294, 0.0142, 0.0252, 0.0244, 0.0235, 0.0013, 0.0043, \\
& 0.0077, 0.0038, 0.0006, 0.0066, 0.0040, 0.0043, 0.0118, 0.0404, 0.0052, 0.0042, 0.0046, \\
& 0.0018, 0.0055, 0.0113, 0.0107, 0.0031, 0.0025, 0.0070, 0.0079, 0.0042, 0.0350, 0.1337, \\
& 0.0035, 0.0073, 0.0091, 0.0101, 0.0037, 0.0019, 0.0006, 0.0069, 0.0118, 0.0054, 0.0061, \\
& 0.0134). \\
(45)
\end{aligned}$$

Since we assume a risk free investor, the margining process is assumed to be unilateral, where by unilateral we mean that only the counterparty posts collateral provided the value of the portfolio is positive for the investor at the margin date. Hence if the portfolio has negative (positive) value for the investor (counterparty), there is no collateral posted by the investor. The margin process is thus assumed to be $\Gamma_\tau = P_{\tau-}^+$ (see Remark 3.1 and equation (25)).

Example 5.1 The scenario is that of

- a. a non defaultable investor,
- b. a defaultable counterparty with a **constant default intensity** such that the market CDS spread of the counterparty κ_0 is either 1, 20, 40, 60, 80, 100, 120 or 300 basis points,
- c. the portfolio of $p = 70$ payer CDS and $r = 30$ receiver CDS based on the description given in this section.

In Table 2 we examine the behavior of the CVA with respect to the default intensity of the counterparty, in three different cases: no netting and no margining, netting and margining, and both netting

⁶We stress that the partition of the portfolio into payer and receiver CDS was not based on the sorted list but on the original list of market CDS spreads given in (44).

and margining. The heading ‘Counterparty risk type’ for the first column is to indicate the classification of the counterparty into one of the homogenous groups $I_1, I_2 \setminus I_1$ or $I_3 \setminus I_2$, whose respective default intensities are modeled using CIR parameters corresponding to high, middle and low credit risk regimes respectively. From the results obtained we see that CVA increases as the counterparty has a higher market CDS spread κ_0 (which corresponds to a higher intensity of default).

Netting appears to have a higher impact as the counterparty becomes riskier. However this does not seem to be the case of margining. Essentially, margining mitigates all the counterparty risk that it can mitigate, regardless of the riskiness of the counterparty, and the remaining risk, due to composition of the credit names and possible joint defaults, simply can't be mitigated by the kind of margining that we use here. In the margining case with extreme collateralization, the CVA is essentially due to joint defaults. As increasing the riskiness of the counterparty does not increase $\lambda_{I_3}(t, \chi)$, so the joint default intensity of the counterparty is constant for all rows in this scenario (except for the last two rows, see end of the paragraph), and the CVA does not really change either. This is of course only true to the extent that the joint default intensity of the counterparty stays unchanged, that is, as long as the counterparty stays in the ‘low’ risk group. So, in the last two rows of the table, with spreads κ_0 of 120 or 300 bps, the counterparty belongs to the ‘middle’ risk group, its joint default intensity is thus higher, and the net effect is an increase of the CVA, even in the case of extreme collateralization.

In Figures 2, 3, 4, 5, 6 and 7, we give the EPE(t) curves in the cases of no netting and no margining, netting and no margining, and netting and margining with extreme collateralization, respectively. In each case the EPE(t) curves are given for a selection of κ_0 , i.e. $\kappa_0 = 20$ basis points, $\kappa_0 = 60$ basis points and $\kappa_0 = 100$ basis points. The curves were obtained by fitting a quadratic polynomial (*non-linear regression*) through the $\xi_{(\tau)}$ values for $\tau = \tau_0 < T$.

Remark 5.1 Another way to calculate EPE(t) is to divide the time interval $[0, T]$ into time buckets and calculate the average loss incurred in each time bucket. But the EPE curves obtained in this way are very oscillatory, which indicates that there is a high variance in this way, even with two million Monte Carlo simulations. When superposing the EPE(t) curve obtained through the bucketing and the nonlinear regression, one obtains that the two results are in good qualitative agreement though (see Figure 1).

Let η_0 denote the constant intensity of default of the counterparty, i.e. $\eta_0(t, X_t^0) = \eta_0$. To explain the pattern of the EPE(t) curves observed in the CDS portfolio case one needs to consider the ratios $\mathbb{1}_{0 \in I_i} \frac{\mathbb{E}[\lambda_{I_i}(t, X_t)]}{\eta_0}$ which gives the probability of a **joint** default of the counterparty, **conditionally to its default** (alone or jointly). Provided that the counterparty risk profile does not change as the CDS spread of the counterparty κ_0 increases, the numerator in this ratio remains the same, whereas the denominator increases. Therefore the ratio decreases, meaning that the conditional probability of a joint default decreases (see Table 3). Since joint defaults are responsible for the most part of the PFED, therefore, as κ_0 increases, the EPE(t) curves become lower, as we can see in the Figures. However if the riskiness of the counterparty increases such that it can default jointly with the group I_2 and/or I_3 , then the joint default probability will generally increase as there are 61 reference names (corresponding to 61 payer CDS) in I_2 which can jointly default with the counterparty. Hence as κ_0 increases such that the counterparty risk profile changes to a more risky type, the EPE(t) curve becomes higher, as we can see in the Figures.

Example 5.2 The scenario is that of

- a. a non defaultable investor,
- b. a defaultable counterparty with a **stochastic intensity** such that it has a market CDS spread κ_0 equal to 10, 20, 60, 100, 120, 300, 400 or 500 basis points,

Counterparty risk type	κ_0	η_0	CVA No Nett. No Marg.	CVA Nett. No Marg.	CVA Nett. with Marg.
low	1	0.00017	50.7 (2.2)	27.1 (1.2)	26.6 (1.2)
low	20	0.00333	808.2 (8.9)	433.8 (4.9)	422.8 (4.8)
low	40	0.00667	822.2 (8.9)	441.9 (4.8)	419.7 (4.8)
low	60	0.0100	834.2 (8.9)	448.6 (4.8)	415.8 (4.8)
low	80	0.01333	847.4 (8.9)	456.1 (4.8)	412.9 (4.8)
low	100	0.01667	860.5 (8.8)	463.2 (4.8)	409.8 (4.8)
middle	120	0.0200	4729.1 (20.1)	3737.2 (16.0)	3675.8 (15.9)
middle	300	0.0500	5348.0 (21.0)	4230.8 (16.9)	4083.4 (16.8)

Table 2: CVA for portfolio of CDS when $\eta_0(t, X_t^0) = \eta_0$ is constant
and $(\alpha_{I_1}, \alpha_{I_2}, \alpha_{I_3}) = (0.3, 0.3, 0.3)$.

Counterparty risk type	κ_0	η_0	$\frac{\mathbb{1}_{0 \in I_1} \times \mathbb{E}[\lambda_{I_1}(t, X_t)]}{\eta_0}$	$\frac{\mathbb{1}_{0 \in I_2} \times \mathbb{E}[\lambda_{I_2}(t, X_t)]}{\eta_0}$	$\frac{\mathbb{1}_{0 \in I_3} \times \mathbb{E}[\lambda_{I_3}(t, X_t)]}{\eta_0}$
low	20	0.0033	0	0	0.2511
low	60	0.0100	0	0	0.0829
low	100	0.01667	0	0	0.0166
middle	120	0.0200	0	0.2808	0.0414
middle	300	0.0500	0	0.1240	0.0166

Table 3: Dependence of EPE on $\frac{\mathbb{E}[\lambda_{I_i}(t, X_t)]}{\eta_0}$, with $t = \frac{1}{12}$ (one month),
for portfolio of CDS when $\eta_0(t, X_t^0) = \eta_0$ is constant.

- c. a portfolio of $p = 70$ payer CDS and $r = 30$ receiver CDS based on the description given in this section.

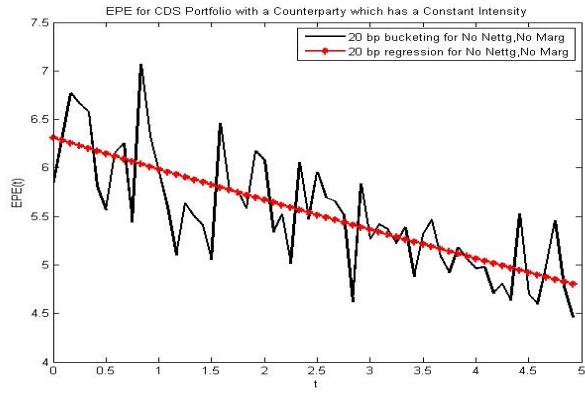
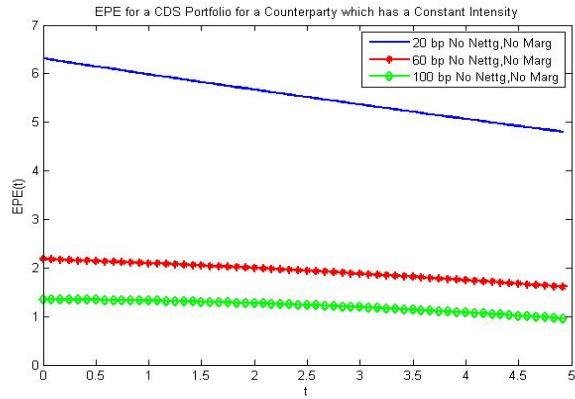
Table 4 is the analog of Table 2 in Example 5.1, it thus gives the behavior of the CVA with respect to the default intensity of the counterparty, in the three cases of no netting and no margining, netting and margining, and both netting and margining. From the results we see that CVA values are larger for a counterparty with a stochastic intensity and increase with the riskiness of the counterparty. In Figures 8, 9, 10, 11, 12, 13, 14, 15, and 16, we give the EPE(t) curves for the no netting with no margining, netting with no margining and netting with margining cases. This time the behavior of the EPE(t) curves is explained by the ratio $\mathbb{E}\left[\frac{\lambda_{I_3}(t, X_t)}{\eta_0(t, X_t^0)}\right]$ (see Table 5).

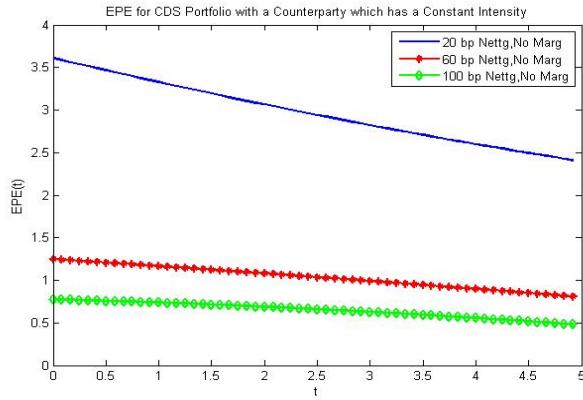
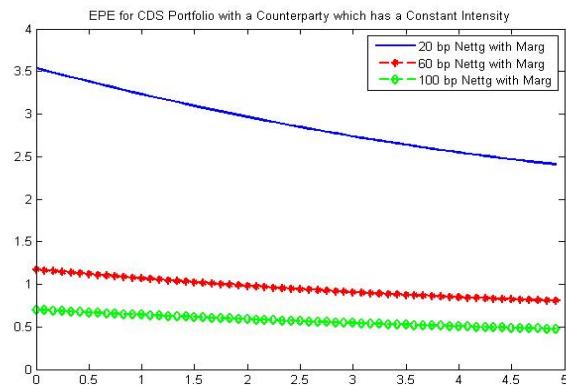
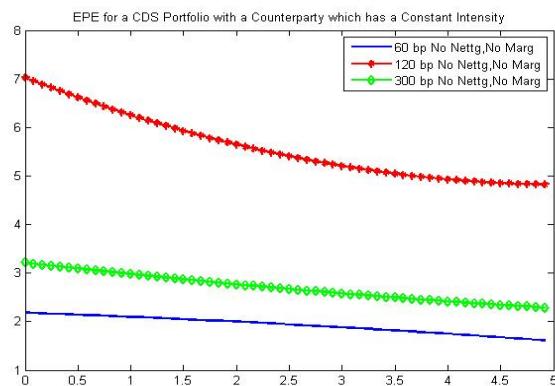
Counterparty risk type	κ_0	CVA No Nett. No Marg.	CVA Nett. No Marg.	CVA Nett. with Marg.
low	10	488.4 (7.0)	262.1 (3.8)	256.7 (3.8)
low	20	808.4 (8.9)	433.9 (4.9)	423.0 (4.8)
low	60	834.2 (8.9)	448.5 (4.8)	415.9 (4.8)
low	100	860.4 (8.8)	463.2 (4.8)	409.8 (4.8)
middle	120	5440.2 (21.4)	4338.1 (17.2)	4256.6 (17.1)
middle	300	5364.1 (21.0)	4243.1 (16.9)	4076.8 (16.8)
high	400	8749.9 (22.1)	7211.3 (18.1)	6943.0 (18.0)
high	500	8543.5 (21.8)	7017.9 (17.8)	6713.8 (17.7)

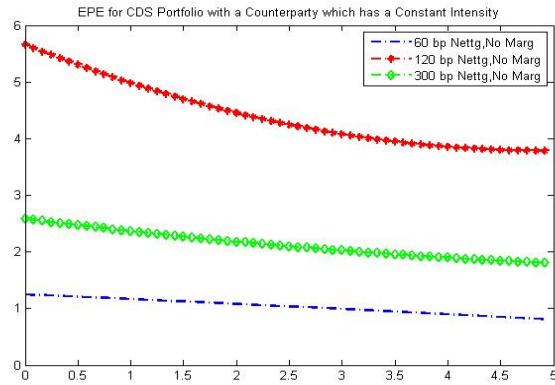
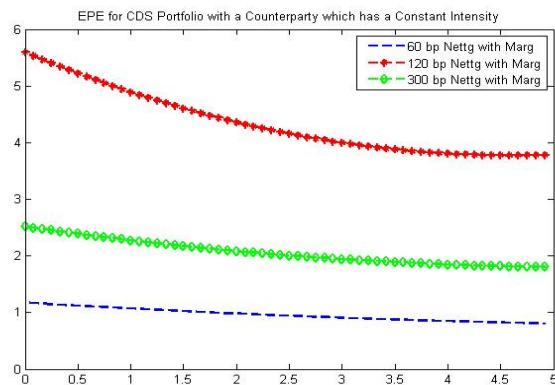
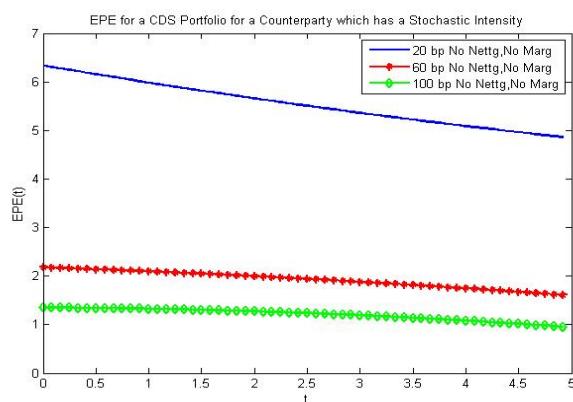
Table 4: CVA for portfolio of CDS when $\eta_0(t, X_t^0)$ is stochastic
and $((\alpha_{I_1}, \alpha_{I_2}, \alpha_{I_3}) = (0.3, 0.3, 0.3))$.

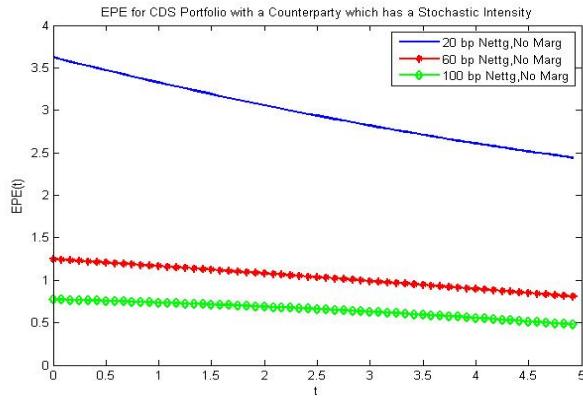
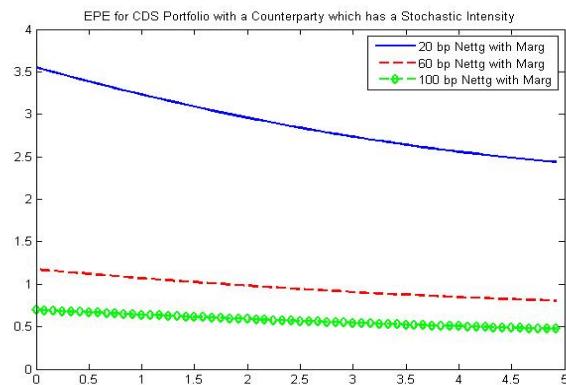
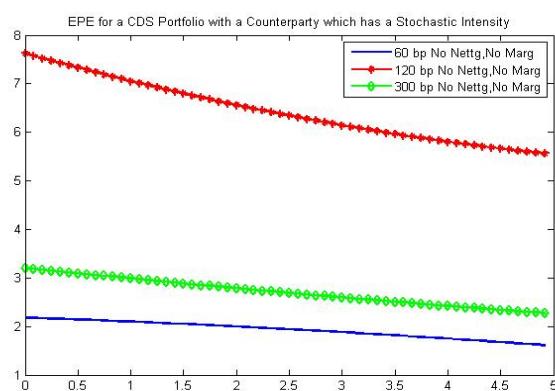
Counterparty risk type	κ_0	$\mathbb{E}\left[\frac{\lambda_{I_1}(t, X_t)}{\eta_0(t, X_t^0)}\right] \times$	$\mathbb{E}\left[\frac{\lambda_{I_2}(t, X_t)}{\eta_0(t, X_t^0)}\right] \times$	$\mathbb{E}\left[\frac{\lambda_{I_3}(t, X_t)}{\eta_0(t, X_t^0)}\right] \times$
low	20	0	0	0.2485
low	60	0	0	0.0828
low	100	0	0	0.0497
middle	120	0	0.3000	0.0429
middle	300	0	0.1222	0.0166
high	400	0.2809	0.0937	0.0126
high	500	0.2247	0.0744	0.0100

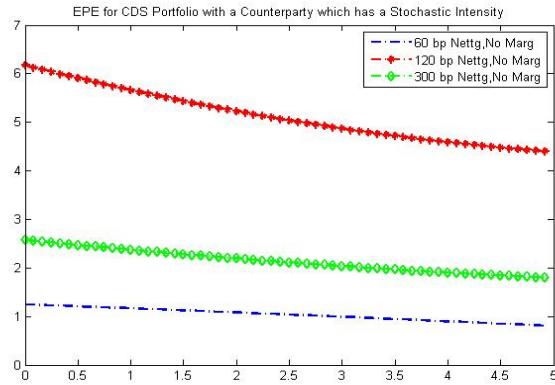
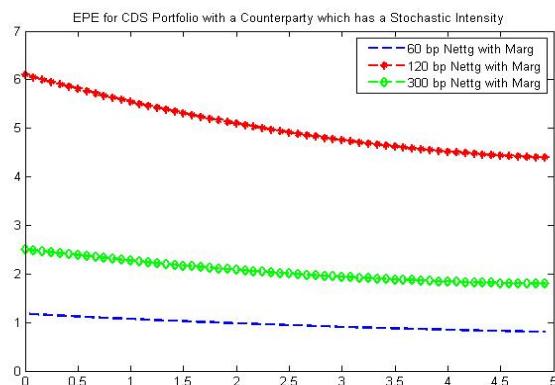
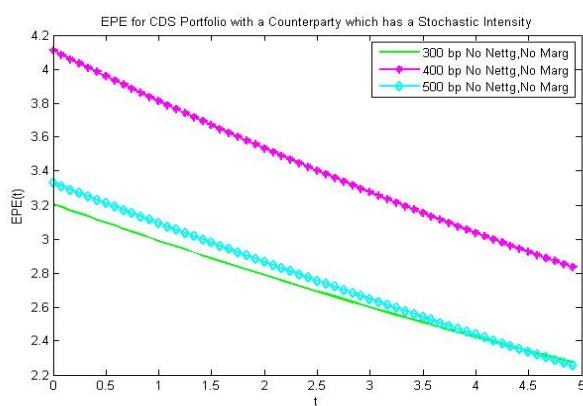
Table 5: Dependence of EPE on $\mathbb{E}\left[\frac{\lambda_{I_i}(t, X_t)}{\eta_0(t, X_t^0)}\right]$, with $t = \frac{1}{12}$, for portfolio
of CDS when $\eta_0(t, X_t^0)$ is stochastic.

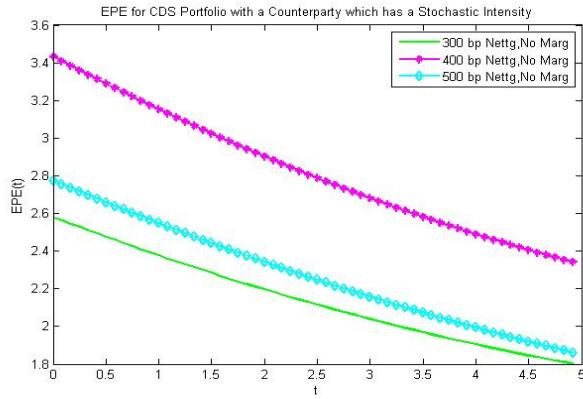
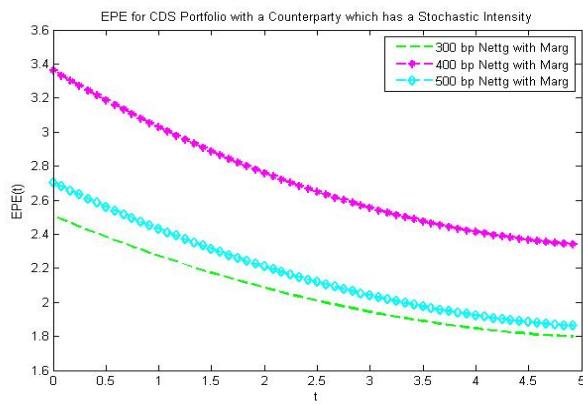
Figure 1: $EPE(t)$ for portfolio.Figure 2: $EPE(t)$ for portfolio.

Figure 3: $EPE(t)$ for portfolio.Figure 4: $EPE(t)$ for portfolio.Figure 5: $EPE(t)$ for portfolio.

Figure 6: $EPE(t)$ for portfolio.Figure 7: $EPE(t)$ for portfolio.Figure 8: $EPE(t)$ for portfolio.

Figure 9: $EPE(t)$ for portfolio.Figure 10: $EPE(t)$ for portfolio.Figure 11: $EPE(t)$ for portfolio.

Figure 12: $EPE(t)$ for portfolio.Figure 13: $EPE(t)$ for portfolio.Figure 14: $EPE(t)$ for portfolio.

Figure 15: $EPE(t)$ for portfolio.Figure 16: $EPE(t)$ for portfolio.

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