Counterparty Risk on a CDS in a Markov Chain Copula Model with Joint Defaults

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†See http://www.cris-creditrisk.com.

Abstract

In this paper we study the counterparty risk on a payer CDS in a Markov chain model of two reference credits, the firm underlying the CDS and the protection seller in the CDS. We first state few preliminary results about pricing and CVA of a CDS with counterparty risk in a general set-up. We then introduce a Markov chain copula model in which wrong way risk is represented by the possibility of joint defaults between the counterpart and the firm underlying the CDS. In the set-up thus specified we derive semi-explicit formulas for most quantities of interest with regard to CDS counterparty risk like price, CVA, EPE or hedging strategies. Model calibration is made simple by the copula property of the model. Numerical results show adequation of the behavior of EPE and CVA in the model with stylized features.

Keywords: Counterparty Credit Risk, CDS, Wrong Way Risk, CVA, EPE.

Contents

1 Introduction

Since the sub-prime crisis, counterparty risk is a crucial issue in connection with valuation and risk management of credit derivatives. Counterparty risk in general is 'the risk that a party to an OTC derivative contract may fail to perform on its contractual obligations, causing losses to the other party' (cf. Canabarro and Duffie [11]). A major issue in this regard is the so-called *wrong way risk*, namely the risk that the value of the contract be particularly high from the perspective of the other party at the moment of default of the counterparty. As classic examples of wrong way risk, one can mention the situations of selling a put option to a company on its own stock, or entering a forward contract in which oil is bought by an airline company (see Redon [22]).

Among papers dealing with general counterparty risk, one can mention, apart from the abovementioned references, Canabarro et al. [12], Zhu and Pykhtin [24], and the series of papers by Brigo et al. [7, 8, 9, 10]. From the point of view of measurement and management of counterparty risk, two important notions emerge:
• The Credit Value Adjustment process (CVA), which measures the depreciation of a contract due to counterparty risk. So, in rough terms, $\mathrm{CVA}_t = P_t - \Pi_t$, where Π and P denote the price process of a contract depending on whether one accounts or not for counterparty risk.
• The Expected Positive Exposure function (EPE), where $\mathrm{EPE}(t)$ is the risk-neutral expectation of the loss on a contract conditional on a default of the counterparty occurring at time t.
Note that the CVA can be given an option-theoretic interpretation, so that counterparty risk can, in principle, be managed dynamically.

1.1 Counterparty Credit Risk

Wrong way risk is particularly important in the case of *credit derivatives* transactions, at least from the perspective of a credit protection buyer. Indeed, via economic cycle and default contagion effects, the time of default of a counterparty selling credit protection is typically a time of higher value of credit protection.

We consider in this paper a ***Credit Default Swap with counterparty risk*** ('risky CDS' in the sequel, as opposed to 'risk-free CDS', without counterparty risk). Note that this topic already received a lot of attention in the literature. It can thus be considered as a benchmark problem of counterparty credit risk. To quote but a few:
• Huge and Lando [15] propose a rating-based approach,
• Hull and White [16] study this problem in the set-up of a static copula model,
• Jarrow and Yu [17] use an intensity contagion model, further considered in Leung and Kwok [19],
• Brigo and Chourdakis [7] work in the set-up of their Gaussian copula and CIR++ intensity model, extended to the issue of bilateral counterparty credit risk in Brigo and Capponi [6],

• Blanchet-Scalliet and Patras [5] or Lipton and Sepp [20] develop structural approaches.

1.2 A Markov Copula Approach

We shall consider a Markovian model of credit risk in which simultaneous defaults are possible. Wrong way risk is thus represented in the model by the fact that at the time of default of the counterparty, there is a positive probability that the firm on which the CDS is written defaults too, in which case the loss incurred to the investor (Exposure at Default ED, cf. (3)) is the loss given default of the firm (up to the recovery on the counterparty), that is a very large amount. Of course, this simple model should not be taken too literally. We are not claiming here that simultaneous defaults can happen in actual practice. The rationale and financial interpretation of our model is rather that at the time of default of the counterparty, there is a positive probability of a high defaults spreads environment, in which case, the value of the CDS for a protection buyer is close to the loss given default of the firm.

More specifically, we shall be considering a four-state Markov Chain model of two obligors, so that all the computations are straightforward, either that there are explicit formulas for all the quantities of interest, or, in case less elementary parameterizations of the model are used, that these quantities can be easily and quickly computed by solving numerically the related Kolmogorov ODEs.

This Markovian set-up makes it possible to address in a dynamic and consistent way the issues of valuing (and also hedging) the CDS, and/or, if wished, the CVA, interpreted as an option as evoked above.

To make this even more practical, we shall work in a *Markovian copula* set-up in the sense of Bielecki et al. [3], in which calibration of the model marginals to the related CDS curves is straightforward. The only really free model parameters are thus the few dependence parameters, which can be calibrated or estimated in ways that we shall explain in the paper.

1.3 Outline of the Paper

In Section 2 we first describe the mechanism and cash flows of a payer CDS with counterparty credit risk. We then state a few preliminary results about pricing and CVA of this CDS in a general set-up. In Section 3 we introduce our Markov chain copula model, in which we derive explicit formulas for most quantities of interest in regard to a risky CDS, like price, EPE, CVA or hedging ratios. Section 4 is about implementation of the model. Alternative model parameterizations and related calibration or estimation procedures are proposed and analyzed. Numerical results are presented and discussed, showing good agreement of model's EPE and CVA with expected features. Section 5 recapitulates our model's main properties and presents some directions for possible extensions of the previous results.

2 General Set-Up

2.1 Cash Flows

As is well known, a CDS contract involves three entities: A reference credit (firm), a buyer of default protection on the firm, and a seller of default protection on the firm. The issue of counterparty risk on a CDS is:

- Primarily, the fact that the seller of protection may fail to pay the protection cash flows to the buyer in case of a default of the firm;
- Also, the symmetric concern that the buyer may fail to pay the contractual CDS spread to the seller.

We shall focus in this paper on the so-called *unilateral counterparty credit risk* involved in a payer CDS contract, namely the risk corresponding to the first bullet point above; however it should be noted that the approach of this paper could be extended to the issue of bilateral credit risk.
We shall refer to the buyer and the seller of protection on the firm as the risk-free *investor* and the defaultable *counterpart*, respectively. Indices 1 and 2 will refer to quantities related to the firm and to the counterpart, first of which, their default times τ_1 and τ_2.

Under a risky CDS (payer CDS with counterparty credit risk), the investor pays to the counterpart a stream of premia with spread κ, or *Fees Cash Flows*, from the inception date (time 0 henceforth) until the occurrence of a credit event (default of the counterpart or the firm) or the maturity T of the contract, whichever comes first.

Let us denote by R_1 and R_2 the recovery of the firm and the counterpart, supposed to be adapted to the information available at time τ_1 and τ_2, respectively. If the firm defaults prior to the expiration of the contract, the *Protection Cash Flows* paid by the counterpart to the investor depends on the situation of the counterpart:

- If the counterpart is still alive, she can fully compensate the loss of investor, i.e., she pays $(1-R_1)$ times the face value of the CDS to the investor;
- If the counterpart defaults at the same time as the firm (note that it is important to take this case into account in the perspective of the model with simultaneous defaults to be introduced later in this paper), she will only be able to pay to the investor a fraction of this amount, namely $R_2(1-R_1)$ times the face value of the CDS.

Finally, there is a *Close-Out Cash Flow* which is associated to clearing the positions in the case of early default of the counterpart. As of today, CDSs are sold over-the-counter (OTC), meaning that the two parties have to negotiate and agree on the terms of the contract. In particular the two parties can agree on one of the following three possibilities to exit (unwind) a trade:

- *Termination:* The contract is stopped after a terminal cash flow (positive or negative) has been paid to the investor;
- *Offsetting:* The counterpart takes the opposite protection position. This new contract should have virtually the same terms as the original CDS except for the premium which is fixed at the prevailing market level, and for the tenor which is set at the remaining time to maturity of the original CDS. So the counterpart leaves the original transaction in place but effectively cancels out its economic effect;
- *Novation* (or *Assignment*): The original CDS is assigned to a new counterpart, settling the amount of gain or loss with him. In this assignment the original counterpart (or *transferor*), the new coun-

terpart (*transferee*) and the investor agree to transfer all the rights and obligations of the transferor to transferee. So the transferor thereby ends his involvement in the contract and the investor thereafter deals with the default risk of the transferee.

In this paper we shall focus on ***termination***. More precisely, if the counterpart defaults in the lifetime of the CDS while the firm is still alive, a 'fair value' $\chi_{(\tau_2)}$ of the CDS is computed at time τ_2 according to a methodology specified in the CDS contract at inception. If this value (from the perspective of the investor) is negative, $(-\chi_{(\tau_2)})$ is paid by the investor to the counterpart, whereas if it is positive, the counterpart is assumed to pay to the investor a portion R_2 of $\chi_{(\tau_2)}$.

Remark 2.1 A typical specification is $\chi_{(\tau_2)} = P_{\tau_2}$, where P_t is the value at time t of a risk-free CDS on the same reference name, with the same contractual maturity T and spread κ as the original risky CDS. The consistency of this rather standard way of specifying $\chi_{(\tau_2)}$ is, in a sense, questionable. Given a pricing model accounting for the major risks in the product at hand, including, if appropriate, counterparty credit risk, with a related price process of the risky CDS denoted by Π, it could be argued that a more consistent specification would be $\chi_{(\tau_2)} = \Pi_{\tau_2}$ (or, more precisely, $\chi_{(\tau_2)} = \Pi_{\tau_2-}$, since $\Pi_{\tau_2} = 0$ in view of the usual conventions regarding the definition of ex-dividend prices). We shall see in section 4 that, at least in the specific model of this paper, adopting either convention makes little difference in practice.

2.2 Pricing

Let us be given a risk-neutral pricing model $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a given filtration making the τ_i's stopping times. In absence of further precision, all the *processes*, first of which, the *discount factor* process β, are supposed to be $\mathbb{F}$-adapted, and all the *random variables* are assumed to be $\mathcal{F}_T$-measurable. The fair value $\chi_{(\tau_2)}$ is supposed to be an $\mathcal{F}_{\tau_2}$-measurable random variable. The recoveries R_1 and R_2 are assumed to be $\mathcal{F}_{\tau_1}$- and $\mathcal{F}_{\tau_2}$- measurable random variables. Let $\mathbb{E}_\tau$ stand for the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_\tau$, for any stopping time τ.

We assume for simplicity that the face value of all the CDSs under consideration (risky or not) is equal to monetary unit and that the spreads are paid continuously in time. All the cash flows and prices are considered from the perspective of the investor. In accordance with the usual convention regarding the definition of *ex-dividend* prices, the integrals in this paper are taken open on the left and closed on the right of the interval of integration. In view of the description of the cash-flows in subsection 2.1, one then has,

Definition 2.2 **(i)** The model *price* process of a risky CDS is given by $\Pi_t = \mathbb{E}_t\big[\pi_T(t)\big]$, where $\pi_T(t)$ corresponds to the *risky CDS cumulative discounted cash flows* on the time interval $(t, T]$, so,

$$\begin{aligned}\beta_t \pi_T(t) &= -\kappa \int_{t\wedge\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge\tau_2\wedge T} \beta_s ds + \beta_{\tau_1}(1-R_1)\mathbb{1}_{t<\tau_1\leq T}\big[\mathbb{1}_{\tau_1<\tau_2} + R_2\mathbb{1}_{\tau_1=\tau_2}\big] \\ &\quad + \beta_{\tau_2}\mathbb{1}_{t<\tau_2\leq T}\mathbb{1}_{\tau_2<\tau_1}\big[R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big] \,. \end{aligned} \tag{1}$$

(ii) The model *price* process of a risk-free CDS is given by $P_t = \mathbb{E}_t[p_T(t)]$, where $p_T(t)$ corresponds to the *risk-free CDS cumulative discounted cash flows* on the time interval $(t, T]$, so,

$$\beta_t p_T(t) = -\kappa \int_{t\wedge\tau_1\wedge T}^{\tau_1\wedge T} \beta_s ds + (1-R_1)\beta_{\tau_1}\mathbb{1}_{t<\tau_1\leq T}. \tag{2}$$

The first, second and third term on the right-hand side of (1) correspond to the fees, protection and close-out cash flows of a risky CDS, respectively. Note that there are no cash flows of any kind after $\tau_1 \wedge \tau_2 \wedge T$ (in the case of the risky CDS) or $\tau_1 \wedge T$ (in the case of the risk-free CDS), so $\pi_T(t) = 0$ for $t \geq \tau_1 \wedge \tau_2 \wedge T$ and $p_T(t) = 0$ for $t \geq \tau_1 \wedge T$.

Remark 2.3 In these definitions it is implicitly assumed that, consistently with the now standard theory of no-arbitrage [13], a primary market of financial instruments (along with the risk-free asset β^{-1}) has been defined, with price processes given as locally bounded $(\Omega, \mathbb{F}, \mathbb{P})$ – local martingales. No-arbitrage on the extended market consisting of the primary assets and a further CDS then motivates the previous definitions. Since the precise specification of the primary market is irrelevant until the question of hedging is dealt with, we postpone it to section 3.3.

Definition 2.4 **(i)** *The* Exposure at Default *(*ED*) is the $\mathcal{F}_{\tau_2}$-measurable random variable $\xi_{(\tau_2)}$ defined by,*

$$\xi_{(\tau_2)} = \begin{cases} (1-R_2)(1-R_1), & \tau_2 = \tau_1 \leq T, \\ P_{\tau_2} - (R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}) & \tau_2 < \tau_1,\, \tau_2 \leq T, \\ 0, & \textit{otherwise} . \end{cases} \tag{3}$$

(ii) *The* Credit Valuation Adjustment *(*CVA*) is the process killed at $\tau_1 \wedge \tau_2 \wedge T$ defined by, for $t \in [0,T]$,*

$$\beta_t CVA_t = \mathbb{1}_{\{t<\tau_2\}}\mathbb{E}_t\left[\beta_{\tau_2}\xi_{(\tau_2)}\right] . \tag{4}$$

(iii) *The* Expected Positive Exposure *(*EPE*) is the function of time defined by, for $t \in [0,T]$,*

$$EPE(t) = \mathbb{E}\left[\xi_{(\tau_2)}|\tau_2 = t\right] . \tag{5}$$

The following proposition justifies the name of Credit Valuation Adjustment which is used for the CVA process defined by (4). In case $\chi_{(\tau_2)} = P_{\tau_2}$ (see Remark 2.1) then

$$\xi_{(\tau_2)} = \xi^0_{(\tau_2)} := (1-R_2) \times \begin{cases} (1-R_1), & \tau_2 = \tau_1 \leq T, \\ P^+_{\tau_2} & \tau_2 < \tau_1,\, \tau_2 \leq T, \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and we essentially recover the basic result that appears in the series of papers by Brigo et alii. Note that as opposed to Brigo et al. we do not exclude simultaneous defaults in our set-up, whence further terms in $\mathbb{1}_{t<\tau_1=\tau_2\leq T}$ in the proof of Proposition 2.1.

Proposition 2.1 *One has $CVA_t = P_t - \Pi_t$ on $\{t < \tau_2\}$.*

Proof. If $\tau_1 \leq t < \tau_2$, then $\Pi_t = P_t = \mathrm{CVA}_t = 0$ in view of (1), (2) and (4).

Assume $t < \tau_1 \wedge \tau_2$. Subtracting $\pi_T(t)$ from $p_T(t)$ yields,

$$\begin{aligned} \beta_t\left(p_T(t) - \pi_T(t)\right) = -\kappa \int_{\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge T} \beta_s ds + \beta_{\tau_1}(1-R_1)\mathbb{1}_{\tau_1\leq T}\mathbb{1}_{\tau_1\geq\tau_2} \\ -\beta_{\tau_1}R_2(1-R_1)\mathbb{1}_{\tau_1\leq T}\mathbb{1}_{\tau_1=\tau_2} - \beta_{\tau_2}\mathbb{1}_{\tau_2<\tau_1}\mathbb{1}_{\tau_2\leq T}\big(R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big) . \end{aligned} \tag{7}$$

Moreover, in view of (2), one has,

$$\beta_{\tau_2} p_T(\tau_2) \mathbb{1}_{\tau_2<\tau_1} \mathbb{1}_{\tau_2 \le T} = -\kappa \int_{\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge T} \beta_s ds + (1-R_1)\beta_{\tau_1} \mathbb{1}_{\tau_2<\tau_1\le T}. \tag{8}$$

Now, using the following identity in the second term on the right-hand-side of (7):

$$\mathbb{1}_{\tau_1\le T}\mathbb{1}_{\tau_1\ge\tau_2} = \mathbb{1}_{\tau_1\le T}\mathbb{1}_{\tau_2<\tau_1} + \mathbb{1}_{\tau_1=\tau_2\le T},$$

and plugging (8) into (7) , it comes (recall $t < \tau_1 \wedge \tau_2$),

$$\begin{aligned}\beta_t\left(p_T(t) - \pi_T(t)\right) &= \beta_{\tau_2}\mathbb{1}_{\tau_2<\tau_1,\,\tau_2\le T}\, p_T(\tau_2) \\ &+\beta_{\tau_2}\mathbb{1}_{\tau_2=\tau_1\le T}(1-R_2)(1-R_1) - \beta_{\tau_2}\mathbb{1}_{\tau_2<\tau_1,\,\tau_2\le T}\big(R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big).\end{aligned}$$

Thus:

- On the set $\{\tau_2 < \tau_1,\, \tau_2 \le T\}$,

$$\beta_t\left(p_T(t) - \pi_T(t)\right) = \beta_{\tau_2} p_T(\tau_2) - \beta_{\tau_2}\big(R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big)$$

As $P_{\tau_2} = \mathbb{E}_{\tau_2}[p_T(\tau_2)]$, we then have, since R_2 and $\chi_{(\tau_2)}$ are $\mathcal{F}_{\tau_2}$-measurable,

$$\beta_t \mathbb{E}_{\tau_2}\big[p_T(t) - \pi_T(t)\big] = \beta_{\tau_2}\left(P_{\tau_2} - (R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)})\right)\ ; \tag{9}$$

- On the set $\{\tau_1 = \tau_2 \le T\}$,

$$\beta_t\left(p_T(t) - \pi_T(t)\right) = \beta_{\tau_2}(1-R_1)(1-R_2)$$

and thus

$$\beta_t \mathbb{E}_{\tau_2}\big[p_T(t) - \pi_T(t)\big] = \mathbb{E}_{\tau_2}\big[\beta_{\tau_2}(1-R_1)(1-R_2)\big]. \tag{10}$$

Using the fact that $\tau_2 < \tau_1,\, \tau_2 \le T$ and $\tau_2 = \tau_1 \le T$ are $\mathcal{F}_{\tau_2}$-measurable, it follows,

$$\begin{aligned}\beta_t P_t - \beta_t \Pi_t &= \beta_t \mathbb{E}_t\big[\mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]\big] \\ &= \beta_t \mathbb{E}_t\Big[\mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]\mathbb{1}_{\tau_2<\tau_1,\,\tau_2\le T} + \mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]\mathbb{1}_{\tau_2=\tau_1\le T}\Big] \\ &= \mathbb{E}_t\big[\beta_{\tau_2}\xi_{(\tau_2)}\big] = \beta_t \mathrm{CVA}_t.\end{aligned}$$

□

2.3 Special Case $\mathbb{F} = \mathbb{H}$

Let $H = (H^1, H^2)$ denote the pair of the default indicator processes of the firm and the counterpart, so $H^i_t = \mathbb{1}_{\tau_i\le t}$. The following proposition gathers a few useful results that can be established in the special case of a model filtration $\mathbb{F}$ given as

$$\mathbb{F} = \mathbb{H} = (\mathcal{H}^1_t \vee \mathcal{H}^2_t)_{t\in[0,T]}\ ,$$

with $\mathcal{H}^i_t = \sigma(H^i_s; 0 \le s \le t)$.

Proposition 2.2 **(i)** *For $t \in [0, T]$, any $\mathcal{H}_t$-measurable random variable Y_t can be written as*

$$Y_t = y_0(t)\mathbb{1}_{t<\tau_1\wedge\tau_2} + y_1(t,\tau_1)\mathbb{1}_{\tau_1\le t<\tau_2} + y_2(t,\tau_2)\mathbb{1}_{\tau_2\le t<\tau_1} + y_3(t,\tau_1,\tau_2)\mathbb{1}_{\tau_2\vee\tau_1\le t}$$

where $y_0(t)$, $y_1(t,u)$, $y_2(t,v)$, $y_3(t,u,v)$ are deterministic functions.
(ii) *For any integrable random variable Z, one has,*

$$\mathbb{1}_{t<\tau_1\wedge\tau_2}\mathbb{E}_t Z = \mathbb{1}_{t<\tau_1\wedge\tau_2}\frac{\mathbb{E}(Z\mathbb{1}_{t<\tau_1\wedge\tau_2})}{\mathbb{P}(t<\tau_1\wedge\tau_2)} . \tag{11}$$

(iii) *The price process of the risky CDS is given by $\Pi_t = \Pi(t, H_t)$, for a* pricing function Π *defined on $\mathbb{R}^+ \times E_1 \times E_1$ with $E_1 = \{0,1\}$, such that $\Pi(t,e) = 0$ for $e \neq (0,0)$. On the set $\{t < \tau_1 \wedge \tau_2\}$, Π_t is given by the deterministic function*

$$\Pi(t,0,0) = u(t) := \frac{\mathbb{E}\big[\pi_T(t)\big]}{\mathbb{P}(\tau_1\wedge\tau_2 > t)} . \tag{12}$$

(iv) *One has, for suitable functions $\widetilde{\chi}(\cdot)$, $v(\cdot)$, $\widetilde{\xi}(\cdot,\cdot)$ and CVA$(\cdot)$,*

$$\mathbb{1}_{\{\tau_2<\tau_1\}}\chi_{(\tau_2)} = \mathbb{1}_{\{\tau_2<\tau_1\}}\widetilde{\chi}(\tau_2)\,,\ \ \mathbb{1}_{\{\tau_2<\tau_1\}}P_{\tau_2} = \mathbb{1}_{\{\tau_2<\tau_1\}}v(\tau_2) \tag{13}$$

$$\xi_{(\tau_2)} = \widetilde{\xi}(\tau_1,\tau_2) := \big(\mathbb{1}_{\tau_2=\tau_1\le T}(1-R_2)(1-R_1) + \mathbb{1}_{\tau_2<\tau_1,\,\tau_2\le T}\big(v(\tau_2) - (R_2\widetilde{\chi}^+(\tau_2) - \widetilde{\chi}^-(\tau_2))\big)\big) \tag{14}$$

$$\mathit{CVA}_t = \mathbb{1}_{t<\tau_1\wedge\tau_2}\mathit{CVA}(t) . \tag{15}$$

(v) *A function CVA$(\cdot)$ satisfying (15) is defined by, for $t \in [0,T]$,*

$$\beta_t \mathit{CVA}(t) := \int_t^T \beta_s \mathit{EPE}(s)\frac{\mathbb{P}(\tau_2\in ds)}{\mathbb{P}(t<\tau_1\wedge\tau_2)} . \tag{16}$$

Proof. **(i)** and **(ii)** are standard (see, e.g., [4]; (ii) in particular is the so-called *Key Lemma*).
(iii) Since there are no cash flows of a risky CDS beyond the first default (cf. (1)), one has $\pi_T(t) = \pi_T(t)\mathbb{1}_{t<\tau_1\wedge\tau_2}$. The Key Lemma then yields,

$$\Pi_t = \mathbb{E}_t\big[\mathbb{1}_{t<\tau_1\wedge\tau_2}\pi_T(t)\big] = (1-H_t^1)(1-H_t^2)\frac{\mathbb{E}\big[\pi_T(t)\big]}{\mathbb{P}(\tau_1\wedge\tau_2>t)} .$$

Thus $\Pi_t = \Pi(t, H_t^1, H_t^2)$, for a pricing function Π defined by

$$\Pi(t,e_1,e_2) = (1-e_1)(1-e_2)u(t) ,$$

where $u(t)$ is defined by the right-hand-side of (12).
(iv) follows directly from part (i), given the definition of P_{τ_2}, $\chi_{(\tau_2)}$, $\xi_{(\tau_2)}$ and of the CVA process.
(v) By (iv), one has, using (ii) again,

$$\begin{aligned}
\beta_t\mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\mathrm{CVA}_t &= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\mathbb{E}_t\big[\beta_{\tau_2}\xi_{(\tau_2)}\big] = \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\mathbb{E}_t\big[\beta_{\tau_2}\widetilde{\xi}(\tau_1,\tau_2)\big] \\
&= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\frac{\mathbb{E}\big[\beta_{\tau_2}\widetilde{\xi}(\tau_1,\tau_2)\mathbb{1}_{t<\tau_1\wedge\tau_2}\big]}{\mathbb{P}(t<\tau_1\wedge\tau_2)} = \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\frac{\mathbb{E}\big[\mathbb{E}\big(\beta_{\tau_2}\widetilde{\xi}(\tau_1,\tau_2)\mathbb{1}_{t<\tau_1\wedge\tau_2}|\tau_2\big)\big]}{\mathbb{P}(t<\tau_1\wedge\tau_2)} \\
&= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\frac{\mathbb{E}\big[\mathbb{E}\big(\beta_{\tau_2}\widetilde{\xi}(\tau_1,\tau_2)\mathbb{1}_{t<\tau_2\le T}|\tau_2\big)\big]}{\mathbb{P}(t<\tau_1\wedge\tau_2)} = \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\frac{\mathbb{E}\big[\beta_{\tau_2}\mathbb{E}\big(\widetilde{\xi}(\tau_1,\tau_2)|\tau_2\big)\mathbb{1}_{t<\tau_2\le T}\big]}{\mathbb{P}(t<\tau_1\wedge\tau_2)} \\
&= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\frac{\mathbb{E}\big[\beta_{\tau_2}\mathrm{EPE}(\tau_2)\mathbb{1}_{t<\tau_2\le T}\big]}{\mathbb{P}(t<\tau_1\wedge\tau_2)} = \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\int_t^T \beta_s\mathrm{EPE}(s)\frac{\mathbb{P}(\tau_2\in ds)}{\mathbb{P}(t<\tau_1\wedge\tau_2)} ,
\end{aligned}$$

whence (v). □

3 Markov Copula Factor Set-Up

3.1 Factor Process Model

We shall now introduce a suitable *Markovian Copula Model* for the pair of default indicator processes $H = (H^1, H^2)$ of the firm and the counterpart. The name 'Markovian Copula' refers to the fact that the model will have prescribed marginals for the laws of H^1 and H^2, respectively (see Bielecki et al. [2, 3] for a general theory). The practical interest of a Markovian copula model is clear with respect to the task of model calibration, since the copula property allows one to decouple the calibration of the marginal and of the dependence parameters in the model (see again section 4.1). More fundamentally, the opinion developed in this paper is that it is also a virtue for a model to 'take the right inputs to generate the right outputs', namely taking as basic inputs the individual default probabilities (individual CDS curves), which correspond to the more reliable information on the market, and are then 'coupled together' in a suitable way (see section 4.1).

An apparent shortcoming of the Markov copula approach is that it is does not allow for default contagion effects in the usual sense (default of a name impacting the default intensities of the other ones). The way we shall introduce dependence between τ_1 and τ_2 is by relaxing the standard assumption of no simultaneous defaults. As we shall see, allowing for simultaneous defaults is a powerful way of modeling defaults dependence.

Specifically, we model the pair $H = (H^1, H^2)$ as an inhomogeneous Markov chain relative to its own filtration $\mathbb{H}$ on a probability space $(\Omega, \mathbb{P})$ (for the σ-algebra $\mathcal{H}_T$), with state space $E = \{(0,0),(1,0),(0,1),(1,1)\}$, and generator matrix at time t given by the following 4×4 matrix $A(t)$, where the first to fourth rows (or columns) correspond to the four possible states $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ of H_t :

$$A(t) = \begin{bmatrix} -l(t) & l_1(t) & l_2(t) & l_3(t) \\ 0 & -q_2(t) & 0 & q_2(t) \\ 0 & 0 & -q_1(t) & q_1(t) \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{17}$$

In (17) the l's and q's denote deterministic functions of time integrable over $[0,T]$, with in particular $l(t) = l_1(t) + l_2(t) + l_3(t)$.

Remark 3.1 The intuitive meaning of '(17) being the generator matrix of H' is the following (see, e.g., Rogers and Williams [23], Vol. I, Chap. III, Sec. 2, for standard definitions and results on Markov Chains):
• *First line:* Conditional on the pair $H_t = (H_t^1, H_t^2)$ being in state $(0,0)$ (firm and counterpart still alive at time t), there is a probability $l_1(t)dt$, (resp. $l_2(t)dt$; resp. $l_3(t)dt$) of a default of the firm alone (resp. of the counterpart alone; resp. of a simultaneous default of the firm and the counterpart) in the infinitesimal time interval $(t, t+dt)$;
• *Second line:* Conditional on the pair $H_t = (H_t^1, H_t^2)$ being in state $(1,0)$ (firm defaulted but

counterpart still alive at time t), there is a probability $q_2(t)dt$ of a further default of the counterpart in the time interval $(t, t+dt)$;
• *Third line:* Conditional on the pair $H_t = (H_t^1, H_t^2)$ being in state $(0,1)$ (firm still alive but counterpart defaulted at time t), there is a probability $q_1(t)dt$ of a further default of the firm in the time interval $(t, t+dt)$.

On each line the diagonal term is then set as minus the sum of the off-diagonal terms, so that the sum of the entries of each line be equal to zero, as should be for $A(t)$ to represent the generator of a Markov process.

Moreover, for the sake of the desired ***Markov copula property*** (Proposition 3.1(iii) below), we impose the following relations between the l's and the q's.

Assumption 3.2 $q_1(t) = l_1(t) + l_3(t)\,,\ q_2(t) = l_2(t) + l_3(t)$.

Observe that in virtue of these relations:
• Conditional on H_t^1 being in state 0, and whatever the state of H_t^2 may be (that is, in the state $(0,0)$ as in the state $(0,1)$ for H_t), there is a probability $q_1(t)dt$ of a default of the firm (alone or jointly with the counterpart) in the next time interval $(t, t+dt)$;
• Conditional on H_t^2 being in state 0, and whatever the state of H_t^1 may be (that is, in the states $(0,0)$ or $(1,0)$ for H_t), there is a probability $q_2(t)dt$ of a default of the counterpart (alone or jointly with the firm) in the next time interval $(t, t+dt)$.
In mathematical terms the default indicator processes H^1 and H^2 are $\mathbb{H}$-Markov processes on the state space $E_1 = \{0,1\}$ with time t generators respectively given by

$$A_1(t) = \begin{bmatrix} -q_1(t) & q_1(t) \\ 0 & 0 \end{bmatrix}\,,\ A_2(t) = \begin{bmatrix} -q_2(t) & q_2(t) \\ 0 & 0 \end{bmatrix}\,. \tag{18}$$

To formalize the previous statements, and in view of the study of simultaneous jumps, let us further introduce the processes $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$ standing for the indicator processes of a default of the firm alone, of the counterpart alone, and of a simultaneous default of the firm and the counterpart, respectively. So

$$H^{\{1,2\}} = [H^1, H^2]\,,\ H^{\{1\}} = H^1 - H^{\{1,2\}}\,,\ H^{\{2\}} = H^1 - H^{\{1,2\}}\,, \tag{19}$$

where $[\cdot,\cdot]$ stands for the quadratic covariation. Equivalently, for $t \in [0,T]$,

$$H_t^{\{1\}} = \mathbb{1}_{\tau_1 \le t, \tau_1 \ne \tau_2}\,,\ H_t^{\{2\}} = \mathbb{1}_{\tau_2 \le t, \tau_1 \ne \tau_2}\,,\ H_t^{\{1,2\}} = \mathbb{1}_{\tau_1 = \tau_2 \le t}\,.$$

Note that the natural filtration of $(H^\iota)_{\iota \in I}$, with here and henceforth $I = \{\{1\},\{2\},\{1,2\}\}$, is equal to $\mathbb{H}$. The proof of the following Proposition is deferred to Appendix A.

Proposition 3.1 **(i)** *The $\mathbb{H}$-intensity of H^ι is of the form $q_\iota(t, H_t)$ for a suitable function $q_\iota(t,e)$ for every $\iota \in I$, namely,*

$$\begin{aligned} q_{\{1\}}(t,e) &= \mathbb{1}_{e_1=0}\left(\mathbb{1}_{e_2=0} l_1(t) + \mathbb{1}_{e_2=1} q_1(t)\right) \\ q_{\{2\}}(t,e) &= \mathbb{1}_{e_2=0}\left(\mathbb{1}_{e_1=0} l_2(t) + \mathbb{1}_{e_1=1} q_2(t)\right) \\ q_{\{1,2\}}(t,e) &= \mathbb{1}_{e=(0,0)} l_3(t)\,. \end{aligned}$$

Put another way, the processes M^ι defined by, for every $\iota \in I$,

$$M_t^\iota = H_t^\iota - \int_0^t q_\iota(s, H_s) ds \;, \tag{20}$$

with

$$\begin{aligned} q_{\{1\}}(t, H_t) &= (1 - H_t^1)\left((1 - H_t^2) l_1(t) + H_t^2 q_1(t)\right) \\ q_{\{2\}}(t, H_t) &= (1 - H_t^2)\left((1 - H_t^1) l_2(t) + H_t^1 q_2(t)\right) \\ q_{\{1,2\}}(t, H_t) &= (1 - H_t^1)(1 - H_t^2) l_3(t) \;, \end{aligned} \tag{21}$$

are $\mathbb{H}$-martingales.
(ii) *The $\mathbb{H}$-intensity process of H^i is given by $(1 - H_t^i) q_i(t)$. In other words, the processes M^i defined by, for $i = 1, 2$,*

$$M_t^i = H_t^i - \int_0^t (1 - H_s^i) q_i(s) ds \;, \tag{22}$$

are $\mathbb{H}$-martingales.
(iii) *The processes H^1 and H^2 are $\mathbb{H}$-Markov processes with generator matrix at time t given by $A_1(t)$ and $A_2(t)$ (cf. (18)).*
(iv) *One has, for $s < t$,*

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = e^{-\int_0^s l(u)du} e^{-\int_s^t q_2(u)du} \;,\; \mathbb{P}(\tau_1 > t, \tau_2 > s) = e^{-\int_0^s l(u)du} e^{-\int_s^t q_1(u)du} \tag{23}$$

and therefore

$$\begin{aligned} &\mathbb{P}(\tau_1 > t) = e^{-\int_0^t q_1(u)du} \;,\; \mathbb{P}(\tau_2 > t) = e^{-\int_0^t q_2(u)du} \\ &\mathbb{P}(\tau_1 > s, \tau_2 \in dt) = q_2(t) e^{-\int_0^s l(u)du} e^{-\int_s^t q_2(u)du} dt \;,\; \mathbb{P}(\tau_1 \in dt, \tau_2 > s) = q_1(t) e^{-\int_0^s l(u)du} e^{-\int_s^t q_1(u)du} dt \\ &\mathbb{P}(\tau_1 > t, \tau_2 \in dt) = q_2(t) e^{-\int_0^t l(u)du} dt \;,\; \mathbb{P}(\tau_1 \in dt, \tau_2 > t) = q_1(t) e^{-\int_0^t l(u)du} dt \\ &\mathbb{P}(\tau_1 \wedge \tau_2 > t) = \exp\left(-\int_0^t l(u)du\right) \;. \end{aligned} \tag{24}$$

(v) *The correlation of H_t^1 and H_t^2 (*default correlation at the time horizon *t) is*

$$\rho_d(t) = \frac{\exp\left(\int_0^t l_3(s)ds\right) - 1}{\sqrt{\left(\exp\left(\int_0^t q_1(s)ds\right) - 1\right)\left(\exp\left(\int_0^t q_2(s)ds\right) - 1\right)}}. \tag{25}$$

Remark 3.3 **(i)** In the Markov copula [3] terminology, the so-called *consistency condition* is satisfied (H^1 and H^2 are $\mathbb{H}$-Markov processes). The bi-variate model H with generator A is thus a *Markovian copula model* with marginal generators A_1 and A_2.
(ii) The default times τ_1 and τ_2 could equivalently be defined by

$$\tau_1 = \eta_1 \wedge \eta_3 \;,\; \tau_2 = \eta_2 \wedge \eta_3$$

where the η_i's are independent inhomogeneous exponential random variables with parameters $l_i(t)$'s. Thus, for every $0 \le s, t$,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{P}(\eta_1 > s)\mathbb{P}(\eta_2 > t)\mathbb{P}(\eta_3 > s \vee t) \ . \tag{26}$$

In the special case of *homogeneous* exponential random variables with (constant) parameters l_i's, one has further (see section 4 of [14] or [21]),

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = C(\mathbb{P}(\eta_1 > s), \mathbb{P}(\eta_2 > t)) \ , \tag{27}$$

where the *Marshall-Olkin survival copula function* C is defined by, for $p, q \in [0,1]$,

$$C(p,q) = pq \min(p^{-\alpha_1}, q^{-\alpha_2}) \tag{28}$$

with $\alpha_i = \frac{l_3}{l_i + l_3}$. Our model is thus an extension of the classical Marshall-Olkin copula model in which *inhomogeneous* exponential random variables are used as model inputs, and where, more importantly, a *dynamic perspective* is shed on the random times τ_1 and τ_2 by introducing the model filtration $\mathbb{H}$.

3.2 Pricing

We use the notation of Proposition 2.2, which applies here since we are in the special case $\mathbb{F} = \mathbb{H}$. Recall in particular $\Pi_t = \Pi(t, H_t) = (1-H_t^1)(1-H_t^2)u(t)$, for a *pricing function* $\Pi(t,0,0) = u(t)$, as well as the identities (13), (15) (16).

We assume henceforth for simplicity that:
- The discount factor writes $\beta_t = \exp(-\int_0^t r(s)ds)$, for a deterministic *short-term interest-rate* function r,
- The recovery rates R_1 and R_2 are constant.

Proposition 3.2 *The pricing function u of the risky CDS is given by*

$$\beta_t u(t) = \int_t^T \beta_s e^{-\int_t^s l(u)du} \pi(s) ds \tag{29}$$

with

$$\pi(s) = (1 - R_1)\big[l_1(s) + R_2 l_3(s)\big] + l_2(s)\big[R_2 \widetilde{\chi}(s)^+ - \widetilde{\chi}(s)^-\big] - \kappa \ . \tag{30}$$

The function u satisfies the following ODE:

$$\begin{cases} u(T) = 0 \\ \frac{du}{dt}(t) - (r(t) + l(t))u(t) + \pi(t) = 0 \,, \ \ t \in [0,T) \ . \end{cases} \tag{31}$$

Proof. Recall (12):

$$u(t) = \frac{\mathbb{E}\big[\pi_T(t)\big]}{\mathbb{P}(\tau_1 \wedge \tau_2 > t)} \ ,$$

where the denominator can be calculated using Proposition 3.1(iv). For computing the numerator, one rewrites the expressions for the cumulative discounted Fee, Protection and Close-out cash flows in terms of integrals with respect to $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$, as follows:

$$\begin{aligned}
\text{Fees Cash Flow} &= \kappa \int_0^T \beta_s (1 - H_s^1)(1 - H_s^2) ds \\
\text{Protection Cash Flow} &= (1 - R_1) \int_0^T \beta_s (1 - H_{s-}^2) dH_s^{\{1\}} + R_2 (1 - R_1) \int_0^T \beta_s dH_s^{\{1,2\}} \\
&= (1 - R_1) \int_0^T \beta_s (1 - H_{s-}^2) dM_s^{\{1\}} + (1 - R_1) \int_0^T \beta_s (1 - H_s^2) q_{\{1\}}(s, H_s) ds \\
&+ R_2 (1 - R_1) \int_0^T \beta_s dM_s^{\{1,2\}} + R_2 (1 - R_1) \int_0^T \beta_s q_{\{1,2\}}(s, H_s) ds \\
\text{Close-out Cash Flow} &= \int_0^T \beta_s \big[R_2 \widetilde{\chi}(s)^+ - \widetilde{\chi}(s)^- \big] (1 - H_{s-}^1) dH_s^{\{2\}} \\
&= \int_0^T \beta_s \big[R_2 \widetilde{\chi}(s)^+ - \widetilde{\chi}(s)^- \big] (1 - H_{s-}^1) dM_s^{\{2\}} \\
&+ \int_0^T \beta_s \big[R_2 \widetilde{\chi}(s)^+ - \widetilde{\chi}(s)^- \big] (1 - H_s^1) q_{\{2\}}(s, H_s) ds
\end{aligned}$$

Taking care of the martingale property of $M^{\{1\}}$, $M^{\{2\}}$ and $M^{\{1,2\}}$ and of the fact that the integrals of bounded predictable processes with respect to these martingales are indeed martingales, it thus comes,

$$\mathbb{E}(\pi_T(t)) = \mathbb{E}(\widetilde{\pi}_T(t)) \tag{32}$$

with

$$\begin{aligned}
\beta_t \widetilde{\pi}_T(t) = &- \kappa \int_t^T \beta_s (1 - H_s^1)(1 - H_s^2) ds \\
&+ (1 - R_1) \int_t^T \beta_s (1 - H_s^2) q_{\{1\}}(s, H_s) ds + R_2 (1 - R_1) \int_t^T \beta_s q_{\{1,2\}}(s, H_s) ds \\
&+ \int_t^T \beta_s \big[R_2 \widetilde{\chi}(s)^+ - \widetilde{\chi}(s)^- \big] (1 - H_s^1) q_{\{2\}}(s, H_s) ds.
\end{aligned}$$

Moreover, in view of the expressions for $q_{\{1\}}$ and $q_{\{2\}}$ in (21), one has

$$\begin{aligned}
(1 - H_s^2) q_{\{1\}}(s, H_s) &= (1 - H_s^1)(1 - H_s^2) l_1(s) \, , \\
(1 - H_s^1) q_{\{2\}}(s, H_s) &= (1 - H_s^1)(1 - H_s^2) l_2(s) \, .
\end{aligned} \tag{33}$$

Plugging this into (32) and using (24), it comes,

$$\begin{aligned}\beta_t \mathbb{E}[\pi_T(t)] &= \mathbb{E}\left[\int_t^T \beta_s (1-H_s^1)(1-H_s^2)\pi(s)ds\right] \\ &= \int_t^T \beta_s \mathbb{E}\left[(1-H_s^1)(1-H_s^2)\right]\pi(s)ds \\ &= \int_t^T \beta_s e^{-\int_0^s l(x)dx}\pi(s)ds\end{aligned}$$

where π is given by (30). One can now check by inspection that the function u satisfies the ODE (31). □

Remark 3.4 The equation (31) can also be interpreted as the Kolmogorov backward equation related to the valuation of a risky CDS in our set-up. This ODE can in fact be derived directly and independently by an application of the Itô formula to the martingale $\Pi(t, H_t^1, H_t^2)$, which results in an alternative proof of Proposition 3.2.

Remark 3.5 In the set-up of the Markov chain copula model, the identity (whenever assumed) $\chi_{(\tau_2)} = \Pi_{\tau_2-}$ (see Remark 2.1) is thus equivalent to

$$\chi_{(\tau_2)} = \Pi_{\tau_2-} = \lim_{t\to\tau_2-} u(t) = u(\tau_2)\ ,$$

by continuity of u. This case thus corresponds to the case where the function $\widetilde{\chi}$ in Proposition 2.2(iv) is in fact given by the function u (case $\widetilde{\chi} = u$). In this case the positive and negative parts of u, i.e., u^+ and u^- are sitting in the expression for π in (30). One thus deals with a non-linear valuation ODE (31), and the formula (29) is not explicit anymore, since u is 'hidden' in π in the right hand side of this formula. However one can still compute u by numerical solution of (31).

Proposition 3.3 *The price of a risk-free CDS with spread κ on the firm admits the representation:*

$$P_t = P(t, H_t^1)\ , \tag{34}$$

for a function P of the form $P(t, e_1) = (1-e_1)v(t)$. The pricing function v is given by

$$\beta_t v(t) = \int_t^T \beta_s e^{-\int_t^s q_1(x)dx} p(s)ds$$

with

$$p(s) = (1-R_1)q_1(s) - \kappa. \tag{35}$$

The pricing function v thus solves the following pricing ODE:

$$\begin{cases} v(T) = 0 \\ \frac{dv}{dt}(t) - (r(t)+q_1(t))v(t) + p(t) = 0\,,\ \ t\in[0,T)\ . \end{cases}$$

Proof. One has,

$$\begin{aligned}\beta_t p_T(t) &= -\kappa \int_t^T \beta_s (1 - H_s^1) ds + (1 - R_1) \int_t^T \beta_s dH_s^1 \\ &= -\kappa \int_t^T \beta_s (1 - H_s^1) ds + (1 - R_1) \int_t^T \beta_s dM_s^1 + (1 - R_1) \int_t^T \beta_s q_1(s)(1 - H_s^1) ds.\end{aligned}$$

As M^1 is an $\mathbb{H}$-martingale and β a bounded continuous function, thus

$$\beta_t \mathbb{E}_t[p_T(t)] = \mathbb{E}_t \left[\int_t^T \beta_s (1 - H_s^1) p(s) ds \right] = \int_t^T \beta_s \mathbb{E}_t[1 - H_s^1] p(s) ds \ , \tag{36}$$

with $p(t)$ defined by (35), and where in virtue of Proposition 3.1(iii) and Proposition 2.2(ii) (Key Lemma), one has for $t < s$,

$$\mathbb{E}_t[1 - H_s^1] = \mathbb{E}[1 - H_s^1 | H_t^1] = (1 - H_t^1) \frac{\mathbb{P}(\tau_1 > s)}{\mathbb{P}(\tau_1 > t)} = (1 - H_t^1) e^{-\int_t^s q_1(x) dx} \ .$$

□

Proposition 3.4 *One has, for* $t \in [0, T]$, *(cf. (13), (15) (16)),*

$$\mathit{EPE}(t) = \left((1 - R_2)(1 - R_1) \frac{l_3(t)}{q_2(t)} \ + \ \big(v(t) - (R_2 \widetilde{\chi}^+(t) - \widetilde{\chi}^-(t)) \big) \frac{l_2(t)}{q_2(t)} \right) e^{-\int_0^t l_1(x) dx} \tag{37}$$

$$\mathit{CVA}(t) = \int_t^T \beta_s \Big((1 - R_2)(1 - R_1) l_3(s) \ + \ \big(v(s) - (R_2 \widetilde{\chi}^+(s) - \widetilde{\chi}^-(s)) \big) \, l_2(s) \Big) e^{-\int_t^s l(x) dx} ds \tag{38}$$

which in the special case where $\chi_{(\tau_2)} = P_{\tau_2}$, $\widetilde{\chi} = v$ *reduce to*

$$\mathit{EPE}(t) = \mathit{EPE}^0(t) := (1 - R_2) \left((1 - R_1) \frac{l_3(t)}{q_2(t)} \ + \ v^+(t) \frac{l_2(t)}{q_2(t)} \right) e^{-\int_0^t l_1(x) dx} \tag{39}$$

$$\mathit{CVA}(t) = \mathit{CVA}^0(t) := \int_t^T (1 - R_2) \beta_s \Big((1 - R_1) l_3(s) \ + \ v^+(s) l_2(s) \Big) e^{-\int_t^s l(x) dx} ds \tag{40}$$

Proof. Set

$$\Phi(\tau_2) = \mathbb{E}(\mathbb{1}_{\tau_1 = \tau_2 \le T} | \tau_2) \quad , \quad \Psi(\tau_2) = \mathbb{E}(\mathbb{1}_{\tau_2 < \tau_1, \, \tau_2 \le T} | \tau_2) \ ,$$

which are characterized by

$$\begin{aligned}\mathbb{E}\big(\Phi(\tau_2) f(\tau_2)\big) &= \mathbb{E}\big(f(\tau_2) \mathbb{1}_{\tau_1 = \tau_2 \le T}\big) \ , \\ \mathbb{E}(\Psi(\tau_2) f(\tau_2)) &= \mathbb{E}(f(\tau_2) \mathbb{1}_{\tau_2 < \tau_1, \, \tau_2 \le T}) \ ,\end{aligned} \tag{41}$$

for every Borel function f. In particular we take $f(x) = \mathbb{1}_{x \le t}$ for some $t \in (0, T]$.

Now using law of τ_2, the left-hand sides of (41) are given by

$$\begin{aligned}\mathbb{E}\big(\Phi(\tau_2) \mathbb{1}_{\tau_2 \le t}\big) &= \int_0^t \Phi(s) q_2(s) e^{-\int_0^s q_2(x) dx} ds \\ \mathbb{E}\big(\Psi(\tau_2) \mathbb{1}_{\tau_2 \le t}\big) &= \int_0^t \Psi(s) q_2(s) e^{-\int_0^s q_2(x) dx} ds\end{aligned}$$

As for the right-hand-sides of (41), thanks to Proposition 3.1(i) and (iv), one has

$$\begin{aligned}\mathbb{E}\big(\mathbb{1}_{\tau_2\le t}\mathbb{1}_{\tau_1=\tau_2\le T}\big) &= \mathbb{E}(\int_0^t dH_s^{\{1,2\}})\\ &= \int_0^t \mathbb{E}\big((1-H_s^1)(1-H_s^2)\big)l_3(s)ds = \int_0^t e^{-\int_0^s l(x)dx} l_3(s)ds\ ,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\big(\mathbb{1}_{\tau_2\le t}\mathbb{1}_{\tau_2<\tau_1,\,\tau_2\le T}\big) &= \mathbb{E}\Big(\int_0^t \mathbb{1}_{s\le\tau_1\wedge T} dH_s^{\{2\}}\Big) = \mathbb{E}\Big(\int_0^t \mathbb{1}_{s\le\tau_1} q_{\{2\}}(s,H_s)ds\Big)\\ &= \mathbb{E}\Big(\int_0^t (1-H_s^1)(1-H_s^2) l_2(s)ds\Big) = \int_0^t e^{-\int_0^s l(x)dx} l_2(s)ds\ ,\end{aligned}$$

where the second identity in the first line uses that $H^{\{2\}}$ does not jump at τ_1.

Thus for $f(x) = \mathbb{1}_{x\le t}$ the identities in (41) can be rewritten as

$$\begin{aligned}\int_0^t \Phi(s)q_2(s)e^{-\int_0^s q_2(x)dx}ds &= \int_0^t l_3(s)e^{-\int_0^s l(x)dx}ds\ ,\\ \int_0^t \Psi(s)q_2(s)e^{-\int_0^s q_2(x)dx}ds &= \int_0^t l_2(s)e^{-\int_0^s l(x)dx}ds\ .\end{aligned}$$

Taking derivative with respect to t of these last equations, leads us to

$$\Phi(t) = \frac{l_3(t)e^{-\int_0^t l(x)dx}}{q_2(t)} e^{\int_0^t q_2(x)dxd}\quad,\quad \Psi(t) = \frac{l_2(t)e^{-\int_0^t l(x)dx}}{q_2(t)} e^{\int_0^t q_2(x)dx}$$

and (37) follows.

Using (16), one then has for $t\in[0,T]$,

$$\begin{aligned}\beta_t\text{CVA}(t) &= \int_t^T \beta_s\text{EPE}(s)e^{\int_0^s l(x)dx}e^{-\int_0^s q_2(x)dx}q_2(s)e^{-\int_t^s l(x)dx}ds\\ &= \int_t^T \beta_s\text{EPE}(s)e^{\int_0^s l_1(x)dx}q_2(s)e^{-\int_t^s l(x)dx}ds\ .\end{aligned}$$

Hence (38) follows from (37). □

Remark 3.6 In view of the option-theoretic interpretation of the CVA, the CVA valuation formula (38) can also established directly, without passing by the EPE, much like formula (29) in Proposition 3.2 above (using a probabilistic computation, or resorting to the related Kolmogorov pricing ODE).

3.3 Hedging

We now give few preliminary results about hedging the risky CDS. We shall mainly consider the issue of delta-hedging, at least partially, the risky CDS, by a risk-free CDS which would also be available on the market (CDS on the firm with the same characteristics, except for the counterparty credit risk). Another perspective on the counterparty credit risk of the risky CDS can thus be given by assessing to which extent the risky CDS could, in principle, be hedged by the risk-free CDS.

3.3.1 Price Dynamics

Let $\widehat{\Pi}$ denote the discounted cum-dividend price of the risky CDS, that is, the local martingale

$$\widehat{\Pi}_t = \beta_t \Pi_t + \pi_t(0).$$

The Itô formula applied to $\Pi_t = \Pi(t, H_t)$ yields, on $[0, \tau_1 \wedge \tau_2 \wedge T]$,)

$$d\widehat{\Pi}_t = \beta_t\big(\delta\Pi_{\{1\}}(t)dM_t^{\{1\}} + \delta\Pi_{\{2\}}(t)dM_t^{\{2\}} + \delta\Pi_{\{1,2\}}(t)dM_t^{\{1,2\}}\big) \tag{42}$$

with

$$\delta\Pi_{\{1\}}(t) = 1 - R_1 - u(t)\,,\ \ \delta\Pi_{\{2\}}(t) = R_2\widetilde{\chi}^+(t) - \widetilde{\chi}^-(t) - u(t)\,,\ \ \delta\Pi_{\{1,2\}}(t) = R_2(1 - R_1) - u(t)\,.$$

Similarly, setting

$$\widehat{P}_t = \beta_t P_t + p_t(0),$$

it comes

$$d\widehat{P}_t = \beta_t \delta P_1(t) dM_t^1 \tag{43}$$

with

$$\delta P_1(t) = 1 - R_1 - v(t)\ .$$

3.3.2 Min-Variance Hedging

Let us denote by ψ a (self-financing) strategy in the risk-free CDS with price process P (and the savings account β_t^{-1}) for tentatively hedging the risky CDS with price process Π.

Recall that $\mathbb{P}$ is the risk neutral probability chosen by market. So the discounted cum-dividend price process $\widehat{P}$ is a $\mathbb{P}$-local martingale (actually in view of (43) $\widehat{P}$ is here a $\mathbb{P}$-martingale). As a result of the Galtchouk-Kunita-Watanabe decomposition, the hedging strategy ψ^{va} which minimizes the $\mathbb{P}$-variance of the hedging error, or *min-variance hedging strategy*, is given by

$$\psi_t^{va} = \frac{d\langle \widehat{\Pi}, \widehat{P}\rangle_t}{d\langle \widehat{P}\rangle_t}\ .$$

Remark 3.7 Note that we only deal with minimization of the risk-neutral variance of the hedging error, here, as opposed to the more difficult problem of minimizing the variance of the hedging error under the historical probability measure.

In view of the price dynamics (42)-(43), one has, for $t \le \tau_1 \wedge \tau_2$,

$$\frac{d\langle \widehat{\Pi}, \widehat{P}\rangle_t}{d\langle \widehat{P}\rangle_t} = \frac{l_1(t)(\delta\Pi_{\{1\}}(t))(\delta P_1(t)) + l_3(t)(\delta\Pi_{\{1,2\}}(t))(\delta P_1(t))}{q_1(t)(\delta P_1(t))^2}\ .$$

So

$$\psi_t^{va} = \frac{l_1(t)}{q_1(t)}\frac{1 - R_1 - u(t)}{1 - R_1 - v(t)} + \frac{l_3(t)}{q_1(t)}\frac{R_2(1 - R_1) - u(t)}{1 - R_1 - v(t)}$$

on $[0, \tau^1 \wedge \tau^2 \wedge T]$ (and $\psi^{va} = 0$ on $(\tau^1 \wedge \tau^2 \wedge T, T]$). The related min-variance *hedging reduction factor* writes:

$$\frac{\mathbb{V}\mathrm{ar}(\widehat{\Pi}_T)}{\mathbb{V}\mathrm{ar}(\widehat{\Pi}_T - \int_0^T \psi_t^{va} d\widehat{P}_t)} = \frac{\mathbb{V}\mathrm{ar}(\widehat{\Pi}_T)}{\mathbb{V}\mathrm{ar}(\widehat{\Pi}_T) + \mathbb{V}\mathrm{ar}(\int_0^T \psi_t^{va} d\widehat{P}_t) - 2\mathbb{C}\mathrm{ov}(\widehat{\Pi}_T, \int_0^T \psi_t^{va} d\widehat{P}_t)}\ , \tag{44}$$

where:

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}(\widehat{\Pi}_T) = \mathbb{E}\langle\widehat{\Pi}\rangle_T = \mathbb{E}\int_0^{\tau_1\wedge\tau_2\wedge T} \left(l_1(t)(\delta\Pi_{\{1\}}(t))^2 + l_2(t)(\delta\Pi_{\{2\}}(t))^2 + l_3(t)(\delta\Pi_{\{1,2\}}(t))^2\right) dt \\
&\mathbb{V}\mathrm{ar}(\int_0^T \psi_t^{va} d\widehat{P}_t) = \mathbb{E}\langle\int_0^\cdot \psi_t^{va} d\widehat{P}_t\rangle_T = \mathbb{E}\int_0^{\tau_1\wedge\tau_2\wedge T} q_1(t)(\psi_t^{va}\delta P_1(t))^2 dt \\
&\mathbb{C}\mathrm{ov}(\widehat{\Pi}_T, \int_0^T \psi_t^{va} d\widehat{P}_t) = \mathbb{E}\langle\widehat{\Pi}, \int_0^\cdot \psi_t^{va} d\widehat{P}_t\rangle_T = \\
&\qquad \mathbb{E}\int_0^{\tau_1\wedge\tau_2\wedge T} \left(l_1(t)\delta\Pi_{\{1\}}(t) + l_3(t)\delta\Pi_{\{1,2\}}(t)\right)\psi_t^{va}\delta P_1(t) dt\ .
\end{aligned} \tag{45}$$

The various quantities that arise in (45), and therefore the hedging reduction factor given by (44), can be computed by Monte Carlo simulation.

Remark 3.8 The previous min-variance hedging strategy can be easily extended to multi-instrument hedging schemes. In case three non-redundant hedging instruments are available, then, in view of (42), the risky CDS can be perfectly replicated.

4 Implementation

4.1 Affine Intensities Model Specification

Note that the Markov chain copula model primitives are the marginal pre-default intensity functions q_1 and q_2 as well as the 'dependence intensity function' l_3 in $A(t)$ (cf. (17)).

Let us specify, for constants a's and b's,

$$q_i(t) = a_i + b_i t\ , l_3(t) = a_3 + b_3 t\ , \tag{46}$$

with

$$a_3 = \alpha \min\{a_1, a_2\}\ ,\ \ b_3 = \alpha \min\{b_1, b_2\}\ ,$$

for a *model dependence parameter* $\alpha \in [0, 1]$ (for the sake of Assumption 3.2).

Remark 4.1 Such an affine specification of intensities was already used by Bielecki et. al. [2] in a context of CDO modeling.

It is immediate to check that under (46), the spread κ_i of a risk-free CDS on name i is given by

$$\kappa_i = (1 - R_i)\frac{\displaystyle\int_0^T \beta_t(a_i + b_i t)\exp(-a_i t - \frac{b_i}{2}t^2) dt}{\displaystyle\int_0^T \beta_t \exp(-a_i t - \frac{b_i}{2}t^2) dt}\ . \tag{47}$$

Also note that one has, by Proposition 3.1(v),

$$\rho_d := \rho_d(T) = \frac{e^{a_3 T + b_3 T^2/2} - 1}{\sqrt{\left(e^{a_1 T + b_1 T^2/2} - 1\right)\left(e^{a_2 T + b_2 T^2/2} - 1\right)}} , \tag{48}$$

or, equivalently,

$$\alpha = \frac{\ln\left(1 + \rho_d \sqrt{\left(e^{a_1 T + b_1 T^2/2} - 1\right)\left(e^{a_2 T + b_2 T^2/2} - 1\right)}\right)}{aT + bT^2/2} \tag{49}$$

where $a = \min\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$.

4.1.1 Calibration Issues

Using (47), the a_i's and b_i's can be calibrated independently in a straightforward way to the market CDS curves of the firm and the counterpart, respectively. Note in this regard that market CDS curves can be considered as 'risk-free CDS curves'.

As for the model dependence parameter α, in case the market price of an instrument sensitive to the dependence structure of default times (basket credit instrument on the firm and the counterpart) is available, one can use it to calibrate α. Admittedly however, this situation is an exception rather than the rule. It is thus important to devise a practical way of setting α in case such a market data is not available. A possible procedure[1] thus consists in 'calibrating' α to a target value for the model probability $p_{1,2}(T) = \mathbb{P}(H^1_T = H^2_T = 1)$ of joint default at the time horizon T. A target value for $p_{1,2}(T)$ can be obtained by plugging a standard static Gaussian copula *asset correlation* ρ into a bivariate normal distribution function, so

$$p_{1,2}(T) = \mathcal{N}_2^\rho \left(\mathcal{N}_1^{-1}(p_1(T)), \mathcal{N}_1^{-1}(p_2(T))\right) , \tag{50}$$

where:
• $\mathcal{N}_1$ denotes the standard Gaussian c.d.f.,
• $\mathcal{N}_2^\rho$ denotes a bivariate centered Gaussian c.d.f. with one-factor Gaussian copula correlation matrix of parameter ρ,
• $p_i(T) = \mathbb{P}(H^i_T = 1)$ for $i = 1, 2$.
Regulatory capital requirements being based on the Vasicek formula, such a static copula correlation ρ can be retrieved from the Basel II correlations per asset class (cf. [1, pages 63 to 66]).

4.1.2 Special Case of Constant Intensities

We now look at a particular case in which $b_1 = b_2 = b_3 = 0$. This case will be referred to henceforth as the case of *constant intensities*, as opposed to the more general case of *affine intensities* introduced in subsection 4.1. In the case of constant intensities, one has,

$$q_1(t) = a_1 \ , \ q_2(t) = a_2 \ , \ l_3(t) = a_3.$$

[1] We thank J.-P. Lardy for the suggestion of this procedure.

The correlation coefficient ρ_d in (48) simplifies to

$$\rho_d = \frac{e^{a_3 T} - 1}{\sqrt{(e^{a_1 T} - 1)\,(e^{a_2 T} - 1)}}$$

from which a_3 can be calculated as

$$a_3 = \frac{1}{T} \ln \left(1 + \rho_d \sqrt{(e^{a_1 T} - 1)\,(e^{a_2 T} - 1)} \right) .$$

As is well known, the price of a risk-free CDS in a constant intensity model is null, i.e., $v(t) \equiv 0$ when $b_1 = 0$. So the EPE formula (37) simplifies to

$$\text{EPE}(t) = (1 - R_1)(1 - R_2)\frac{a_3}{a_2} e^{-(a_1 - a_3)t} \; .$$

Also in this case, the pricing formula (29) for the risky CDS reduces to (assuming here $r(t) = r$),

$$u(t) = -(1 - R_1)(1 - R_2)a_3 \frac{1 - e^{-(r + a_1 + a_2 - a_3)(T - t)}}{r + a_1 + a_2 - a_3} .$$

Finally, from Proposition 2.1, one gets,

$$\text{CVA}(t) = -u(t) \; .$$

In particular, for low values of the coefficients,

$$CVA(0) \simeq (1 - R_1)(1 - R_2)a_3 T = (1 - R_1)(1 - R_2) \ln \left[1 + \rho_d \sqrt{(e^{a_1 T} - 1)\,(e^{a_2 T} - 1)} \right] \; ,$$

so, finally,

$$CVA(0) \simeq (1 - R_1)(1 - R_2)\sqrt{a_1 a_2}\, T \rho_d \; . \tag{51}$$

4.2 Numerical Results

Our aim is to assess by means of numerical experiments the impact of ρ (the asset correlation between the firm and the counterpart, cf. (50)) on one hand, and of κ_2 (the risk-free CDS fair spread of the counterparty as of (47)) on the other hand, on the counterparty risk exposure of the investor.

Towards this end we fix the general data of Table 1 (case with affine intensities) or 3 (case with constant intensities, all b's equal to 0), and we further consider twelve alternative sets of values for a_2, b_2, and ρ given in columns one, two and four of Table 2 (case with affine intensities), resp. for a_2 and ρ given in columns one and three of Table 4 (case with constant intensities).

r	R_1	R_2	T	a_1	b_1	κ_1
5%	40%	40%	10 years	.0095	.0010	84 bp

Table 1: Fixed Data — Affine Intensities.

a_2	b_2	κ_2	ρ	ρ_d	α	$p_{1,2}$	CVA(0)
.0056	.0006	50 bp	10%	.0378	.0520	.0147	.0013
.0085	.0009	75 bp	10%	.0418	.0472	.0211	.0018
.0122	.0010	100 bp	10%	.0444	.0522	.0269	.0021
.0189	.0014	150 bp	10%	.0476	.0702	.0376	.0028
.0056	.0006	50 bp	40%	.1859	.2531	.0286	.0056
.0085	.0009	75 bp	40%	.1998	.2230	.0388	.0074
.0122	.0010	100 bp	40%	.2074	.2406	.0472	.0087
.0189	.0014	150 bp	40%	.2145	.3107	.0616	.0110
.0056	.0006	50 bp	70%	.4020	.5406	.0489	.0119
.0085	.0009	75 bp	70%	.4256	.4673	.0640	.0153
.0122	.0010	100 bp	70%	.4336	.4937	.0754	.0178
.0189	.0014	150 bp	70%	.4306	.6100	.0925	.0214

Table 2: Variable Data — Affine Intensities.

r	R_1	R_2	T	a_1	κ_1
5%	40%	40%	10 years	.0140	84 bp

Table 3: Fixed Data — Constant Intensities.

a_2	κ_2	ρ	ρ_d	α	$p_{1,2}$	CVA(0)
.0083	50 bp	10%	.0372	.0510	.0138	.0011
.0125	75 bp	10%	.0411	.0464	.0198	.0015
.0167	100 bp	10%	.0438	.0515	.0254	.0018
.0250	150 bp	10%	.0470	.0690	.0355	.0023
.0083	50 bp	40%	.1839	.2501	.0272	.0054
.0125	75 bp	40%	.1977	.2207	.0368	.0070
.0167	100 bp	40%	.2056	.2387	.0451	.0084
.0250	150 bp	40%	.2128	.3073	.0587	.0104
.0083	50 bp	70%	.3998	.5372	.0469	.0117
.0125	75 bp	70%	.4231	.4650	.0613	.0150
.0167	100 bp	70%	.4315	.4921	.0726	.0175
.0250	150 bp	70%	.4288	.6063	.0889	.0210

Table 4: Variable Data — Constant Intensities.

In the case of affine intensities the corresponding spreads κ_2 at time 0, default correlation ρ_d, model dependence parameter α and joint default probabilities $p_{1,2} = \mathbb{P}(H^1_T = H^2_T = 1)$ are displayed respectively in the third, fifth, sixth and seventh column of Table 2, whereas the last column of Table 2 (which will be commented later in the text) gives the corresponding CVA's at time 0. The risky and risk-free CDS pricing functions u and v corresponding to each of our twelve sets of parameters are displayed in Figure 1. On each graph three curves are represented (see Remark 3.5):

• $v(t)$ (dashed blue curve),
• $u(t)$ with $\widetilde{\chi} = v$ therein, denoted by $u^0(t)$ (doted red curve),
• $u(t)$ with $\widetilde{\chi} = u$ therein, denoted by $u^1(t)$ (black curve).

The analogous results in the case of constant intensities are displayed in Table 4 and Figure 2. Note that on each graph in Figure 2 the function v is equal to 0, as must be in the case of constant intensities.

In all the cases u^0 and u^1 are rather close to each other, and one can check numerically that using either one makes little difference regarding the related EPEs and CVAs. We present henceforth the results for $u = u^0$.

Figures 3, 4 and 5 show the graphs of the Expected Positive Exposure as a function of time, of the Credit valuation Adjustment as a function of time, and of the Credit Valuation Adjustment at time 0 as a function of ρ, in the cases of affine (left graphs) or constant (right graphs) intensities.

One can see on Figure 3 the impact on the counterparty risk exposure of the investor of the default risk (as measured by the risk-free spread κ_2) of the counterpart. On each graph the asset correlation ρ is fixed, with from top to down $\rho = 10\%$, 40% and 70%. The four curves on each graph of Figure 3 correspond to EPE(t) for $\kappa_2 = 50$, 75, 100 and 150bps. Observe that as κ_2 decreases the counterparty risk exposure increases. This is in line with the stylized features and the financial intuition regarding the EPE: EPE(t) is the expectation of the investor's loss, given the default of the counterpart at time t. A default of a counterpart with a lower spread is interpreted by the markets as a worse news than a default of a counterpart with a higher spread. The related EPE is thus larger.

Figure 4 shows the graphs of the Credit Valuation Adjustment as a function of time, for affine (left column) or constant (right column) intensities. One can thus see the impact of κ_2 on the CVA. In each graph the asset correlation ρ is fixed, with from top to down $\rho = 10\%$, 40% and 70%. The four curves on each graph of Figure 4 correspond to CVA(t) for $\kappa_2 = 50$, 75, 100 and 150bps. Observe that as opposed to the EPE, the CVA is increasing in κ_2, in line with stylized features. Also note that the CVA is a decreasing function of time, in accordance again with expected features: less time to maturity, less risk.

Finally Figure 5 represents the graphs of CVA(0) as a function of the asset correlation ρ for $\kappa_2 = 50$, 75, 100 and 150bps. Note for comparison that CVA(0) grows essentially linearly in the default correlation ρ_d, at least in the case of constant coefficients (cf. formula (51)).

5 Concluding Remarks and Perspectives

In this article we propose a model of CDS with counterparty credit risk, with the following desirable properties:

• Adequation of the behavior of EPE and CVA in the model with expected features (see Section

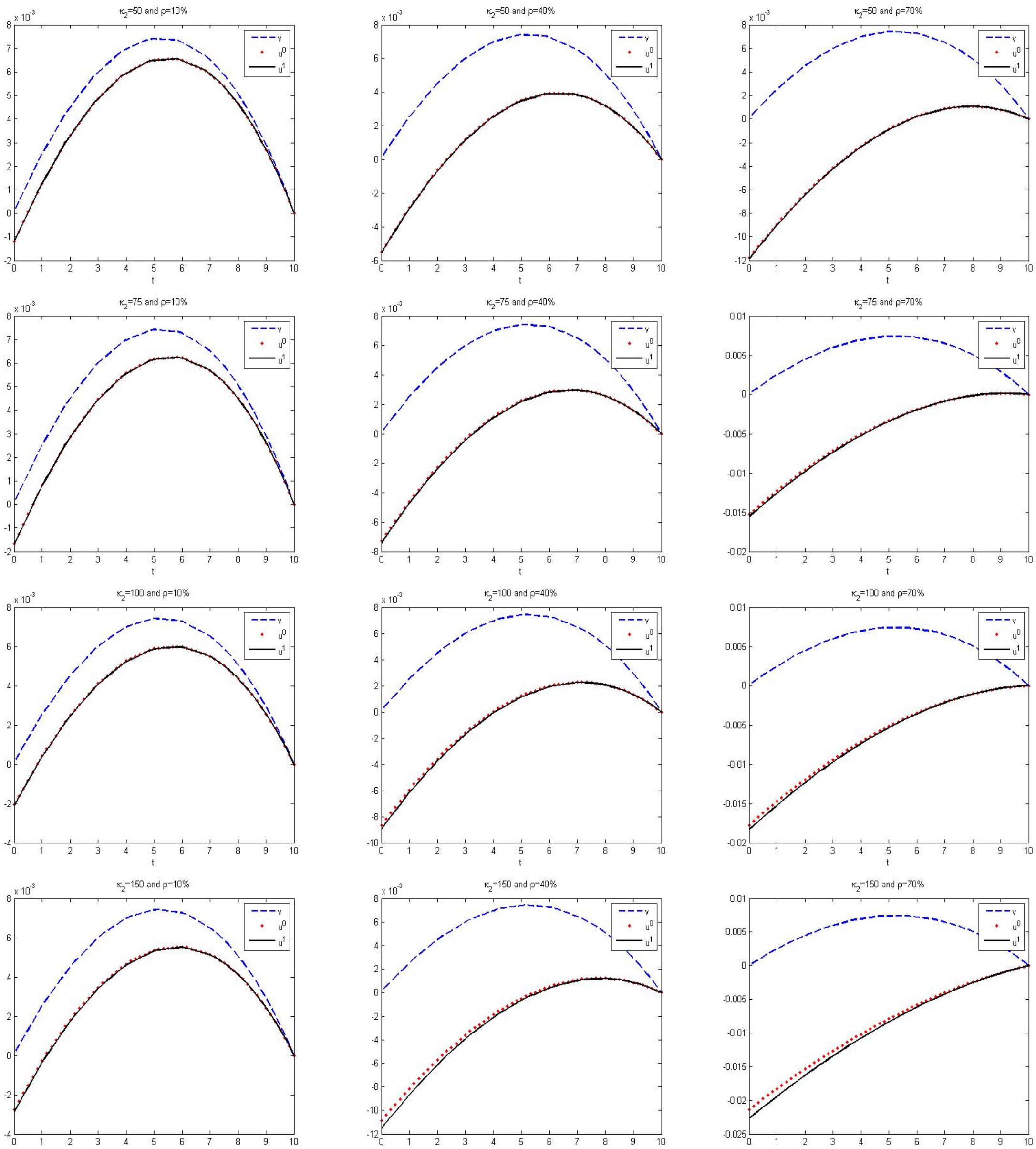

Figure 1: Pricing functions in the case of affine intensities — $v(t)$ (dashed blue curve), $u^0(t)$ (doted red curve) and $u^1(t)$ (black curve).

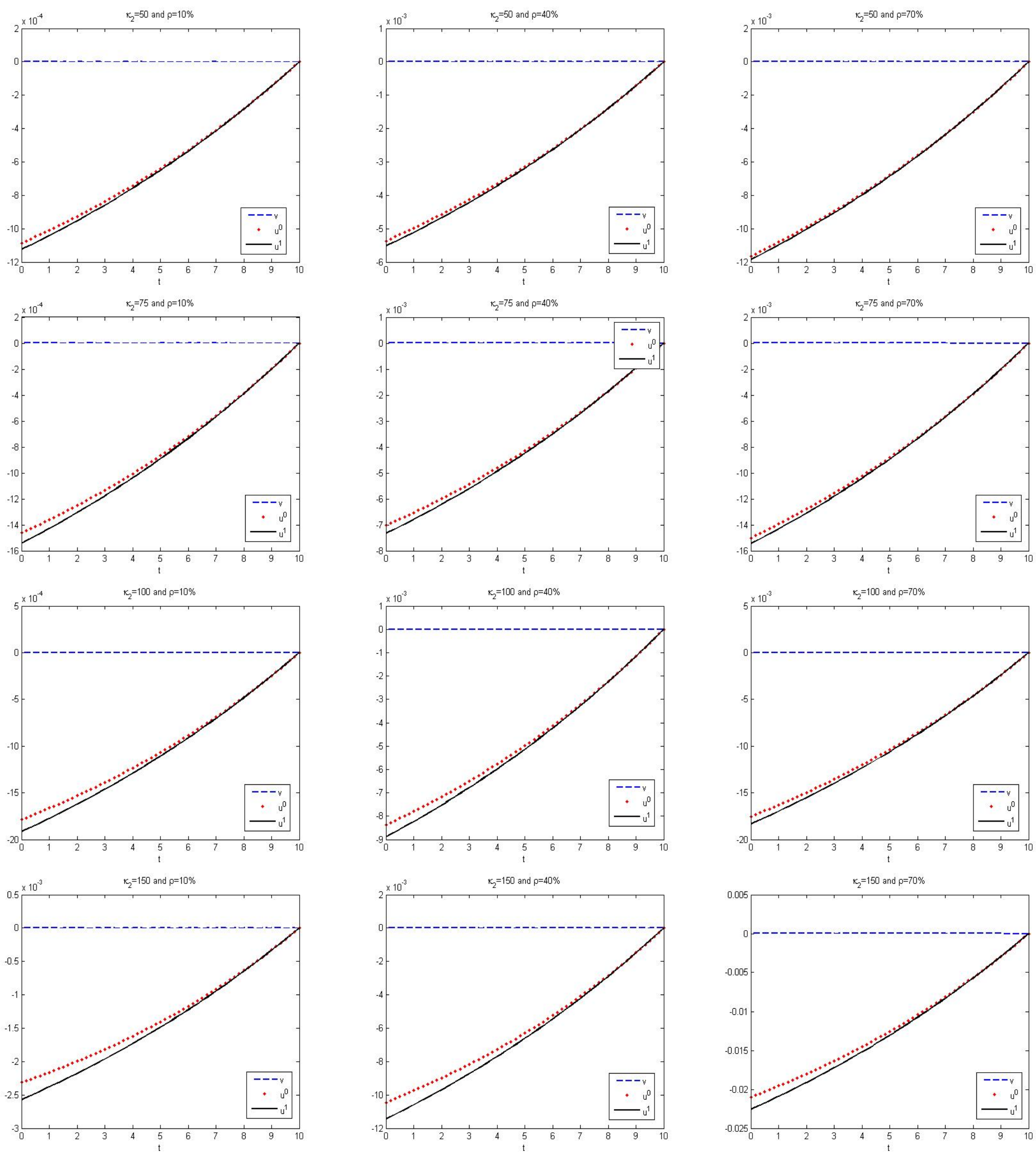

Figure 2: Pricing functions in the case of constant intensities — $v(t)$ (dashed blue curve), $u^0(t)$ (doted red curve) and $u^1(t)$ (black curve).

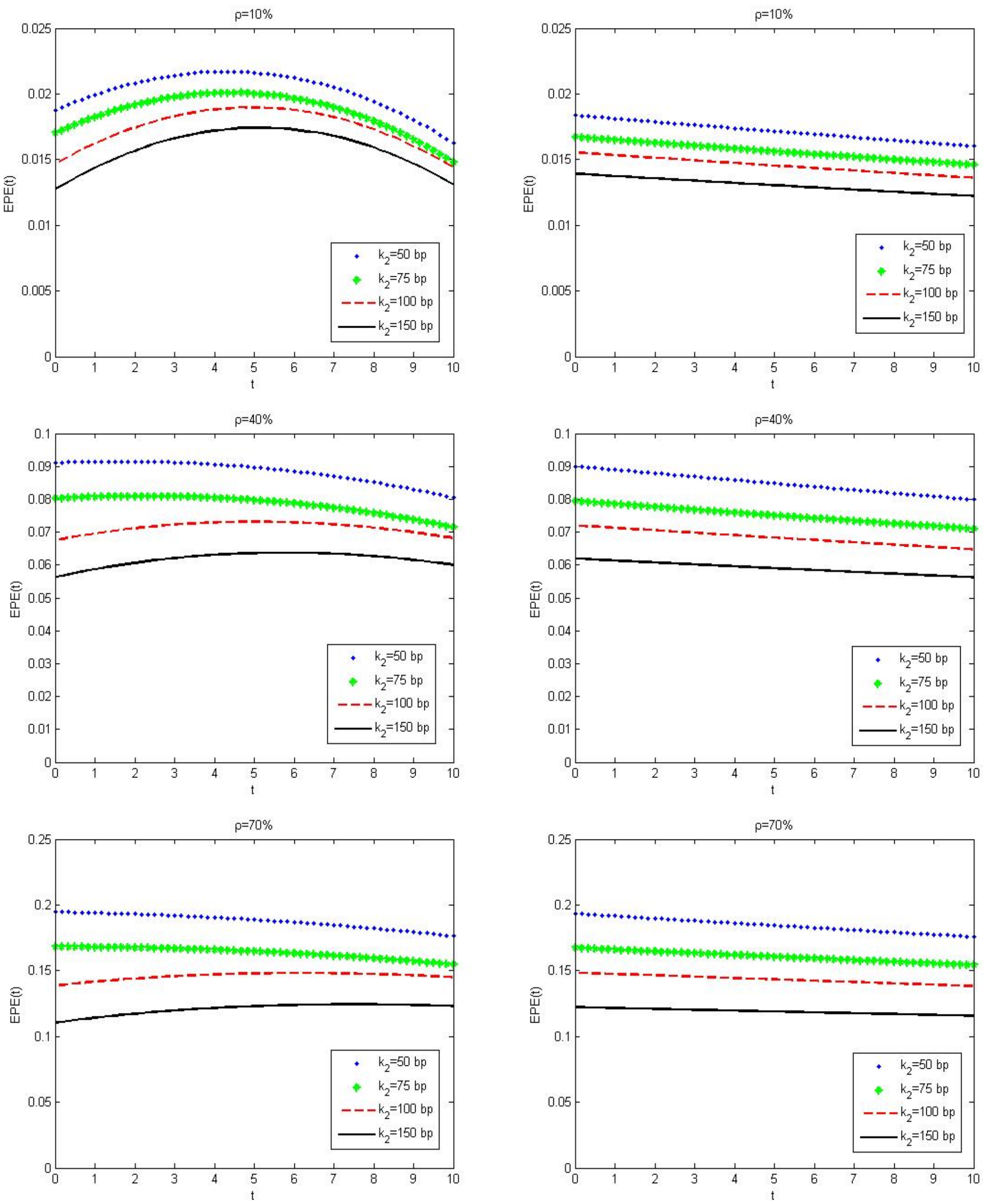

Figure 3: EPE(t) ($\widetilde{\chi} = v$, $u = u^0$). In each graph ρ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. Left column: affine intensities. Right column: constant intensities.

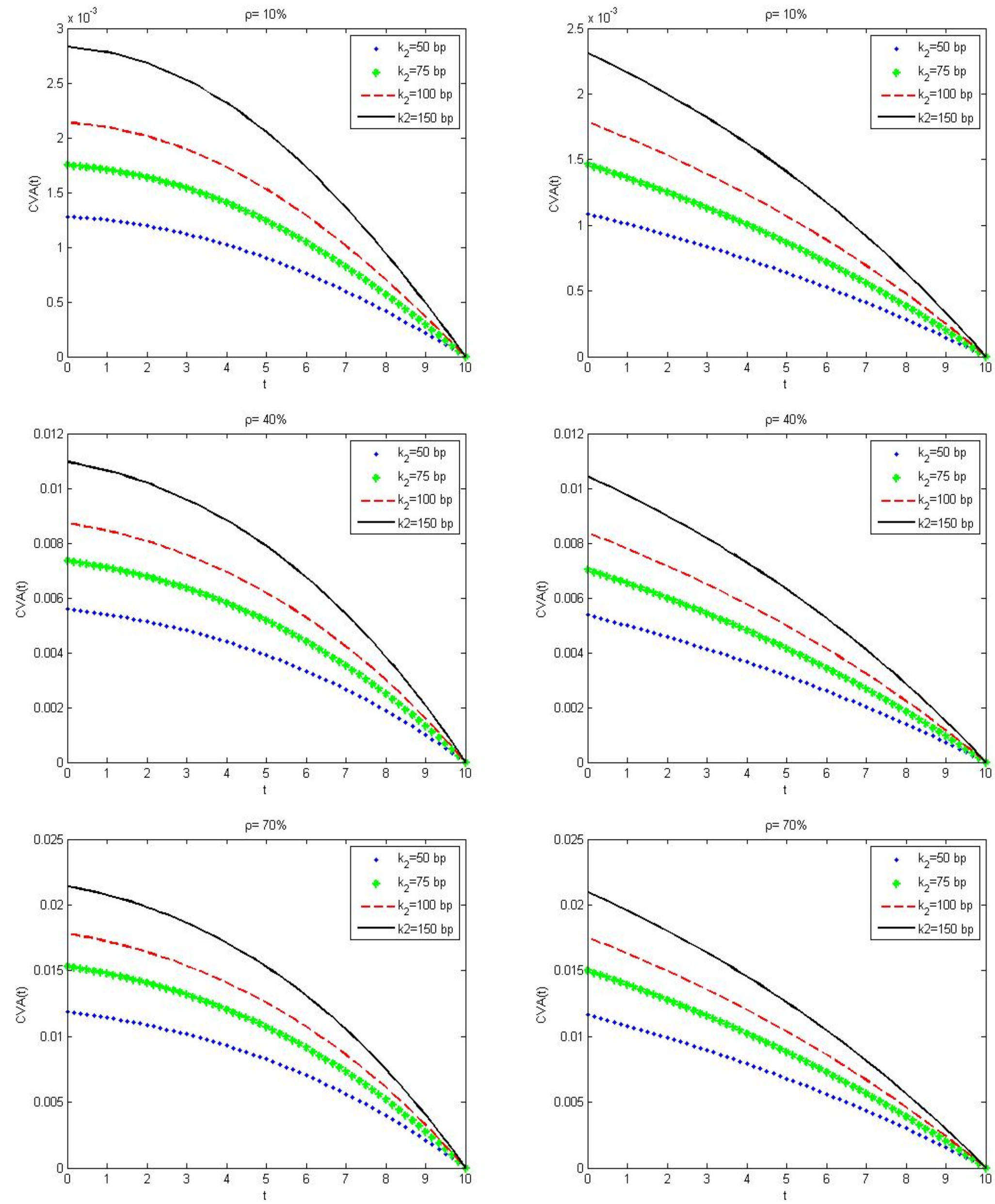

Figure 4: CVA(t) ($\widetilde{\chi} = v$, $u = u^0$). In each graph ρ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. Left column: affine intensities. Right column: constant intensities.

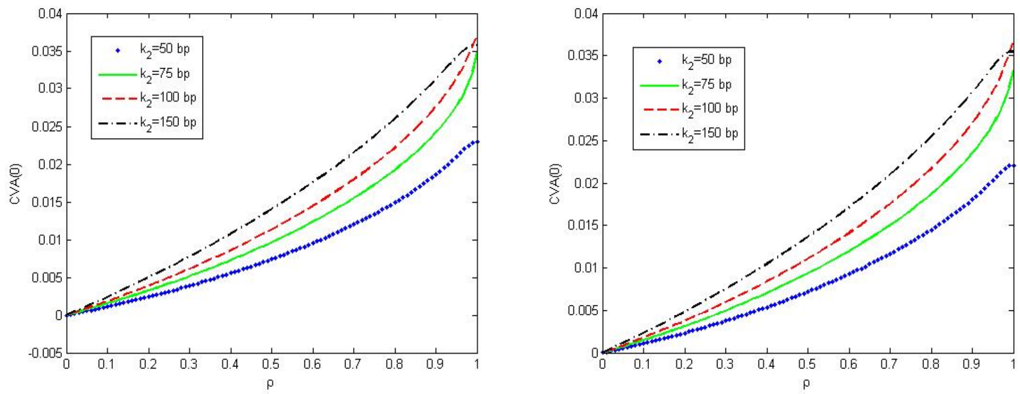

Figure 5: CVA(0) as a function of ρ for $\kappa_2 = 50$ bp, 75 bp, 100 bp and 150 bp ($\widetilde{\chi} = v$, $u = u^0$). Left: Affine intensities. Right: Constant intensities.

4.2),
• Wrong way risk (via joint defaults, specifically),
• Simplicity, since the model is a four-state Markov chain of two credit names, with one-name marginals automatically calibrated to the individual CDS curves,
• Fact, related to the previous one, that the model 'takes the right inputs to generate the right outputs', namely it takes as basic inputs the individual default probabilities (individual CDS curves), which correspond to the more reliable information on the market, which are then 'coupled' in a suitable way,
• Consistency, in the sense that it is a dynamic model with replication-based valuation and hedging arguments.

The present work might be extended in at least three directions.
First, it would be desirable to add credit spread volatility into the model. This could be achieved by adding a *reference filtration* $\widetilde{\mathbb{F}}$ so that the model filtration $\mathbb{F}$ be given as $\widetilde{\mathbb{F}} \vee \mathbb{H}$, and the intensities l, q are non-negative $\widetilde{\mathbb{F}}$-adapted processes.
A second related issue is that of merging the CDS-CVA pricing tool of this paper into a more general, real-life CVA engine, including the following features:
• Netting, that is, aggregation in a suitable way of all the contracts (as opposed to only one CDS in this paper) relative to a given counterpart,
• Market (other than credit) risk factors,
• Margin agreements.
Finally, at the stage of implementation (see, e.g., Zhu and Pykhtin [24]), such real-life CVA engines pose interesting challenges from the numerical point of view of Monte Carlo simulations.

A Proof of Proposition 3.1

We shall need the following (essentially classic) Lemma.

Lemma A.1 *Let $\mathcal{X}$ be a right-continuous process with a finite state space $\mathcal{E}$ and adapted to some filtration $\mathbb{F}$. Condition (i), (ii) or (iii) below are necessary and sufficient conditions for $\mathcal{X}$ to be an $\mathbb{F}$ – Markov chain with infinitesimal generator $\mathcal{A}(t) = \mathcal{A}_t = [\mathcal{A}_t^{i,j}]_{i,j\in\mathcal{E}}$:*
(i) *For every function h over $\mathcal{E}$,*

$$\mathcal{M}_t^h = h(\mathcal{X}_t) - \int_0^t (\mathcal{A}_s h)(\mathcal{X}_s)ds \tag{52}$$

is an $\mathbb{F}$ – local martingale;
(ii) *For every $j \in \mathcal{E}$, the process $\mathcal{M}^j$ defined by*

$$\mathcal{M}_t^j = \mathbb{1}_{\mathcal{X}_t=j} - \int_0^t \mathcal{A}_s^{\mathcal{X}_s,j} ds$$

is an $\mathbb{F}$ – local martingale;
(iii) *For every $i, j \in \mathcal{E}$ the process $\mathcal{M}^{i,j}$ given by*

$$\mathcal{M}_t^{i,j} = \mathbb{1}_{\mathcal{X}_{t-}=i,\mathcal{X}_t=j} - \int_0^t \mathbb{1}_{\mathcal{X}_s=i}\mathcal{A}_s^{i,j} ds$$

is an $\mathbb{F}$ – local martingale.

Proof. **(i)** is the usual local martingale characterization of Markov chains (see, e.g., Proposition 11.2.2 in [4]).
(ii) Since $\mathcal{E}$ is finite, the set of the indicator functions $\mathbb{1}_{\cdot=j}$ spans linearly the set of all functions over $\mathcal{E}$. The condition of part (ii) is thus equivalent to that of (i).
(iii) Necessity follows by combination of Proposition 11.2.2 and Lemma 11.2.3 in [4]. As for sufficiency, note that the $\mathcal{M}^{i,j}$'s being $\mathbb{F}$ – local martingales implies the same property for the $\mathcal{M}^j$'s in (ii), by summation over i. We thus conclude by the sufficiency in part (ii). □

Let us proceed with the proof of Proposition 3.1. First, note the processes H^ι can also be written as

$$H_t^{\{1\}} = \sum_{0<s\le t} \mathbb{1}_{\Delta H_s=(1,0)}\ ,\ H_t^{\{2\}} = \sum_{0<s\le t} \mathbb{1}_{\Delta H_s=(0,1)}\ ,\ H_t^{\{1,2\}} = \sum_{0<s\le t} \mathbb{1}_{\Delta H_s=(1,1)}\ .$$

(i) Let us verify that the M^ι's in (20) are $\mathbb{H}$ – local martingales. As bounded $\mathbb{H}$ – local martingales, $M^{\{1\}}$, $M^{\{2\}}$ and $M^{\{1,2\}}$ will thus be $\mathbb{H}$-martingales. For $I = \{1, 2\}$, one has,

$$\begin{aligned} M_t^{\{1,2\}} &= H_t^{\{1,2\}} - \int_0^t q_{\{1,2\}}(s, H_s)ds \\ &= \sum_{0<s\le t} \mathbb{1}_{\Delta H_s=(1,1)} - \int_0^t \mathbb{1}_{H_s=(0,0)} l_3(s)ds \\ &= \sum_{0<s\le t} \mathbb{1}_{H_{s-}=(0,0),H_s=(1,1)} - \int_0^t \mathbb{1}_{H_s=(0,0)} l_3(s)ds\ . \end{aligned}$$

Thus Lemma A.1 with $i = (0, 0)$ and $j = (1, 1)$, implies the local martingale property of $M^{\{1,2\}}$.

For $M^{\{1\}}$, one has,

$$
\begin{aligned}
M_t^{\{1\}} &= H_t^{\{1\}} - \int_0^t q_{\{1\}}(s, H_s)ds \\
&= \sum_{0<s\le t} \mathbb{1}_{\Delta H_s=(1,0)} - \int_0^t \mathbb{1}_{H_s^1=0}\left[\mathbb{1}_{H_s^2=0} l_1(s) + \mathbb{1}_{H_s^2=1} q_1(s)\right] ds \\
&= \left\{ \sum_{0<s\le t} \mathbb{1}_{H_{s-}=(0,0),H_s=(1,0)} - \int_0^t \mathbb{1}_{H_s=(0,0)} l_1(s)ds \right\} \\
&\quad + \left\{ \sum_{0<s\le t} \mathbb{1}_{H_{s-}=(0,1),H_s=(1,1)} - \int_0^t \mathbb{1}_{H_s=(0,1)} q_1(s)ds \right\} .
\end{aligned}
$$

Now we apply Lemma A.1 to the two terms in the last equation, with $i = (0, 0)$ and $j = (1, 0)$ for the first term and $i = (0, 1)$ and $j = (1, 1)$ for the second term. Thus $M^{\{1\}}$ being the sum of two $\mathbb{H}$ – local martingales is an $\mathbb{H}$ – local martingale. In the same way, $M^{\{2\}}$ is an $\mathbb{H}$ – local martingale. As bounded $\mathbb{H}$ – local martingales, $M^{\{1\}}$, $M^{\{2\}}$ and $M^{\{1,2\}}$ are thus $\mathbb{H}$–martingales.
(ii) As $q_i = l_i + l_3$ and $H^i = H^{\{i\}} + H^{\{1,2\}}$, one has $M^i = M^{\{i\}} + M^{\{1,2\}}$, so the M^i's are in turn $\mathbb{H}$-martingales.
(iii) Since the M^i's are $\mathbb{H}$-martingales, this follows easily from the sufficiency in Lemma A.1(ii).
(iv) Formulas (24) follow directly from (23), in which we shall now show the first identity. One has for $t > s$ (see the end of the proof of Proposition 3.3),

$$
\mathbb{P}(\tau_2 > t|\mathcal{H}_s) = \mathbb{P}(\tau_2 > t|H_s^2) = (1 - H_s^2)e^{-\int_s^t q_2(u)du} .
$$

Thus

$$
\begin{aligned}
\mathbb{P}(\tau_1 > s, \tau_2 > t) &= \mathbb{E}(\mathbb{1}_{\tau_1>s}\mathbb{E}(\mathbb{1}_{\tau_2>t}|\mathcal{H}_s)) \\
&= \mathbb{E}\left\{(1 - H_s^1)(1 - H_s^2)e^{-\int_s^t q_2(u)du}\right\} ,
\end{aligned}
$$

and the result follows.
(v) Since H_t^i is a Bernoulli random variable with (cf. Proposition 3.1(iv))

$$
\mathbb{P}(H_t^i = 1) = \mathbb{P}(\tau_i \le t) = 1 - \exp(-\int_0^t q_i(s)ds) := p_i(t),
$$

one has

$$
\mathbb{V}\mathrm{ar}(H_t^i) = p_i(t)(1 - p_i(t))
$$

Also

$$\begin{aligned}\mathbb{C}\mathrm{ov}(H_t^1, H_t^2) &= \mathbb{C}\mathrm{ov}(1 - H_t^1, 1 - H_t^2)\\ &= \mathbb{E}\left[(1 - H_t^1)(1 - H_t^2)\right] - \mathbb{E}(1 - H_t^1)\mathbb{E}(1 - H_t^2)\\ &= \mathbb{P}(\tau_1 > t, \tau_2 > t) - \mathbb{P}(\tau_1 > t)\mathbb{P}(\tau_2 > t)\\ &= \exp\left(-\int_0^t l(s)ds\right) - \exp\left(-\int_0^t q_1(s)ds\right)\exp\left(-\int_0^t q_2(s)ds\right).\end{aligned}$$

Thus, after some algebraic simplifications,

$$\rho_d(t) = \frac{\mathbb{C}\mathrm{ov}(H_t^1, H_t^2)}{\sqrt{\mathbb{V}\mathrm{ar}(H_t^1)\mathbb{V}\mathrm{ar}(H_t^2)}} = \frac{\exp\left(\int_0^t l_3(s)ds\right) - 1}{\sqrt{\left(\exp\left(\int_0^t q_1(s)ds\right) - 1\right)\left(\exp\left(\int_0^t q_2(s)ds\right) - 1\right)}}.$$

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