The Cost-of-Capital XVA Approach in Continuous Time

Part I: Positive XVAs

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Abstract

Since the 2008 crisis, derivative dealers charge to their clients various add-ons, dubbed XVAs, meant to account for counterparty risk and its capital and funding implications.

As banks cannot replicate jump-to-default related cash flows, deals trigger wealth transfers and shareholders need to set capital at risk. We devise an XVA policy, whereby so called contra-liabilities and cost of capital are sourced from bank clients at trade inceptions, on top of the fair valuation of counterparty risk, in order to guarantee to the shareholders a hurdle rate h on their capital at risk.

The resulting all-inclusive XVA formula reads (CVA + FVA + KVA), where C sits for credit, F for funding, and where the KVA is a cost of capital risk premium. All these XVA metrics are portfolio-wide, nonnegative and, despite the fact that we include the default of the bank itself in our modeling, they are ultimately unilateral. This makes them naturally in line with the requirement that capital at risk and reserve capital should not decrease simply because the credit risk of the bank has worsened.

Keywords: Counterparty risk, market incompleteness, credit valuation adjustment (CVA), funding valuation adjustment (FVA), capital valuation adjustment (KVA), wealth transfer.

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1 The Sustainable Pricing and Dividends Problem

We devise a pricing and dividend policy for a dealer bank, **sustainable** in the sense of ensuring to its shareholders a constant instantaneous return rate h on their capital at risk, even in the limiting case of a portfolio held on a run-off basis, i.e. without future deals.

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Moreover, the corresponding policy of the bank should satisfy several **regulatory constraints**. Firstly, in order to comply with the Volcker rule that bans proprietary trading for a dealer bank, the market risk of the bank should be hedged as much as possible. As a result, mainly counterparty risk remains. Secondly, reserve capital should be maintained by the bank at the level of its *expected* counterparty credit losses, along two lines: the credit valuation adjustment (CVA) of the bank, meant to cope with the counterparty risk of the bank clients, i.e. with the expected losses of the bank due to client defaults; and the funding valuation adjustment (FVA), meant to cope with the counterparty risk of the bank itself, i.e. with its expected risky funding expenses. Thirdly, capital should be set at risk by the bank to deal with its *exceptional* (above expected) *losses*. The above return rate h is then meant at a hurdle rate for the bank shareholders, i.e. a risk premium for their capital at risk within the bank.

Reserve capital, like capital at risk, should obviously be ***nonnegative***. Furthermore, it should not decrease simply because the credit risk of the bank itself has worsened, a property which we refer to as ***monotonicity***: See Albanese and Andersen (2014, Section 3.1) for the relevant wordings from Basel Committee on Banking Supervision (2012) and Federal Register (2014).

Further requirements on a solution to the above sustainable pricing and dividend release policy problem are **economic interpretability** and **logical consistency** (for intellectual adhesion by market participants), **numerical feasibility** and **robustness** at the level of a realistic banking portfolio (for practicality), and **minimality** in the sense of being, all things equal, as cheap as possible (for competitiveness).

The design of a pricing and dividend policy satisfying all the above requirements is the main achievement of this article. Although we can not claim for uniqueness, we will see in Section 5.2 that alternative XVA approaches in the literature breach several of the above requirements. For instance, it is not uncommon in the XVA literature to see some *possibly negative FVA metrics, or CVA and KVA metrics that tend to 0 when the default risk of the bank goes to infinity.*

The cost-of-capital XVA approach has been introduced in Albanese et al. (2016) and developed in various directions in a stream of papers including Albanese et al. (2017) or Crépey et al. (2020). The intent of this work is to clarify the foundations of this approach, i.e. the cost-of-capital XVA conceptual framework. This is done in an abstract setup, which is then specified to bilateral and centrally cleared trading setups in the follow-up paper (Crépey, 2022). Related numerics are provided in Albanese, Crépey, Hoskinson, and Saadeddine (2021)[1] and Abbas-Turki et al. (2021) for the bilateral trading case and by Albanese et al. (2020) for the centrally cleared case.

1.1 Solution Setup

The starting point of our solution to the sustainable pricing and dividends problem is an organizational and accounting separation between three kinds of business units within the bank: the ***CA desks***, the ***trading desks***, and the ***management*** of the bank.

The CA desks are themselves split between the CVA desk and the FVA desk (or

[1] abbreviated herefater as Albanese et al. (2021).

Treasury) of the bank, respectively in charge of the default risk triggered by clients and of the risky funding expenses of the bank. The corresponding cash flows are collectively called the contra-assets of the bank. In an informal sense made precise by this paper, contra-assets (-liabilities)[2] are bank liabilities (assets) that arise as a feedback effect of counterparty credit risk on the counterparty-risk-free assets (liabilities) of the bank.

The CA desks guarantee the trading of the trading desks against client and bank defaults, through an trading margin account, which also funds the trading of the trading desks at the risk-free rate. Thanks to this work accomplished by the CA desks, the trading desks can focus on the market risk of the contracts in their respective business lines, as if there was no counterparty risk (even if some of their positions are liquidated, this will occur at no loss from their perspective). We denote by MtM the amount on the trading margin account of the bank (counted positively when posted by the CA desks) and we write

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} \tag{1.1}$$

for the overall amount of reserve capital of the bank, which will correspond to the valuation of its contra-assets.

The management of the bank is in charge of its dividend release policy. We consider a level of capital at risk (CR) sufficient to make the bank resilient to a forty-year adverse event, i.e. greater than an economic capital (EC) defined as the expected shortfall of the losses of the bank over one year at the confidence level $\alpha = 97.5\% = 1 - \frac{1}{40}$. The implementation of a sustainable dividend remuneration policy requires a dedicated risk margin account, on which bank profits are initially retained so that they can then be gradually released as dividends at a hurdle rate h on shareholder capital at risk (as opposed to being readily distributed as day-one profit). Counterparty default losses, as also funding payments, are materialities for default if not paid. By contrast, risk margin payments are at the discretion of the bank management, hence they do not represent an actual liability to the bank. As a consequence, the capital valuation adjustment (KVA) amount on the risk margin account is also loss-absorbing, i.e. part of capital at risk (CR). With minimality in view (see Section 1, Corollary 3.4 and Section 5.4), we thus set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}). \tag{1.2}$$

All bank accounts are marked-to-model, i.e. continuously and instantaneously readjusted to theoretical target levels, which will be defined in Sections 2–3 in view of yielding a solution to the sustainable pricing and dividends problem. All cash accounts of the bank, as well as all the collateral (assumed all cash for simplicity) posted as a guarantee for the trades, are remunerated at the risk-free rate.

In line with the sustainability requirement edicted in Section 1, the portfolio is supposed to be held on a run-off basis between inception time 0 and its final maturity. At the portfolio inception time 0, the trading desks pay MtM_0 to the clients; the CA desks add an amount MtM_0 on the trading margin account if $\mathrm{MtM}_0 > 0$, whereas the trading desks put an amount $(-\mathrm{MtM}_0)$ on the trading margin account if $\mathrm{MtM}_0 < 0$; the CA desks charge to the clients an amount CA_0 and add it on the reserve capital account;

[2] detailed in Table 4.1.

the management of the bank charges the amount KVA_0 to the clients and adds it on the risk margin account. Between time 0 and the bank default time τ (both excluded), mark-to-model readjustments of all bank accounts are on bank shareholders. If the bank defaults, any residual amount on the reserve capital and risk margin accounts, as well as any remaining trading cash flows, are transferred to the estate of the defaulted bank, dubbed creditors of the bank hereafter, which is mandated to deal with the liquidation of the bank. These liquidation costs to be born by the creditors are outside the scope of the model, as is also the primary business of the clients of the bank, which motivates their deals with the bank.

See Table 1.1 for a list of the main valuation acronyms used in the paper.

CA	Contra-assets valuation	(1.1) and first row in Table 4.1
CL	Contra-liabilities valuation	Table 4.1
CR	Capital at risk	Sect. 1 and (1.2)
CVA	Credit valuation adjustment	Sect. 1, (2.12), (3.10), and (II.5.29)
DVA	Debt valuation adjustment	Table 4.1 and (II.5.33)
EC	Economic capital	Sect. 1 and Definition 3.3
FDA	Funding debt adjustment	Table 4.1 and (II.5.34)
FV	Fair valuation of counterparty risk	Table 4.1 and Lemma 4.1(ii)
FVA	Funding valuation adjustment	Sect. 1, (2.13), (3.11), and (II.5.30)
KVA	Capital valuation adjustment	Sect. 1.1 and (2.17), (3.12), (3.23)
MtM	Mark-to-market	Sect. 1.1, (2.11), (3.9), and (II.3.5)
SCR	Shareholder capital at risk	Definition 2.3

Table 1.1: Main valuation acronyms and place where they are introduced conceptually in this paper and/or concretely specified in the follow-up paper (Crépey, 2022), then referred to in the table as (II.$\cdots$).

2 The Cost-of-Capital XVA Equations

2.1 Probabilistic Setup

Implicit to the above-sketched XVA framework is a probabilistic structure, i.e. a measurable space $(\Omega, \mathcal{A})$ endowed with a stochastic basis $(\mathfrak{G}, \mathbb{Q})$, with respect to which all the involved conditional expectations and risk measures are defined. The filtration $\mathfrak{G} = (\mathfrak{G}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions. All the processes in the paper are $\mathfrak{G}$ adapted and all the random times of interest are $\mathfrak{G}$ stopping times. The probability measure $\mathbb{Q}$ is used for the linear valuation of cash flows[3], using the risk-free asset as our numéraire everywhere[4].

The XVA matter also crucially entails nontraded assets (or trading constraints). The default risk of most of the clients of the bank is not liquidly priced in the market.

[3]For simplicity, we only consider European derivatives.

[4]This choice of a numéraire simplifies equations by removing all terms related to the (risk-free, see after (1.2)) remuneration of the cash accounts and of the collateral.

Even if a liquid CDS market on the bank is avaible, this market is not accessible to the bank itself, which, in particular, cannot sell jump-to-default protection on itself.

These hedging limitations of the bank lead us to introduce the financial sub-σ-field $\mathcal{B} \subseteq \mathcal{A}$, on which a risk-neutral measure, equivalent to the restriction to $\mathcal{B}$ of the physical probability measure (itself defined on $\mathcal{A}$), is given. We then define $\mathbb{Q}$ to be the uniquely defined probability measure on $\mathcal{A}$, provided by Artzner et al. (2020, Proposition 2.1), such that *(i)* $\mathbb{Q}$ coincides with the risk-neutral measure on $\mathcal{B}$ and *(ii)* $\mathbb{Q}$ and the physical measure coincide conditionally on $\mathcal{B}$. The probability measure $\mathbb{Q}$ that emerges from this construction is a hybrid of the underlying risk-neutral and physical measures, with respect to which all the XVA computations are then conducted.

Example 2.1 *$\mathbb{Q}$ here and $\mathbb{P}$ in Section 3.1 should not be confused with the underlying (aforementioned) risk-neutral and physical measures. In the special (unrealistic) case of a bank that would not resort to dynamic hedging, then our probability measure $\mathbb{Q}$ would just be the physical one and we would have $\mathcal{H} = 0$ in the trading loss $\mathcal{L}$ of the bank in* (2.9) *below. In the opposite (equally unrealistic, cf. Remark 4.2) case of a bank that would be perfectly hedged, our probability measure $\mathbb{Q}$ would coincide with the reference risk-neutral probability measure on $\mathcal{B} = \mathcal{A}$ and the trading loss process $\mathcal{L}$ of the bank would vanish (cf. Proposition 4.2), as in turn the economic capital and the KVA of the bank. In practice it is mostly jump-to-default risk that cannot be hedged by the bank and should therefore be assessed on a physical ground in our setup.*

We denote by T the sum between the final maturity of all claims in the portfolio (e.g. 50 years) and an upper bound $\delta \geq 0$ on the time of liquidating defaulted positions (typically considered in practice to be of the order of one to two weeks). All our cash flow and price processes are modeled as semimartingales, which are all taken in a càdlàg version. All our cash flow processes are cumulative starting from 0 at time 0, stopped at T, and integrable.

For any left-limited process Y, we denote by $Y^{\tau-}$ and ${}^{\tau-}Y$ the respective processes Y stopped before the bank default time τ and starting before τ, i.e.

$$Y^{\tau-} = JY + (1-J)Y_{\tau-},\ {}^{\tau-}Y = Y - Y^{\tau-}, \tag{2.1}$$

where $J = \mathbb{1}_{[\![0,\tau[\![}$ is the survival indicator process of the bank.

The $\mathbb{Q}$ expectation and $(\mathfrak{G}_t, \mathbb{Q})$ conditional expectation are denoted by $\mathbb{E}$ and $\mathbb{E}_t$. Given counterparty risk and its funding and capital consequences, the pricing of a derivative portfolio by the bank will depart from the $\mathbb{Q}$ valuation of its contractually promised cash flows. The following distinction between valuation and shareholder valuation below will be important in this regard.

Definition 2.1 *Given an optional, integrable process $\mathcal{Y}$ stopped at T (cumulative cash flow stream in the financial interpretation), we call:*
(i) *value process Z of $\mathcal{Y}$, the optional projection of $(\mathcal{Y}_T - \mathcal{Y})$, i.e.*

$$Z_t = \mathbb{E}_t(\mathcal{Y}_T - \mathcal{Y}_t),\ t \leq T, \tag{2.2}$$

and Z vanishes on $[T, +\infty)$;
(ii) *shareholder value process Y of $\mathcal{Y}$, any process Y vanishing on $[T, +\infty)$ if $T < \tau$*

and such that

$$Y_t = \mathbb{E}_t(\mathcal{Y}_{\tau-} - \mathcal{Y}_t + Y_{\tau-}),\ t < \tau. \tag{2.3}$$

Note that the shareholder value equation (2.3), for a process Y vanishing on $[T, +\infty)$ if $T < \tau$, is equivalent to

$$Y_t^{\tau-} = \mathbb{E}_t(\mathcal{Y}_{\tau\wedge T}^{\tau-} - \mathcal{Y}_t^{\tau-} + \mathbb{1}_{\{\tau\leq T\}} Y_\tau^{\tau-}),\ t \leq \tau \wedge T. \tag{2.4}$$

In particular, $(\mathcal{Y} + Y)^{\tau-}$ is then a martingale (stopped before τ).

This makes it apparent that the shareholder value equation (2.3) is actually an equation for $Y^{\tau-}$ and, in fact, a backward stochastic differential equation (BSDE) for $Y^{\tau-}$. This is a nonstandard BSDE, stopped before the bank default time τ. Such a BSDE is tantamount to the notion of recursive valuation of defaultable securities in Collin-Dufresne et al. (2004, Section 3.2), in the special case where $R_t(x) = x$ there. This notion is shown to be well posed in their Proposition 2, based on Schönbucher (2004)'s tool of the bank survival pricing measure (see around (3.1) below). We will deal with shareholder valuation by a more comprehensive reduction of filtration methodology in Section 3.2 (yielding a more complete grasp on the related integrability issues).

2.2 Abstract Trading Cash Flows

To avoid blurring the XVA conceptual picture by the combinatorial complexity of the financial network of the bank, in the present paper, we simply denote by $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{F}$ the trading cash flows to the trading desks and from the CVA and FVA desks, respectively[5]. These abstract cash flows $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{F}$ are specified in the follow-up paper (Crépey, 2022). On top of $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{F}$, we also consider the dynamic hedging cash flows (inclusive of the cost of setting up the hedges)

$$\mathcal{H} = \mathcal{H}^{mtm} - \mathcal{H}^{cva} - \mathcal{H}^{fva}, \tag{2.5}$$

where $\mathcal{H}^{mtm}$ is the dynamic hedging loss of the trading desks, whilst $\mathcal{H}^{cva}$ and $\mathcal{H}^{fva}$ are the dynamic hedging gains of the CVA and FVA desks.

Example 2.2 *Assuming a hedge of the trading desks implemented through a repo market on a Black-Scholes stock S with volatility σ, then, supposing no dividends and no repo basis on S, we have until the liquidation of the position that is hedged (with $\mathcal{H}^{mtm}$ and S in units of the risk-free asset that we use as our numéraire):*

$$d\mathcal{H}_t^{mtm} = \zeta_t dS_t = \zeta_t \sigma S_t dW_t. \tag{2.6}$$

Here W is a $(\mathfrak{G}, \mathbb{Q})$ Brownian motion driving S and ζ is the number of stocks shorted by the trading desks as their hedge. Given our choice of the risk-free asset as numéraire, the risk-free cost of funding the hedge is already included in (2.6).

[5]Unless explicitly specified, an amount paid means effectively paid if positive, received if negative. A similar convention applies to the notions of loss and gain or cost and benefit.

Risky funding for the bank means borrowing at a nonnegative risky spread over the risk-free rate. We assume the funding debt of the bank instantaneously liquidated at the bank default time τ. Accordingly:

Assumption 2.1 *The process $\mathcal{F}$ is a martingale nondecreasing before τ and stopped at $\tau \wedge T$. Each of the processes $\Xi = \mathcal{H}^{mtm}, \mathcal{H}^{cva}, \mathcal{H}^{fva}$ (hence, the aggregated hedging loss $\mathcal{H}$) is such that the processes $\Xi^{\tau-}$ and ${}^{\tau-}\Xi$ are martingales.*

Martingale assumptions on the hedging cash flows Ξ and on the risky funding cash flows $\mathcal{F}$ are in line with the definition of $\mathbb{Q}$ provided in Section 2.1. As bank shareholders only perceive bank pre-default cash flows, assuming that the $\Xi^{\tau-}$ processes are also martingales makes it consistent with a bank shareholder centric viewpoint of the different desks of the bank, in line with the fact that the shareholders have the control of the bank as long at it is nondefault (see the end of Section 1.1). A shareholder risk premium will be introduced in a second step, through KVA dividends.

Remark 2.1 *Martingales with martingale $(\cdot)^{\tau-}$ component obviously include all martingales without jump at τ, in particular all continuous martingales. They also include all the $\mathfrak{F}$ (càdlàg) martingales in an immersed reduction of filtration framework $\mathfrak{F} \subseteq \mathfrak{G}$, corresponding to the special case where $\mathbb{P} = \mathbb{Q}$ in the setup of Section 3.1, provided the $\mathfrak{F}$ Azéma supermartingale of τ is continuous and nonincreasing (Crépey, 2015, Lemma 2.1(ii)).*

Remark 2.2 *In (Crépey, 2022, Sections 3–5), nonnegative losses triggered by client defaults accumulate before the default of the bank itself, so that $\mathcal{C}^{\tau-}$ is nondecreasing, if not for minor corrective terms related to a mismatch of temporality between some cash flows and their hedge during the liquidation periods that separate the defaults from their settlements. In particular, in the limiting case where defaults are instantaneously settled, $\mathcal{C}^{\tau-}$ is nondecreasing (Crépey, 2022, Remark 5.1).*

2.3 Loss Processes

In our marked-to-model framework (see the end of Section 1.1), the gain process of the trading desks is given, inclusive of the corresponding hedging loss $\mathcal{H}^{mtm}$, by

$$\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \mathcal{H}^{mtm}, \tag{2.7}$$

for some theoretical target MtM to be devised later in view of addressing the requirements of Section 1. Likewise, the loss processes of the CA desks are given, inclusive of their respective hedging gains $\mathcal{H}^{cva}$ (for the CVA desk) and $\mathcal{H}^{fva}$ (for the FVA desk), by

$$\begin{aligned} &\mathcal{C} + \mathrm{CVA} - \mathrm{CVA}_0 - \mathcal{H}^{cva} \\ &\mathcal{F} + \mathrm{FVA} - \mathrm{FVA}_0 - \mathcal{H}^{fva}, \end{aligned} \tag{2.8}$$

for some to-be-suitably-devised theoretical target CVA and FVA levels. Denoting CA=CVA+FVA (cf. (1.1)), the overall loss of the bank is (recalling (2.5))

$$\mathcal{L} = \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 - (\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0) + \mathcal{H}. \tag{2.9}$$

A dealer bank should not do proprietary trading (cf. Section 1). But the risk of financial loss as a consequence of client default is hard to hedge, because single name credit default swaps that could in principle be used for that purpose are illiquid. The possibility for the bank of hedging its own jump-to-default is even more questionable. Indeed, for the bank, hedging its default is tantamount to selling jump-to-default protection on itself, which is unfeasible (see Section 2.1). Accordingly, our *reference dynamic hedging case* is when the trading desks are perfectly hedged ($\mathcal{P}+\mathrm{MtM}-\mathrm{MtM}_0-\mathcal{H}^{mtm}=0$) whilst the CA desks are not hedged ($\mathcal{H}^{cva}=\mathcal{H}^{fva}=0$), hence

$$\mathcal{H}=\mathcal{H}^{mtm}=\mathcal{P}+\mathrm{MtM}-\mathrm{MtM}_0 \text{ and } \mathcal{L}=\mathcal{C}+\mathcal{F}+\mathrm{CA}-\mathrm{CA}_0. \tag{2.10}$$

Example 2.3 *Continuing with Example 2.2, the gain process of a trading desk long a delta-hedged option position with payoff* $(S_\Theta-K)^+$ *(for some fixed maturity* Θ*) is given, until the liquidation of the position, by:*

$$\mathcal{P}+\mathrm{MtM}-\mathrm{MtM}_0-\int_0^{\cdot}\zeta_t dS_t,$$

where $\mathcal{P}=\mathbb{1}_{[\Theta,+\infty)}(S_\Theta-K)^+$ *and* MtM *is the Black-Scholes price process of the option (vanishing from time* Θ *onward). This trading gain process is therefore 0 if the trader uses the Black-Scholes delta as his hedge, which he should do in order to conform to the Volcker rule.*

Remark 2.3 *Even if* (2.10) *is our reference dynamic hedging case, we do* not *assume* (2.10) *throughout the paper. Indeed, on the one hand, the feasibility of the hedging loss* $\mathcal{H}^{mtm}=\mathcal{P}+\mathrm{MtM}-\mathrm{MtM}_0$ *by the trading desks is subject to the depth of the dynamic hedging markets, so supposing this hedging loss would be too restrictive. On the other hand, CVA traders do tentatively hedge their market (if not jump-to-default) risk, hence supposing* $\mathcal{H}^{cva}=\mathcal{H}^{fva}=0$ *(i.e.* $\mathcal{H}=\mathcal{H}^{mtm}$*) would also be too restrictive.*

2.4 MtM, CVA, and FVA

Definition 2.2 MtM, CVA, *and* FVA *are shareholder value processes*[6] *of* $\mathcal{P}$, $\mathcal{C}$, *and* $\mathcal{F}$.

That is, MtM, CVA, and FVA are killed at T on $\{T<\tau\}$ and, for $t<\tau$,

$$\mathrm{MtM}_t=\mathbb{E}_t\big(\mathcal{P}_{\tau-}-\mathcal{P}_t+\mathrm{MtM}_{\tau-}\big), \tag{2.11}$$
$$\mathrm{CVA}_t=\mathbb{E}_t\big(\mathcal{C}_{\tau-}-\mathcal{C}_t+\mathrm{CVA}_{\tau-}\big), \tag{2.12}$$
$$\mathrm{FVA}_t=\mathbb{E}_t\big(\mathcal{F}_{\tau-}-\mathcal{F}_t+\mathrm{FVA}_{\tau-}\big). \tag{2.13}$$

Hence, by Assumption 2.1 and the sentence following (2.4):

[6] cf. Definition 2.1(ii). Explicit assumptions ensuring existence of the corresponding shareholder value processes, and also their uniqueness within suitable spaces of square integrable solutions stopped before τ, will be provided by Corollaries 3.1-3.2 and Theorem 3.2(ii).

Remark 2.4 *The processes* MtM *and* CA = CVA + FVA *(cf.* (1.1)*) are such that each of the trading gains and losses in* (2.7)-(2.8)*, stopped before* τ*, is a martingale. So is therefore*

$$\mathcal{L}^{\tau-} = \mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathrm{CA}^{\tau-} - \mathrm{CA}_0 - (\mathcal{P}^{\tau-} + \mathrm{MtM}^{\tau-} - \mathrm{MtM}_0) + \mathcal{H}^{\tau-} \tag{2.14}$$

(cf. (2.9)*), the trading loss of the bank shareholders.* ∎

Given Assumption 2.1 on $\mathcal{H}$, the reference dynamic hedging case (2.10) is only attainable provided not only $(\mathcal{P} + \mathrm{MtM})^{\tau-}$ (as dictated by (2.11), cf. again the line following (2.4)), but also ${}^{\tau-}(\mathcal{P} + \mathrm{MtM})$, hence $\mathcal{P} + \mathrm{MtM}$ as a whole, is a martingale. Even if we do not assume (2.10) throughout the paper (cf. Remark 2.3), this motivates the following:

Assumption 2.2 MtM *is the value process of* $\mathcal{P}$ *(in particular,* $\mathrm{MtM}_T = 0$*).* ∎

Remark 2.5 *In further support of Assumption 2.2, note that, even if the trading desks were able to find clients accepting to deal with the bank on the basis of an* MtM *process that would be the bank shareholder value of* $\mathcal{P}$ *but not its value process, the corresponding discrepancy between valuation and shareholder valuation of* $\mathcal{P}$ *would be an indication of extreme dependence between the mark-to-market risk of the bank and its own default risk, such as the bank trading its own default risk, which should be considered with caution (cf. Remark 4.2).*

The processes CVA and FVA are so far unconstrained on $[\![\tau, +\infty[\![\bigcap \big(\{\tau \le T\} \times \mathbb{R}_+\big)$. We define them as zero there. As they already vanish on $[T, +\infty)$ if $T < \tau$, either of them, say Y, is in fact killed at $\tau \wedge T$, hence such that

$${}^{\tau-}Y = \mathbb{1}_{[\![\tau,+\infty[\![}(Y - Y_{\tau-}) = -\mathbb{1}_{[\![\tau,+\infty[\![} Y_{\tau-}. \tag{2.15}$$

2.5 Shareholder Capital at Risk and KVA

Since default risk can hardly be hedged, capital needs be set at risk by shareholders, who therefore deserve, in the cost-of-capital pricing approach of this paper, a further KVA add-on as a risk premium.

Economic capital (EC) is the level of capital at risk (CR) that a regulator would like to see on an economic basis. We assume that EC is killed at $\tau \wedge T$, as will in turn be CR from what follows. In view of (1.2), where KVA is provided by the clients in the first place (see Section 1.1):

Definition 2.3 *We define the shareholder capital at risk (*SCR*), to be remunerated at a constant and nonnegative hurdle rate* h*, as*

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA} = \max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA} = (\mathrm{EC} - \mathrm{KVA})^+, \tag{2.16}$$

where KVA *is a shareholder value process of* $\int_0^\cdot h\mathrm{SCR}_s ds$ *killed at* $\tau \wedge T$*, i.e.*[7]

$$\mathrm{KVA}_t = \mathbb{E}_t\Big[\int_t^\tau h\big(\mathrm{EC}_s - \mathrm{KVA}_s\big)^+ ds + \mathrm{KVA}_{\tau-}\Big],\ t < \tau,\ \textit{and} \tag{2.17}$$

KVA *is killed at* $\tau \wedge T$*.* ∎

[7] Explicit assumptions ensuring that such a KVA process is well defined will be provided by Theorem 3.3.

Remark 2.6 *The process* $\mathrm{KVA}^{\tau-}$ *is a supermartingale with drift coefficient*

$$-h\mathrm{SCR} = -h\big(\mathrm{EC} - \mathrm{KVA}\big)^+.$$

Note the following differential form of (2.17) (cf. (2.4)):

$$\begin{aligned}
&\mathrm{KVA}^{\tau-}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \le \tau \wedge T,\\
&d\mathrm{KVA}^{\tau-}_t = -h\mathrm{SCR}_t dt + d\nu_t,\\
&\text{for some martingale } \nu.
\end{aligned} \tag{2.18}$$

This formulation makes it apparent that the KVA corresponds to the amount to be maintained by the bank on its risk margin account in order to be in a position to deliver to its shareholders, dynamically into the future, a hurdle rate h on their capital at risk (SCR). In this sense our KVA addresses the sustainability requirement in Section 1. Moreover the amount on the risk margin account should land off at $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$. Indeed, ending up in the negative would mean an insufficient risk margin for ensuring the hurdle rate h to the shareholders. Conversely, ending up in the positive at $T < \tau$ would mean that the bank is unnecessarily expensive to its clients, which would contradict the minimality requirement in Section 1. See Corollary 3.4 for more on the minimality features of our KVA.

Proposition 2.1 *Shareholder dividends*

$$\mathcal{D} = -(\mathcal{L}^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0) \tag{2.19}$$

are a submartingale stopped before τ*, with drift coefficient* $h\mathrm{SCR}$*.*

Proof. Shareholder trading gains ($-\mathcal{L}^{\tau-}$) and KVA risk margin payments result in a dividend stream (2.19) to shareholders. The stated properties of $\mathcal{D}$ follow from the observations made in Remarks 2.4 and 2.6. ∎

3 Well-Posedness and Comparison Results

This section yields a technical specification of the setup of Section 2, in which the shareholder valuation equation (2.3) (in fact, a slight extension of it required for the XVA applications that follow) and the abstract XVA equations (2.12), (2.13) and (2.17), are shown to be well posed.

3.1 Reduced Stochastic Basis

We introduce a subfiltration $\mathfrak{F}$ of $\mathfrak{G}$, complete, right-continuous, quasi-left-continuous, and such that the bank default time τ is a $\mathfrak{G}$, but not an $\mathfrak{F}$, stopping time.

Assumption 3.1 *For any* $\mathfrak{G}$ *predictable, resp. optional process* Y*, there exists an* $\mathfrak{F}$ *predictable, resp. optional process* Y'*, dubbed* $\mathfrak{F}$ *reduction of* Y*, that coincides with* Y *on* $]\!]0,\tau]\!]$*, resp. on* $[\![0,\tau[\![$. ∎

In particular, any $\mathfrak{G}$ stopping time θ admits an $\mathfrak{F}$ stopping time θ', dubbed $\mathfrak{F}$ reduction of θ, such that $\theta \wedge \tau = \theta' \wedge \tau$.

Remark 3.1 *Regarding the relation between $\mathfrak{F}_\infty$ and the financial sub-σ-field $\mathcal{B}$ of $\mathcal{A}$ introduced in Section 2.1, as $\mathfrak{F}_\infty$ is meant to exclude the default of the bank, whereas $\mathcal{B}$ is meant to exclude nontraded risk factors starting with bank default, the big picture (even if not explicitly used in the paper) is: $\mathcal{B} \subset \mathfrak{F}_\infty \subset \mathcal{A}$.*

Definition 3.1 *We call invariance probability measure, a probability measure $\mathbb{P}$ on $\mathfrak{F}_T$ equivalent to the restriction of $\mathbb{Q}$ to $\mathfrak{F}_T$, such that*
(i) *stopping before τ turns $(\mathfrak{F}, \mathbb{P})$ local martingales on $[0, T]$ into $(\mathfrak{G}, \mathbb{Q})$ local martingales on $[\![0, \tau \wedge T]\!]$, and*
(ii) *the $\mathfrak{F}$ optional reductions of $(\mathfrak{G}, \mathbb{Q})$ local martingales on $[\![0, \tau \wedge T]\!]$ without jump at τ are $(\mathfrak{F}, \mathbb{P})$ local martingales on $[0, T]$.*

Remark 3.2 *The "non-immersion" case where $\mathbb{P} \neq \mathbb{Q}$ corresponds to situations of extreme dependence (hard wrong/right way risk) between the defaults of the bank and a client, or between the default of the bank and its portfolio exposure with a client (see e.g. Crépey and Song (2017a) and Crépey and Song (2016)).*

We denote by $\mathsf{S} = \mathbb{Q}(\tau > \cdot \,|\, \mathfrak{F}_\cdot)$ the $\mathfrak{F}$ Azéma supermartingale of τ, i.e. the $(\mathfrak{F}, \mathbb{Q})$ optional projection of $J = \mathbb{1}_{[\![0,\tau[\![}$. By bank survival probability measure associated with $\mathbb{Q}$, we mean the probability measure on $(\Omega, \mathcal{A})$ with $(\mathfrak{G}, \mathbb{Q})$ density process $Je^{\int_0^\cdot \gamma_s ds}$ (Schönbucher, 2004; Collin-Dufresne et al., 2004).

(Crépey and Song, 2017b, Lemma 2.3, Theorem 3.5, and Section 4.2) *Under Assumption 3.1, if $\mathsf{S}_T > 0$ a.s., then*

$$\mathfrak{F} \text{ optional reductions are uniquely defined on } [0, T]. \tag{3.1}$$

Additionally assuming τ endowed with a $(\mathfrak{G}, \mathbb{Q})$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^\tau \gamma_s ds}$ is $\mathbb{Q}$ integrable, then there exists a unique invariance probability measure $\mathbb{P}$ on $\mathfrak{F}_T$. The measure $\mathbb{P}$ coincides with the restriction to $\mathfrak{F}_T$ of the bank survival probability measure associated with $\mathbb{Q}$. ∎

By reduction of filtration setup hereafter, we mean the setup of the above-recalled result, so that $\mathfrak{F}$ optional or predictable reductions of processes are uniquely defined on $[0, T]$, and $\mathfrak{F}$ reductions of $[0, T]$ valued $\mathfrak{G}$ stopping times are uniquely defined.

Lemma 3.1 *For any $\mathfrak{G}$ optional process Y, Y' is nondecreasing on $[0, T]$ if $Y^{\tau-}$ is nondecreasing.*

Proof. If $Y^{\tau-}$ nondecreasing, then, for any constants $0 \leq t \leq s \leq T$,

$$Y'_t \mathbb{1}_{\{s<\tau\}} = Y^{\tau-}_t \mathbb{1}_{\{s<\tau\}} \leq Y^{\tau-}_s \mathbb{1}_{\{s<\tau\}} = Y'_s \mathbb{1}_{\{s<\tau\}},$$

hence $\mathbb{E}(Y'_t \mathbb{1}_{\{s<\tau\}} | \mathfrak{F}_s) \leq \mathbb{E}(Y'_s \mathbb{1}_{\{s<\tau\}} | \mathfrak{F}_s)$, i.e. $Y'_t \mathsf{S}_s \leq Y'_s \mathsf{S}_s$, where $\mathsf{S}_s > 0$—recalling from Dellacherie and Meyer (1980, Chapitre VI n° 17) that the set $\{\mathsf{S} > 0\}$ is a random

interval starting from 0 included, which we assumed contains T. ∎

As can classically be established by section theorem, for any $\mathfrak{G}$ progressive Lebesgue integrand X such that the $\mathfrak{G}$ predictable projection ${}^pX$ exists,[8] the indistinguishable equality $\int_0^\cdot {}^pX_s ds = \int_0^\cdot X_s ds$ holds. As a consequence, one can actually consider the $\mathfrak{F}$ predictable reduction X' of any $\mathfrak{G}$ progressive Lebesgue integrand X (even if this means replacing X by ${}^pX$).

Unless explicitly mentioned, probabilistic statements hereafter still refer to the original stochastic basis $(\mathfrak{G}, \mathbb{Q})$, with $\mathbb{Q}$ expectation and $(\mathfrak{G}_t, \mathbb{Q})$ conditional expectation denoted by $\mathbb{E}$ and $\mathbb{E}_t$ as before. The $\mathbb{P}$ expectation and $(\mathfrak{F}_t, \mathbb{P})$ conditional expectation will be denoted by $\mathbb{E}'$ and $\mathbb{E}'_t$. We will need the following spaces of processes:

- $\mathbb{S}_2$, the space of càdlàg $\mathfrak{G}$ adapted processes Y over $[\![0, \tau \wedge T]\!]$ without jump at time τ and such that, denoting $Y^*_t = \sup_{s\in[0,t]} |Y_s|$:

$$\mathbb{E}\Big[Y_0^2 + \int_0^T J_s e^{\int_0^s \gamma_u du} d(Y^*)_s^2\Big] = \mathbb{E}'\Big[\sup_{t\in[0,T]} (Y')_t^2\Big] < \infty, \tag{3.2}$$

  where the equality is established as Lemma 5.2 in Crépey et al. (2020). Note that, for $Y \in \mathbb{S}_2$,

$$\mathbb{E}\Big[\sup_{t\in[0,\tau\wedge T]} Y_t^2\Big] \leq \mathbb{E}\Big[Y_0^2 + \int_0^T J_s\ e^{\int_0^s \gamma_u du} d\big(Y^*\big)_s^2\Big] < \infty; \tag{3.3}$$

- $\mathbb{L}_2$, the space of $\mathfrak{G}$ progressive processes X over $[0, T]$ such that

$$\mathbb{E}\big[\int_0^{\tau\wedge T} e^{\int_0^s \gamma_u du} X_s^2 ds\big] = \mathbb{E}'\big[\int_0^T (X'_s)^2 ds\big] < +\infty, \tag{3.4}$$

  where the equality follows by an application of the formula (36) in Crépey et al. (2020) to the process $\int_0^\cdot e^{\int_0^t \gamma'_s ds} X'_t dt$;

- $\mathbb{S}'_2 \subset \mathbb{L}'_2$, the respective spaces of $\mathfrak{F}$ adapted càdlàg and $\mathfrak{F}$ progressive processes Y' and X' over $[0, T]$ making the right-hand side expectation finite in (3.2), respectively (3.4).

Lemma 3.2 *The $\mathfrak{F}$ optional reduction operator is an isometry from $\mathbb{S}_2$ onto $\mathbb{S}'_2$, with stopping before τ as the reciprocal operator.*

Proof. In view of (3.2), $\mathfrak{F}$ optional reduction applies $\mathbb{S}_2$ isometrically into $\mathbb{S}'_2$. Conversely, given $Z \in \mathbb{S}'_2$, then $Z^{\tau-} \in \mathbb{S}_2$, by (3.2) again, and $Z^{\tau-} = Z$ before τ, hence $Z = (Z^{\tau-})'$, by definition, existence and uniqueness of optional reductions. ∎

[8] For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

Remark 3.3 *In contrast with the setup of Crépey et al. (2020) (see Section 1.1 there), where more general anticipated XVA BSDEs are considered, in this and the follow-up paper no specific structure of the* $(\mathfrak{F}, \mathbb{P})$ *martingales is required. We do not even need to specify the risk drivers in the model. In practice (see e.g. Abbas-Turki et al. (2021, Section B)), those would typically consist of Brownian motions, driving the market risk factors, and of a continuous-time Markov chain (possibly modulated by the Brownian motions), used for driving the client default events.*

3.2 Shareholder Valuation Equation

Recall that the shareholder value equation (2.3), for a process Y vanishing on $[T, +\infty)$ if $T < \tau$, is nothing but the BSDE (2.4) for $Y^{\tau-}$. This applies to each of the MtM, CVA, and FVA equations (2.11), (2.12), and (2.13). In the case of the KVA equation (2.17), the drift coefficient in the equation also depends on the KVA itself. To include this case, as well as the one of risky funding cash flows $\mathcal{F}$ encompassing the FVA process itself later in the paper, we slightly extend the notion of shareholder valuation in Definition 2.1(ii), to cash flows including a component of the form

$$\int_0^{\cdot} J_t j_t(Y_t) dt,$$

for some random function $j = j_t(y)$ measurable with respect to the product of the $\mathfrak{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$. We thus consider the following shareholder value equation, which generalizes (2.4):

$$Y_t^{\tau-} = \mathbb{E}_t\big(\mathcal{Y}_{\tau\wedge T}^{\tau-} - \mathcal{Y}_t^{\tau-} + \int_t^{\tau\wedge T} j_s(Y_s) ds + \mathbb{1}_{\{\tau\leq T\}} Y_\tau^{\tau-}\big),\ t \leq \tau \wedge T. \tag{3.5}$$

As this equation says nothing on ${}^{\tau-}Y$ (cf. after (2.4)), we will consider it as an equation for the process $Z = Y^{\tau-}$. Likewise we consider the following reduced valuation equation for a process $Z = Y'$:

$$Y_t' = \mathbb{E}_t'\big(\mathcal{Y}_T' - \mathcal{Y}_t' + \int_t^T j_s(Y_s') ds\big), t \leq T, \tag{3.6}$$

and Y' vanishes on $[T, +\infty)$.

Definition 3.2 *We call* $\mathbb{S}_2$ *solution to the shareholder valuation equation* (3.5) *for* $Y^{\tau-}$*, any* $(\mathfrak{G}, \mathbb{Q})$ *semimartingale solution* $Y^{\tau-}$ *in* $\mathbb{S}_2$ *to* (3.5) *with* $(Y + \mathcal{Y} + \int_0^{\cdot} j_s(Y_s) ds)^{\tau-}$ *in* $\mathbb{S}_2$*. We call* $\mathbb{S}_2'$ *solution to the reduced equation* (3.6) *for* Y'*, any* $(\mathfrak{F}, \mathbb{P})$ *semimartingale solution* Y' *in* $\mathbb{S}_2'$ *to* (3.6) *with* $(Y' + \mathcal{Y}' + \int_0^{\cdot} j_s(Y_s') ds)$ *in* $\mathbb{S}_2'$*.*

By well-posedness of an equation in a certain space of solutions, we mean existence and uniqueness of a solution in this class.

First we show an equivalence between the following differential variations on (3.5) and (3.6):

$$\begin{aligned} &Y_T^{\tau-} = 0 \text{ on } \{T < \tau\} \text{ and, for } t \leq \tau \wedge T, \\ &dY_t^{\tau-} = -d\mathcal{Y}_t^{\tau-} - j_t(Y_t) dt + d\nu_t, \\ &\text{for some } (\mathfrak{G}, \mathbb{Q}) \text{ martingale } \nu \text{ in } \mathbb{S}_2, \end{aligned} \tag{3.7}$$

respectively

$$\begin{aligned} &Y'_T = 0 \text{ and, for } t \le T, \\ &dY'_t = -d\mathcal{Y}'_t - j_t(Y'_t)dt + d\mu_t, \\ &\text{for some } (\mathfrak{F}, \mathbb{P}) \text{ martingale } \mu \text{ in } \mathbb{S}'_2. \end{aligned} \tag{3.8}$$

Lemma 3.3 *If $Y^{\tau-}, \nu$ in $\mathbb{S}_2$ solve* (3.7)*, then $Y', \mu = \nu'$ in $\mathbb{S}'_2$ solve* (3.8)*; Conversely, if Y', μ in $\mathbb{S}'_2$ solve* (3.8)*, then $Y^{\tau-} = (Y')^{\tau-}, \nu = \mu^{\tau-}$ in $\mathbb{S}_2$ solve* (3.7)*.*

Proof. By definition of $\mathfrak{F}$ optional reductions, the terminal condition in (3.8) implies the one in (3.7). Conversely, taking the $(\mathfrak{F}_T, \mathbb{Q})$ conditional expectation of the terminal condition in (3.7) yields

$$0 = \mathbb{E}[Y_T^{\tau-}\mathbb{1}_{\{T<\tau\}}|\mathfrak{F}_T] = \mathbb{E}[Y'_T\mathbb{1}_{\{T<\tau\}}|\mathfrak{F}_T] = Y'_T\mathbb{Q}(\tau > T \,|\, \mathfrak{F}_T) = Y'_T\mathsf{S}_T,$$

hence $Y'_T = 0$ (as by assumption $\mathsf{S}_T > 0$, see above (3.1)), which is the terminal condition in (3.8).

For $Y^{\tau-}$ in $\mathbb{S}_2$, the martingale condition in (3.8) implies the one in (3.7), by stopping before τ and application to $\nu = \mu^{\tau-}$ of Lemma 3.2 and of the condition (i) in Definition 3.1. Conversely, the martingale condition in (3.7) implies that $(Y', \mu = \nu')$ satisfies the second line in (3.8) on $[\![0, \tau \wedge T]\!]$, hence on $[0, T]$, by (3.1). Moreover, by application of the condition (ii) in Definition 3.1 and of Lemma 3.2, $\mu = \nu'$ is an $(\mathfrak{F}, \mathbb{P})$ martingale in $\mathbb{S}'_2$. ∎

Theorem 3.1 **(i)** *The shareholder value equation* (3.5) *in $\mathbb{S}_2$ for $Y^{\tau-}$ is equivalent, through the bijection of Lemma 3.2, to the reduced valuation equation* (3.6) *in $\mathbb{S}'_2$ for Y'.*
(ii) *In the case where $\mathcal{Y}'$ is in $\mathbb{S}_2$, if the random function $v \mapsto j_t(v - \mathcal{Y}'_t)$ is continuous and nonincreasing in v, if*

$$\mathbb{E}' \int_0^T \sup_{|v| \le r} |j_t(v - \mathcal{Y}'_t) - j_t(-\mathcal{Y}'_t)| dt < \infty \text{ for all } r > 0,$$

and if $j_\cdot(-\mathcal{Y}'_\cdot)$ is in $\mathbb{L}'_2$, then the reduced valuation equation (3.6) *for Y' is well-posed in $\mathbb{S}'_2$, and so is the shareholder value equation* (3.5) *in $\mathbb{S}_2$ for $Y^{\tau-}$.*

Proof. **(i)** If $Y^{\tau-}$ is an $\mathbb{S}_2$ solution to (3.5), then $Y^{\tau-}, \nu$ in $\mathbb{S}_2$ solve (3.7) (for some ν). Hence $Y', \mu = \nu'$ in $\mathbb{S}'_2$ solve (3.8), by Lemma 3.3. Therefore Y' is an $\mathbb{S}'_2$ solution to (3.6). Conversely, if Y' is an $\mathbb{S}'_2$ solution to (3.6), then Y', μ in $\mathbb{S}'_2$ solve (3.8) (for some μ). Hence $Y^{\tau-} = (Y')^{\tau-}, \nu = \mu^{\tau-}$ in $\mathbb{S}_2$ solve (3.7), by the converse in Lemma 3.3. Thus $Y^{\tau-}$ is an $\mathbb{S}_2$ solution to (3.5) (noting that $\nu \in \mathbb{S}_2$ is $\mathbb{Q}$ square integrable over $[\![0, \tau \wedge T]\!]$, by (3.3)).
(ii) Under the additional assumptions made in the second part of the theorem, the well-posedness in $\mathbb{S}'_2$ of the reduced valuation equation (3.6) follows from Kruse and Popier (2016, Theorem 1) applied to the $(\mathfrak{F}, \mathbb{P})$ BSDE for $V' = Y' + \mathcal{Y}'$, i.e. to the $(\mathfrak{F}, \mathbb{P})$ BSDE with terminal condition $\mathcal{Y}'_T$ and coefficient $v \mapsto j(v - \mathcal{Y}')$. The well-posedness in $\mathbb{S}_2$ of the shareholder value equation (3.5) for $Y^{\tau-}$ then follows from the first part of the theorem. ∎

Corollary 3.1 *The equations* (2.11), (2.12), (2.13), *and* (2.17) *for* $\mathrm{MtM}^{\tau-}$, $\mathrm{CVA}^{\tau-}$, $\mathrm{FVA}^{\tau-}$, *and* $\mathrm{KVA}^{\tau-}$ *in* $\mathbb{S}_2$ *are respectively equivalent to the following equations for* MtM', CVA', FVA', *and* KVA' *in* $\mathbb{S}_2'$*: For* $t \leq T$,

$$\mathrm{MtM}_t' = \mathbb{E}_t'\big(\mathcal{P}_T' - \mathcal{P}_t'\big), \tag{3.9}$$

$$\mathrm{CVA}_t' = \mathbb{E}_t'\big(\mathcal{C}_T' - \mathcal{C}_t'\big), \tag{3.10}$$

$$\mathrm{FVA}_t' = \mathbb{E}_t'\big(\mathcal{F}_T' - \mathcal{F}_t'\big), \tag{3.11}$$

$$\mathrm{KVA}_t' = \mathbb{E}_t' \int_t^T h(\mathrm{EC}_s' - \mathrm{KVA}_s')^+ ds, \tag{3.12}$$

and each of the four processes vanishes on $[T, +\infty)$.

Assuming these reduced equations well posed in $\mathbb{S}_2'$, FVA *and* KVA *are nonnegative. So is also* CVA *provided* $\mathcal{C}^{\tau-}$ *is nondecreasing*[9].

Proof. The equivalences hold by application of Theorem 3.1(i). The remaining statements are immediate consequences of Theorem 3.1(ii) and Equations (3.10)–(3.12), noting from Lemma 3.1 that $\mathcal{F}'$ is nondecreasing on $[0, T]$, by Assumption 2.1, as is also $\mathcal{C}'$ provided $\mathcal{C}^{\tau-}$ is nondecreasing. ∎

Remark 3.4 *The reduced* MtM *equation* (3.9) *may not be needed in practice, given the availability in applications of more explicit formulas in concrete cases such as (Crépey, 2022,* (5.8)*), based on the alternative characterization of* MtM *as the value process* (2.2) *of* $\mathcal{P}$*, by Assumption 2.2. Still, the ability to formulate all equations (MtM included) with respect to a common stochastic basis can be useful for certain XVA numerical schemes.* ∎

3.3 CVA and FVA

We consider the CVA and FVA equations (2.12) and (2.13), as well as their reduced forms (3.10) and (3.11).

Regarding the risky funding cash flows, we postulate

$$d\mathcal{F}_t' = f_t(\mathrm{FVA}_t')dt, \tag{3.13}$$

for some nonnegative random function $f = f_t(y)$ measurable with respect to the product of the $\mathfrak{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$. The corresponding reduced FVA equation is

$$\mathrm{FVA}_t' = \mathbb{E}_t' \int_t^T f_s(\mathrm{FVA}_s')ds,\ t \leq T. \tag{3.14}$$

Corollary 3.2 **(i)** *If* $\mathcal{C}' \in \mathbb{S}_2'$*, then the formula* (3.10) *yields a well defined* CVA' *process in* $\mathbb{S}_2'$*, which is nonnegative provided* $\mathcal{C}^{\tau-}$ *is nondecreasing.*
(ii) *Assuming* f *continuous and nonincreasing in* y, $\mathbb{E}' \int_0^T \sup_{|y|\leq r} |f_t(y) - f_t(0)|dt < \infty$ *for all* $r > 0$*, and* $f.(0)$ *in* $\mathbb{L}_2'$*, then the* FVA' *equation* (3.14) *is well posed in* $\mathbb{S}_2'$ *and* FVA *is nonnegative.*

[9] cf. Remark 2.1.

Proof. By two successive applications of Theorem 3.1(ii), with $j = 0$ in the CVA case and $\mathcal{Y} = 0$ and $j = f$ in the FVA case, and by Corollary 3.1. ∎

As detailed in (Crépey, 2022), the bank posts collateral to its counterparties in the form of rehypothecable collateral, which is fungible across netting sets, and segregated collateral, which is not. We denote by $\mathcal{R}$ and $\mathcal{S}$ two $\mathfrak{G}$ optional processes, with $\mathcal{S} \geq 0$, corresponding to the overall amounts of rehypothecable collateral and segregated collateral posted by the bank Treasury.

Assuming the bank default time τ endowed with an intensity $\gamma = J_{-}\gamma$, let $R \leq \bar{R}$ denote two predictable, $[0, 1]$ valued recovery rates processes, corresponding to two bonds of different seniorities issued by the bank. We consider a market of risky funding assets defined by $U_0 = \bar{U}_0 = 1$ and, for $t \geq 0$,

$$\begin{aligned} dU_t &= \lambda_t U_t dt + (1 - R_t) U_{t-} dJ_t = (1 - R_t) U_{t-} (dJ_t + \gamma_t dt) \\ d\bar{U}_t &= \bar{\lambda}_t \bar{U}_t dt + (1 - \bar{R}_t) \bar{U}_{t-} dJ_t, = (1 - \bar{R}_t) \bar{U}_{t-} (dJ_t + \gamma_t dt), \end{aligned}$$

where $dJ_t + \gamma_t dt$ is the compensated jump-to-default martingale of the bank and $\lambda = (1 - R)\gamma, \bar{\lambda} = (1 - \bar{R})\gamma$:

Assumption 3.2 *$\bar{U}$ and U are the martingale price processes of two risky funding assets used by the bank for its respective segregated collateral and residual*[10] *borrowing purposes. As the bank cannot sell default protection on itself, it can only take short positions in U and $\bar{U}$.*

The motivation for Assumption 3.2 is that, for reasons explained in Albanese et al. (2017, Section 3.2), reserve capital is a possible source of funding for rehypothecable collateral, whereas segregated collateral must be borrowed entirely. However, a recovery rate $\bar{R} \geq R$ may be available on the borrowing debt for segregated collateral, via the funding scheme described in Albanese et al. (2020, Section 5).

Remark 3.5 *Assumption 3.2 implicitly supposes that the funders of the bank, i.e. the entities that fund the different segments of the debt of the bank (corresponding to its different funding sets), are default-free, and that the positions of the bank with its funders are instantaneously liquidated in case of default of the bank, which is all assumed for simplicity.*

Remark 3.6 *In practice, capital at risk (CR) can also be used by the bank, on top of reserve capital (CA), for its rehypothecable collateral funding purposes. This induces an intertwining of the FVA and the KVA, which is the topic of Crépey et al. (2020). For simplicity in this paper, we assume that the bank does not use capital at risk for funding purposes.*

[10] Once everything else has already been accounted for, as prescribed by an overall budget condition.

Theorem 3.2 **(i)** *We have*

$$\begin{aligned}\mathcal{F} &= \int_0^{\cdot} \lambda_t(\mathcal{R}-\mathrm{CA})_t^+ dt - (1-R_\tau)(\mathcal{R}-\mathrm{CA})_{\tau-}^+ \mathbb{1}_{[\![\tau,\infty[\![} \\ &\quad + \int_0^{\cdot} \bar{\lambda}_t \mathcal{S}_t dt - (1-\bar{R}_\tau)\mathcal{S}_{\tau-}\mathbb{1}_{[\![\tau,\infty[\![} \\ &= \int_0^{\cdot} \big((1-R_t)(\mathcal{R}-\mathrm{CA})_{t-}^+ + (1-\bar{R}_t)\mathcal{S}_{t-}\big)(dJ_t + \gamma_t dt), \end{aligned} \tag{3.15}$$

i.e.

$$\begin{aligned}\mathcal{F}^{\tau-} &= \int_0^{\cdot} \lambda_t(\mathcal{R}-\mathrm{CA})_t^+ dt + \int_0^{\cdot} \bar{\lambda}_t \mathcal{S}_t dt, \\ {}^{\tau-}(-\mathcal{F}) &= (1-R_\tau)(\mathcal{R}-\mathrm{CA})_{\tau-}^+ \mathbb{1}_{[\![\tau,\infty[\![} + (1-\bar{R}_\tau)\mathcal{S}_{\tau-}\mathbb{1}_{[\![\tau,\infty[\![}. \end{aligned} \tag{3.16}$$

(ii) *Assuming the reduction of filtration setup of Section 3.1 and* $\mathcal{C}' \in \mathbb{S}_2'$*, then*

$$\mathcal{F}' = \int_0^{\cdot} \lambda_t'(\mathcal{R}' - \mathrm{CVA}' - \mathrm{FVA}')_t^+ dt + \int_0^{\cdot} \bar{\lambda}_t' \mathcal{S}_t' dt, \tag{3.17}$$

which is of the form (3.13) *for*

$$f_t(y) = \lambda'(\mathcal{R}' - \mathrm{CVA}' - y)^+ + \bar{\lambda}'\mathcal{S}'; \tag{3.18}$$

if $\mathbb{E}' \int_0^T \lambda_t' dt < \infty$ *and* $\lambda'(\mathcal{R}' - \mathrm{CVA}')^+ + \bar{\lambda}'\mathcal{S}' \in \mathbb{L}_2'$*, then the* FVA *equation* (2.13) *is well-posed in* $\mathbb{S}_2$ *and we have, for* $t \le T$*,*

$$\mathrm{FVA}_t' = \mathbb{E}_t' \int_t^T \lambda_s'(\mathcal{R}' - \mathrm{CVA}' - \mathrm{FVA}')_s^+ ds + \mathbb{E}_t' \int_t^T \bar{\lambda}_s' \mathcal{S}_s' ds \tag{3.19}$$

as well as

$$\mathcal{L}' = \mathcal{C}' + \mathcal{F}' + \mathrm{CA}' - \mathrm{CA}_0 - (\mathcal{P}' + \mathrm{MtM}' - \mathrm{MtM}_0) + \mathcal{H}' \in \mathbb{S}_2'$$

provided $\mathcal{P}' + \mathrm{MtM}' - \mathcal{H}' \in \mathbb{S}_2'$[11]*.*

Proof. **(i)** By Assumption 3.2, the risky funding strategy of the bank consists in maintaining short positions (consistent with the condition concluding Assumption 3.2) of $\frac{\mathcal{S}^{\tau-}}{\bar{U}^{\tau-}}$ units of the asset $\bar{U}$ and $\frac{((\mathcal{R}-\mathrm{CA})^+)^{\tau-}}{U^{\tau-}}$ units of U. Given our use of the risk-free asset as numéraire, the self-financing condition on the funding strategy of the FVA desk is then written as

$$d\mathcal{F}_t = \frac{((\mathcal{R}-\mathrm{CA})^+)_t^{\tau-}}{U_t^{\tau-}} dU_t + \frac{\mathcal{S}_t^{\tau-}}{\bar{U}_t^{\tau-}} d\bar{U}_t. \tag{3.20}$$

This yields (3.15), which splits as (3.16).
(ii) Assuming the reduction of filtration setup of Section 3.1, (3.17) follows from the first line in (3.16) by definition, existence and uniqueness of optional reductions viewed

[11] e.g. in the reference dynamic hedging case (2.10), where $\mathcal{P}' + \mathrm{MtM}' - \mathrm{MtM}_0 - \mathcal{H}' = 0$.

in Section 3.1 (noting that $\mathrm{CA}' = \mathrm{CVA}' + \mathrm{FVA}'$, where CVA' is well defined under the stated hypotheses, by Corollary 3.2(i)). Hence the expression (3.17) is of the form (3.13), for f there given by (3.18). The remaining FVA statements follow by an application of Corollary 3.2(ii).

The process $\mathcal{L}'$ belongs to $\mathbb{S}_2'$ as the sum (modulo a constant) between $\mathrm{CVA}' + \mathcal{C}'$, the $(\mathfrak{F}, \mathbb{P})$ martingale part of FVA', and $-(\mathcal{P}' + \mathrm{MtM}') + \mathcal{H}'$ (assumed in $\mathbb{S}_2'$). ∎

Remark 3.7 *The industry terminology distinguishes a strict FVA, in the sense of the cost of funding rehypothecable collateral, from an MVA, corresponding to the cost of funding segregated collateral. In this paper, to spare one "VA" notation, we merge the two in an overall FVA meant in the broad sense of the cost of funding the derivative trading of the bank. The strict FVA and the MVA correspond to the first and the second terms on the right-hand side of* (3.19).

3.4 Economic Capital

We denote by $q_a(\ell)$ the left quantile of level $a \in (0,1)$ of a random loss ℓ. The value-at-risk and expected shortfall of ℓ, both at some confidence level α (with in practice α "close to 1"), respectively denote $q_\alpha(\ell)$ and $(1-\alpha)^{-1}\int_\alpha^1 q_a(\ell)da$. As is well known, the expected shortfall operator is $(1-\alpha)^{-1}$ Lipschitz from the space of integrable losses ℓ to $\mathbb{R}$, and to $\mathbb{R}_+$ when restricted to centered losses ℓ.

Capital requirements are focused on the solvency issue, because it is when a regulated firm becomes insolvent that the regulator may choose to intervene, take over, or restructure a firm. Specifically, Basel II Pillar II defines economic capital as the $\alpha = 99\%$ value-at-risk of the depletion of core equity tier I capital (CET1) over one year. Moreover, the Fundamental Review of the Trading Book (cf. Basel Committee on Banking Supervision (2013)) required a shift from 99% value-at-risk to 97.5% expected shortfall as the reference risk measure in capital calculations.

In our setup, before bank default, CET1 depletions correspond to the shareholder trading loss process $\mathcal{L}^{\tau-}$.[12] In addition, economic capital calculations are typically made by a bank "on a going concern", hence assuming that the bank itself does not default. Accordingly (cf. the last sentence in the result recalled after (3.1)), assuming the process $(\mathcal{L}')^T$ integrable under $\mathbb{P}$[13]:

Definition 3.3 *The economic capital (*EC*) of the bank is defined as* $\mathrm{EC} = J\,\mathrm{ES}$*, where, for* $t \geq 0$*,* ES_t *is the* $(\mathfrak{F}_t, \mathbb{P})$ *conditional expected shortfall*[14] *of the random variable* $(\mathcal{L}')^T_{t+1} - (\mathcal{L}')^T_t$ *at the confidence level* $\alpha = 97.5\%$*, in the following (*$\mathfrak{F}$ *predictable) sense*[15]*:*

$$\mathrm{ES} = \inf_{\textit{rationals } k} \Big(k + (1-\alpha)^{-1}\, {}^p\big[((\mathcal{L}')^T_{\cdot+1} - (\mathcal{L}')^T_{\cdot} - k)^+\big]\Big),$$

[12] See Albanese et al. (2021, Proposition 4.1) for a detailed balance sheet perspective on that matter.

[13] See Remark 3.8.

[14] see Crépey et al. (2020, Lemma 2.1 and following comments), whose proof also works in the present context by the assumed quasi-left continuity of $\mathfrak{F}$.

[15] making $\mathrm{EC}' = \mathrm{ES}$ a suitable input to the BSDE (3.12).

where ^{p} *denotes the* $(\mathbb{F}, \mathbb{P})$ *predictable projection operator*[16].

Remark 3.8 *Remark 2.4 and the converse part in Definition 3.1 imply that the process* $\mathcal{L}'$ *is an* $(\mathfrak{F}, \mathbb{P})$ *local martingale. Assuming its* $\mathbb{P}$ *integrability is not a practical restriction as, in concrete setups such as the one of Theorem 3.2(ii),* $\mathcal{L}'$ *is even in* $\mathbb{S}'_2$.

In particular, the economic capital process is killed at $\tau \wedge T$ and, since the expected shortfall of a centered random variable is nonnegative:

Remark 3.9 EC *is nonnegative.*

3.5 KVA in the Case of a Default-Free Bank

In this section we temporarily suppose the bank default free, i.e., formally,

$$\text{“}\tau = +\infty,\ (\mathfrak{F}, \mathbb{P}) = (\mathfrak{G}, \mathbb{Q})\text{.”}$$

The results are then extended to the case of a defaultable bank in Section 3.6.

So in this section we use the "$\cdot'$" notation, not in the sense of $\mathfrak{F}$ reduction (as $\mathfrak{F} = \mathfrak{G}$ here), but simply in order to distinguish the present equations from the ones in Section 3.6, where $\mathfrak{F} \neq \mathfrak{G}$. In Section 3.6 the present data will then be interpreted as the $\mathfrak{F}$ reductions of the corresponding data there.

In particular, the process EC' is defined just like EC in Definition 3.3, except that $\tau = +\infty$ here. Given $C' \geq \mathrm{EC}' \geq 0$ representing a putative capital at risk process for the bank, we consider the auxiliary BSDE

$$K'_t = \mathbb{E}'_t \int_t^T h\big(C'_s - K'_s\big) ds,\ t \leq T, \tag{3.21}$$

with the same interpretation as the KVA (cf. the comment following (2.18)), but relative to any putative capital at risk process C', and simplified to the present setup of a risk-free bank.

Corollary 3.3 *If* C' *is in* $\mathbb{L}'_2$*, then the equation* (3.21) *for* K' *has for unique* $\mathbb{S}'_2$ *solution*

$$K'_t = \mathbb{E}'_t \int_t^T h e^{-h(s-t)} C'_s ds,\ 0 \leq t \leq T. \tag{3.22}$$

If $\mathcal{L}'$ *is in* $\mathbb{L}'_2$*, then* EC' *is in* $\mathbb{L}'_2$ *and the* KVA' *equation* (3.12) *has a unique* $\mathbb{S}'_2$ *solution, such that*

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T h e^{-h(s-t)} \max(\mathrm{EC}'_s, \mathrm{KVA}'_s) ds,\ 0 \leq t \leq T. \tag{3.23}$$

[16] which applies to any raw, non necessarily adapted, càdlàg process, such as $((\mathcal{L}')^T_{\cdot+1} - (\mathcal{L}')^T_{\cdot} - k)^+$ for any constant k.; see e.g. He et al. (1992, Theorem V.5.2).

Proof. If $\mathcal{L}'$ is in $\mathbb{L}'_2$, then EC' is in $\mathbb{L}'_2$, by Definition 3.3 and $(1-\alpha)^{-1}$ Lipschitz property of the expected shortfall operator recalled in the beginning of Section 3.4. Moreover, the KVA' BSDE (3.12) has a nonincreasing coefficient

$$k_t(y) = h\big(\mathrm{EC}'_t - y\big)^+,\ y \in \mathbb{R}. \tag{3.24}$$

By Theorem 3.1(ii) applied with $\mathcal{Y} = 0$ and nonincreasing coefficient $j_t(y) = h(\mathrm{EC}'_t - y)^+$, the KVA' equation (3.12) has therefore a unique $\mathbb{S}'_2$ solution. This also holds for the linear BSDE (3.21), by even simpler considerations. Moreover, the $\mathbb{S}'_2$ solution K' to (3.22) solves (3.21).

The process KVA' is in $\mathbb{S}'_2$ with martingale part in $\mathbb{S}'_2$ and, by (3.12), we have, for $0 \le t \le T$,

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T h\big(\mathrm{EC}'_s - \mathrm{KVA}'_s\big)^+ ds = \mathbb{E}'_t \int_t^T h\Big(\max(\mathrm{EC}'_s, \mathrm{KVA}'_s) - \mathrm{KVA}'_s\Big) ds.$$

Hence the process KVA' solves in $\mathbb{S}'_2$ the linear BSDE (3.21) corresponding to the implicit data $C' = \max(\mathrm{EC}', \mathrm{KVA}') \in \mathbb{L}'_2$. Equation (3.23) is the corresponding instantiation of (3.22). ∎

Assuming $\mathcal{L}'$ in $\mathbb{L}'_2$, let

$$\mathrm{CR}' = \max(\mathrm{EC}', \mathrm{KVA}'), \tag{3.25}$$

where KVA' is the $\mathbb{S}'_2$ solution to (3.12). Note that CR' is nonnegative, as this is already the case for EC' as seen in Remark 3.9. To emphasize the dependence on C', we henceforth denote by $K' = K'(C')$ the solution (3.22) to the linear BSDE (3.21). In particular, (3.23) and (3.25) read as

$$\mathrm{KVA}' = K'(\mathrm{CR}'). \tag{3.26}$$

We define the set of admissible capital at risk processes as

$$\mathrm{Adm}' = \{C' \in \mathbb{L}'_2; C' \ge \max\big(\mathrm{EC}', K'(C')\big)\}. \tag{3.27}$$

Here $C' \ge \mathrm{EC}'$ is the risk acceptability condition, while $C' \ge K'(C')$ expresses that the risk margin $K'(C')$, which would correspond through the hurdle rate h to the tentative capital at risk process C' (cf. the comment regarding the KVA made after (2.18)), is part of capital at risk (cf. the comment above (1.2)).

Lemma 3.4 *Assuming that $\mathcal{L}'$ is in $\mathbb{L}'_2$, then*
(i) $\mathrm{CR}' = \min \mathrm{Adm}', \mathrm{KVA}' = \min_{C' \in \mathrm{Adm}'} K'(C')$;
(ii) *The process* KVA' *is nondecreasing in the target hurdle rate h.*

Proof. **(i)** By (3.26),

$$\mathrm{CR}' = \max(\mathrm{EC}', \mathrm{KVA}') = \max\big(\mathrm{EC}', K'(\mathrm{CR}')\big).$$

Therefore $\mathrm{CR}' \in \mathrm{Adm}'$. Moreover, for any $C' \in \mathrm{Adm}'$, we have (cf. (3.24)):

$$k_t(K'_t(C')) = h\big(\mathrm{EC}'_t - K'_t(C')\big)^+ \le h(C'_t - K'_t(C')).$$

Hence the coefficient of the KVA$'$ BSDE (3.12) never exceeds the coefficient of the linear BSDE (3.21) when both coefficients are evaluated at the solution $K'_t(C')$ of (3.21). Since these are BSDEs with equal (null) terminal condition, the BSDE comparison principle of Proposition 4 in Kruse and Popier (2016)[17] applied to the BSDEs (3.21) and (3.12) yields $\mathrm{KVA}' \leq K'(C')$. Consequently, $\mathrm{KVA}' = \min_{C' \in \mathrm{Adm}'} K'(C')$ and, for any $C' \in \mathrm{Adm}'$,

$$C' \geq \max(\mathrm{EC}', K'(C')) \geq \max(\mathrm{EC}', \mathrm{KVA}') = \mathrm{CR}'.$$

Thus $\mathrm{CR}' = \min \mathrm{Adm}'$.
(ii) The coefficient (3.24) of the KVA$'$ BSDE (3.12) is nondecreasing in h. So is therefore the $\mathbb{S}'_2$ solution KVA$'$ to (3.12), by the BSDE comparison theorem of Kruse and Popier (2016, Proposition 4) applied to the BSDE (3.12) for different h. ∎

3.6 KVA in the Case of a Defaultable Bank

In the case of a defaultable bank, "$\cdot'$" now denoting $\mathfrak{F}$ reduction (predictable, optional, or progressive, as relevant), we have by applications of Theorem 3.1(i) (with $\mathcal{Y} = 0$ there):

Lemma 3.5 *The equation*

$$K_t^{\tau-} = \mathbb{E}_t\Big(\int_t^{\tau \wedge T} h\big(C_s - K_s\big) ds + \mathbb{1}_{\{\tau \leq T\}} K_\tau^{\tau-} \Big),\ t \leq \tau \wedge T \tag{3.28}$$

in $\mathbb{S}_2$ for $K^{\tau-}$ is equivalent, through the bijection of Lemma 3.2, to the equation (3.21) *in $\mathbb{S}'_2$ for K'sky.*

Hence, given also Corollaries 3.1 and 3.3:

Theorem 3.3 *If $C' \in \mathbb{L}'_2$, then the equation* (3.28) *for $K^{\tau-}$ is well posed in $\mathbb{S}_2$ and the $\mathfrak{F}$ optional reduction K' of its $\mathbb{S}_2$ solution K is the $\mathbb{S}'_2$ solution to* (3.21)*.*

If $\mathcal{L}'$ is in $\mathbb{L}'_2$[18], then the equation (2.17) *for* $\mathrm{KVA}^{\tau-}$ *is well posed in $\mathbb{S}_2$ and the $\mathfrak{F}$ optional reduction* KVA$'$ *of its $\mathbb{S}_2$ solution* $\mathrm{KVA}^{\tau-}$ *is the $\mathbb{S}'_2$ solution to* (3.12) *or, equivalently,* (3.23)*.*

In the case of a defaultable bank, writing $K = K(C)$ for the $\mathbb{S}_2$ solution to (3.28), the set of admissible capital at risk processes is defined by (cf. (3.27) and the following comments)

$$\mathrm{Adm} = \{C \in \mathbb{L}_2; C \geq \max\big(\mathrm{EC}, K(C)\big)\}.$$

The following result shows that $\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA})$ is in fact *the minimal and cheapest* capital at risk process C satisfying the risk admissibility condition $C \geq \mathrm{EC}$ and consistent with the target hurdle rate h on shareholder capital at risk.

[17] Note that jumps are not an issue for comparison in our setup, where the coefficient k "only depends on y"; cf. Kruse and Popier (2016, Assumption (H3')).
[18] e.g. in the setup of Theorem 3.2(ii).

Corollary 3.4 *Assuming that $\mathcal{L}'$ is in $\mathbb{L}_2'$:*
(i) *We have* $\mathrm{CR} = \min \mathrm{Adm}, \mathrm{KVA} = \min_{C \in \mathrm{Adm}} K(C)$;
(ii) *The process* KVA *is nondecreasing in h.*

Proof. This follows by application of Lemma 3.4 and Theorem 3.3. ∎

The fact that KVA is continuous and nondecreasing in h as just seen (and in fact, unless the processes $\mathcal{L}'$ and EC would vanish, increasing) allows one to define the *implied hurdle rate* as the value of the target hurdle rate $h \in \mathbb{R}_+$ calibrated, through the KVA formula (3.23) valued at time $t = 0$, to the actual amount on the risk margin account of the bank (if this amount cannot be reached by any $h \in \mathbb{R}_+$, then the implied hurdle rate is deemed infinite).

See the concluding paragraph of Albanese et al. (2021, Section 3.3) for a simple example in a stylized, one-period XVA model, where the implied hurdle rate can be characterized quite explicitly, shedding light on the impact thereon of the portfolio of the bank, the default intensities for the bank and its counterparty, and the risk aversion of the bank shareholders.

4 Wealth Transfer Analysis

We now bring to light the symmetrical companions of the contra-assets, i.e. the contra-liabilities. Put together, contra-assets and contra-liabilities allow analyzing the wealth transfers triggered by the trading of the bank, which occur without giving rise to arbitrage opportunities to shareholders. A view on DVA and FDA as wealth transfers is consistent with the conclusions drawn in a structural default model of the bank by Andersen et al. (2019) (who however do not deal with the KVA).

Using (2.15) that applies to $Y = \mathrm{CVA}$ and FVA, Figure 4.1 details the split of the overall loss process of the bank, $\mathcal{L}$ in (2.9), as the difference between the pre-bank default loss process, i.e. the shareholder loss process $\mathcal{L}^{\tau-}$ as per (2.14), and the creditor gain process

$$^{\tau-}(-\mathcal{L}) = {}^{\tau-}(-\mathcal{C}) + {}^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CA}_{\tau-} - {}^{\tau-}(-\mathcal{P} - \mathrm{MtM}) + {}^{\tau-}(-\mathcal{H}). \tag{4.1}$$

Table 4.1 identifies various cash flows (column 3) and the related value processes[19] (columns 1 and 2) involved in the foregoing wealth transfer analysis.

Lemma 4.1 *We have*
(i) $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, *which is the value process of both* $^{\tau-}(-\mathcal{L})$ *and* $(-\mathcal{L})$*;*
(ii) $\mathrm{FVA} = \mathrm{FDA}$, $\mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}$;
(iii) $\mathrm{KVA}_t^{sh} = J_t \mathbb{E}_t \int_t^\tau h\big(\mathrm{EC}_s - \mathrm{KVA}_s\big)^+ ds$ *and* $\mathrm{KVA}_t^{cr} = J_t \mathbb{E}_t \mathrm{KVA}_{\tau-}$, *for* $t \in \mathbb{R}_+$.

Proof. **(i)** holds by the definitions of Table 4.1, the formula (4.1) for $^{\tau-}(-\mathcal{L})$, and the martingale (hence, zero-value) properties of $^{\tau-}(\mathcal{P} + \mathrm{MtM})$, $^{\tau-}\mathcal{H}$, and $\mathcal{L}^{\tau-}$ (see Assumption 2.1 and Remark 2.4).
(ii) holds by the definitions of Table 4.1, by (2.12), (2.13), (1.1), and the fact that $\mathcal{F}$ is a martingale (hence, zero-valued) stopped at $\tau \wedge T$, by Assumption 2.1.

[19] cf. Definition 2.1(i).

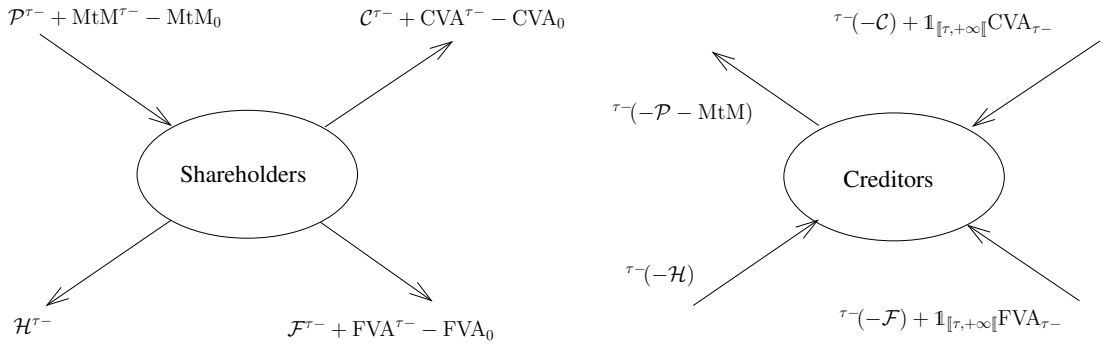

Figure 4.1: *Left*: Pre-bank-default trading cash flows $\mathcal{L}^{\tau-}$. *Right*: Trading cash flows from bank default onward ${}^{\tau-}(-\mathcal{L})$.

CA	Contra-assets valuation	$\mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CA}_{\tau-}$
CL	Contra-liabilities valuation	${}^{\tau-}(-\mathcal{L})$
DVA	Debt valuation adjustment	${}^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CVA}_{\tau-}$
FDA	Funding debt adjustment	${}^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{FVA}_{\tau-}$
FV	Fair valuation of counterparty risk	$\mathcal{C} + \mathcal{F}$
KVA^{sh}	Shareholders' KVA	$(\mathrm{KVA}_0 - \mathrm{KVA}^{\tau-})$
KVA^{cr}	Creditors' KVA	$\mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}$

Table 4.1: Valuation acronym and name (columns 1 and 2) of various cash flows (column 3) involved in the XVA wealth transfer analysis.

(iii) At $t < \tau$, the value processes of $(\mathrm{KVA}_0 - \mathrm{KVA}^{\tau-})$ and $\mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}$ are respectively worth $\mathbb{E}_t(\mathrm{KVA}_t - \mathrm{KVA}_{\tau-}) = \mathbb{E}_t \int_t^\tau h\big(\mathrm{EC}_s - \mathrm{KVA}_s\big)^+ ds$ (by (2.17)) and $\mathbb{E}_t\mathrm{KVA}_{\tau-}$. From τ onward these value processes vanish, as well as KVA^{sh} and KVA^{cr}. ∎

We assume that the shareholders have no other business than their involvement within the bank. The bank creditors have to face the liquidation costs of the bank, which are outside the scope of our model.

Definition 4.1 *We call wealth transferred to the bank shareholders and creditors by the derivative trading of the bank, denoted by $\mathcal{W}^{sh}$ and $\mathcal{W}^{cr}$, the sum between their respective (received) cash flows and the value process of their cash flows.* ∎

Proposition 4.1 The shareholder and creditor wealth transfer processes are the martingales

$$\mathcal{W}^{sh} = -(\mathcal{L}^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0) + \mathrm{KVA}^{sh} \text{ and} \tag{4.2}$$

$$\mathcal{W}^{cr} = {}^{\tau-}(-\mathcal{L}) + \mathrm{CL} + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-} + \mathrm{KVA}^{cr}, \tag{4.3}$$

which start from the respective values KVA_0^{sh} and $\mathrm{CL}_0 + \mathrm{KVA}_0^{cr}$ at time 0.

Proof. Shareholder cash flows are given by $\mathcal{D}$ in (2.19), where $\mathcal{L}^{\tau-}$ is zero-valued (as a martingale starting from zero) and $\mathrm{KVA}_0 - \mathrm{KVA}^{\tau-}$ is valued by KVA^{sh} (by definition of the latter in Table 4.1). Creditor cash flows are given by ${}^{\tau-}(-\mathcal{L}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}$, where ${}^{\tau-}(-\mathcal{L})$ is valued by CL, by Lemma 4.1(i), and $\mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}$ by KVA^{cr} (by definition of the latter in Table 4.1). ∎

Should the shareholders decide to put the bank in default at time 0 right after the portfolio has been set up, they should not make any profit or loss, otherwise this would be an arbitrage for the shareholders. The fact that the shareholder wealth process $\mathcal{W}^{sh}$ starts from $\mathrm{KVA}_0^{sh} > 0$ (positive initial wealth transfer to shareholders, unless the KVA vanishes) suggests that our XVA framework may entail shareholder arbitrage. Yet, given the rules of default settlement stated in Section 1.1, upon bank default, the residual value on the (reserve capital and) risk margin account of the bank goes to creditors. So, even though KVA_0^{sh} is part of the wealth of the shareholders, the shareholders would not monetize KVA_0^{sh} by putting the bank in default at time 0 right after the portfolio has been set up. Hence the positive initial wealth transfer to shareholders does not entail such a shareholder arbitrage.

Likewise, the fact that the creditor wealth transfer martingale $\mathcal{W}^{cr}$ starts from $\mathrm{CL}_0 + \mathrm{KVA}_0^{cr}$ suggests that the derivative trading of the bank may entail a riskless profit to creditors. However, the scope of the model does not include the liquidation costs of the bank. For the creditors to monetize the wealth transfer triggered to them by the derivative portfolio of the bank, the bank has to default and there is a substantial cost associated to that to creditors.

4.1 What-If Analysis

In this section we examine the consequences of an assumption that the bank could hedge out its risks completely, including default risk. As explained in Section 2.3, this

assumption is counterfactual, for both practical and legal reasons. However, we endorse it here for the sake of the argument.

We recall from Lemma 4.1(i) that the value process of $(-\mathcal{L})$ is CL. To "complete the market", we now assume that the bank has access to a new deal with a client, insuring the payment of a cash flow stream $\mathcal{L}$ to the bank, along with a time 0 premium CL_0 (initiating the strategy at time 0 where the portfolio is settled). The deal would be fully collateralized, in the sense that the above cash flows would occur independently of the default status of the bank and its client.

Proposition 4.2 *Assume that the bank has access to the new deal as defined above, coming on top of the derivative portfolio of the bank and its dynamic hedge considered earlier in the paper. Then*

$$\mathrm{MtM} - \mathrm{FV} = \mathrm{MtM} - (\mathrm{CVA} - \mathrm{DVA}) = \mathrm{MtM} - (\mathrm{CA} - \mathrm{CL})$$

(by Lemma 4.1(ii)) is a replication price process for the derivative portfolio of the ban: the resulting loss process of the bank (impact of all the hedges included) vanishes, as do in turn the economic capital of the bank, its KVA, and the shareholder and creditor wealth transfer processes.

Proof. Here is the corresponding replication strategy, starting from time 0 where the portfolio is settled. The trading and CVA desks act exactly as before. The FVA desk passes to the client (at time 0) and to the bank shareholders (on $(0, \tau)$) a diminished add-on $\mathrm{FVA} - \mathrm{CL}$, instead of FVA before without the new deal. In addition, a new CL desk puts the upfront premium CL_0 of the new deal on a dedicated cash account, along with a matching liability of CL_0 on the bank balance sheet. This CL account, like all the other ones within the bank, is market-to-model, i.e. reset in continuous time to the value process of the corresponding liability (see Section 1.1), namely to the value CL of the cash flow $(-\mathcal{L})$ due by the bank under the terms of the new deal. Before bank default, the resets to the CL account, which accumulate to $\mathrm{CL}_0 - \mathrm{CL}$, are passed to bank shareholders, as is the $\mathcal{L}^{\tau-}$ component from the cash flows of the new deal (which thus do not stay on the balance sheet of the bank). Finally, from time τ onward, the ${}^{\tau-}\mathcal{L}$ component of the cash flows of the new deal hedges out the ${}^{\tau-}(-\mathcal{L})$ cash flows that were previously falling into the hands of the bank creditors.

As a result, the loss of the bank starting before τ (i.e. the process "${}^{\tau-}$(loss)") vanishes, whereas before τ the loss of the bank (i.e. the process "(loss)${}^{\tau-}$") is given by

$$\begin{aligned}
&\underbrace{\mathcal{C} + \mathcal{F} + \mathrm{FV} - \mathrm{FV}_0 - (\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0) + \mathcal{H}}_{\text{modified loss of the CA desks}} \\
&\qquad + \qquad \underbrace{-\mathcal{L}^{\tau-} + \mathrm{CL} - \mathrm{CL}_0}_{\text{new hedging loss components passed to shareholders}} \\
&= \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 - (\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0) + \mathcal{H} - \mathcal{L} = 0,
\end{aligned}$$

where Lemma 4.1(ii) (and $\mathcal{L} = \mathcal{L}^{\tau-}$ before τ) was used in the first equality and (2.9) in the second one.

Shareholders bear no more trading risk, hence economic capital and the KVA vanish, as do in turn the shareholder and creditor wealth transfer processes. ∎

Remark 4.1 *Before τ, the amount available free of charge to the bank for its rehypothecable collateral funding purposes is* $\mathrm{FV} + \mathrm{CL} = \mathrm{CA}$ *as before. Hence the risky funding cash flows for collateral, $\mathcal{F}$, are not modified by the new deal. The cash flows $\mathcal{C}$ are not affected by the new deal either.*

Due to the new deal, the creditors are left without any resource to adress the liquidation costs of the bank. The clients pay $(\mathrm{CA}_0 - \mathrm{CL}_0) - \mathrm{MtM}_0$ instead of $(\mathrm{CA}_0 + \mathrm{KVA}_0) - \mathrm{MtM}_0$ before, hence they are better off by the amount $\mathrm{CL}_0 + \mathrm{KVA}_0$. As before, the bank shareholders are indifferent to the portfolio at the accordingly adjusted price paid by the bank. But this price is now $\mathrm{MtM}_0 - (\mathrm{CA}_0 - \mathrm{CL}_0)$, instead of $\mathrm{MtM}_0 - (\mathrm{CA}_0 + \mathrm{KVA}_0)$ before.

Remark 4.2 *The reason why the new deal may alter so deeply the XVA picture is that this deal is different in nature from the contracts eligible of the new deal to the bank derivative portfolio or its dynamic hedge earlier in the paper. In fact, as $\mathcal{L}^{\tau-}$ is a martingale, the only case where the value process* $\mathrm{CL} = \mathrm{DVA} + \mathrm{FVA}$ *of $(-\mathcal{L})$ could also be its shareholder value (as required for eligibility to the derivative portfolio or its dynamic hedge earlier in the paper, cf. the combined requirement of* (2.11) *and Assumption 2.2) would be when*

$$\mathrm{CL}_t = \mathbb{E}_t\big(- \mathcal{L}^{\tau-}_{\tau-} + \mathcal{L}^{\tau-}_t + \mathrm{CL}_{\tau-}\big) = \mathbb{E}_t \mathrm{CL}_{\tau-},\ t < \tau, \tag{4.4}$$

e.g. if $\mathrm{CL} = \mathrm{DVA} + \mathrm{FVA} = 0$ *(and in this case only, under the reduction of filtration setup of Section 3.1, by Theorem 3.1(i)). Hence the new deal cannot be seen as part of the derivative portfolio of the bank or its dynamic hedge earlier in the paper.*

Instead, one may interpret the trading loss

$$-\mathcal{L} + \mathrm{CL} - \mathrm{CL}_0$$

triggered by the new deal as an additional line of risky funding cash flows, coming on top of the risky funding cash flows $\mathcal{F}$ earlier in the paper. As opposed to $\mathcal{F}$, this new line of risky funding does not satisfy (3.13) *for some nonnegative coefficient f, and it entails a loss of the bank at its own default, hence a benefit prior to τ, which is how the new deal allows the shareholders to monetize the default of the bank.*

In any case, again, the new deal is unfeasible for the bank. In particular, a risky funding benefit of the bank at its own default means that the bank is effectively selling default protection on itself, which is even illegal (cf. the condition concluding Assumption 3.2).

Even if the new deal is unfeasible, the what-if analysis of Proposition 4.2 is enlightening on the nature of XVAs:

Corollary 4.1 *The shareholder and creditor wealth transfers* (4.2) *and* (4.3) *can be interpreted as the wealth transferred to them by the trading of the bank,* due to the inability of the bank to hedge, in particular, jump-to-default risk. ∎

Proof. Without the new deal, the wealth transfers to shareholders and creditors are given by (4.2) and (4.3), whereas Proposition 4.2 shows that these wealth transfers would vanish if the bank had access to the new deal. ∎

5 Discussion

We revisit, in the light of the requirements of Section 1, our cost-of-capital XVA solution to the sustainable pricing and dividend release policy problem.

5.1 Regulatorily Admissible and Sustainable

As stated in Proposition 2.1, shareholder trading gains and KVA risk margin payments result in a $-(\mathcal{L}^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0)$ dividend stream, turning the shareholder *equity* process into a submartingale with growth rate h on their capital at risk (yet the shareholder *wealth* process is a martingale and the setup is nonarbitrable in the sense explained after Proposition 4.1). This holds even in the case of a portfolio held on a run-off basis, i.e. without the need to enter new deals for generating new profits. This feature addresses the sustainability requirement in Section 1. Moreover, Albanese et al. (2021, Section 4.2) shows that the sustainability property of Proposition 2.1 is still valid in the (realistic) case of a trade incremental portfolio, provided the there-defined trade incremental XVA policy is applied at every new deal.

Our setup crucially includes the default time τ of the bank itself, which is the essence of the contra-liabilities wealth transfer issue detailed in Section 4. However, accounting for all wealth transfers involved (cf. Proposition 4.1), we end up with portfolio-wide nonnegative[20] and ultimately unilateral CVA, FVA, and KVA, which price the related (nondecreasing) cash flows until the final horizon T of the XVA problem (cf. (3.10)–(3.12)), as opposed to $\tau \wedge T$. This makes our approach naturally in line with the monotonicity requirement of Section 1, that capital at risk and reserve capital should not decrease simply because the credit risk of the bank has worsened.

5.2 Economically Credible and Logically Consistent

Whereas counterparty jump-to-default risk risk can fundamentally not be hedged, a large part of the XVA literature relies on a replication paradigm. However, as established in Proposition 4.2, in a theoretical, complete counterparty risk market, the all-inclusive XVA formula would simply be $\mathrm{CVA} - \mathrm{DVA}$, instead of $\mathrm{CVA} + \mathrm{FVA} + \mathrm{KVA}$ when market incompleteness is accounted for.

In particular, the Burgard and Kjaer (2011, 2013, 2017) FVA approach was pioneering, but it breaches several of the requirements stated in Section 1, namely: nonnegativity (with an FVA that may become negative in the limiting case of a deeply out-of-the-money portfolio), monotonicity (with CVA and KVA tending to 0 when the default risk of the bank goes to infinity), and economic realism (which is lacking to an "XVA replication paradigm"). Likewise, the Green et al. (2014) KVA approach was pioneering but it breaches monotonicity, economic realism, and (see below) minimality. The XVA metrics of Bichuch et al. (2018) also breach the nonnegativity and monotonicity requirements.

With respect to the so called XVA replication framework, our cost-of-capital XVA approach results in materially different XVA formulas and balance sheet implications. In particular:

[20] Essentially, cf. Remark 2.2.

- Despite the fact that we include the default of the bank itself in our modeling, our (portfolio-wide) XVA metrics are, ultimately, unilateral (hence do not tend to decrease simply because the credit risk of the bank has worsened), and they are nonnegative (portfolio-wide);

- Our KVA is loss-absorbing: by contrast with the KVA of Green et al. (2014), it does not belong to the loss process $\mathcal{L}$ of the bank (it is not a liability like the CVA and the FVA, it would make no sense to try and replicate it);

- As a consequence of the previous point, our KVA discounts future capital at risk projections at the hurdle rate h (cf. (3.23)-(3.25)), which makes a big difference at the very long time horizon T (such as 50 years) of XVA computations.

In addition, instead of working with economic capital, Green et al. (2014) use approximations in the form of scriptural regulatory capital specifications. This is meant for simplicity but it is less satisfying economically. It is also less self-consistent: under the cost-of-capital, economic capital based, XVA approach, counterparty-risk-free valuations flow into CVA computations, which in turn flow into FVA computations, which all flow into KVA computations. These connections make the counterparty-risk-free valuation, $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$, and KVA equations, thus the derivative pricing problem as a whole, a self-contained and self-consistent problem.

5.3 Numerically Feasible and Robust

These connections can also be exploited for numerical purposes. Albanese et al. (2021, Section 5) and Abbas-Turki et al. (2021) present numerical applications on realistically large bilateral trade portfolios, based on neural net regression computational strategies. See also Albanese et al. (2017, Section 5) and Abbas-Turki et al. (2018) for an alternative, nested Monte Carlo computational strategy. These papers demonstrate the numerical feasibility and scalability of the cost-of-capital XVA approach.

The model risk inherent to XVA computations in general, and to economic capital based KVA computations more specifically, can be addressed by a Bayesian variant of our baseline cost-of-capital XVA approach. This is achieved by combining, in a global simulation, paths of the risk factors obtained in several "good" models, all econometrically realistic and calibrated to the market in counterparty-risk-free valuation terms (cf. Albanese, Crépey, and Iabichino (2021)). Drawing scenarios equally from each makes tails more leptokurtotic and risk measures greater as they are when one picks just a single (even good) model. The difference between the resulting enhanced KVA and a baseline, reference KVA, can be used as a reserve against model risk.

5.4 Minimal

An FVA for rehypothecable collateral computed at the level of a unique funding set, as in the first term of (3.19) (detailed further in the follow-up paper as the first line of Crépey (2022, (5.30))), avoids the over-conservatism of FVAs for rehypothecable collateral sometimes calculated for simplicity by netting set and aggregated. Indeed, such simplification misses the FVA markdown corresponding to the rehypothecability

of (eligible) collateral across the different netting sets of the bank. One should also account for the further FVA markdown due to the possibility for a bank to use its capital at risk as variation margin, which is done in Crépey et al. (2020).

Our KVA is minimal in the sense of Corollary 3.4. An even cheaper KVA as in (Albanese et al., 2017, Proposition 4.2(v)) results from the following variation on our approach in this paper: Upon bank default, notwithstanding the bankruptcy rules recalled in the last paragraph of Section 1.1, the residual risk margin flows back into shareholder equity instead of going to creditors. Likewise, a cheaper FVA as in (Albanese et al., 2017, Proposition 4.2(i)) follows from asserting that, upon bank default, the residual reserve capital of the FVA desk flows back into shareholder equity instead of going to creditors. However, these violations of the usual bankruptcy rules induce shareholder arbitrage, in the sense of a riskless profit strategy consisting for shareholders in letting the bank default instantaneously at time 0, right after the client portfolio has been setup and the corresponding reserve capital and risk margin amounts have been sourced from the clients. By contrast, as explained after Proposition 4.1, our approach in this paper excludes such arbitrage opportunities.

Hence, "local departures" from our cost-of-capital XVA solution to the sustainable pricing and dividends problem of Section 1 may be a bit cheaper, but they are less self-consistent. As seen in Section 5.2, more radically different approaches to the problem suffer from severe shortcomings with respect to the requirements of Section 1. In an intuitive formulation, we conclude that the cost of capital XVA solution to the sustainable pricing and dividends problem may not be the only solution, nor is it necessarily "globally minimal", but it has some "locally minimizing properties, at least in certain directions of the search space", and we are not aware of any other "distant solution".

Conclusion: From Replication to Balance Sheet Optimization

The KVA formula (3.23), where $\max(\mathrm{EC}', \mathrm{KVA}') = \mathrm{CR}'$ represents the capital at risk, appears as a continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology: See Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88). More broadly, as detailed in Table 5.1, the cost-of-capital XVA approach can be seen as an investment banking, genuinely dynamic and continuous-time version of the Solvency II insurance methodology, driven by the same motivation for a sustainable (financial or insurance) system and economy.

The KVA formula (3.23) can be used either in the direct mode, for computing the KVA corresponding to a given target hurdle rate h set by the management of the bank, or in the reverse-engineering mode, like the Black–Scholes model with volatility, for defining the implied hurdle rate associated with the actual amount on the risk margin account of the bank (see after Corollary 3.4). Cost of capital proxies have always been used to estimate return-on-equity. Whether it is used in the direct or in the implied mode, the KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base concept is far older than even the CVA.

In the current state of the market, even when they are computed, the KVA and

contra-assets CA = **CVA** + **FVA**	**liabilities** best estimate, also called market consistent valuation
priced by conditional expectation of related future cash flows	priced by conditional expectation of related future cash flows
economic capital EC	solvency capital requirement
sized as a conditional expected shortfall of future losses over one year	sized as unconditional expected shortfall of future losses over each successive year
capital valuation adjustment **KVA**	market value margin or **risk margin**
sized as a supermartingale with drift coeff. hSCR and zero terminal condition	sized as summed future (deterministic) hSCR

Table 5.1: *Left*: Cost-of-capital XVA banking approach; *Right*: Solvency II insurance methodology (with $\mathrm{SCR} = (\mathrm{EC} - \mathrm{KVA})^+$ for shareholder capital at risk everywhere).

even the MVA (which is included in the FVA in this paper, see Remark 3.7) are not necessarily passed into entry prices. But they are strategically used for collateral and capital optimization purposes. This reflects a switch of paradigm in derivative management, from replication to balance sheet optimization.

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