When Capital is a Funding Source: The Anticipated Backward Stochastic Differential Equations of X-Value Adjustments

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January 21, 2020

Abstract

XVAs refer to various financial derivative pricing adjustments accounting for counterparty risk (CVA) and its funding (FVA) and capital (KVA) implications for a bank. In this paper we show that the XVA equations are well posed, including in the realistic case where capital is deemed fungible as a source of funding for variation margin. This intertwining of capital at risk and the FVA, added to the fact that the KVA is part of capital at risk, lead to a system of backward SDEs of the McKean type (anticipated BSDEs) for the FVA and the KVA, with coefficients entailing a conditional risk measure of the one-year-ahead increment of the martingale part of the FVA. This is first considered in the case of a hypothetical bank without debt. In the practical case of a defaultable bank, the resulting anticipated BSDEs, which are stopped before the default of the bank, are solved likewise after reduction to a reference market filtration.

Keywords: Credit valuation adjustment (CVA), funding valuation adjustment (FVA), capital valuation adjustment (KVA), anticipated (or McKean) BSDE, progressive enlargement of filtration, invariance time.

Mathematics Subject Classification: 91B30, 91G20, 91G40, 60G07, 60G44.

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Acknowledgement: We are thankful to Romuald Élie for useful discussions. The research of Stéphane Crépey has been conducted with the support of the Research Initiative "Modélisation des marchés actions, obligations et dérivés" financed by HSBC France under the aegis of the Europlace Institute of Finance. The views and opinions expressed in this paper are those of the authors alone and do not necessarily reflect the views or policies of HSBC Investment Bank, its subsidiaries or affiliates. The research of Wissal Sabbagh has been conducted with the support of the Fondation du Risque, program "XVA analysis and applications," under the aegis of Louis Bachelier laboratory and of the ANR project 11-LABX-0019.

1 Introduction

XVAs, where VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, D for debt, F for funding, M for margin, or K for capital, denote various pricing adjustments applied to financial derivatives since the 2008 crisis, in order to account for counterparty risk and its capital and funding implications for a bank.

The valuation of securities in a defaultable environment has started with the pricing of risky bonds or credit default swaps (CDS). These only involve a single default time, the one of the issuer of the bond or of the reference firm in a CDS: See Pykhtin (2005) for a collection of early CVA papers. But the 2008 crisis clearly showed that banks are themselves risky. As a consequence, new regulations have been established on the functioning of financial institutions (see Basel Committee on Banking Supervision (2011)), with a great impact on capital, margin, and clearing requirements for investment banks, making it vital for them to understand how they should operate in this new environment.

The question has first been studied in a setup where the default risk of the bank and a counterparty are treated symmetrically, in a CVA/DVA mindset (see e.g. Brigo and Capponi (2010) and cf. previously Duffie and Huang (1996) or Bielecki and Rutkowski (2002, (14.25) p. 448)). However, the view of the industry evolved quite dramatically, banks reacting to the Basel III regulatory changes by pricing further the FVA, the MVA, and the KVA, which are the respective costs of funding the so called variation margin (VM), initial margin (IM), and of capital: see Brigo and Pallavicini (2014), Bichuch, Capponi, and Sturm (2018), and Burgard and Kjaer (2011, 2013, 2017) (without KVA) or, with a KVA meant as a liability like the CVA and the FVA (as opposed to a risk premium in our case), Green, Kenyon, and Dennis (2014) and Elouerkhaoui (2016). See also Andersen, Duffie, and Song (2019) and Bielecki and Rutkowski (2015) for different focuses on the funding side of the problem, with respective emphases on the related wealth transfer and nonlinear arbitrage issues.

The main dividing line in this literature is between an XVA replication approach and a cost-of-capital, incomplete market approach. Following up on the Hull and White (2012) prompted FVA debate, Albanese and Andersen (2015), Albanese, Caenazzo, and Crépey (2016), and Albanese and Crépey (2019) have delineated the specific implications of the default risk of the bank itself, providing a better insight on counterparty risk considered not only in the strict sense, but also through its consequences in terms of capital and funding. A key point in Albanese and Crépey (2019) is that, in order to account for the defaultability of the bank itself, all the cumulative cash flow and value processes must be stopped before the bank default time τ in the XVA equations. Indeed, given the impossibility for bank shareholders to hedge these cash flows and monetize them before τ, later cash flows only benefit bank bondholders.

In Albanese and Crépey (2019, Section 5) (see their Remark 4.3), the XVA equations are studied in the base case where capital at risk is not used by the bank for its funding purposes. Then, in the context of partially or uncollateralized transactions, the FVA can seem very large. For instance, in January 2014, JP Morgan has recorded a \$1.5 billion FVA loss[1]. However, in practice, capital at risk can be used for funding the so called variation margin (cf. the parameter ϕ representing "the fraction of capital used for funding" in Green, Kenyon, and Dennis (2014)). This may cause a material FVA reduction, as high as one half or more on a real banking portfolio, as demonstrated numerically in Albanese, Caenazzo, and Crépey (2017, Section 5.2) (see also Albanese, Crépey, Hoskinson, and Saadeddine (2019, Section 5.2)).

In this paper, we provide a mathematical analysis of the FVA and its KVA implications when capital at risk is a possible funding source. Our results thus complement the XVA analysis of Albanese and Crépey (2019, Section 5), which had been made under the assumption that capital at risk is not used for funding purposes, as well as the numerical studies of Albanese, Caenazzo, and Crépey (2017, Section 5.2) and Albanese, Crépey, Hoskinson, and Saadeddine (2019, Section 5), which are based on the equations of the present paper.

The fungibility of capital at risk as a source of funding for variation margin leads to a system of anticipated backward stochastic differential equations (ABSDEs, or BSDEs "of the McKean type"). Peng and Yang (2009) have introduced ABSDEs in relation with a problem of SDEs with delay. They established the well-posedness of a multivariate ABSDE on a fixed time horizon with Lipschitz coefficients in a Brownian setup. As usual, in the univariate case, the Lipschitz condition can be relaxed into continuous coefficients with linear growth, which was done in Elliot and Yang (2013). In parallel to our work, an ABSDE involving a conditional expected shortfall as anticipated term (by contrast with a conditional expectation in the previous ABSDE literature) has been considered in Agarwal et al. (2019) to study the cost of initial margin, in a univariate, Brownian, and Lipschitz setup, over a fixed time horizon. In this paper, we solve a system of ABSDEs with jumps (because of the counterparty defaults), monotone coefficients (this is in fact for the sake of generality: our XVA coefficients are even Lipschitz), a more general anticipated dependence of the coefficient (on the integrand components of the solution), and stopped before a random time (the default time of the bank itself, in the XVA context).

The ***outline*** of the paper is as follows. Section 2 revisits the XVA equations from Albanese and Crépey (2019) when capital is deemed fungible as a source of funding for variation margin, first assuming the bank default-free. Section 3 establishes a general ABSDE well-posedness result, which is applied in Section 4 to show the well-posedness of the XVA ABSDEs in the theoretical case of a default-free bank. Section 5, which is of independent interest, extends beyond the basic immersion setup the classical credit risk intensity pricing formulas (see e.g. Bielecki, Jeanblanc, and Rutkowski (2009, Chapter 3)). This framework is used in Section 6 for stating and solving the XVA ABSDEs in the realistic case of a defaultable bank. Section 7 wraps up the paper.

The ***main contributions*** of the paper are Theorems 3.1 (an extension with jumps, monotone coefficient, and a more general anticipated dependence, of the ABSDE result of Peng and Yang (2009)), 4.1 (well-posedness of the XVA ABSDEs in the

[1] see https://www.risk.net/derivatives/2322843/jp-morgan-takes-15-billion-fva-loss.

theoretical case of a default-free bank), 5.1 (extension to the invariance time setup of the classical credit risk intensity pricing formulas), and 6.1 (well-posedness of the XVA ABSDEs in the realistic case of a defaultable bank).

1.1 Mathematical Setup

We denote by

- $\cdot^{\mathsf{T}}$, vector or matrix transpose;
- $|\cdot|$ and $\langle\cdot,\cdot\rangle$, Euclidean norms and scalar products in the dimension of their arguments (vectors or vectorized matrices);
- $(\Omega, \mathfrak{A}, \mathbb{F}, \mathbb{P})$, a filtered probability space, for a complete and right continuous filtration $\mathbb{F} = (\mathfrak{F}_t)_{t\geq 0}$ of sub-σ fields of a reference σ field $\mathfrak{A}$ and for a probability measure $\mathbb{P}$;
- $\mathbb{E}$ and $\mathbb{E}_t$, the $\mathbb{P}$ expectation and the $(\mathfrak{F}_t, \mathbb{P})$ conditional expectation;
- $\mathfrak{P}$, the $\mathbb{F}$ predictable sigma-field;
- $\mathfrak{B}(E)$, the Borel σ field on any metrizable space E;
- $m(S)$, the $(\mathbb{F}, \mathbb{P})$ canonical Doob–Meyer local martingale component of an $(\mathbb{F}, \mathbb{P})$ special semimartingale S (with $m(S_0) = S_0$);
- C, a positive constant, the value of which may change from line to line.

Stochastic integrals of $\mathfrak{P}$ measurable ($\mathbb{F}$ predictable) processes against semimartingales and stochastic integrals of $\widehat{\mathfrak{P}}$ measurable random functions with respect to jump measures or their compensations are defined as in Jacod (1979). Stochastic integrals are sometimes written in $\centerdot$ notation, using the precedence convention $KL \centerdot X = (KL) \centerdot X$.

As can be classically established by section theorem, for any progressive process (Lebesgue integrand) X such that the predictable projection ${}^pX$ exists,[2] the indistinguishable equality $\int_0^\cdot {}^pX_s ds = \int_0^\cdot X_s ds$ holds. As a consequence, we only consider predictable Lebesgue integrands (even if this means replacing X by ${}^pX$).

We denote by B an $(\mathbb{F}, \mathbb{P})$ standard d variate Brownian motion, for some nonnegative integer d. We denote by E an Euclidean space, by π a σ finite measure on $(E, \mathfrak{B}(E))$ such that $\displaystyle\int_E (1 \wedge |e|^2)\pi(de) < \infty$, and we write $\widehat{\mathfrak{P}} = \mathfrak{P} \otimes \mathfrak{B}(E)$. We consider an $\mathbb{F}$ optional integer valued random measure $j(dt, de)$ on $\mathbb{R}_+ \times E$, with $\mathbb{P}$ compensator $\eta(t, e)\pi(de)dt$ and compensated martingale measure $M(dt, de)$, for some nonnegative and bounded $\widehat{\mathfrak{P}}$ measurable random function η.

Given a positive integer l, we introduce

[2] For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

- $\mathcal{S}_2^l$, the space of $\mathbb{R}^l$ valued $\mathbb{F}$ adapted càdlàg processes Y such that

$$\|Y\|^2_{\mathcal{S}_2^l} = \mathbb{E}\big[\sup_{0\le t\le T} |Y_t|^2\big] < +\infty, \tag{1}$$

  and $\mathcal{M}_2^l$, the space of martingales (componentwise) in $\mathcal{S}_2^l$;

- $\mathcal{H}_2^l$, the space of $\mathbb{R}^{l\otimes d}$ valued $\mathbb{F}$ predictable processes Z such that

$$\|Z\|^2_{\mathcal{H}_2^l} = \mathbb{E}\big[\int_0^T |Z_t|^2 dt\big] < +\infty;$$

- $\mathcal{L}_0^l$, the space of l-variate $\mathfrak{B}(E)$ measurable functions endowed with the topology of convergence in measure induced by π and we write, for any time t and $u \in \mathcal{L}_0^l$,

$$|u|_t = \Big(\int_E |u(e)|^2 \eta(t,e)\pi(de)\Big)^{1/2};$$

- $\widehat{\mathcal{H}}_2^l$, the space of l-variate $\widehat{\mathfrak{P}}$ measurable random functions U such that

$$\|U\|^2_{\widehat{\mathcal{H}}_2^l} = \mathbb{E}\Big[\int_0^T \int_E |U_t(e)|^2 \eta(t,e)\pi(de)dt\Big] = \mathbb{E}\Big[\int_0^T |U_t|_t^2 dt\Big] < \infty.$$

In the case where $l = 1$ we drop the index l, e.g. we write $\mathcal{S}_2$ instead of $\mathcal{S}_2^1$. We introduce likewise the space $\mathcal{H}_1$ of real valued $\mathbb{F}$ predictable processes X such that

$$\|X\|_{\mathcal{H}_1} = \mathbb{E}\big[\int_0^T |X_t| dt\big] < +\infty. \tag{2}$$

We assume that every $(\mathbb{F},\mathbb{P})$ square integrable martingale null at time 0 has a representation of the form

$$\int_0^t Z_s dB_s + \int_0^t \int_E U_s(e) M(ds,de),\ 0 \le t \le T, \tag{3}$$

for suitable integrands $Z \in \mathcal{H}_2$ and $U \in \widehat{\mathcal{H}}_2$, uniquely defined modulo $d[B,B]$ and $\eta(t,e)\pi(de)dt$ negligible sets, respectively. The left and right terms in (3) represent the corresponding vector and random measure stochastic integrals.

Given this square integrable martingale representation assumption, one can readily check that all the results in Kruse and Popier (2016, Sect. 4),[3] derived there in the case of a Poisson random measure, are valid in our setup.

[3] At least, the part of their results derived under square integrable assumptions, including their Theorem 1, which we use in the paper.

2 The XVA Equations in the Case of a Default-Free Bank

Let there be given

- r, an $\mathbb{F}$ progressive risk-free interest rate process, with related risk-free discount factor $\beta = e^{-\int_0^\cdot r_t dt}$;
- λ, an $\mathbb{F}$ progressive bank funding (borrowing) spread process,

both assumed bounded (possibly after truncation by a large positive constant of standard interest rate or credit spread models).

We consider a derivative portfolio with final maturity T between a bank and a client. In order to focus on counterparty risk and XVA analysis, we assume that the bank has setup a perfectly collateralized market hedge of its client portfolio (see Section 3.2 in Albanese and Crépey (2019) for a more detailed description). Hence only the counterparty risk related cash flows remain.

In this section, we assume that only the client is default prone (the defaultability of the bank itself will be added in Sect. 6). The probability measure $\mathbb{P}$ is then interpreted as a risk-neutral pricing measure, calibrated to prices of liquid derivatives as standard. In the context of a cost-of-capital XVA approach, the historical probability measure required for capital at risk computations is then taken equal to $\mathbb{P}$ (see Remark 3.5 in Albanese and Crépey (2019)), leaving the discrepancy between the two to model risk.

Remark 2.1 In the hypothetical case of a default-free bank, the bank borrowing spread λ is interpreted as a liquidity spread. However, in reality, banks are defaultable and λ is, essentially, a credit spread process (liquidity spreads are typically in the order of a handful of basis points while banks funding spreads can run into the hundreds of basis points). ∎

Collateral means cash or liquid assets that are posted to guarantee a netted set of transactions against defaults of the counterparties. Collateral may include variation margin (VM), which tracks the mark-to-market of the client portfolio and is typically rehypothecable, and initial margin (IM) set as a cushion against gap risk, i.e. the risk of slippage between the mark-to-market of the portfolio and its variation margin during the liquidation period that follows a default. Moreover, by regulation, the bank needs to earmark capital at risk (CR ≥ 0) devoted to cope with unexpected losses. For simplicity, we assume cash only collateral and capital at risk. Posted collateral is remunerated at the risk-free rate r by the receiving party.

As explained Section 3.2 in Albanese et al. (2017), the bank can use reserve capital and capital at risk as variation margin, but not as initial margin. As CR can only be used for funding variation margin and because this feature is our main funding focus in this paper, we assume no initial margin in the sequel. Initial margin can be added without harm as done in Albanese et al. (2019, Section A) (see also Albanese et al. (2017)).

Remark 2.2 Bilateral transactions under SIMM, which is the initial margin standard for non-cleared (vanilla) derivatives, are subject to even higher levels of initial margining than centrally cleared transactions. For such transactions, the MVA issue dominates the FVA one. The FVA is dominant in the case of deals with clients subject to little or partial collateralization, so that the bank needs to borrow in order to fund the variation margin required on its hedge. ∎

We denote by P the difference between the variation margin posted by the bank on its market hedge and the variation margin received by the bank on its client portfolio. The market exposure of the bank to the default of its client is modeled as $Q_t\boldsymbol{\delta}(dt)$, where $\boldsymbol{\delta}$ is a Dirac measure at the (positive) default time of the client and where Q is some $\mathbb{F}$ optional nonnegative loss process of the bank given the client default.

Example 2.1 Let MtM (for mark-to-market) denote the value process of the client portfolio ignoring counterparty risk and risky funding costs, i.e. the conditional expectation of the future contractual cash flows D promised by the client to the bank, discounted at the risk-free rate. In line with our assumption of a perfectly collateralized market hedge of its client portfolio by the bank, the bank posts MtM as variation margin on its hedge. Let VM denote the variation margin exchanged between the client and the bank, counted positively when received by the bank. Let R denote the recovery rate of the client. Then, assuming instantaneous liquidation of the bank portfolio in case the client defaults (and no initial margins), we have[4]

$$\begin{aligned} P &= (\mathrm{MtM} - \mathrm{VM}) \text{ killed at the default time of the client,} \\ Q &= (1-R)\big(\mathrm{MtM} + D - D_- - \mathrm{VM}\big)^+. \end{aligned}$$

The jump $(D - D_-)$ of the contractual cash flows contributes to the exposure Q of the bank to the default of its client, consistent with the fact that it fails to be paid by the client if the latter defaults. ∎

Remark 2.3 The XVA setup of this paper can be readily extended to a bank engaged into bilateral trade portfolios with several clients, as considered in Albanese et al. (2017) and Albanese et al. (2019, Section A), by summing the $Q\boldsymbol{\delta}$ and P processes over the different clients of the bank in all equations. ∎

All our XVA processes are a priori sought for as semimartingales (meant componentwise in the multivariate case), which will all happen to be nonnegative special semimartingales.

2.1 Contra-Assets Valuation

We assume that the process λP^+ is in $\mathcal{H}_1$ and that $\int_0^T Q_s\boldsymbol{\delta}(ds)$ is $\mathbb{P}$ integrable. We define

$$\mathrm{CVA}_t = \mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s Q_s\boldsymbol{\delta}(ds)\Big],\ t \in [0,T], \tag{4}$$

[4] See Lemmas 5.2–5.3 in Albanese and Crépey (2019) for detailed derivations.

which is a nonnegative special semimartingale, like $\int_0^{\cdot} \beta_t^{-1}\beta_s Q_s \boldsymbol{\delta}(ds)$ (cf. He et al. (1992, Corollary 11.26)). We define the local martingale

$$\mu = m(\mathrm{CVA}) + m(Q \centerdot \boldsymbol{\delta}). \tag{5}$$

Hence, for $t \in [0, T]$,

$$\mathrm{CVA}_0 + \beta \centerdot \mu_t = m(\beta\mathrm{CVA}) + m(\beta Q \centerdot \boldsymbol{\delta}) = \beta\mathrm{CVA} + \beta Q \centerdot \boldsymbol{\delta} = \mathbb{E}_t\Big[\int_0^T \beta_s Q_s \boldsymbol{\delta}(ds)\Big],$$

by (4). That is, $\mu_0 = \mathrm{CVA}_0$ (recall that the client default time is positive, hence $Q \centerdot \boldsymbol{\delta}_0 = 0$) and, for $t \in [0, T]$,

$$\beta_t d\mu_t = d(\beta\mathrm{CVA})_t + \beta_t Q_t \boldsymbol{\delta}(dt),$$

i.e.

$$d\mu_t = d\mathrm{CVA}_t - r_t\mathrm{CVA}_t dt + Q_t \boldsymbol{\delta}(dt). \tag{6}$$

Remark 2.4 If $\int_0^T Q_s \boldsymbol{\delta}(ds)$ is $\mathbb{P}$ square integrable, then CVA is in $\mathcal{S}_2$, by (4), and so is μ, by (6) (having assumed r bounded). ∎

The nonnegative contra-asset value process (CA ≥ 0) corresponds to the conditionally expected future counterparty default and risky funding losses, i.e. CA $=$ CVA $+$ FVA, computed under the risk-neutral pricing measure $\mathbb{P}$. A well-grounded, refined specification of the FVA is in fact part of the objectives of the paper. The ensuing CA amount is assumed to be charged to the client at time 0 by the CVA desk and the FVA desk (or Treasury) of the bank, which put it into a reserve capital (RC) account dedicated to cope with these expected losses as time goes on. From an accounting perspective, the CVA and the FVA represent special liabilities, which arise from the feedback impact of counterparty risk on financial receivables to the bank, hence the name of contra-assets for the cash-flows valued by CA (cf. Figure 1 in Albanese et al. (2019, Section 2)).

We assume that all the losses and earnings of the bank are marked to the model and realize immediately. In particular, the RC amount is reset to its theoretical CA value at all times. Reserve capital can also be used as variation margin. Accounting for this feature and since RC $=$ CA, the variation margin borrowing needs of the bank are reduced from P^+ otherwise to $(P - \mathrm{CA})^+$.

Remark 2.5 The identity RC $=$ CA does of course not mean that reserve capital and contra-assets are one and the same thing: RC is the amount on a cash account which is part of the assets of the bank (and can, in particular, be pledged as variation margin), whereas the contra-assets valuation CA is the matching liability in the bank balance sheet (cf. again Figure 1 in Albanese et al. (2019, Section 2)). ∎

On top of reserve capital, capital at risk ($\mathrm{CR} \geq 0$) can also be used by the bank for its funding purposes (provided the bank pays to its shareholders a risk-free rate on CR that they would gain by depositing it otherwise). The funding needs of the bank are then reduced further from $(P - \mathrm{CA})^+$ to $(P - \mathrm{CA} - \mathrm{CR})^+$.

Rephrasing the above qualitative descriptions in mathematical terms, the trading loss process L of the bank satisfies

$$\begin{aligned}
&L_0 = z \text{ and, for } t \in (0, T] \\
&dL_t = \underbrace{Q_t \boldsymbol{\delta}(dt)}_{\text{loss in case of client default}} \\
&\quad + \underbrace{(r_t + \lambda_t)\big(P_t - \mathrm{CA}_t - \mathrm{CR}_t\big)^+ dt}_{\text{portfolio funding costs}} - \underbrace{r_t\big(P_t - \mathrm{CA}_t - \mathrm{CR}_t\big)^- dt}_{\text{portfolio funding benefits}} \\
&\quad + \underbrace{r_t \mathrm{CR}_t dt}_{\text{remuneration to shareholders for the CR funding resource}} \\
&\quad - \underbrace{r_t P_t dt}_{\text{remuneration of the collateral between the bank and the client}} \\
&\quad + \underbrace{d\mathrm{CA}_t}_{\text{appreciation of the contra-assets of the bank}} \\
&\quad = d\mathrm{CA}_t - r_t \mathrm{CA}_t dt + Q_t \boldsymbol{\delta}(dt) + \lambda_t\big(P_t - \mathrm{CA}_t - \mathrm{CR}_t\big)^+ dt
\end{aligned} \tag{7}$$

or, equivalently (recall $\beta = e^{-\int_0^\cdot r_t dt}$),

$$\beta_t dL_t = d(\beta_t \mathrm{CA}_t) + \beta_t Q_t \boldsymbol{\delta}(dt) + \beta_t \lambda_t \big(P_t - \mathrm{CA}_t - \mathrm{CR}_t\big)^+ dt \tag{8}$$

(and L is constant beyond T). Note that the above dynamics for L are well defined under the postulated integrability conditions on P and Q, as well as $\mathrm{CA} \geq 0, \mathrm{CR} \geq 0$. In the case where the stochastic integral $\int_0^\cdot \beta_t dL_t$ is a uniformly integrable martingale, the formula (8), together with our integrability conditions on P and Q and a terminal condition $\mathrm{CA}_T = 0$, lead to the following equation for the CA process: For $t \in [0, T]$,

$$\mathrm{CA}_t = \underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1} \beta_s Q_s \boldsymbol{\delta}(ds)\Big]}_{\mathrm{CVA}_t} + \underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1} \beta_s \lambda_s \big(P_s - \mathrm{CA}_s - \mathrm{CR}_s\big)^+ ds\Big]}_{\mathrm{FVA}_t}. \tag{9}$$

2.2 KVA

As visible in (9), capital at risk CR is FVA reducing. However, under a cost-of-capital XVA approach, before being FVA reducing as a side effect, capital at risk entails a charge for the bank, which is the cost, called KVA (capital valuation adjustment), of remunerating shareholders at some hurdle rate (dividend rate) for their capital at risk.

This cost is sourced from the client at time 0 and put into the so-called risk margin (RM) account, from where it is gradually released, as dividends, to bank shareholders.

Under our continuous reset assumption on all bank accounts, the RM amount is reset to its theoretical KVA value at all times. Moreover, under our XVA approach, the risk margin is loss absorbing (see Sections 3.6 and 6.2 in Albanese and Crépey (2019)), so that it is part of capital at risk. Hence, the inequality

$$\mathrm{KVA} \leq \mathrm{CR} \tag{10}$$

holds and *shareholder* capital at risk only corresponds to the difference (CR − KVA). Assuming a constant hurdle rate $h \geq 0$ (including a risk-free remuneration of the RM account to shareholders), this leads to the KVA equation

$$-d\mathrm{KVA}_t + r_t \mathrm{KVA}_t dt = h(\mathrm{CR}_t - \mathrm{KVA}_t)dt - d\nu_t, \quad 0 \leq t \leq T,$$

for some local martingale ν, or, equivalently[5],

$$-d(\beta_t \mathrm{KVA}_t) = h\beta_t(\mathrm{CR}_t - \mathrm{KVA}_t)dt - \beta_t d\nu_t, \quad 0 \leq t \leq T.$$

In the case where the stochastic integral $\int_0^{\cdot} \beta_t d\nu_t$ is a uniformly integrable martingale, the above formula, together with the terminal condition $\mathrm{KVA}_T = 0$ (as the portfolio expires at time T), lead to

$$0 \leq \beta_t \mathrm{KVA}_t = h\mathbb{E}_t\Big[\int_t^T \beta_s(\mathrm{CR}_s - \mathrm{KVA}_s)ds\Big], \quad 0 \leq t \leq T. \tag{11}$$

2.3 Economic Capital and Capital at Risk Specifications

Economic capital is the (minimum) level of capital at risk that a regulator would like to see on an economic, structural basis. We use the following dynamic specification of economical capital:

$$\mathbb{ES}_t\big(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s\big),\ t \in [0,T], \tag{12}$$

where L is the trading loss and profit process of the bank in (8) and where $\mathbb{ES}_t(\ell)$ denotes the $\mathfrak{F}_t$ conditional expected shortfall, at some level α (e.g. $\alpha = 97.5\%$), of an $\mathfrak{F}_T$ measurable, $\mathbb{P}$ integrable random variable ℓ. That is, denoting by $q_t^a(\ell)$ the $(\mathfrak{F}_t, \mathbb{P})$ conditional value at risk (left quantile) of level a of ℓ (cf. Artzner, Delbean, Eber, and Heath (1999)): For $t \leq T$,

$$\begin{aligned}\mathbb{ES}_t(\ell) &= (1-\alpha)^{-1}\int_\alpha^1 q_t^a(\ell)da \\ &= \inf_{x\in\mathbb{R}}\Big(x + (1-\alpha)^{-1}\mathbb{E}_t\big[(\ell - x)^+\big]\Big) \\ &= \sup\big\{\mathbb{E}_t[\ell\chi]\ ;\ \chi \text{ is } \mathfrak{F}_T \text{ measurable, } 0 \leq \chi \leq (1-\alpha)^{-1}, \text{ and } \mathbb{E}_t[\chi] = 1\big\}.\end{aligned} \tag{13}$$

[5] All the Lebesgue integrals that appear in these expressions are well defined over $[0,T]$, having assumed semimartingale XVAs.

Note that we will only deal with martingale loss and profit processes L and therefore centered loss variables ℓ, for which $\mathbb{ES}_t(\ell) \geq 0$ holds in view of the third line in (13). Moreover, for any $\mathfrak{F}_T$ measurable, $\mathbb{P}$ integrable random variables ℓ and ℓ', we have (cf. Barrera et al. (2019, Lemma 6.10, Eq. (6.20)) and its proof):

$$|\mathbb{ES}_t(\ell) - \mathbb{ES}_t(\ell')| \;\leq\; (1-\alpha)^{-1}\mathbb{E}_t[|\ell - \ell'|],\, 0 \leq t \leq T. \tag{14}$$

We will need to plug economic capital processes such as (12) into the coefficient of BSDEs. Before doing so (see e.g. (27)), let us show that any such economic capital process is the image, in the sense of (16) below, of a predictable process. Namely, for every (raw, non necessarily adapted) process Λ admitting[6] an $(\mathbb{F},\mathbb{P})$ predictable projection ${}^p\Lambda$, let

$$\mathrm{EC}(\Lambda) = \inf_{\text{rationals } k} \Big(k + (1-\alpha)^{-1}\, {}^p\big[(\Lambda - k)^+\big]\Big). \tag{15}$$

Lemma 2.1 *For every raw process Λ endowed with an $(\mathbb{F},\mathbb{P})$ predictable projection, we have, for every constant $t \geq 0$,*

$$\mathrm{EC}(\Lambda)_t = \mathbb{ES}_t(\Lambda_t). \tag{16}$$

Proof. By definition,

$$\mathrm{EC}(\Lambda)_t = \inf_{\text{rationals } k} \Big(k + (1-\alpha)^{-1}\mathbb{E}_{t-}[(\Lambda_t - k)^+]\Big).$$

By Jacod (1979, Lemma 4.48),

$$\mathfrak{F}_t = \mathfrak{F}_{t-} \vee \sigma(\Delta_t X : \text{all uniformly integrable martingale } X).$$

By the martingale representation property (3), the martingales can have no jump at the predictable stopping time t. Consequently, $\mathfrak{F}_t = \mathfrak{F}_{t-}$ and

$$\mathrm{EC}(\Lambda)_t = \inf_{\text{rationals } k} \Big(k + (1-\alpha)^{-1}\mathbb{E}_t[(\Lambda_t - k)^+]\Big) = \mathbb{ES}_t(\Lambda_t),$$

by the second line in (13). ∎

Hence, by application of Lemma 2.1 (to $\Lambda = \int_\cdot^{\cdot+1} \beta_\cdot^{-1}\beta_s dL_s$ in (16)), it is harmless to use processes such as $\mathbb{ES}_\cdot\big(\int_\cdot^{\cdot+1} \beta_\cdot^{-1}\beta_s dL_s\big)$ (cf. (12)) instead of the corresponding predictable expression $\mathrm{EC}(\int_\cdot^{\cdot+1} \beta_\cdot^{-1}\beta_s dL_s)$ in our XVA equations below (where L is a local martingale, as implied by the requirements for a solution to these XVA equations).

In view of the inequality (10) and of the risk admissibility condition

$$\mathrm{CR} \geq \mathbb{ES}_\cdot(\int_\cdot^{\cdot+1} \beta_\cdot^{-1}\beta_s dL_s)$$

(see before (12)), we adopt the following "minimal" specification of capital at risk:

$$\mathrm{CR} = \max\big(\mathbb{ES}_\cdot(\int_\cdot^{\cdot+1} \beta_\cdot^{-1}\beta_s dL_s), \mathrm{KVA}\big). \tag{17}$$

Table 1 reviews the main protagonists of the cost-of-capital XVA approach.

[6] For which Λ càdlàg is enough.

CA	Contra-assets valuation	Sect. 2.1 and (27)
CR	Capital at risk	Paragraph following Remark 2.1 and (17)
CVA	Credit valuation adjustment	First paragraph of Sect. 1 and (4)
EC	Economic capital	(12) and preceding lines
FVA	Funding valuation adjustment	First paragraph of Sect. 1 and (9), (27)
KVA	Capital valuation adjustment	Sect. 2.2 and (11), (27)
MtM	Mark-to-market	Example 2.1
RC	Reserve capital	Sect. 2.1, Remark 2.5 in particular
RM	Risk margin	Sect. 2.2
XVA	Generic "X" valuation adjustment	First paragraph of Section 1

Table 1: Main financial acronyms and place where they are introduced conceptually and/or specified mathematically in the paper, as relevant.

3 A General ABSDE Well-Posedness Result

In this section, we extend to anticipated BSDEs the (square integrable part of the) monotone coefficient BSDE results in Kruse and Popier (2016).

Let there be given a map ρ from $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ into the space of $\mathbb{F}$ predictable processes satisfying the following:

Assumption 3.1 There exists a constant c_ρ such that, for any (Y, Z, U), (Y', Z', U') in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, for any $t \in [0, T]$,

$$|\rho_t(Y, Z, U) - \rho_t(Y', Z', U')|^2 \leq \\ c_\rho^2 \mathbb{E}_t\Big[\sup_{t\leq s\leq T} |Y_s - Y'_s|^2 + \int_t^T \big(|Z_s - Z'_s|^2 + |U_s - U'_s|_s^2\big) ds\Big]. \blacksquare \tag{18}$$

Let there additionally be given an $\mathfrak{F}_T$ measurable terminal condition $\xi \in \mathbb{R}^l$ and a $\mathfrak{P} \otimes \mathfrak{B}(\mathbb{R}^l) \otimes \mathfrak{B}(\mathbb{R}^{l\otimes d}) \otimes \mathfrak{B}(\mathcal{L}_0^l) \otimes \mathfrak{B}(\mathbb{R})$ measurable coefficient f, such that:

Assumption 3.2 (i) The function $y \mapsto f(t, y, z, u, \varrho)$ is continuous. Moreover, there exists a positive constant c_m such that

$$\langle f(t, y, z, u, \varrho) - f(t, y', z, u, \varrho), y - y' \rangle \leq c_m |y - y'|^2;$$

(ii) There exists a positive constant c_f such that

$$|f(t, y, z, u, \varrho) - f(t, y, z', u', \varrho')| \leq c_f (|z - z'| + |u - u'|_t + |\varrho - \varrho'|);$$

(iii) The processes $\sup_{|y|\leq c} |f(\cdot, y, 0, 0, \rho_\cdot(0, 0, 0)) - f(\cdot, 0, 0, 0, \rho_\cdot(0, 0, 0))|$ (for every $c > 0$), as well as $|f(\cdot, 0, 0, 0, \rho_\cdot(0, 0, 0))|^2$, are in $\mathcal{H}_1$;

(iv) $\mathbb{E}\big[|\xi|^2\big] < +\infty$. ∎

We consider the following l-variate ABSDE with data ξ, f, ρ:

$$\begin{cases} \quad Y_T = \xi \text{ and, for } t \le T, \\ -dY_t = f\left(t, Y_t, Z_t, U_t, \rho_t(Y,Z,U)\right) dt - Z_t dB_t - \displaystyle\int_E U_t(e) M(dt, de), \end{cases} \tag{19}$$

to be solved for (Y, Z, U) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$.

Lemma 3.1 *If (Y, Z, U) is a solution in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ to the ABSDE* (19)*, then*

$$\|Y\|^2_{\mathcal{S}_2^l} + \|Z\|^2_{\mathcal{H}_2^l} + \|U\|^2_{\widehat{\mathcal{H}}_2^l} \le C\, \mathbb{E}\left[|\xi|^2 + \int_0^T |f(s, 0, 0, 0, \rho_\cdot(0, 0, 0))|^2 ds\right]. \tag{20}$$

Proof. See Sect. A.1. ∎

Lemma 3.2 *For any given (X, V, W) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, the BSDE*

$$\begin{cases} Y_T = \xi \text{ and, for } t \le T, \\ -\, dY_t = f\left(t, Y_t, Z_t, U_t, \rho_t(X, V, W)\right) dt - Z_t dB_t - \displaystyle\int_E U_t(e) M(dt, de) \end{cases} \tag{21}$$

has a unique solution (Y, Z, U) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$.

Proof. Given (X, V, W) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, the equation (21) for (Y, Z, U) is a monotone BSDE, which is well posed in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, by Kruse and Popier (2016, Theorem 1). ∎

Theorem 3.1 *The ABSDE* (19) *has a unique solution (Y, Z, U) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, which is the limit in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, with a geometrical convergence rate, of the Picard iteration defined by $(Y^{(0)}, Z^{(0)}, U^{(0)}) = (0, 0, 0)$ and, for $n \ge 1$,*

$$\begin{cases} Y_T^{(n)} = \xi \text{ and, for } t \le T \\ -\, dY_t^{(n)} = f\big(t, Y_t^{(n)}, Z_t^{(n)}, U_t^{(n)}, \rho_t(Y^{(n-1)}, Z^{(n-1)}, U^{(n-1)})\big) dt \\ \qquad\quad - Z_t^{(n)} dB_t - \displaystyle\int_E U_t^{(n)}(e) M(dt, de). \end{cases} \tag{22}$$

Proof. Lemma 3.2 allows defining a map Φ from $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ into itself by $\Phi(X, V, W) = (Y, Z, U)$. Classical BSDE computations detailed in Sect. A.2 show that Φ is a contraction on the space $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ (endowed with a suitably weighted norm still making it a Banach space), so that Φ has a unique fix-point (Y, Z, U) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$. Hence, the equation (19) has a solution (Y, Z, U) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, unique by Lemma 3.1.

Moreover, the sequence $(Y^{(n)}, Z^{(n)}, U^{(n)})$, is well defined, by iterated application of Lemma 3.2, and the majoration (69) implies the stated geometrical convergence of this sequence to (Y, Z, U). ∎

Remark 3.1 In the case where f is not only monotone but Lipschitz in y, then one can check likewise that the explicit Picard iteration, with $Y_t^{(n-1)}$ instead of $Y_t^{(n)}$ as second argument of f in (22), also converges in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ to (Y, Z, U). ∎

3.1 A Special Case

In this subsection, we assume that $\rho(Y,Z,U)$ only depends on (Z,U) through $\int_0^\cdot Z_s dB_s + \int_0^\cdot \int_E U_s(e) M(ds,de)$ and that f does not depend on (z,u). More precisely:

$$\rho(Y,Z,U) = \bar{\rho}\big(Y, Y_0 + \int_0^\cdot Z_s dB_s + \int_0^\cdot \int_E U_s(e) M(ds,de)\big),\ f(t,y,z,u,\varrho) = \bar{f}(t,y,\varrho),$$

where $\bar{f}$ satisfies the amended form of Assumption 3.2 obtained by replacing all missing arguments by 0 there, while $\bar{\rho}$ is a map from $\mathcal{S}_2^l \times \mathcal{M}_2^l$ into the space of $\mathbb{F}$ predictable processes, which satisfies the following:

Assumption 3.3 There exists a constant $c_{\bar{\rho}}$ such that, for any (Y,N), (Y',N') in $\mathcal{S}_2^l \times \mathcal{M}_2^l$, for any $t \in [0,T]$,

$$\begin{aligned} &|\bar{\rho}_t(Y,N) - \bar{\rho}_t(Y',N')|^2 \le \\ &\qquad c_{\bar{\rho}}^2 \mathbb{E}_t\Big[\sup_{t\le s\le T} |Y_s - Y'_s|^2 + \sum_{k=1}^l \big(\langle N^k - (N')^k\rangle_T - \langle N^k - (N')^k\rangle_t\big)\Big]. \ \blacksquare \end{aligned} \tag{23}$$

Then $|\rho_t(Y,Z,U) - \rho_t(Y',Z',U')|^2$ in (18) is rewritten as $|\bar{\rho}\big(Y,N\big) - \bar{\rho}\big(Y',N'|^2$, where

$$\begin{aligned} N &= Y_0 + \int_0^\cdot Z_s dB_s + \int_0^\cdot \int_E U_s(e) M(ds,de) \text{ and} \\ N' &= Y'_0 + \int_0^\cdot Z'_s dB_s + \int_0^\cdot \int_E U'_s(e) M(ds,de) \end{aligned}$$

are such that

$$\sum_{k=1}^l \big(\langle N^k - (N')^k\rangle_T - \langle N^k - (N')^k\rangle_t\big) = \int_t^T \big(|Z_s - Z'_s|^2 + |U_s - U'_s|_s^2\big) ds.$$

Hence (23) is rewritten as (18) (for c_ρ there equal to $c_{\bar{\rho}}$), which shows that Assumption 3.1 is satisfied. Moreover, by inspection:

1 Assumption 3.2 holds;

2 The ABSDE (19) for (Y,Z,U) in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ is equivalent, via the martingale representation property (3), to the following equation to be solved for a (special) semimartingale Y in $\mathcal{S}_2^l$ with $m(Y)$ in $\mathcal{S}_2^l$:

$$Y_t = \mathbb{E}_t\Big[\xi + \int_t^T \bar{f}\Big(s, Y_s, \bar{\rho}_s\big(Y, m(Y)\big)\Big) ds\Big],\ t \le T, \tag{24}$$

which is in turn equivalent to the following system of equations for a (special) semimartingale Y in $\mathcal{S}_2^l$ and a martingale $N(= m(Y))$ in $\mathcal{S}_2^l$:

$$\begin{aligned} &N_0 = Y_0 \text{ and, for } t \in (0,T], \\ &dN_t = dY_t - \bar{f}\,(t, Y_t, \bar{\rho}_t(Y,N))\,dt \\ &Y_t = \mathbb{E}_t\Big[\xi + \int_t^T \bar{f}\big(s, Y_s, \bar{\rho}_s(Y,N)\big) ds\Big]; \end{aligned} \tag{25}$$

3 The Picard iteration of Theorem 3.1 for (Y, Z, U) is equivalent to the following Picard iteration for $(Y, N = m(Y))$ in (25): $Y^{(0)} = N^{(0)} = 0$ and, for $n \geq 1$,

$$\begin{aligned} &Y_t^{(n)} = \mathbb{E}_t\Big[\xi + \int_t^T \bar{f}\big(s, Y_s^{(n)}, \bar{\rho}_s(Y^{(n-1)}, N^{(n-1)})\big)ds\Big],\ 0 \leq t \leq T, \\ &N_0^{(n)} = Y_0^{(n)} \text{ and, for } t \in (0, T], \\ &dN_t^{(n)} = dY_t^{(n)} - \bar{f}\big(t, Y_t^{(n)}, \bar{\rho}_t(Y^{(n-1)}, N^{(n-1)})\big)dt. \end{aligned} \tag{26}$$

4 The XVA Equations in the Case of a Default-Free Bank Are Well-Posed

4.1 The (FVA,KVA) Anticipated BSDE System

Note that L only intervenes via CR as per (17) in the FVA and KVA equations (7), (9), (11), and (17). As a consequence, using in this paragraph a superscript z in reference to the value of the initial condition z for L in (7), if $(L^z, \mathrm{FVA}^z, \mathrm{KVA}^z)$ solves the FVA and KVA equations, then, for every real u, $(L^z + u - z, \mathrm{FVA}^z, \mathrm{KVA}^z)$ solves the same equations with z replaced by u in (7). Hence, the value of the initial condition z for L is immaterial in the XVA problem.

Hereafter, in order to be in line with the convention that the martingale part of a special semimartingale S starts from S_0 (in this case we target $L = m(\mathrm{CA} + Q \bullet \boldsymbol{\delta})$), we set $z = \mathrm{CA}_0$. The FVA and KVA problem (7), (9), (11), and (17) is then rewritten as (with CA used as a shorthand for $\mathrm{CVA} + \mathrm{FVA}$, hence $L = \mu + m(\mathrm{FVA})$, by (5)):

$$\begin{aligned} &L_0 = \mathrm{CA}_0 \text{ and, for } t \in (0, T], \\ &dL_t = d\mathrm{CA}_t - r_t\mathrm{CA}_t dt + Q_t\boldsymbol{\delta}(dt) + \\ &\qquad \lambda_t\big(P_t - \mathrm{CA}_t - \max(\mathbb{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s), \mathrm{KVA}_t)\big)^+ dt, \\ &\mathrm{FVA}_t = \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CA}_s - \max\big(\mathbb{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u), \mathrm{KVA}_s\big)\Big)^+ ds, \\ &\mathrm{KVA}_t = h\mathbb{E}_t\left[\int_t^T \beta_t^{-1}\beta_s\left(\mathbb{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u) - \mathrm{KVA}_s\right)^+ ds\right]. \end{aligned} \tag{27}$$

As will be detailed in the proof of Theorem 4.1 below, the system (27) belongs formally to the class of equations considered in Sect. 3.1, in the form (25), for $Y = (\mathrm{FVA}, \mathrm{KVA})$ and its martingale part N (having noted that $L = \mu + m(\mathrm{FVA})$ in (27)).

Remark 4.1 As λ is bounded (and h is constant), the system (27) is Lipschitz in $(\beta\mathrm{FVA}, \beta\mathrm{KVA})$, irrespective of the boundedness assumption made on r. However, we prefer to look at the system in terms of $(\mathrm{FVA}, \mathrm{KVA})$, even if this is at the cost of assuming r bounded (which is not a real restriction in practice, as explained at the beginning of Sect. 2). Indeed, in Markov setups with numerical solutions in mind, working with $(\beta\mathrm{FVA}, \beta\mathrm{KVA})$ would require to introduce an additional factor process (at least mathematically) to account for the path-dependence induced by β. ∎

Definition 4.1 We call square integrable solution to (27), any special semimartingale solution (componentwise) $(\mathrm{FVA}, \mathrm{KVA})$ (with $L = \mu + m(\mathrm{FVA})$ in (27)), with components (i.e. the XVA processes themselves) and their martingale parts in $\mathcal{S}_2$. ∎

4.2 The (FVA,KVA) ABSDE System (27) is Well-Posed

Lemma 4.1 *The map defined by, for (Y,N)in $\mathcal{S}_2^2 \times \mathcal{M}_2^2$,*

$$\bar{\rho}_t(Y,N) = \mathbb{ES}_t\left[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s \left(d\mu_s + dN_s^1\right)\right], t \in [0,T] \tag{28}$$

(which only depends on (Y,N) through the first component N^1 of N), satisfies Assumption 3.3 (for $l=2$ there).

Proof. By (14), we have for any (Y,N), (Y',N') in $\mathcal{S}_2^2 \times \mathcal{M}_2^2$,

$$\begin{aligned}|\bar{\rho}_t(Y,N) - \bar{\rho}_t(Y',N')|^2 &\leq (1-\alpha)^{-2}\left(\mathbb{E}_t\Big[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\big(N_s^1 - (N')_s^1\big)\Big]\right)^2 \\ &\leq (1-\alpha)^{-2}\mathbb{E}_t\Big[\Big(\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\big(N_s^1 - (N')_s^1\big)\Big)^2\Big],\end{aligned}$$

by the (conditional) Jensen inequality. Moreover, as a local martingale in $\mathcal{S}_2$, the process $(N^1 - (N')^1)$ is a square integrable martingale. As $\beta_t^{-1}\beta_s, t \leq s \leq T$, is bounded, the process $s \mapsto \int_t^{\cdot} \beta_t^{-1}\beta_s d\big(N_s^1 - (N')_s^1\big)$ is therefore a square integrable martingale over $[t,T]$. The (conditional) Burkholder inequality applied to this process then yields

$$|\bar{\rho}_t(Y,N) - \bar{\rho}_t(Y',N')|^2 \leq (1-\alpha)^{-2}C\mathbb{E}_t\Big(\langle N^1 - (N')^1\rangle_T - \langle N^1 - (N')^1\rangle_t\Big).$$

In particular, Assumption 3.3 holds (with $l=2$ there). ∎

We introduce the following Picard iteration, where CVA and μ are given by (4) and (5): $\mathrm{FVA}^{(0)} = \mathrm{KVA}^{(0)} = 0, L^{(0)} = \mu$ and, for $n \geq 1$, $(\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)})$ given as the unique square integrable solution [7] to, with $\mathrm{CA}^{(n)}$ as a shorthand for $\mathrm{CVA} + \mathrm{FVA}^{(n)}$:

$$\begin{aligned}
\mathrm{KVA}^{(n)}_t &= h\mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\Big(\mathbb{ES}_s\big(\int_s^{s+1}\beta_s^{-1}\beta_u dL_u^{(n-1)}\big) - \mathrm{KVA}^{(n)}_s\Big)^+ ds,\ t\in[0,T],\\
\mathrm{FVA}^{(n)}_t &= \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CA}^{(n)}_s\\
&\quad - \max\Big(\mathbb{ES}_s\big(\int_s^{s+1}\beta_s^{-1}\beta_u dL_u^{(n-1)}\big), \mathrm{KVA}^{(n)}_s\Big)\Big)^+ ds,\ t\in[0,T],\\
L_0^{(n)} &= \mathrm{CA}_0^{(n)} \text{ and, for } t\in[0,T], dL_t^{(n)} = Q_t\boldsymbol{\delta}(dt) + d\mathrm{CA}_t^{(n)} - r_t\mathrm{CA}_t^{(n)}dt\\
&\quad + \lambda_t\Big(P_t - \mathrm{CA}_t^{(n)} - \max\Big(\mathbb{ES}_t\big(\int_t^{t+1}\beta_t^{-1}\beta_s dL_s^{(n-1)}\big), \mathrm{KVA}_t^{(n)}\Big)\Big)^+ dt.
\end{aligned} \tag{29}$$

Theorem 4.1 *Suppose that (r and λ are bounded,) $(P^+)^2 \in \mathcal{H}_1$ and $\int_0^T Q_s\boldsymbol{\delta}(ds)$ is $\mathbb{P}$ square integrable. Then the system (27) admits a unique square integrable solution* $(\mathrm{FVA}, \mathrm{KVA})$*, which is also the limit in $\mathcal{S}_2^2$, with a geometrical convergence rate, of the sequence* $((\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)}))_{n\in\mathbb{N}}$ *in* (29)*. Moreover,* $L = \mu + m(\mathrm{FVA}) \in \mathcal{M}_2$.

Proof**.** The system (27) is nothing but the integral form of the equation (25) with

$l = 2$, $\bar\rho$ as per (28), $\xi = (0,0)^\mathsf{T}$, and $\bar f = (\bar f_1, \bar f_2)^\mathsf{T}$ such that, for any $t, y = (y_1, y_2)^\mathsf{T}$, ϱ,

$$\begin{aligned}
\bar f_1 &= \bar f_1(t,y,\varrho) = \lambda_t\Big(P_t - \mathrm{CVA}_t - y_1 - \max(\varrho, y_2)\Big)^+ - r_t y_1,\\
\bar f_2 &= \bar f_2(t,y,\varrho) = h\big(\varrho - y_2\big)^+ - r_t y_2.
\end{aligned} \tag{30}$$

As shown by Lemma 4.1, $\bar\rho$ in (28) satisfies Assumption 3.3. Moreover, for any $t\in[0,T]$, for $y = (y_1,y_2)^\mathsf{T}$ and $y' = (y_1', y_2')^\mathsf{T}$ in $\mathbb{R}^2$, as r and λ are bounded from below, we have

$$\begin{aligned}
&\langle \bar f(t,y,\varrho) - \bar f(t,y',\varrho), y - y'\rangle\\
&= \big(\bar f_1(t,y,\varrho) - \bar f_1(t,y',\varrho)\big)(y_1 - y_1') + \big(\bar f_2(t,y,\varrho) - \bar f_2(t,y',\varrho)\big)(y_2 - y_2')\\
&= \lambda_t\Big(\big(P_t - \mathrm{CVA}_t - y_1 - \max(\varrho, y_2)\big)^+ - \big(P_t - \mathrm{CVA}_t - y_1' - \max(\varrho, y_2')\big)^+\Big)(y_1 - y_1')\\
&\qquad - r_t(y_1 - y_1')^2 + h\Big(\big(\varrho - y_2\big)^+ - \big(\varrho - y_2'\big)^+\Big)(y_2 - y_2') - r_t\big(y_2 - y_2'\big)^2\\
&\le C|y - y'|^2.
\end{aligned}$$

Hence, the ABSDE coefficient $\bar f$ in (30) satisfies the monotonicity condition of Assumption 3.2(i) (it is in fact even Lipschitz, as our processes r and λ are actually assumed

[7] In the sense immediately analogous to the one of Definition 4.1, as justified at the end of the proof of Theorem 4.1.

bounded). Next, for any arguments $t, y = (y_1, y_2)^\mathsf{T}$, and ϱ, ϱ', we have

$$
\begin{aligned}
&|\bar f(t,y,\varrho) - \bar f(t,y,\varrho')|^2 \\
&\qquad = \big(\bar f_1(t,y,\varrho) - \bar f_1(t,y,\varrho')\big)^2 + \big(\bar f_2(t,y,\varrho) - \bar f_2(t,y,\varrho')\big)^2 \\
&\qquad = \lambda_t^2 \Big| \big(P_t - \mathrm{CVA}_t - y_1 - \max\left(\varrho, y_2\right)\big)^+ - \big(P_t - \mathrm{CVA}_t - y_1 - \max\left(\varrho', y_2\right)\big)^+ \Big|^2 \\
&\qquad\qquad + h^2 |(\varrho - y_2)^+ - (\varrho' - y_2)^+|^2 \le C|\varrho - \varrho'|^2,
\end{aligned}
$$

which implies that Assumption 3.2(ii) holds. Finally, writing

$$
\widehat\rho_t = \bar\rho_t\big((0,0)^\mathsf{T}, (0,0)^\mathsf{T}\big) = \mathbb{ES}_t\Big[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s\Big],
$$

we have, for each t and $y = (y_1, y_2)^\mathsf{T}$ (simply denoting by 0 the origin of $\mathbb{R}^2$ in what follows)

$$
\begin{aligned}
\bar f_1(t,y,\widehat\rho_t) &= \lambda_t\Big(P_t - \mathrm{CVA}_t - y_1 \\
&\qquad\qquad - \max(\mathbb{ES}_t[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s], y_2)\Big)^+ - r_t y_1, \\
\bar f_1(t,0,\widehat\rho_t) &= \lambda_t\Big(P_t - \mathrm{CVA}_t - \max(\mathbb{ES}_t[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s])\Big)^+, \\
\bar f_2(t,y,\widehat\rho_t) &= h\Big(\mathbb{ES}_t[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s] - y_2\Big)^+ - r_t y_2, \\
\bar f_2(t,0,\widehat\rho_t) &= h\mathbb{ES}_t[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s],
\end{aligned}
\tag{31}
$$

hence

$$
|\bar f(t,y,\widehat\rho_t) - \bar f(t,0,\widehat\rho_t)| \le (|\lambda_t| + |r_t|)(|y_1| + |y_2|) + h|y_2|.
$$

By assumption, r and λ are bounded and $(P^+)^2 \in \mathcal{H}_1$. In addition,

$$
\begin{aligned}
\|\widehat\rho^2\|_{\mathcal{H}_1} &= \mathbb{E}\Big[\int_0^T \Big(\mathbb{ES}_t\big[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s\big]\Big)^2 dt\Big] \\
&\le (1-\alpha)^{-2}\mathbb{E}\Big[\int_0^T \Big(\mathbb{E}_t\big[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s\big]\Big)^2 dt\Big] \\
&\le (1-\alpha)^{-2}\mathbb{E}\Big[\int_0^T \Big(\mathbb{E}_t\big[\int_t^{(t+1)\wedge T} \beta_t^{-2}\beta_s^2 d[\mu]_s\big]\Big) dt\Big].
\end{aligned}
$$

Hence, since $\beta_t^{-2}\beta_s^2 \le e^{CT}$,

$$
\|\widehat\rho^2\|_{\mathcal{H}_1} \le (1-\alpha)^{-2} e^{CT}\mathbb{E}\Big[\int_0^T \int_0^T d[\mu]_s dt\Big] \le (1-\alpha)^{-2} e^{CT} T\mathbb{E}[\mu_T^2],
$$

which is finite, by Remark 2.4 (having assumed that $\int_0^T Q_s \boldsymbol{\delta}(ds)$ is $\mathbb{P}$ square integrable). From the above (starting with (31) onward), we deduce that Assumption 3.2(iii) is satisfied. Hence, an application of Theorem 3.1, via the observations 1 and 2 in Sect. 3.1, shows that the equation (25) with data (30) has a unique solution in $\mathcal{S}_2^2 \times \mathcal{M}_2^2$, i.e. the $(\mathrm{FVA}, \mathrm{KVA})$ system (27) admits a unique square integrable solution.

The well-posedness of the Picard iteration (29) among square integrable solutions $(\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)})$, for each n, and the geometrical convergence of $(\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)})$ when $n \to \infty$, follow, through the observation 3 in Sect. 3.1, from the second part of Theorem 3.1.

Finally, we have $L = \mu + m(\mathrm{FVA})$ in (27), where μ is in $\mathcal{M}_2$, by Remark 2.4, and so is $m(\mathrm{FVA})$, by Definition 4.1 of a square integrable solution to (27). ∎

Remark 4.2 As $L = \mu + m(\mathrm{FVA}) \in \mathcal{S}_2$ (last statement in Theorem 4.1) and β is bounded, the stochastic integral $\int_0^{\cdot} \beta_t dL_t$ is a uniformly integrable martingale, in line with the corresponding requirement before (9). ∎

5 Invariance Times Transfer Properties

What precedes was done in the theoretical case of a default-free bank. However, in reality, banks are defaultable and counterparty risk is related to cash flows or valuations linked to either counterparty default or the default of the bank itself. In particular, the bank funding spread λ, which we introduced in the above as liquidity, is essentially related to the credit spread of the bank (see Remark 2.1). Hence it is crucial to understand the case of a defaultable bank.

Then, as detailed in Albanese and Crépey (2019, Section 4.1) and recalled in what follows, we consider not only one pricing basis, i.e. $(\mathbb{F}, \mathbb{P})$ above, but actually two, i.e. also $(\mathbb{G}, \mathbb{Q})$ below. These pricing bases are connected by suitable consistency conditions (B) and (A), meaning that the bank default time τ is an invariance time in the sense of Crépey and Song (2017) (see Section B for a survey of the main results there).

In this transition section, which is general and of independent interest, we establish a transfer between expectations and martingale properties in the original and changed stochastic bases, assuming an invariance time τ endowed with an intensity and a positive Azéma supermartingale.

5.1 Reduction of Filtration Setup

We suppose that a complete and right-continuous filtration $\mathbb{G} = (\mathfrak{G}_t)_{t \in \mathbb{R}_+}$ of sub-σ fields of $\mathfrak{A}$ is an enlargement of our previous filtration $\mathbb{F} = (\mathfrak{F}_t)_{t \in \mathbb{R}_+}$, such that:

Condition (B)[8] $\forall t \geq 0$ and $B \in \mathfrak{G}_t$, $\exists B' \in \mathfrak{F}_t$ such that $B \cap \{t < \tau\} = B' \cap \{t < \tau\}$∎

[8] cf. Crépey and Song (2017, Eq. (2.1)).

This holds, in particular, under a standard progressive enlargement of filtration setup, i.e. when $\mathfrak{G}_t = \mathfrak{F}_t \vee \sigma(\tau \wedge t), t \geq 0$.

As seen in Crépey and Song (2017, Lemma 2.2), under the condition (B), any $\mathbb{G}$ optional process Y admits an $\mathbb{F}$ optional reduction Y' such that $\mathbb{1}_{[0,\tau)} Y = \mathbb{1}_{[0,\tau)} Y'$; any $\mathbb{G}$ predictable process Y admits an $\mathbb{F}$ predictable reduction Y' such that $\mathbb{1}_{(0,\tau]} Y = \mathbb{1}_{(0,\tau]} Y'$.

For any process Y, we denote by $Y^\tau = \mathbb{1}_{[0,\tau)} Y + \mathbb{1}_{[\tau,+\infty)} Y_\tau$ and[9] by $Y^{\tau-} = \mathbb{1}_{[0,\tau)} Y + \mathbb{1}_{[\tau,+\infty)} Y_{\tau-}$ the processes Y stopped at and before τ, respectively. A process Y is said to be stopped at τ if $Y = Y^\tau$ and stopped before τ if $Y = Y^{\tau-}$.

Given a positive constant T, we assume further that a full model probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathfrak{F}_T$ satisfies the following:

Condition (A) For any $(\mathbb{F}, \mathbb{P})$ local martingale P on $[0, T]$, $P^{\tau-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$. ∎

The full model probability measure $\mathbb{Q}$ is then interpreted as the prevailing risk-neutral pricing measure, whereas the reduced stochastic pricing basis $(\mathbb{F}, \mathbb{P})$ plays a technical role analogous to the one of the survival probability measure associated with $\mathbb{Q}$ (cf. Schönbucher (2004)), whilst avoiding the singularity issue of the latter (see Crépey and Song (2017, Section 4.2)).

The Azéma supermartingale S of τ is defined as $\mathsf{S}_t = \mathbb{Q}(\tau > t \,|\, \mathfrak{F}_t), t > 0$.

Remark 5.1 The situation where $\mathbb{P} = \mathbb{Q}$ in the condition (A) corresponds to the special case where S is nonincreasing and predictable. The subcase where S is also continuous corresponds to the class of pseudo-stopping times with the avoidance property (see Nikeghbali and Yor (2005) and Crépey and Song (2017, Sect. 4.1)). Pseudo-stopping times include Cox times, the family of default times the most commonly used in the credit and counterparty risk literatures: see e.g. Bielecki et al. (2009, Chapter 3), Brigo and Pallavicini (2014), or Bichuch, Capponi, and Sturm (2018). ∎

We assume $\mathsf{S}_T > 0$ a.s., so that, by Crépey and Song (2017, Lemma 2.3):

$$\text{Two } \mathbb{F} \text{ optional processes that coincide before } \tau \text{ coincide on } [0, T]. \tag{32}$$

In particular, $\mathbb{F}$ optional reductions are uniquely defined on $[0, T]$.

Finally, we assume that τ has a $(\mathbb{G}, \mathbb{Q})$ intensity γ, i.e. the $(\mathbb{G}, \mathbb{Q})$ compensator of $\mathbb{1}_{[\tau,\infty)}$ is given as $\int_0^\cdot \gamma_t dt$, for some $\mathbb{G}$ predictable process γ (vanishing beyond time τ). Summarizing, we suppose in the remaining of Sect. 5–6 the following:

Condition (C). The conditions (B) and (A) are satisfied, $\mathsf{S}_T > 0$, and τ has a $(\mathbb{G}, \mathbb{Q})$ intensity γ. ∎

We then have the following converse to the condition (A):

[9] Assuming the involved left-limit exists and using the convention that $Y_{0-} = Y_0$.

Lemma 5.1 *For any $(\mathbb{G}, \mathbb{Q})$ local martingale M stopped before τ (i.e. such that $M = M^{\tau-}$), M' is an $(\mathbb{F}, \mathbb{P})$ local martingale on $[0, T]$.*

Proof. The process $\mathsf{S}_{-} \centerdot M' + [\mathsf{S}, M']$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{\mathsf{S}_{-} > 0\}$, by (71), where $\{\mathsf{S}_{-} > 0\} = \{\mathsf{S} > 0\} \supseteq [0, T]$ by (75) and our assumption that $\mathsf{S}_T > 0$. The conclusion then follows from (76). ∎

5.2 Expectation Transfer Formulas

The following result provides an extension of the classical credit risk intensity pricing formulas (see e.g. Bielecki, Jeanblanc, and Rutkowski (2009, Chapter 3)) to the invariance time setup, i.e. beyond the basic immersion setup where $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ local martingales are $(\mathbb{G}, \mathbb{Q})$ local martingales without jump at τ.

We denote the $\mathbb{Q}$ expectation by $\widetilde{\mathbb{E}}$, whereas $\mathbb{P}$ expectation is denoted as before by $\mathbb{E}$.

Theorem 5.1 *For any $\mathbb{F}$ stopping time $\sigma \leq T$ and $\mathfrak{F}_\sigma$ measurable nonnegative random variable χ, for any $\mathbb{F}$ predictable nonnegative process K, for any $\mathbb{F}$ optional nondecreasing process A starting from 0, we have, respectively,*

$$\widetilde{\mathbb{E}}[\chi \mathbb{1}_{\{\sigma < \tau\}}] = \mathbb{E}[\chi e^{-\int_0^\sigma \gamma'_s ds}], \tag{33}$$

$$\widetilde{\mathbb{E}}[K_\tau \mathbb{1}_{\{\tau \leq T\}}] = \mathbb{E}[\int_0^T K_s e^{-\int_0^s \gamma'_u du} \gamma'_s \, ds], \tag{34}$$

$$\widetilde{\mathbb{E}}[A_T^{\tau-}] = \mathbb{E}[\int_0^T e^{-\int_0^s \gamma'_u du} \, dA_s]. \tag{35}$$

Proof. Let $\Gamma = \int_0^\cdot \gamma'_s ds$. Since χ is $\mathfrak{F}_\sigma$ measurable, $\widetilde{\mathbb{E}}[\chi \mathbb{1}_{\{\sigma < \tau\}}] = \widetilde{\mathbb{E}}[\chi \mathsf{S}_\sigma]$. As $S_T > 0$, (75) implies that $\{\mathsf{S}_{-} > 0\} = \{{}^p\mathsf{S} > 0\} \supseteq [0, T]$. Then (72) implies that

$$\widetilde{\mathbb{E}}[\chi S_\sigma] = \widetilde{\mathbb{E}}[\chi \mathsf{S}_0 \mathcal{E}(-\frac{1}{\mathsf{S}_{-}} \centerdot \mathsf{D})_\sigma \mathcal{E}(\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})_\sigma] = \mathbb{E}[\chi \mathsf{S}_0 \mathcal{E}(-\frac{1}{\mathsf{S}_{-}} \centerdot \mathsf{D})_\sigma], \tag{36}$$

by (74). In view of (73), we obtain (33).

For (34), we compute

$$\begin{aligned}\widetilde{\mathbb{E}}[K_\tau \mathbb{1}_{\{\tau \leq T\}}] &= \widetilde{\mathbb{E}}[\int_0^T K_s \mathbb{1}_{\{s \leq \tau\}} \gamma'_s \, ds] = \int_0^T \widetilde{\mathbb{E}}[K_s \mathbb{1}_{\{s < \tau\}} \gamma'_s] \, ds \\ &= \int_0^T \mathbb{E}[K_s e^{-\Gamma_s} \gamma'_s] \, ds = \mathbb{E}[\int_0^T K_s e^{-\Gamma_s} \gamma'_s \, ds],\end{aligned}$$

where (33) was used for passing to the second line.

Regarding (35), an application of (34) yields

$$\widetilde{\mathbb{E}}[A_{\tau-} \mathbb{1}_{\{\tau \leq T\}}] = \mathbb{E}[\int_0^T A_s e^{-\Gamma_s} \gamma'_s \, ds] = -\mathbb{E}[A_T e^{-\Gamma_T}] + \mathbb{E}[\int_0^T e^{-\Gamma_s} \, dA_s].$$

Using (33), we deduce

$$\begin{array}{rcl}\widetilde{\mathbb{E}}[A_T^{\tau-}] & = & \widetilde{\mathbb{E}}[A_T \mathbb{1}_{\{T<\tau\}}] + \widetilde{\mathbb{E}}[A_{\tau-}\mathbb{1}_{\{\tau\le T\}}] \\ & = & \mathbb{E}[A_T e^{-\Gamma_T}] - \mathbb{E}[A_T e^{-\Gamma_T}] + \mathbb{E}[\int_0^T \ e^{-\Gamma_s} \ dA_s] \\ & = & \mathbb{E}[\int_0^T \ e^{-\Gamma_s} \ dA_s]. \ \blacksquare\end{array}$$

Denoting $X_t^* = \sup_{t\in[0,T]} |X_s|$, we introduce the space $\widetilde{\mathcal{S}}_2^l$ (simply renoted $\widetilde{\mathcal{S}}_2$ when $l = 1$) of the $\mathbb{R}^l$ valued càdlàg $\mathbb{G}$ adapted processes $\widetilde{Y}$ such that $\widetilde{Y} = \widetilde{Y}^{\tau-}$ and

$$\|\widetilde{Y}\|^2_{\widetilde{\mathcal{S}}_2^l} = \widetilde{\mathbb{E}}\Big[|\widetilde{Y}_0|^2 + \int_0^T \mathbb{1}_{\{s<\tau\}} \ e^{\int_0^s \gamma_u du} d(\widetilde{Y}^*)^2_s\Big] < \infty. \tag{37}$$

Lemma 5.2 *For any real valued càdlàg $\mathbb{F}$ adapted process Y, we have*

$$\|Y\|_{\mathcal{S}_2} = \|Y\|_{\widetilde{\mathcal{S}}_2}. \tag{38}$$

Proof. For $A = \int_0^\cdot e^{\int_0^s \gamma'_u du} d(Y^*)^2_s$, the expectation transfer formula (35) yields

$$\widetilde{\mathbb{E}}\Big[\int_0^T \ \mathbb{1}_{\{s<\tau\}} e^{\int_0^s \gamma_u du} d(Y^*)^2_s\Big] = \mathbb{E}[(Y^*)^2_T] - \mathbb{E}[Y_0^2] = \mathbb{E}[(Y^*)^2_T] - \widetilde{\mathbb{E}}[Y_0^2]$$

(noting that $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathfrak{F}_0$, by (74)), which is (38). $\blacksquare$

6 The Realistic Case of a Defaultable Bank

We now revisit the analysis of Sect. 2 in the realistic case of a defaultable bank, with default time τ satisfying the condition (C) of Sect. 5.1. We can then identify λ, the funding spread of the bank, with the instantaneous credit spread process $(1 - \bar{R})\gamma'$ of the bank, where $\bar{R}$ is a constant recovery rate (cf. Remark 2.1). The time horizon of the XVA problem is now $\bar{\tau} = \tau \wedge T$, where T is the final maturity of the derivative portfolio of the bank.

Accounting for the defaultability of the bank, a key distinction appears between the cash flows received by the bank prior τ and the cash flows received by the bank during the default resolution period starting at τ. Indeed, the first stream of cash flows affects the bank shareholders, whereas the second stream of cash flows only affects bondholders. For accepting a new deal, bank shareholders need to be at least indifferent given the first stream of cash flows only. Accordingly, as can be seen from the general CVA, FVA, and KVA equations (10), (11), and (18)–(20) in Albanese and Crépey (2019),[10] the "recipe" to obtain the XVA equations of a defaultable bank under $(\mathbb{G}, \mathbb{Q})$ is to stop all cash flows before τ in the $(\mathbb{F}, \mathbb{P})$ equations (or, more precisely but to the same result, to stop all cash flows before τ in the extension of these equations that

[10]The notation in the present paper differs from the one in Albanese and Crépey (2019). In particular, $\widetilde{\mathbb{E}}$ and $\mathbb{E}$ here correspond to $\mathbb{E}$ and $\mathbb{E}'$ there; $\widetilde{\mathcal{S}}_2$ and $\mathcal{S}_2$ here correspond to $\mathcal{S}_2$ and $\mathcal{S}'_2$ there; "$\widetilde{\cdot}$" for "$(\mathbb{G}, \mathbb{Q})$ equations" is only used here; The risk-free asset is used as numéraire there.

also include the cash flows received by the bank from time τ onward). See below for concrete illustrations.

In all $(\mathbb{G},\mathbb{Q})$ equations, we write $\widetilde{\mathrm{XVA}}$ for the corresponding (defaultable bank) XVA process *stopped before* τ. Likewise, we denote by $\widetilde{L}$ (instead of L before) the now $\mathbb{G}$ adapted bank trading loss process L *stopped before* τ.

6.1 CVA

The differential form of the CVA equation (4) is written as

$$\begin{aligned}
&\mathrm{CVA}_T = 0 \text{ and, for } t \in (0,T],\\
&d\mathrm{CVA}_t = r_t \mathrm{CVA}_t dt - Q_t \boldsymbol{\delta}(dt) + d\mu_t,\\
&\text{for some } (\mathbb{F},\mathbb{P}) \text{ martingale } \mu \text{ in } \mathcal{S}_2.
\end{aligned} \tag{39}$$

On the other hand, in the present defaultable bank setup, the differential form of the general $\widetilde{\mathrm{CVA}}$ equation (10) in Albanese and Crépey (2019) is

$$\begin{aligned}
&\widetilde{\mathrm{CVA}}_T = 0 \text{ on } \{T<\tau\} \text{ and, for } t \in [0,\bar{\tau}],\\
&d\widetilde{\mathrm{CVA}}_t = r_t \widetilde{\mathrm{CVA}}_t dt - \mathbb{1}_{\{t<\tau\}} Q_t \boldsymbol{\delta}(dt) + d\widetilde{\mu}_t,\\
&\text{for some } (\mathbb{G},\mathbb{Q}) \text{ martingale } \widetilde{\mu} \text{ in } \widetilde{\mathcal{S}}_2.
\end{aligned} \tag{40}$$

Observe that (40) is nothing but the $(\mathbb{G},\mathbb{Q})$ equation obtained by formally ignoring all cash flows from τ onward in (39).

Now, (39) and (40) are in fact equivalent via $\mathbb{F}$ (optional) reduction $\cdot'$ (cf. Sect. 5.1):

Proposition 6.1 *The equation* (39) *for* CVA *in* $\mathcal{S}_2$ *and the equation* (40) *for* $\widetilde{\mathrm{CVA}}$ *in* $\widetilde{\mathcal{S}}_2$ *are equivalent via* $\mathbb{F}$ *optional reduction, i.e. through the relation* $\mathrm{CVA} = \widetilde{\mathrm{CVA}}'$.

If $\int_0^T Q_s \boldsymbol{\delta}(ds)$ *is* $\mathbb{P}$ *square integrable, then the equation* (39) *for* CVA *in* $\mathcal{S}_2$ *has the unique solution* (4).

Proof. Assuming that $(\widetilde{\mathrm{CVA}},\widetilde{\mu}) \in \widetilde{\mathcal{S}}_2^2$ satisfies (40), then $(\mathrm{CVA},\mu) = (\widetilde{\mathrm{CVA}}',\widetilde{\mu}') \in \mathcal{S}_2^2$, by (38), and (CVA,μ) thus defined satisfies the second line in (39) on $[0,\bar{\tau}]$, hence on $[0,T]$, by (32), while the martingale condition in the third line holds by Lemma 5.1. Moreover, taking the $(\mathfrak{F}_T,\mathbb{Q})$ conditional expectation of the first line in (40) yields

$$0 = \widetilde{\mathbb{E}}[\widetilde{\mathrm{CVA}}_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = \widetilde{\mathbb{E}}[\widetilde{\mathrm{CVA}}'_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = \widetilde{\mathrm{CVA}}'_T \mathsf{S}_T,$$

where $\mathsf{S}_T > 0$, hence $\mathrm{CVA}_T = \widetilde{\mathrm{CVA}}'_T = 0$.

Conversely, assuming that $(\mathrm{CVA},\mu) \in \mathcal{S}_2^2$ satisfies (39), then

$$(\widetilde{\mathrm{CVA}},\widetilde{\mu}) = (\mathrm{CVA}^{\tau-},\mu^{\tau-}) \in \widetilde{\mathcal{S}}_2^2,$$

by (38), and $(\widetilde{\mathrm{CVA}},\widetilde{\mu}) = (\mathrm{CVA}^{\tau-},\mu^{\tau-})$ satisfies the conditions of the first two lines in (40), while $\widetilde{\mu} = \mu^{\tau-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale, by virtue of the condition (A).

This proves the first part in Proposition 6.1. The second part is an immediate consequence of Remark 2.4. ∎

6.2 FVA and KVA

The differential form of the (FVA, KVA) ABSDE system (27) is written as (with CA as a shorthand for CVA + FVA):

$$\begin{aligned}
&\mathrm{CA}_T = 0 \text{ and, for } t \in (0, T],\\
&d\mathrm{CA}_t = r_t \mathrm{CA}_t dt - Q_t \boldsymbol{\delta}(dt)\\
&\qquad - \lambda_t \Big(P_t - \max\big(\mathbb{ES}_t(\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s dL_s), \mathrm{KVA}_t\big) - \mathrm{CA}_t\Big)^+ dt + dL_t,\\
&\text{for some } (\mathbb{F}, \mathbb{P}) \text{ martingale } L \text{ in } \mathcal{S}_2
\end{aligned} \tag{41}$$

(L was interpreted in Sect. 2 as the trading loss of a default-free bank), along with

$$\begin{aligned}
&\mathrm{KVA}_T = 0 \text{ and, for } t \in (0, T],\\
&d\mathrm{KVA}_t = (r_t + h)\mathrm{KVA}_t dt - h \max\big(\mathbb{ES}_t(\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s dL_s), \mathrm{KVA}_t\big) dt + d\nu_t,\\
&\text{for some } (\mathbb{F}, \mathbb{P}) \text{ martingale } \nu \text{ in } \mathcal{S}_2.
\end{aligned} \tag{42}$$

On the other hand, in the present defaultable bank setup, the general $\widetilde{\mathrm{FVA}}$ and $\widetilde{\mathrm{KVA}}$ equations (11) and (18)–(20) in Albanese and Crépey (2019) yield the following system of $\widetilde{\mathrm{XVA}}$ equations (with $\widetilde{\mathrm{CA}}$ used as a shorthand for $\widetilde{\mathrm{CVA}} + \widetilde{\mathrm{FVA}}$):

$$\begin{aligned}
&\widetilde{\mathrm{CA}}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \in (0, \bar{\tau}],\\
&d\widetilde{\mathrm{CA}}_t = r_t \widetilde{\mathrm{CA}}_t dt - \mathbb{1}_{\{t<\tau\}} Q_t \boldsymbol{\delta}(dt)\\
&\qquad - \lambda_t \Big(P_t - \widetilde{\mathrm{CA}}_t - \max\big(\mathbb{ES}_t(\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\widetilde{L}'_s), \widetilde{\mathrm{KVA}}_t\big)\Big)^+ dt + d\widetilde{L}_t,\\
&\text{for some } (\mathbb{G}, \mathbb{Q}) \text{ martingale } \widetilde{L} \text{ in } \widetilde{\mathcal{S}}_2
\end{aligned} \tag{43}$$

($\widetilde{L}$ is interpreted as the bank shareholders' trading loss of the now defaultable bank), along with:

$$\begin{aligned}
&\widetilde{\mathrm{KVA}}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \in (0, \bar{\tau}],\\
&d\widetilde{\mathrm{KVA}}_t = (r_t + h)\widetilde{\mathrm{KVA}}_t dt - h \max\big(\mathbb{ES}_t(\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\widetilde{L}'_s), \widetilde{\mathrm{KVA}}_t\big) dt + d\widetilde{\nu}_t,\\
&\text{for some } (\mathbb{G}, \mathbb{Q}) \text{ martingale } \widetilde{\nu} \text{ in } \widetilde{\mathcal{S}}_2.
\end{aligned} \tag{44}$$

Remark 6.1 We recall from Albanese and Crépey (2019, Section 4.3) that capital calculations are typically performed "on a going-concern basis," i.e. disregarding the default of the bank itself. In view of the comments preceding Remark 5.1, this grounds our specification $\mathbb{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_s d\widetilde{L}'_s)$ in (43)–(44) for the economic capital of a defaultable bank. ∎

Lemma 6.1 *The equations* (41)–(42) *for* $(\mathrm{FVA}, \mathrm{KVA})$ *in* $\mathcal{S}_2^2$ *and the equations* (43)–(44) *for* $(\widetilde{\mathrm{FVA}}, \widetilde{\mathrm{KVA}})$ *in* $\widetilde{\mathcal{S}}_2^2$ *are equivalent via* $\mathbb{F}$ *optional reduction, i.e. through the relation* $(\mathrm{FVA}, \mathrm{KVA}) = (\widetilde{\mathrm{FVA}}', \widetilde{\mathrm{KVA}}')$.

Proof. Similar to the proof of the first part in Proposition 6.1, hence omitted. ∎

Square integrable solutions to the $(\widetilde{\mathrm{FVA}}, \widetilde{\mathrm{KVA}})$ equations are defined in reference to the $\widetilde{\mathcal{S}}_2$ space.

Theorem 6.1 *Under the assumptions of Theorem 4.1, the* $(\widetilde{\mathrm{FVA}}, \widetilde{\mathrm{KVA}})$ *and* $(\mathrm{FVA}, \mathrm{KVA})$ *equations are well posed (and equivalent via* $\mathbb{F}$ *optional reduction) in their respective spaces of square integrable solutions.*

Proof. This is the conclusion of Lemma 6.1 and Theorem 4.1 (first part), also noting that, under the assumptions of Theorem 4.1, the differential and the integral formulations of the $(\mathrm{FVA}, \mathrm{KVA})$ system are equivalent in their space of square integrable solutions. ∎

Remark 6.2 Under the assumptions of Theorem 4.1, one also has the equivalence, within square integrable solutions, between the differential formulation (43)–(44) of the $(\widetilde{\mathrm{FVA}}, \widetilde{\mathrm{KVA}})$ equations and the corresponding integral formulation (not displayed in the paper for length sake). This can be shown by the same argument as the one used for concluding the proof of the first part of Theorem 5.1 in Albanese and Crépey (2019). ∎

7 Synthesis

To conclude, we reread the paper upside down, which reveals the overall logic of our XVA approach.

By application of the general CVA, FVA, and KVA equations in Albanese and Crépey (2019, Sections 3–4), we derive in Section 6 the differential formulations (40) and (43)-(44) of these equations in the specific setup that we want to address in this paper, with possible use of capital at risk as variation margin by the bank. In particular, the FVA and KVA equations (43)-(44) form a bivariate system of anticipated BSDEs "of the Mc Kean type" (ABSDEs), stopped before the bank default time τ.

Using an enlargement of filtration methodology of independent interest, finalized in Section 5 following up on Crépey and Song (2017), we show in Section 6 the equivalence between these ABSDEs, natively stated with respect to the "full" pricing model including the bank default time as a stopping time, and reduced ABSDEs (41)-(42) stated with respect to a smaller filtration and an equivalently changed probability measure.

These reduced equations are none but the ones that were derived in Section 2 in the irrealistic case of a default-free bank, at the cost of abusively interpreting the bank funding spread there as liquidity (cf. Remark 2.1). By ignoring the default of the bank, the XVA setup of Section 2 was far-fetched and, in a sense, internally inconsistent financially. But the related mathematics of Sections 3-4, showing that the corresponding bank default-free XVA equations were well-posed, are of course valid. Hence, we were able to conclude in Theorem 6.1 that the bank default-prone FVA and KVA equations (43)–(44) are well-posed mathematically.

Putting together Theorems 6.1 and 4.1, we obtain schematically, under the assumptions of Theorem 4.1:

$$(\widetilde{\mathrm{FVA}}, \widetilde{\mathrm{KVA}}) \sim (\mathrm{FVA}, \mathrm{KVA}) \leftarrow (\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)}). \tag{45}$$

Moreover, as demonstrated numerically in Albanese et al. (2017, Section 5) and Albanese et al. (2019, Section 5), the Picard iteration $(\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)})$ in (29) is amenable to Monte-Carlo approximation, including at the scale of a real banking portfolio with hundreds of counterparties and hundreds of thousands of contracts. Hence, the XVA approach of this paper is not only theoretically well posed, but also workable in practice for a bank.

A more extensive numerical study of the XVA ABSDEs is left for future research.

A Detail of the ABSDE Proofs

In this section, we give the details of the computations used to establish the well-posedness of the l-variate ABSDE (19).

First we note that, if (Y, Z, U) solves (19), then

$$(\mathcal{Y}_t, \mathcal{Z}_t, \mathcal{U}_t) = (e^{c_m t} Y_t, e^{c_m t} Z_t, e^{c_m t} U_t)$$

solves, for $0 \leq t \leq T$,

$$\begin{cases} \mathcal{Y}_t = e^{c_m T}\xi + \displaystyle\int_t^T \Big(e^{c_m s} f\left(s, e^{-c_m s}\mathcal{Y}_s, e^{-c_m s}\mathcal{Z}_s, e^{-c_m s}\mathcal{U}_s, \rho_s(e^{-c_m \cdot}\mathcal{Y}_\cdot, e^{-c_m \cdot}\mathcal{Z}_\cdot, e^{-c_m \cdot}\mathcal{U}_\cdot)\right) \\ \qquad\qquad - c_m \mathcal{Y}_s\Big) ds - \displaystyle\int_t^T \int_E \mathcal{U}_s(e) M(ds, de) - \int_t^T \mathcal{Z}_s dB_s. \end{cases} \tag{46}$$

Namely, $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$ solves an l-variate ABSDE of the general form (19), but for modified data that satisfy Assumptions 3.1 and 3.2, with $c_m = 0$ in the latter. Moreover,

$$(Y, Z, U) \in \mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l \Longleftrightarrow (\mathcal{Y}, \mathcal{Z}, \mathcal{U}) \in \mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l.$$

Hence we may and do suppose that $c_m = 0$ in Assumption 3.2, without loss of generality in what follows.

A.1 Proof of Lemma 3.1

For $t \in [0,T]$, set $\Gamma_t = e^{\kappa t}$, for some to-be-determined positive constant κ. An application of the Itô formula to $\Gamma|Y|^2$ yields

$$\begin{aligned}\Gamma_t|Y_t|^2 + \int_t^T \Gamma_s|Z_s|^2 ds + \int_t^T \Gamma_s|U_s|_s^2 ds = \Gamma_T|Y_T|^2 - \kappa\int_t^T \Gamma_s|Y_s|^2 ds \\ + 2\int_t^T \Gamma_s\langle Y_s, f\big(s, Y_s, Z_s, U_s, \rho_s(Y,Z,U)\big)\rangle ds \\ - 2\int_t^T \Gamma_s Y_s^{\mathsf{T}} Z_s dB_s - \int_t^T\int_E \Gamma_s\big(|Y_{s-}+U_s(e)|^2 - |Y_{s-}|^2\big)M(ds,de).\end{aligned} \tag{47}$$

The Burkholder's inequality yields

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T \Gamma_s Y_s^{\mathsf{T}} Z_s dB_s\big|\big] &\le C\mathbb{E}\Big[\big(\int_0^T \Gamma_s^2|Y_s|^2|Z_s|^2 ds\big)^{1/2}\Big] \\ &\le \mathbb{E}\Big[\big(\sup_{s\in[0,T]}\Gamma_s|Y_s|^2\big)^{1/2}\big(C\int_0^T \Gamma_s|Z_s|^2 ds\big)^{1/2}\Big]\end{aligned}$$

Then, by the Young's inequality ($ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$) with $\epsilon = 4$, we have

$$\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T \Gamma_s Y_s^{\mathsf{T}} Z_s dB_s\big|\big] \le \frac{1}{8}\mathbb{E}\big(\sup_{s\in[0,T]}\Gamma_s|Y_s|^2\big) + 2C^2\mathbb{E}\big[\int_0^T \Gamma_s|Z_s|^2 ds\big]. \tag{48}$$

Similarly, we obtain

$$\begin{aligned}&\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T \Gamma_s\int_E Y_{s-}U_s(e)M(ds,de)\big|\big] \\ &\qquad \le C\mathbb{E}\Big[\big(\int_0^T \Gamma_s^2|Y_{s-}|^2\int_E |U_s(e)|^2 j(ds,de)\big)^{1/2}\Big] \\ &\qquad \le \mathbb{E}\Big[\big(\sup_{s\in[0,T]}\Gamma_s|Y_s|^2\big)^{1/2}\big(C\int_0^T \Gamma_s\int_E |U_s(e)|^2 j(ds,de)\big)^{1/2}\Big] \\ &\qquad \le \frac{1}{8}\mathbb{E}\big(\sup_{s\in[0,T]}\Gamma_s|Y_s|^2\big) + 2C^2\mathbb{E}\big[\int_0^T \Gamma_s|U_s|_s^2 ds\big].\end{aligned} \tag{49}$$

In particular, the $(\mathfrak{F}_t,\mathbb{P})$ conditional expectation of the last line in (47) is equal to 0.

Moreover, by Assumption 3.2 on f and the Young inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned}&2\langle Y_s, f\big(s, Y_s, Z_s, U_s, \rho_s(Y,Z,U)\big)\rangle \\ &\le 2|Y_s||f(s,0,0,0,\rho_s(0,0,0))| + 2c_f|Y_s|\big(|Z_s| + |U_s|_s + |\rho_s(Y,Z,U) - \rho_s(0,0,0)|\big) \\ &\le \big(1 + 8c_f^2 + c_f^2\epsilon^{-1}\big)|Y_s|^2 + |f(s,0,0,0,\rho_s(0,0,0))|^2 \\ &\qquad + \frac{1}{4}\big(|Z_s|^2 + |U_s|_s^2\big) + \epsilon|\rho_s(Y,Z,U) - \rho_s(0,0,0)|^2.\end{aligned} \tag{50}$$

Besides,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Gamma_s\Big(\sup_{s\le u\le T}|Y_u|^2+\int_s^T(|Z_u|^2+|U_u|_u^2)du\Big)ds\Big]\\
&\le \mathbb{E}\Big[\int_0^T \Gamma_s\big(\sup_{s\le u\le T}|Y_u|^2\big)ds\big]+\mathbb{E}\Big[\int_0^T\int_s^T\Gamma_s(|Z_u|^2+|U_u|_u^2)duds\Big]\\
&\le \mathbb{E}\Big[\int_0^T\big(\sup_{s\le u\le T}\Gamma_u|Y_u|^2\big)ds\big]+\mathbb{E}\big[\int_0^T\int_0^u\Gamma_s(|Z_u|^2+|U_u|_u^2)dsdu\big] \qquad (51)\\
&\le T\,\mathbb{E}\Big[\sup_{0\le u\le T}\Gamma_u|Y_u|^2\big]+T\mathbb{E}\big[\int_0^T\Gamma_u(|Z_u|^2+|U_u|_u^2)du\big]\\
&\le T\,\mathbb{E}\big[\sup_{0\le u\le T}\Gamma_u|Y_u|^2+\int_0^T\Gamma_s(|Z_s|^2+|U_s|_s^2)ds\big].
\end{aligned}
$$

In addition, Assumption 3.1 implies

$$
\begin{aligned}
&\int_t^T\Gamma_s|\rho_s(Y,Z,U)-\rho_s(0,0,0)|^2ds\le\\
&\qquad c_\rho^2\int_t^T\Gamma_s\mathbb{E}_s\Big(\sup_{s\le u\le T}|Y_u|^2+\int_s^T(|Z_u|^2+|U_u|_u^2)du\Big)ds,
\end{aligned} \qquad (52)
$$

hence

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T\Gamma_s|\rho_s(Y,Z,U)-\rho_s(0,0,0)|^2ds\Big]\le\\
&\qquad c_\rho^2T\,\mathbb{E}\big[\sup_{0\le u\le T}\Gamma_u|Y_u|^2+\int_0^T\Gamma_s(|Z_s|^2+|U_s|_s^2)ds\big].
\end{aligned} \qquad (53)
$$

Plugging (50) into (47), taking expectations there and using (53), we therefore obtain, for $\epsilon=\frac{1}{4c_\rho^2T}$,

$$
\begin{aligned}
&\mathbb{E}\big[\Gamma_t|Y_t|^2\big]+\frac{3}{4}\mathbb{E}\big[\int_t^T\Gamma_s(|Z_s|^2+|U_s|_s^2)ds\big]+\mathbb{E}\big[\int_t^T\big(\kappa-(1+8c_f^2+4c_f^2c_\rho^2)\big)\,\Gamma_s|Y_s|^2ds\big]\\
&\le\mathbb{E}\big[\Gamma_T|Y_T|^2\big]+\mathbb{E}\big[\int_t^T\Gamma_s|f(s,0,0,0,\rho_s(0,0,0))|^2ds\big]\\
&+\frac{1}{4}\mathbb{E}\big[\sup_{0\le u\le T}\Gamma_u|Y_u|^2+\int_0^T\Gamma_s(|Z_s|^2+|U_s|_s^2)ds\big]. \qquad (54)
\end{aligned}
$$

In particular, for $\kappa=1+8c_f^2+4c_f^2c_\rho^2$, (54) implies that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\big[|Y_t|^2\big]+\mathbb{E}\big[\int_0^T(|Z_s|^2+|U_s|_s^2)ds\big]\\
&\qquad\le\ C\mathbb{E}\Big(|\xi|^2+\int_0^T|f(s,0,0,0,\rho_s(0,0,0))|^2ds+\sup_{0\le s\le T}|Y_s|^2\Big).
\end{aligned} \qquad (55)
$$

In order to show that the sup and the $\mathbb{E}$ can be switched in the left hand side of (55), we rearrange the Itô formula (47) as

$$\begin{aligned}\Gamma_t|Y_t|^2 = \Gamma_T|\xi|^2 &+ 2\int_t^T \Gamma_s\langle Y_s, f\big(s, Y_s, Z_s, U_s, \rho_s(Y,Z,U)\big)\rangle ds - \kappa\int_t^T \Gamma_s|Y_s|^2 ds \\ &- \int_t^T \Gamma_s|Z_s|^2 ds - \int_t^T\int_E \Gamma_s|U_s(e)|^2 j(ds,de) \\ &- 2\int_t^T \Gamma_s Y_s^\intercal Z_s dB_s - 2\int_t^T \Gamma_s \int_E Y_{s-}U_s(e)M(ds,de).\end{aligned} \tag{56}$$

Plugging (50) and (52) into (56), we obtain

$$\begin{aligned}\Gamma_t|Y_t|^2 \le \Gamma_T|\xi|^2 &+ \int_t^T \Gamma_s|f(s,0,0,0,\rho_s(0,0,0))|^2 ds + \frac{1}{4}\Big[\int_t^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds\Big] \\ &+ (1 + 8c_f^2 + 4c_f^2c_\rho^2 - \kappa)\int_t^T \Gamma_s|Y_s|^2 ds \\ &+ \epsilon c_\rho^2\int_t^T \Gamma_s\mathbb{E}_s\Big(\sup_{s\le u\le T}|Y_u|^2 + \int_s^T(|Z_u|^2 + |U_u|_u^2)du\Big)ds \\ &- 2\int_t^T \Gamma_s Y_s^\intercal Z_s dB_s - 2\int_t^T \Gamma_s\int_E Y_{s-}U_s(e)M(ds,de).\end{aligned} \tag{57, 58}$$

Setting $\kappa = 1 + 8c_f^2 + 4c_f^2c_\rho^2$ yields

$$\begin{aligned}\Gamma_t|Y_t|^2 \le &\Gamma_T|\xi|^2 + \int_t^T \Gamma_s|f(s,0,0,0,\rho_s(0,0,0))|^2 ds + \frac{1}{4}\Big[\int_t^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds\Big] \\ &+ \epsilon c_\rho^2\int_t^T \Gamma_s\mathbb{E}_s\Big(\sup_{s\le u\le T}|Y_u|^2 + \int_s^T(|Z_u|^2 + |U_u|_u^2)du\Big)ds \\ &- 2\int_t^T \Gamma_s Y_s^\intercal Z_s dB_s - 2\int_t^T \Gamma_s\int_E Y_{s-}U_s(e)M(ds,de).\end{aligned} \tag{59}$$

Taking the supremum over $t \in [0,T]$ and expectation on both sides, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\Gamma_t|Y_t|^2\right) \le &\Gamma_T\mathbb{E}\left(|\xi|^2\right) + \mathbb{E}\left(\int_0^T \Gamma_s|f(s,0,0,0,\rho_s(0,0,0))|^2 ds\right) \\ &+ \frac{1}{4}\mathbb{E}\left(\int_0^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds\right) \\ &+ \epsilon c_\rho^2\mathbb{E}\left(\int_0^T \Gamma_s\mathbb{E}_s\Big(\sup_{s\le u\le T}|Y_u|^2 + \int_s^T(|Z_u|^2 + |U_u|_u^2)du\Big)ds\right) \\ &+ 2\mathbb{E}\left(\sup_{t\in[0,T]}\int_t^T \Gamma_s Y_s^\intercal Z_s dB_s\right) + 2\mathbb{E}\left(\sup_{t\in[0,T]}\int_t^T \Gamma_s\int_E Y_{s-}U_s(e)M(ds,de)\right).\end{aligned} \tag{60}$$

Then, by using $\epsilon = \frac{1}{4c_\rho^2 T}$, (48), (49), (51), and the estimates on Z and U in (55), we deduce from (60)

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^2\Big) \leq C\mathbb{E}\left(|\xi|^2 + \int_0^T |f(s,0,0,0,\rho_s(0,0,0))|^2 ds\right).$$

In conjunction with the estimates on Z and U in (55), this concludes the proof of (20).

A.2 Detail of the proof of Theorem 3.1

Given (X,V,W) and (X',V',W') in $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$, let $(Y,Z,U) = \Phi(X,V,W)$ and $(Y',Z',U') = \Phi(X',V',W')$. For $L = X,V,W,Y,Z,U$, we denote $\delta L = L' - L$. As in the proof of Lemma 3.1, letting $\Gamma_t = e^{\kappa t}$ and applying Ito's formula to $\Gamma|\delta Y|^2$ yields

$$\mathbb{E}[\Gamma_t|\delta Y_t|^2] + \mathbb{E}[\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2 + \kappa|\delta Y_s|^2)ds] = 2\int_t^T \mathbb{E}[\Gamma_s\langle \delta Y_s, \delta f_s\rangle]ds, \quad (61)$$

where

$$\delta f_s = f(s, Y_s, Z_s, U_s, \rho_s(X,V,W)) - f(s, Y_s', Z_s', U_s', \rho_s(X',V',W')).$$

Given Assumptions 3.1 on ρ and 3.2 on f, applying Young's inequality with $c_1, c_2 \neq 0$, we have

$$\begin{aligned}
2\int_t^T &\Gamma_s\langle \delta Y_s, \delta f_s\rangle ds \leq \int_t^T \left(2c_1^2 c_f + c_2^2 c_f\right)\Gamma_s|\delta Y_s|^2 ds \\
&+ \frac{c_f}{c_1^2}\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds + \frac{c_f}{c_2^2}\int_t^T \Gamma_s|\rho_s(X,V,W) - \rho_s(X',V',W')|^2 ds \\
\leq &\int_t^T \left(2c_1^2 c_f + c_2^2 c_f\right)\Gamma_s|\delta Y_s|^2 ds + \frac{c_f}{c_1^2}\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds \\
&+ \frac{c_f}{c_2^2}c_\rho^2\int_t^T \Gamma_s\mathbb{E}_s\big[\sup_{s\leq u\leq T}|\delta X_u|^2 + \int_s^T (|\delta V_u|^2 + |\delta W_u|_u^2)du\big]ds.
\end{aligned} \quad (62)$$

Applying this inequality in (61) and proceeding as in (53) yields

$$\begin{aligned}
&\mathbb{E}[\Gamma_t|\delta Y_t|^2] + \mathbb{E}[\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2 + \kappa|\delta Y_s|^2)ds] \\
&\leq \left(2c_1^2 c_f + c_2^2 c_f\right)\mathbb{E}\big[\int_t^T \Gamma_s|\delta Y_s|^2 ds\big] + \frac{c_f}{c_1^2}\mathbb{E}\big[\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds\big] \\
&\qquad + \frac{c_f c_\rho^2 T}{c_2^2}\mathbb{E}\big[\sup_{0\leq s\leq T}\Gamma_s|\delta X_s|^2 + \int_0^T \Gamma_s(|\delta V_s|^2 + |\delta W_s|_s^2)ds\big].
\end{aligned} \quad (63)$$

In particular, we obtain

$$\begin{aligned}\sup_{0\le t\le T}\mathbb{E}[\Gamma_t|\delta Y_t|^2]+(1-\frac{c_f}{c_1^2})\mathbb{E}[\int_0^T\Gamma_s(|\delta Z_s|^2+|\delta U_s|_s^2)ds]\\ +(\kappa-2c_1^2c_f-c_2^2c_f)\mathbb{E}[\int_0^T\Gamma_s|\delta Y_s|^2ds]\\ \le\frac{c_fc_\rho^2T}{c_2^2}\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2+\int_0^T\Gamma_s(|\delta V_s|^2+|\delta W_s|_s^2)ds\big].\end{aligned}\tag{64}$$

Choosing $c_1^2>c_f$ and $\kappa\ge 2c_1^2c_f+c_2^2c_f$, we deduce from (64) that

$$\begin{aligned}&\mathbb{E}[\int_0^T\Gamma_s(|\delta Z_s|^2+|\delta U_s|_s^2)ds]\le\\ &\qquad\frac{c_1^2}{(c_1^2-c_f)}\frac{c_fc_\rho^2T}{c_2^2}\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2+\int_0^T\Gamma_s(|\delta V_s|^2+|\delta W_s|_s^2)ds\big].\end{aligned}\tag{65}$$

For estimating the δY term, we write the Itô formula that applies to $\Gamma|\delta Y|^2$ in the manner of (56), i.e.

$$\begin{aligned}\Gamma_t|\delta Y_t|^2=&\,2\int_t^T\Gamma_s\langle\delta Y_s,\delta f_s\rangle ds-\kappa\int_t^T\Gamma_s|\delta Y_s|^2ds\\ &-\int_t^T\Gamma_s|\delta Z_s|^2ds-\int_t^T\int_E\Gamma_s|\delta U_s(e)|^2j(ds,de)\\ &-2\int_t^T\Gamma_s\delta Y_s^\intercal\delta Z_sdB_s-2\int_t^T\Gamma_s\int_E\delta Y_{s-}\delta U_s(e)M(ds,de).\end{aligned}$$

Taking the supremum over $t\in[0,T]$ and expectation on both sides, using (62) and proceeding as in (53), we obtain

$$\begin{aligned}&\mathbb{E}[\sup_{0\le t\le T}\Gamma_t|\delta Y_t|^2]+(\kappa-2c_1^2c_f-c_2^2c_f)\mathbb{E}[\int_0^T\Gamma_s|\delta Y_s|^2ds]\le\frac{c_f}{c_1^2}\mathbb{E}[\int_0^T\Gamma_s(|\delta Z_s|^2+|\delta U_s|_s^2)ds\\ &\quad+\frac{c_fc_\rho^2T}{c_2^2}\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2+\int_0^T\Gamma_s(|\delta V_s|^2+|\delta W_s|_s^2)ds\big]\\ &\quad+2\mathbb{E}\big[\sup_{0\le t\le T}\big|\int_t^T\Gamma_s\delta Y_s^\intercal\delta Z_sdB_s\big|\big]+2\mathbb{E}\big[\sup_{0\le t\le T}\big|\int_t^T\Gamma_s\int_E\delta Y_{s-}\delta U_s(e)M(ds,de)\big|\big].\end{aligned}\tag{66}$$

By the Burkholder-Davis-Gundy inequality, there exists a positive constant $\bar{c}$ such that

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T\Gamma_s\delta Y_s^\intercal\delta Z_sdB_s\big|\big]&\le\bar{c}\mathbb{E}\Big[\Big(\int_0^Te^{2\kappa s}|Y_s|^2|Z_s|^2ds\Big)^{1/2}\Big]\\ &\le\bar{c}\mathbb{E}\big[\Big(\sup_{t\in[0,T]}\Gamma_t|Y_t|^2\Big)^{1/2}\Big(\int_0^T\Gamma_s|Z_s|^2ds\Big)^{1/2}\big].\end{aligned}$$

Therefore, by Young's inequality with $c_3 > 0$ and (65) we have

$$
\begin{aligned}
&2\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T \Gamma_s \delta Y_s^{\mathsf{T}} \delta Z_s dB_s\big|\big] \le \frac{\bar{c}}{c_3^2}\mathbb{E}\big(\sup_{t\in[0,T]}\Gamma_s|\delta Y_t|^2\big) + c_3^2\bar{c}\mathbb{E}\big[\int_0^T \Gamma_s |Z_s|^2 ds\big] \\
&\le \frac{\bar{c}}{c_3^2}\mathbb{E}\big(\sup_{t\in[0,T]}\Gamma_s|\delta Y_t|^2\big) + \\
&\qquad c_3^2\bar{c}\frac{c_1^2}{(c_1^2-c_f)}\frac{c_f c_\rho^2 T}{c_2^2}\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2 + \int_0^T \Gamma_s(|\delta V_s|^2 + |\delta W_s|_s^2)ds\big].
\end{aligned} \tag{67}
$$

Similarly, we can bound the last term in (66) as follows:

$$
\begin{aligned}
&2\mathbb{E}[\sup_{0\le t\le T}\big|\int_t^T \Gamma_s \int_E \delta Y_{s-}\delta U_s(e) M(ds,de)\big|] \le \frac{\bar{c}}{c_3^2}\mathbb{E}\big(\sup_{t\in[0,T]}\Gamma_s|\delta Y_t|^2\big) \\
&\quad + c_3^2\bar{c}\frac{c_1^2}{(c_1^2-c_f)}\frac{c_f c_\rho^2 T}{c_2^2}\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2 + \int_0^T \Gamma_s(|\delta V_s|^2 + |\delta W_s|_s^2)ds\big].
\end{aligned} \tag{68}
$$

Combining inequalities (65) through (68) gives

$$
\begin{aligned}
&(1-\frac{2\bar{c}}{c_3^2})\mathbb{E}[\sup_{0\le t\le T}\Gamma_t|\delta Y_t|^2] + (1-\frac{c_f}{c_1^2})\mathbb{E}[\int_0^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds] \\
&\le \big(\frac{c_f c_\rho^2 T}{c_2^2} + (1+2c_3^2\bar{c})\frac{c_1^2}{(c_1^2-c_f)}\frac{c_f c_\rho^2 T}{c_2^2}\big)\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2 + \int_0^T \Gamma_s(|\delta V_s|^2 + |\delta W_s|_s^2)ds\big].
\end{aligned}
$$

Finally, we can choose $c_1^2 = 2c_f$, $c_3^2 = 4\bar{c}$, fix c_2 large enough so that $K = \frac{2(3+16\bar{c}^2)c_f c_\rho^2 T}{c_2^2} < 1$, and we then have for $\kappa \ge 2c_1^2 c_f + c_2^2 c_f$ in $\Gamma = e^{\kappa\cdot}$:

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T}\Gamma_t|\delta Y_t|^2] + \mathbb{E}[\int_0^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds] \\
&\qquad \le K\mathbb{E}\big[\sup_{0\le s\le T}\Gamma_s|\delta X_s|^2 + \int_0^T \Gamma_s(|\delta V_s|^2 + |\delta W_s|_s^2)ds\big].
\end{aligned}
$$

Hence Φ is a contraction on the Banach space $\mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$ endowed with the norm defined by the square root of

$$
\mathbb{E}\big[\sup_{0\le t\le T}\Gamma_t|X_t|^2 + \int_0^T \Gamma_s(|V_s|^2 + |W_s|_s^2)ds\big],
$$

for any $(X,V,W) \in \mathcal{S}_2^l \times \mathcal{H}_2^l \times \widehat{\mathcal{H}}_2^l$.

In order to prove the geometrical convergence of the Picard iteration, we can apply the previous computations with the following correspondence:

$$
\delta Y = Y^{(n+1)} - Y^{(n)} \quad , \quad \delta X = Y^{(n)} - Y^{(n-1)},
$$

$$\delta Z = Z^{(n+1)} - Z^{(n)} \quad , \quad \delta V = Z^{(n)} - Z^{(n-1)},$$
$$\delta U = U^{(n+1)} - U^{(n)} \quad , \quad \delta W = U^{(n)} - U^{(n-1)},$$

which shows that

$$\|Y^{(n+1)} - Y^{(n)}\|^2_{\mathcal{S}^l_2} + \|Z^{(n+1)} - Z^{(n)}\|^2_{\mathcal{H}^l_2} + \|U^{(n+1)} - U^{(n)}\|^2_{\widehat{\mathcal{H}}^l_2} \leq CK^n. \tag{69}$$

B Invariance Times

For completeness we recall the results of Crépey and Song (2017) that are used in the proofs of Theorem 5.1 and Lemma 5.1.

We consider the enlargement of filtration setup corresponding to the condition (B) in Section 5.1, which we reproduce here for the reader's convenience:

Condition (B). $\forall t \geq 0$ and $B \in \mathfrak{G}_t$, $\exists B' \in \mathfrak{F}_t$ such that $B \cap \{t < \tau\} = B' \cap \{t < \tau\}$ ∎

The $\mathbb{F}$ Azéma supermartingale S of τ, with $\mathbb{F}$ predictable projection ${}^p\mathsf{S}$ (see Dellacherie and Meyer (1975)), admits the canonical Doob–Meyer decomposition $\mathsf{S} = \mathsf{Q} - \mathsf{D}$, where Q (with $\mathsf{Q}_0 = \mathsf{S}_0$) and D (with $\mathsf{D}_0 = 0$) are the $(\mathbb{F}, \mathbb{Q})$ local martingale component and the $(\mathbb{F}, \mathbb{Q})$ drift of the $(\mathbb{F}, \mathbb{Q})$ supermartingale S.

We work with $\mathbb{F}$ semimartingales and $(\mathbb{F}, \mathbb{P})$ and $(\mathbb{F}, \mathbb{Q})$ local martingales on a predictable set of interval type $\mathcal{I}$ as defined in He et al. (1992, Sect. VIII.3). We recall that, for any semimartingale X and for any predictable X integrable process L, X is a local martingale on $\mathcal{I}$ (respectively $Y = L \centerdot X$ on $\mathcal{I}$) means that

$$X^{\iota_n} \text{ is a local martingale (respectively } Y^{\iota_n} = L \centerdot (X^{\iota_n})) \tag{70}$$

holds for at least one, or equivalently any, nondecreasing sequence of stopping times $(\iota_n)_{n\geq 0}$ such that $\cup[0, \iota_n] = \mathcal{I}$.

The stochastic exponential of a semimartingale X is denoted by $\mathcal{E}(X)$.

Lemmas 2.2 and A.1 in Crépey and Song (2017) *Under the condition (B):*

1 *Let M be a $(\mathbb{G}, \mathbb{Q})$ local martingale stopped before τ. For any $\mathbb{F}$ optional reduction M' of M, M' is an $\mathbb{F}$ semimartingale on $\{\mathsf{S}_- > 0\}$ and*

$$\mathsf{S}_- \centerdot M' + [\mathsf{S}, M'] \text{ is an } (\mathbb{F}, \mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_- > 0\}; \tag{71}$$

Conversely, for any $\mathbb{F}$ semimartingale K on $\{\mathsf{S}_- > 0\}$ such that $\mathsf{S}_- \centerdot K + [\mathsf{S}, K]$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{\mathsf{S}_- > 0\}$, $K^{\tau-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $\mathbb{R}_+$.

2 *The Azéma supermartingale S of τ admits the multiplicative decomposition*

$$\mathsf{S} = \mathsf{S}_0 \mathcal{Q}\mathcal{D} \text{ on } \{{}^p\mathsf{S} > 0\}, \tag{72}$$

where $\mathcal{Q} = \mathcal{E}(\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})$ *is an* $(\mathbb{F}, \mathbb{Q})$ *local martingale on* $\{{}^p\mathsf{S} > 0\}$ *and* $\mathcal{D} = \mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D})$ *is an* $\mathbb{F}$ *predictable nonincreasing process on* $\{{}^p\mathsf{S} > 0\}$*. Moreover, if* τ *has a* $(\mathbb{G}, \mathbb{Q})$ *intensity* γ*, then* D *is continuous and*

$$\mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D}) = e^{-\int_0^{\cdot} \gamma'_t dt} \tag{73}$$

holds on $\{\mathsf{S}_- > 0\}$. ∎

Likewise, we reproduce here the condition (A) from Section 5.1:

Condition (A) For any $(\mathbb{F}, \mathbb{P})$ local martingale P on $[0, T]$, $P^{\tau-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$. ∎

Theorem 3.2 in Crépey and Song (2017) *The condition (A) holds if and only if the* $(\mathbb{F}, \mathbb{Q})$ *density process of* $\mathbb{P}$ *coincides with*

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})_{\cdot \wedge T} \tag{74}$$

on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$. ∎

Theorem 3.7 in Crépey and Song (2017) *If the condition (A) holds and* τ *has a* $(\mathbb{G}, \mathbb{Q})$ *intensity, then*

$$\{\mathsf{S}_- > 0\} = \{{}^p\mathsf{S} > 0\} = \{\mathsf{S} > 0\}. \tag{75}$$

In addition:

$$\begin{gathered} \text{A process } P \text{ is an } (\mathbb{F}, \mathbb{P}) \text{ local martingale on } \{\mathsf{S}_- > 0\} \cap [0, T] \\ \text{if and only if} \\ \mathsf{S}_- \centerdot P + [\mathsf{S}, P] \text{ is an } (\mathbb{F}, \mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_- > 0\} \cap [0, T]. \end{gathered} \tag{76}$$ ∎

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