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INFORMATIONALLY DYNAMIZED GAUSSIAN COPULA

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In order to dynamize the static Gaussian copula model of portfolio credit risk, we introduce a model filtration made of a reference Brownian filtration progressively enlarged by the default times. This yields a multidimensional density model of default times, where, as opposed to the classical situation of the Cox model, the reference filtration is not immersed into the enlarged filtration. In mathematical terms this lack of immersion means that martingales in the reference filtration are not martingales in the enlarged filtration. From the point of view of financial interpretation this means default contagion, a good feature in the perspective of modeling counterparty wrong-way risk on credit derivatives. Computational tractability is ensured by invariance of multivariate Gaussian distributions through conditioning by some components, the ones corresponding to past defaults. Moreover the model is Markov in an augmented state-space including past default times. After a discussion of different notions of deltas, the model is applied to the valuation of counterparty risk on credit derivatives.

Keywords: Gaussian copula; dynamic copula; credit derivatives; counterparty risk; CVA; hedging.

1. Introduction

This work is an attempt, a bit in the spirit of Fermanian and Vigneron [18], to dynamize the static Gaussian copula model of portfolio credit risk [20]. As in [18], one could talk of informational dynamization in the sense that our dynamization takes the route of introducing a filtration with respect to which conditional expectations are computed to give prices at future times. However, whereas [18] uses a Brownian filtration, which is tantamount to "not observing" the defaults as they occur, we use a Brownian filtration progressively enlarged by the default times. Moreover, the construction and presentation of the model is completely different from [18], where a structural approach is used. In our case we rely on a conditional

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density approach as introduced in El Karoui *et al.* in [17]. An aside contribution of this paper is thus to provide a concrete and workable example of a conditional density model. The model of this paper can be used for dynamic valuation and hedging of counterparty risk on credit derivatives. This allows one to assess the related model risk by comparing the CVA (credit valuation adjustment accounting for the counterparty risk) computed in this model with the one in the common shocks model of [7], on the benchmark problem of counterparty risk on one CDS.

Section 2 presents the model. Section 3 deals with the pricing of CDS and CDO tranches and examines the issue of hedging a CDO tranche by CDS, comparing this model deltas with the static Gaussian copula bump-sensitivities. Section 4 applies this model to CVA computations on credit derivatives.

2. Model

In this section, we introduce some notation that will be used all along the paper, and we specify the model we are working with.

2.1. *Exchangeable Gaussian distributions*

Let us introduce some notation:

- I denotes a generic subset of $N = \{1, \ldots, n\}$ with complement set $J = N \backslash I$ and cardinality $|I|$,
- Φ (respectively ϕ) is the standard Gaussian survival function (respectively density function),
- $\Phi_{\rho,\sigma}((z_j)_{j\in J})$ is the survival function evaluated at $(z_j)_{j\in J}$ of a $|J|$-dimensional (ρ, σ)-exchangeable distribution, or distribution of $|J|$-dimensional centered Gaussian vector with homogenous variances σ^2 and pairwise correlations ρ.

The exchangeable terminology stands in reference to the fact that $\Phi_{\rho,\sigma}((z_j)_{j\in J})$ is invariant by permutation of $(z_j)_{j\in J}$. The following straightforward result is the key of the static Gaussian copula model [20].

Lemma 2.1 *If $(Z_j)_{j\in J}$ is (ρ, σ)-exchangeable, then*

$$(Z_j, j \in J) \stackrel{\mathcal{L}}{=} (\sigma(\sqrt{\rho}Y + \sqrt{1-\rho}Y_j), j \in J)$$

where $(Y, Y_1, \ldots, Y_{|J|})$ are i.i.d. standard Gaussian random variables.

Therefore

$$\Phi_{\rho,\sigma}((z_j)_{j\in J}) = \int_{\mathbb{R}} \prod_{j\in J} \Phi\left(\frac{z_j - \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\right) \phi(y)\, dy,$$

$$\partial_{z_k}\Phi_{\rho,\sigma}((z_j)_{j\in J}) = \int_{\mathbb{R}} \frac{-1}{\sigma\sqrt{1-\rho}} \phi\left(\frac{z_k - \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\right) \prod_{j\in J\backslash\{k\}} \Phi\left(\frac{z_j - \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\right) \phi(y)\, dy. \tag{2.1}$$

The following stability result of exchangeable distributions through conditioning components by others belongs to the folklore of Gaussian distributions. Since this is key in this paper, we provide a proof in Appendix A.1.

Lemma 2.2 *If $(X_l)_{l\in N}$ is $(\varrho,1)$-exchangeable, then*

$$((X_j)_{j\in J}\,|\,(X_i)_{i\in I}) \overset{\mathcal{L}}{=} (\mu + Z_j)_{j\in J}$$

where $(Z_j)_{j\in J}$ is (ρ,σ)-exchangeable, with

$$\rho = \frac{\varrho}{|I|\varrho+1}, \quad \sigma^2 = \frac{(|I|-1)\varrho+1-\varrho^2|I|}{(|I|-1)\varrho+1}, \quad \mu = \frac{\varrho\sum_{i\in I}X_i}{(|I|-1)\varrho+1}. \tag{2.2}$$

2.2. *Model of default times*

One considers a probability space $(\Omega,\mathbb{F},\mathbb{P})$ where $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ is the completed filtration of an n-dimensional Brownian motion $B=(B^1,B^2,\ldots,B^n)$, while $\mathbb{P}$ represents a pricing measure chosen by the market. The components of B are mutually correlated with a constant ϱ, that is $d\langle B^l,B^k\rangle_t = \varrho dt$ for any two different indices l and k. For any $l\in N$, let h_l be a differentiable increasing function from $\mathbb{R}_+$ to $\mathbb{R}$ with $\lim_0 h_l(s)=-\infty$ and $\lim_{+\infty} h_l(s)=+\infty$. We define n random times (positive random variables) on $(\Omega,\mathcal{F}_\infty,\mathbb{P})$ by, for every $l\in N$,

$$\tau_l = h_l^{-1}\left(\int_0^{+\infty}\varsigma(v)\,dB_v^l\right), \tag{2.3}$$

where $\varsigma(\cdot)$ is a square integrable function with unit L_2-norm. So the τ_l jointly follow a standard (static) Gaussian copula model [20] with correlation parameter ϱ and with marginal survival function $\Phi\circ h_l$ of τ_l. We denote $\sigma^2(t)=\int_t^{+\infty}\varsigma^2(v)$, assumed positive for every t (and $\sigma(0)=1$). Letting $m_t=(m_t^l)_{l\in N}$ with $m_t^l=\int_0^t\varsigma(v)\,dB_v^l$, we introduce for fixed t the $(\varrho,1)$-exchangeable vector $X_t=(X_t^l)_{l\in N}$ with

$$X_t^l := \frac{1}{\sigma(t)}\int_t^{+\infty}\varsigma(v)\,dB_v^l = \frac{h_l(\tau_l)-m_t^l}{\sigma(t)}.$$

Hence, for every real t, t_l,

$$\{\tau_l > t_l\} = \left\{X_t^l > \frac{h_l(t_l)-m_t^l}{\sigma(t)}\right\} \tag{2.4}$$

where m_t^l is $\mathcal{F}_t$-measurable. Note that X_t is independent from $\mathcal{F}_t$. This allows one to derive the following formula for the joint survival probability given $\mathcal{F}_t$, which at time 0 reduces to the well-known Gaussian copula formula [20].

Lemma 2.3 *For every t and $(t_l)_{l\in N}$,*

$$\mathbb{P}(\tau_l > t_l, l\in N\,|\,\mathcal{F}_t) = \int_{\mathbb{R}}\prod_{l=1}^n \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_l>t_l\,|\,\mathcal{F}_t))-\sqrt{\varrho}y}{\sqrt{1-\varrho}}\right)\phi(y)\,dy > 0.$$

Proof. In view of (2.4) where X_t is $(\varrho, 1)$-exchangeable and independent from $\mathcal{F}_t$, one gets from (2.1) that

$$\mathbb{P}(\tau_l > t_l, l \in N \,|\, \mathcal{F}_t) = \int_{\mathbb{R}} \prod_{l=1}^{n} \Phi\left(\frac{h_l(t_l) - m_t^l - \sigma(t)\sqrt{\varrho}y}{\sigma(t)\sqrt{1-\varrho}}\right)\phi(y)\,dy$$

whereas by (2.4)

$$\mathbb{P}(\tau_l > t_l \,|\, \mathcal{F}_t) = \Phi\left(\frac{h_l(t_l) - m_t^l}{\sigma(t)}\right) > 0$$

in which the positivity results from $\sigma^2(t) > 0$. □

For any $I \subseteq N$, we define the filtration $\mathbb{G}^I = (\mathcal{G}_t^I, t \geq 0)$ as the initial enlargement of $\mathbb{F}$ by the τ_i for $i \in I$, so

$$\mathcal{G}_t^I = \mathcal{F}_t \vee \bigvee_{i \in I} \sigma(\tau_i \wedge t).$$

Note that this filtration is right-continuous by a straightforward multidefault extension of the results of [1], and it is also complete since $\mathbb{F}$ is the completed filtration of B and the τ_i are in $\mathcal{F}_\infty$.

Let $\tau(I) = (\tau_l(I))_{l \in N}$ with $\tau_l(I) = \tau_l \mathbb{1}_{\{l \in I\}}$. Let also $\mathrm{supp}(\theta) = \{l \in N : \theta_l \neq 0\}$, for every θ in $\mathbb{R}_+^n$.

Lemma 2.4 *For every bounded Borel function φ from $\mathbb{R}^{|J|}$ to $\mathbb{R}$, one has*

$$\mathbb{E}[\varphi(h_j(\tau_j), j \in J) \,|\, \mathcal{G}_t^I] = \Gamma_\varphi(t, m_t, \tau(I))$$

in which for every s in $\mathbb{R}_+$, $m = (m_l)_{l \in N}$ in $\mathbb{R}^n$ and $\theta = (\theta_l)_{l \in N}$ in $\mathbb{R}_+^n$

$$\Gamma_\varphi(s, m, \theta) = \mathbb{E}[\varphi(m_j + \sigma(s)(\mu + Z_j), j \notin \mathrm{supp}(\theta))]$$

where $(Z_j)_{j \notin \mathrm{supp}(\theta)}$ is (ρ, σ)-exchangeable; In these expressions ρ, σ and μ are determined as in (2.2) for I and X_i in (2.2) respectively given by $\mathrm{supp}(\theta)$ and $\frac{h_i(\theta_i) - m_i}{\sigma(s)}$, which we denote henceforth

$$\rho = \rho(s, m, \theta), \quad \sigma = \sigma(s, m, \theta), \quad \mu = \mu(s, m, \theta) \tag{2.5}$$

meaning that ρ, σ and μ are deterministic functions of (s, m, θ). In particular (recall $J = N \backslash I$)

$$\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I) = \Phi_{\rho,\sigma}\left(\frac{h_j(s) - m_j}{\sigma(s)} - \mu, j \notin \mathrm{supp}(\theta)\right)\bigg|_{(s,m,\theta)=(t,m_t,\tau(I))} > 0. \tag{2.6}$$

Proof. By $\mathcal{F}_t$-measurability of m_t, one has,

$$
\begin{aligned}
&\mathbb{E}[\varphi(h_j(\tau_j), j \in J) \,|\, \mathcal{G}_t^I] \\
&\quad = \mathbb{E}\left[\varphi\left(\int_0^{+\infty} \varsigma(u)\, dB_u^j, j \in J\right) \middle| \mathcal{F}_t \vee \bigvee_{i \in I} \sigma(\tau_i)\right] \\
&\quad = \mathbb{E}\left[\varphi\left(m_j + \int_t^{+\infty} \varsigma(u)\, dB_u^j, j \in J\right) \middle| \mathcal{F}_t \vee \bigvee_{i \in I} \left(\frac{h_i(\tau_i) - m_t^i}{\sigma(t)}\right)\right]\Bigg|_{m_j = m_t^j, j \in J} \\
&\quad = \mathbb{E}\left[\varphi(m_j + \sigma(t) X_t^j, j \in J) \middle| \mathcal{F}_t \vee \bigvee_{i \in I} X_t^i\right]\Bigg|_{m_j = m_t^j, j \in J}
\end{aligned}
$$

which, by independence of the X_t^i from $\mathcal{F}_t$, boils down to

$$
\mathbb{E}[\varphi(m_j + \sigma(t) X_j, j \in J) \,|\, X_i, i \in I]\big|_{m_j = m_t^j, j \in J}.
$$

The result then follows by an application of Lemma 2.2. As a special case one gets (2.6), where the positivity in the right-hand side results from $\sigma(t) > 0$. □

The natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ of $B = (B^l)_{l \in N}$ is used as a reference filtration. The full model filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is then defined as the progressive enlargement of $\mathbb{F}$ by the τ_l, that is

$$
\mathcal{G}_t = \mathcal{F}_t \vee \bigvee_{l \in N} \sigma(\tau_l \wedge t).
$$

Note that this filtration can be shown to be right-continuous by a combination of the arguments of [1] and of the Appendix[1] of [3], see [16].

2.2.1. *Conditional survival distribution*

For pricing purposes, one needs to compute the conditional expectation of cash flow given the model information $\mathcal{G}_t$. To this end we recall the following classical result (see p. 143 of [6]) which is sometimes referred to as "the key lemma" in a single default credit risk modeling framework: If X is an integrable random variable, then for every t

$$
\mathbb{1}_{\{\tau > t\}} \mathbb{E}[X \,|\, \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau > t\}} \,|\, \mathcal{F}_t)}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)}.
$$

The multidimensional counterpart of this result is stated in the following lemma. Let $\theta_t = (\theta_t^l)_{l \in N}$ with $\theta_t^l = \tau_l \mathbb{1}_{\{\tau_l \leq t\}}$; let

$$
I_t = \{i \in N \,|\, \tau_i \leq t\} = \operatorname{supp}(\theta_t), \quad \text{resp. } J_t = N \backslash I_t
$$

[1] Available online, not present in the *Mathematical Finance* published version of the paper.

denote the random set of the indices of the obligor in default, resp. alive, at time t, so for every fixed $I \subseteq N$

$$\{I_t = I\} = \{\tau_i \le t, i \in I; \tau_j > t, j \in J\}. \tag{2.7}$$

We also denote, for every random function f_t of I

$$\mathbb{E}(f_t(I_t) \,|\, \mathcal{G}_t^{I_t}) := \sum_{I \subseteq N} \mathbb{1}_{\{I_t = I\}} \mathbb{E}(f_t(I) \,|\, \mathcal{G}_t^I). \tag{2.8}$$

Lemma 2.5 *For every integrable random variable X,*

$$\mathbb{E}[X \,|\, \mathcal{G}_t] = \frac{\mathbb{E}(X \mathbb{1}_{\{\tau_j > t, j \in J_t\}} \,|\, \mathcal{G}_t^{I_t})}{\mathbb{P}(\tau_j > t, j \in J_t \,|\, \mathcal{G}_t^{I_t})}.$$

Proof. On the set $\{I_t = I\}$, any $\mathcal{G}_t$-measurable random variable is equal to a $\mathcal{G}_t^I$-measurable random variable X_t^I, so

$$\mathbb{E}[\mathbb{1}_{\{I_t = I\}} X \,|\, \mathcal{G}_t] = \mathbb{1}_{\{I_t = I\}} \mathbb{E}[X \,|\, \mathcal{G}_t] = \mathbb{1}_{\{I_t = I\}} X_t^I. \tag{2.9}$$

Taking conditional expectation given $\mathcal{G}_t^I$ both sides and multiplying by $\{I_t = I\}$, one obtains

$$\mathbb{1}_{\{I_t = I\}} \mathbb{E}[\mathbb{1}_{\{I_t = I\}} X \,|\, \mathcal{G}_t^I] = \mathbb{1}_{\{I_t = I\}} X_t^I \mathbb{E}[\mathbb{1}_{\{I_t = I\}} \,|\, \mathcal{G}_t^I]. \tag{2.10}$$

By (2.7) where $\{\tau_i \le t, i \in I\}$ is $\mathcal{G}_t^I$-measurable, (2.10) boils down to

$$\mathbb{1}_{\{I_t = I\}} \mathbb{E}[\mathbb{1}_{\{\tau_j > t, j \in J\}} X \,|\, \mathcal{G}_t^I] = \mathbb{1}_{\{I_t = I\}} X_t^I \mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I).$$

We saw in (2.6) that $\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I) > 0$. One can thus substitute

$$\mathbb{1}_{\{I_t = I\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau_j > t, j \in J\}} X \,|\, \mathcal{G}_t^I]}{\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I)}$$

for $\mathbb{1}_{\{I_t = I\}} X_t^I$ in (2.9), which yields the result (in notation (2.8)). □

In the following proposition, we compute the conditional survival probability of $(\tau_l)_{l \in N}$ defined by, for every positive t, t_l,

$$G_t(t_1, t_2, \ldots, t_n) = \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_n > t_n \,|\, \mathcal{G}_t),$$

and in particular the individual pre-default conditional survival probability $G_t^l(v)$ which satisfies, for $v \ge t$

$$\mathbb{P}(\tau_l > v \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau_l > t\}} G_t^l(v).$$

Recalling (2.5), let

$$\rho_t, \sigma_t, \mu_t = \rho, \sigma, \mu(t, m_t, \theta_t).$$

We denote (the second equality arising from (2.6))

$$D_t := \mathbb{P}(\tau_j > t, j \in J_t \,|\, \mathcal{G}_t^{I_t}) = \Phi_{\rho_t,\sigma_t}((Z_t^j(t))_{j\in J_t}) \tag{2.11}$$

with for every $j \in J_t$ and $v \geq t$

$$Z_t^j(v) = \frac{h_j(v) - m_t^j}{\sigma(t)} - \mu_t.$$

Proposition 2.1 *One has for every t and $(t_l)_{l\in N}$*

$$D_t G_t(t_1, t_2, \ldots, t_n) = \mathbb{1}_{\{\tau_i > t_i, i \in I_t\}} \Phi_{\rho_t,\sigma_t}((Z_t^j(t \vee t_j))_{j\in J_t}) \tag{2.12}$$

and for every l and $v \geq t$

$$D_t G_t^l(v) = \Phi_{\rho_t,\sigma_t}(Z_t^l(v), (Z_t^j(t))_{j\in J_t\setminus\{l\}}) \tag{2.13}$$

with the abuse of notation that an argument $(z_l, (z_j)_{j\in J_t\setminus\{l\}})$ of Φ is to be understood as a $|J_t|$-dimensional vector (the ordering of the components does not matter by exchangeability of the distributions Φ).

Proof. By Lemma 2.5, one has

$$G_t(t_1, t_2, \ldots, t_n) = \sum_{I\subseteq N} \mathbb{1}_{\{I_t = I\}} \frac{\mathbb{P}(\tau_i > t_i, i \in I; \tau_j > t \vee t_j, j \in J \,|\, \mathcal{G}_t^I)}{\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I)}$$

$$= \sum_{I\subseteq N} \mathbb{1}_{\{I_t = I\}} \mathbb{1}_{\{\tau_i > t_i, i \in I\}} \frac{\mathbb{P}(\tau_j > t \vee t_j, j \in J \,|\, \mathcal{G}_t^I)}{\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I)}$$

where by Lemma 2.4 the numerator $\mathbb{P}(\tau_j > t \vee t_j, j \in J \,|\, \mathcal{G}_t^I)$ equals $\Phi_{\rho_t,\sigma_t}(Z_t^j(t \vee t_j), j \in J)$ and the denominator $\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I)$ equals $\Phi_{\rho_t,\sigma_t}(Z_t^j(t), j \in J)$. This proves (2.12), from which (2.13) follows by an application of (2.12) for $t_l = v \geq t$ and $t_j = 0$ for $j \neq l$. □

That the "effective" Gaussian copula parameter ρ_t reacts dynamically to defaults is a sign of credit risk contagion, a realistic feature of a credit portfolio model.

2.3. *Fundamental martingales*

We denote by $H_t^l = \mathbb{1}_{\{\tau_l \leq t\}}$ the lth default indicator process.

2.3.1. *Univariate case*

Let us first consider the case $n = 1$ of one default time. We drop any index $l = 1$ in the notation, e.g., $\tau_1 = \tau$. The following result shows that our model is an example of a density model in the sense of El Karoui *et al.* [17], as Eq. (2.14) provides a continuous version of the $\mathbb{F}$-conditional density of τ with respect to the Lebesgue measure on the half-line.

Lemma 2.6 *The $\mathbb{F}$-conditional density of τ is given by, for every t and v in $\mathbb{R}_+$,*

$$f_t(v) := \frac{\mathbb{P}(\tau \in dv \,|\, \mathcal{F}_t)}{dv} = \phi\left(\frac{h(v) - m_t}{\sigma(t)}\right)\frac{h'(v)}{\sigma(t)}. \tag{2.14}$$

Proof. In view of (2.4), the $\mathbb{F}$-conditional survival probability of τ is given as

$$F_t(v) = \mathbb{P}(\tau > v \,|\, \mathcal{F}_t) = \Phi\left(\frac{h(v) - m_t}{\sigma(t)}\right)$$

which differentiates with respect to v into (2.14). □

As $F_t(v) \neq F_v(v)$ for $t > v$, the reference filtration $\mathbb{F}$ is not immersed into the full model filtration $\mathbb{G}$. Also, under immersion, the Azéma supermartingale F of τ would be non-increasing in t, which cannot be in view of the right-hand side in the following Eq. (2.15).

Lemma 2.7 *Let F be the Azéma supermartingale of τ, i.e., $F_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$. The dynamics of $f(v)$ and F are*

$$df_t(v) = f_t(v)\alpha_t(v)dB_t, \quad dF_t = -f_t(t)dt + \beta_t dB_t \tag{2.15}$$

with

$$\alpha_t(v) = -\frac{(m_t - h(v))}{\sigma(t)}\frac{\varsigma(t)}{\sigma(t)}$$

$$\beta_t = \phi\left(\frac{m_t - h(t)}{\sigma(t)}\right)\frac{\varsigma(t)}{\sigma(t)}.$$

Proof. Noting that $\phi'(x) = -x\phi(x)$, an application of Itô's formula to the right-hand side of (2.14) yields the left-hand side in (2.15); the right-hand side follows from the Itô-Ventcell formula $dF_t = d_t F_t(v)_{|t=v} + d_v F_t(v)_{|t=v}$. □

An application of a result of Jeanblanc and Le Cam [19] (specified to the case of a density model with F and f continuous) shows that every $\mathbb{F}$-local martingale $\tilde{X}$ is a $\mathbb{G}$-special semimartingale with the following canonical decomposition:

$$\tilde{X}_t = X_t + \int_0^{t\wedge\tau}\frac{d\langle\tilde{X}, F\rangle_u}{F_u} + \left(\int_{t\wedge v}^{t}\frac{d\langle\tilde{X}, f(v)\rangle_u}{f_u(v)}\right)_{|v=\tau} \tag{2.16}$$

where X is a $\mathbb{G}$-local martingale. In particular, the following $\mathbb{G}$-canonical decomposition of the $\mathbb{F}$-Brownian motion B follows from (2.16):

$$\begin{aligned} B_t &= W_t + \int_0^{t\wedge\tau}\frac{d\langle B, F\rangle_u}{F_u} + \left(\int_{t\wedge v}^{t}\frac{d\langle B, f(v)\rangle_u}{f_u(v)}\right)_{|v=\tau} \\ &= W_t + \int_0^{t\wedge\tau}\frac{\beta_u}{F_u}du + \left(\int_{t\wedge v}^{t}\alpha_u(v)du\right)_{|v=\tau} \end{aligned} \tag{2.17}$$

where W is a continuous $\mathbb{G}$-martingale with the same predictable bracket t as the Brownian motion B, hence a $\mathbb{G}$-Brownian motion; and α and β were defined in Lemma 2.7.

Besides, by application of the results of Sec. 4 of [17], the $\mathbb{G}$-compensated martingale of the default indicator process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ is given by

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_v \, dv \tag{2.18}$$

for a pre-default intensity of τ given as $\lambda_t = \frac{f_t(t)}{F_t}$.

2.3.2. *Portfolio case*

In the portfolio case with an arbitrary number n of obligors, an immediate multi-default extension of the results of Jeanblanc and Le Cam [19] shows that one has for every l a $\mathbb{G}$-Brownian motion and a compensated jump-to-default $\mathbb{G}$-martingale of the form

$$W_t^l = B_t^l - \int_0^t \gamma_v^l \, dv, \quad M_t^l = H_t^l - \int_0^{t\wedge\tau_l} \lambda_v^l \, dv.$$

Moreover the family of the W^l and M^l has the $\mathbb{G}$-martingale representation property. We denote by I_{t-}, resp. J_{t-}, the left-limit of I_t, resp. J_t (random set of obligors in default, resp. alive, "right before t"). By order set, we mean any subset of the state space $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+^n$ made of all the triplets (s, m, θ) corresponding to a given support I of θ and with the components of θ in I in a certain order (strict order, say, which is enough for our purpose since there are no joint defaults in a Lebesgue-density model).

Proposition 2.2 (i) *Denoting* $\gamma_t = (\gamma_t^l)_{l\in N}, \lambda_t = (\lambda_t^l)_{l\in N}$, *one has*

$$\gamma_t = \gamma(t, m_t, \theta_t), \quad \lambda_t = \lambda(t, m_t, \theta_t)$$

for Borel functions γ *and* λ, *differentiable in* m *on every order set.*

(ii) *The following Itô formula holds, for every function* $u = u(s, m, \theta)$ *of class* $\mathcal{C}^{1,2}$ *in* (s, m)

$$du(t, m_t, \theta_t) = \varsigma(t) \sum_{l\in N} \partial_{m_l} u(t, m_t, \theta_t) \, dW_t^l + \sum_{j\in J_{t-}} \delta_j u(t, m_t, \theta_{t-}) \, dM_t^j$$
$$+ \mathcal{A}u(t, m_t, \theta_t) \, dt \tag{2.19}$$

where

$$\mathcal{A}u = \partial_s u + \varsigma \sum_{l\in N} \gamma_l \partial_{m_l} u + \frac{\varsigma^2}{2} \left(\sum_{l\in N} \partial^2_{m_l^2} u + \varrho \sum_{l,k\in N, l\neq k} \partial^2_{m_l, m_k} u \right) + \sum_{l\in N} \lambda_l \delta_l u$$

and

$$\delta_l u(s,m,\theta) = u(s,m,\theta^{l,s}) - u(s,m,\theta)$$

in which $\theta^{l,s}$ *stands for* θ *with* θ_l *replaced by* s.

(iii) *The process* (t, m_t, θ_t) *is* $\mathbb{G}$*-Markov with generator* $\mathcal{A}$.

Proof. Part (i) follows from arguments similar to the ones developed in Cousin *et al.* [11]. The Itô Formula (2.19) in part (ii) is a standard Itô formula between the τ_l, amended in the obvious way to account for the jumps of θ_t at the τ_l (which in this model cannot occur simultaneously). This Itô formula where $\mathcal{A}$ is, by part (i), deterministic, implies the Markov property of part (iii). □

We refer the reader to [16] for explicit formulas for the functions $\gamma(s,m,\theta)$ and $\lambda(s,m,\theta)$ in the case of two obligors.

3. Valuation and Hedging of Credit Derivatives

In this section we address the issues of valuation and hedging of credit derivatives in the above dynamized Gaussian copula setup. Note this is in a counterparty risk-free environment, so clean valuation and hedging in this sense, as opposed to the counterparty risky setup of the next section.

For notational convenience, we assume zero interest rates. The extension of the theoretical results to deterministic interest rates is straightforward but cumbersome notionally. Deterministic interest rates $r(t)$ will be used however in the numerical applications.

In a zero interest rates environment, the (ex-dividend) price process of an asset is simply given by the risk neutral conditional expectation of future cash flow. The martingale cumulative value process is the sum of the price process and of the cumulative cash flow process.

3.1. *Pricing of a CDS*

The cumulative cash-flow of a credit default swap (CDS) on firm l with maturity T and recovery R_l (assumed constant) is, assuming swapped and continuously paid fees,

$$\int_0^T (1-R_l)\,dH_v^l - \int_0^{T\wedge\tau_l} \kappa_l\,dv$$

where the constant κ_l represents the contractual spread.

Proposition 3.1 (i) *The price process of a CDS on firm* l *is given as* $\mathbb{1}_{\{\tau_l>t\}}C_t^l$ *for a pre-default price*

$$C_t^l = C_l(t,m_t,\theta_t) = (1-R_l)(1-G_t^l(T)) - \kappa_l \int_t^T G_t^l(v)\,dv \tag{3.1}$$

where the explicit form of $G_t^l(v) = G_l(v;t,m_t,\theta_t)$ *is given in* (2.13);

(ii) *The dynamics of the cumulative price of a CDS on firm l is given as*

$$d\hat{C}^l_t = \mathbb{1}_{\{\tau_l > t\}} \left[\varsigma(t) \sum_{k \in N} \partial_{m_k} C_l(t, m_t, \theta_t) dW^k_t + (1 - R_l - C_l(t, m_t, \theta_{t-})) dM^l_t \right.$$
$$\left. + \sum_{j \in J_{t-} \setminus \{l\}} \delta_j C_l(t, m_t, \theta_{t-}) dM^j_t \right]. \tag{3.2}$$

Proof. One has

$$\begin{aligned}
\mathbb{1}_{\{\tau_l > t\}} C^l_t &= \mathbb{E}\left[\int_t^T (1 - R_l) dH^l_v - \int_t^{T \wedge \tau_l} \kappa_l \, dv \,|\, \mathcal{G}_t \right] \\
&= (1 - R_l)\mathbb{P}(t < \tau_l \le T \,|\, \mathcal{G}_t) - \kappa_l \int_t^T \mathbb{P}(\tau_l > v \,|\, \mathcal{G}_t) dv \\
&= \mathbb{1}_{\{\tau_l > t\}} \left[(1 - R_l)(1 - G^l_t(T)) - \kappa_l \int_t^T G^l_t(v) \, dv \right]
\end{aligned}$$

where the third equality follows from (2.13) and $G^l_t(t) = 1$. This proves part (i). Since a cumulative price can be seen as the martingale component of a (special semimartingale) price process, part (ii) immediately follows from part (i) by an application of the Itô-Markov Formula (2.19) to process $C_l(t, m_t, \theta_t)$. □

3.2. *Pricing of a CDO*

We denote by $N_v = \sum_{l=1}^n \mathbb{1}_{\{\tau_l \le v\}}$ the number of defaults at time v (or cardinality of I_v). The conditional distribution

$$\Gamma^\ell_t(v) = \mathbb{P}(N_v = \ell \,|\, \mathcal{G}_t)$$

is the key in the pricing of CDO tranches. It can be computed thanks to the next lemma, in which the c_ℓ can be efficiently computed for by standard recursive procedures (see, e.g., [2]).

Lemma 3.1 *One has for every $v \ge t$ and $|I_t| \le \ell \le n$*

$$D_t \Gamma^\ell_t(v) = \int_{\mathbb{R}} c^y_\ell(v; t, m_t, \theta_t) \phi(y) dy =: \Gamma_\ell(v; t, m_t, \theta_t) \tag{3.3}$$

where $c^y_\ell(v; s, m, \theta)$ is the order $(\ell - |I_t|)$-coefficient of the polynomial P^y in x parameterized by a real y (and its other arguments v and s, m, θ) given as

$$P^y(x, v; s, m, \theta) = \prod_{j \notin \mathrm{supp}(\theta)} (p^y_j(v) x + q^y_j(v))$$

in which $p_j^y(v) = p_j^y(v; s, m, \theta)$ *and* $q_j^y(v) = q_j^y(v; s, m, \theta)$ *are shorthand notation for*

$$\begin{aligned} p_j^y(v) &= \Phi\left(\frac{h_j(s) - m_j - \sigma(s)\mu - \sigma(s)\sigma\sqrt{\rho}y}{\sigma(s)\sigma\sqrt{1-\rho}}\right) \\ &\quad - \Phi\left(\frac{h_j(v) - m_j - \sigma(s)\mu - \sigma(s)\sigma\sqrt{\rho}y}{\sigma(s)\sigma\sqrt{1-\rho}}\right) \\ q_j^y(v) &= \Phi\left(\frac{h_j(v) - m_j - \sigma(s)\mu - \sigma(s)\sigma\sqrt{\rho}y}{\sigma(s)\sigma\sqrt{1-\rho}}\right) \end{aligned} \tag{3.4}$$

with ρ, σ *and* μ *as in Lemma* 2.4.

Proof. An application of Lemma 2.5 yields

$$\mathbb{P}(N_v = \ell \,|\, \mathcal{G}_t) = \sum_{I \subseteq N} \mathbb{1}_{\{I_t = I\}} \frac{\mathbb{P}(N_v = \ell; \tau_j > t, j \in J \,|\, \mathcal{G}_t^I)}{\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I)}$$

in which the denominator $\mathbb{P}(\tau_j > t, j \in J \,|\, \mathcal{G}_t^I)$ is given by (2.11). For the numerator, setting $N_v^J = \sum_{j \in J} \mathbb{1}_{\{\tau_j \le v\}}$, one has, on $\{I_t = I\}$,

$$\mathbb{P}(N_v = \ell; \tau_j > t, j \in J \,|\, \mathcal{G}_t^I) = \mathbb{P}(N_v^J = \ell - |I|; \tau_j > t, j \in J \,|\, \mathcal{G}_t^I).$$

Recalling (2.4), one can thus use Lemma 2.4, choosing φ such that

$$\varphi(h_j(\tau_j), j \in J) = \mathbb{1}_{\{N_v^J = \ell - |I|; \tau_j > t, j \in J\}}$$

in order to get

$$\mathbb{P}(N_v^J = \ell - |I|; \tau_j > t, j \in J \,|\, \mathcal{G}_t^I) = \Gamma_\varphi(v; t, m_t, \theta_t)$$

with

$$\begin{aligned} \Gamma_\varphi(v; s, m, \theta) &= \mathbb{E}[\varphi(m_j + \sigma(s) Z_j, j \in J)] \\ &= \mathbb{P}\left(\sum_{j \in J} \mathbb{1}_{\{h_j(s) < m_j + \sigma(s)\mu + \sigma(s) Z_j \le h_j(v)\}} = \ell - |I|; h_j(s) < m_j \right. \\ &\qquad \left. + \sigma(s)\mu + \sigma(s) Z_j, j \in J \right) \end{aligned}$$

for some (ρ, σ)-exchangeable $(Z_j, j \in J)$. We apply Lemma 2.1 to represent $(Z_j, j \in J)$ in terms of an independent standard Gaussian vector $(Y, Y_j, j \in J)$, which yields with X_j^Y as a shorthand for $m_j + \sigma(s)\mu + \sigma(s)\sigma(\sqrt{\rho}Y + \sqrt{1-\rho}Y_j)$

$$\begin{aligned} \Gamma_\varphi(v; s, m, \theta) &= \mathbb{P}\left(\sum_{j \in J} \mathbb{1}_{\{h_j(s) < X_j^Y \le h_j(v)\}} = \ell - |I|; h_j(s) < X_j^Y, j \in J\right) \\ &= \int_{\mathbb{R}} \mathbb{P}\left(\sum_{j \in J} \varepsilon_j^y = \ell - |I|\right) \phi(y)\, dy \end{aligned}$$

where the random variables ε_j^y are defined, for every real y,

$$\varepsilon_j^y = \begin{cases} \infty, & X_j^y \le h_j(s) \\ 1, & h_j(s) < X_j^y \le h_j(v) \\ 0, & X_j^y > h_j(v). \end{cases}$$

so that $\mathbb{P}(\sum_{j\in J} \varepsilon_j^y = \ell - |I|) = c_\ell^y(v; s, m_t, \theta)$ as introduced in the statement of the proposition. □

Assuming a common (and constant) recovery R on the n firms, the cumulative loss on the portfolio at time t writes

$$L_t = (1 - R)N_t.$$

The cumulative cash flow of a CDO tranche of maturity T, attachment point a, detachment point b and contractual spread κ, is given by

$$\int_0^T [\kappa(b - a - L_v^{a,b})\,dv - dL_v^{a,b}]$$

where the tranche cumulative loss process $L_t^{a,b}$ is given as

$$L_t^{a,b} = (L_t - a)^+ - (L_t - b)^+ =: L_{a,b}(N_t).$$

Proposition 3.2 (i) *The price process of a CDO tranche* $[a, b]$ *is given by*

$$\begin{aligned} C_t^{a,b} &:= C_{a,b}(t, m_t, \theta_t) \\ &= \kappa(b - a)(T - t) - \sum_{\ell=N_t}^{n} L_{a,b}(\ell)\left(\kappa \int_t^T \Gamma_t^\ell(v)\,dv + \Gamma_t^\ell(T)\right) + L_{a,b}(N_t) \end{aligned} \quad (3.5)$$

for $\Gamma_t^\ell(v) = \mathbb{P}(N_v = \ell \,|\, \mathcal{G}_t) = \Gamma_\ell(v; t, m_t, \theta_t)$ *as in* (3.3).

(ii) *The dynamics of the cumulative price of a CDO tranche* $[a, b]$ *is given as*

$$d\hat{C}_t^{a,b} = \varsigma(t) \sum_{k\in N} \partial_{m_k} C_{a,b}(t, m_t, \theta_t)\,dW_t^k + \sum_{j\in J_{t-}} \delta_j C_{a,b}(t, m_t, \theta_{t-})\,dM_t^j. \quad (3.6)$$

Proof. One has

$$\begin{aligned} C_t^{a,b} &= \mathbb{E}\left[\int_t^T \left[\kappa(b - a - L_u^{a,b})\,dv - dL_v^{a,b}\right] \Big|\, \mathcal{G}_t\right] \\ &= \kappa(b - a)(T - t) - \kappa \int_t^T \mathbb{E}[L_v^{a,b} \,|\, \mathcal{G}_t]\,dv - \mathbb{E}[L_T^{a,b} \,|\, \mathcal{G}_t] + L_t^{a,b} \end{aligned}$$

where

$$\mathbb{E}[L_v^{a,b} \,|\, \mathcal{G}_t] = \mathbb{E}[L_{a,b}(N_v) \,|\, \mathcal{G}_t] = \sum_{\ell=|I_t|}^{n} L_{a,b}(\ell)\Gamma_\ell(v; t, m_t, \theta_t),$$

which proves part (i), from which part (ii) follows as in the proof of Proposition 3.1. □

3.3. *Hedging CDO with CDS*

In the dynamized Gaussian copula setup of this paper, one can consider the issue of dynamic hedging of a CDO tranche by individual CDS, in various senses. Mimicking the pre-crises market practice of hedging the spread risk of a CDO tranche (not caring about default risk), one can thus get rid of the dW-exposures in (3.6) through suitable dynamic positions in individual CDS (this was also one of the motivations of the related paper [18]). In view of (3.2) and (3.6), until the first default in the portfolio, this objective is achieved by the following row-vector ζ_t^{spd} of dynamic positions in CDS on all the names underlying the tranche:

$$\zeta_t^{spd} = (\partial_m C_{a,b}(\partial_m C)^{-1})(t, m_t, \theta_{t-}) \tag{3.7}$$

where $C = (C_l)_{l \in N}$. In general these deltas are obtained by numerical solution of a linear system. However at $t = 0$ this is a diagonal system so the numerical solution is elementary. These time-0 deltas will be found below very similar numerically to standard "static" Gaussian copula bump-sensitivities. Note that after the first default time in the portfolio, say τ_k, the matrix $\partial_m C$ becomes degenerate as the corresponding C_t^k and its sensitivities vanish, whereas $d\hat{C}_t^{a,b}$ in (3.6) still depends on dW_t^k; So another non-redundant instrument (e.g., another non-redundant CDS on one of the surviving names) must be substituted to the CDS on the defaulted name if one wants to sustain a perfect hedge of the spread risk of the tranche.

Alternatively, it is also possible to compute in our dynamized Gaussian copula setup the min-variance deltas ζ_t^{va} which minimize the risk-neutral variance of the hedging error (spread risk and jump-to-default risk altogether [5, 12, 15] as opposed to a focus on spread risk only with ζ_t^{spd}). Moreover for comparison we shall also compute min-variance deltas in the dynamized Marshall-Olkin copula model of [5]. DGC and DMO will be used as acronyms for dynamized Gaussian copula (the model of this paper) and dynamized Marshall-Olkin (the model of [5]). Note that a DGC model can only be fitted to one tranche quote at a time (as it has a unique correlation parameter ϱ), whereas a DMO model has a richer dependence structure which can be jointly fitted to all the tranches. A DGC model is sufficient to deal with, for instance, counterparty risk (see Sec. 4) on CDS, but a DMO setup is necessary for the calibration sake if CDO tranches are also present in the portfolio.

We shall use, as a common data set for all deltas, the North American CDX 17 December 2007 data set, a set of credit data on 125 underlying credit names, including [0–3%], [3–7%], [7–10%], [10–15%] and [15–30%] CDO tranches market quotes. This data set was already used for numerical purposes in the dynamized Marshall-Olkin setup of [5], which will make it possible to draw comparisons between DGC and DMO results. We refer the reader to [10] for the classical notions of compound correlation and base correlation of a CDO tranche. The notion of base correlation is an alternative to the compound correlation for cases in which the latter is not well-defined, as happens on our data set with the junior-mezzanine tranche 3%–7%. See Fig. 1.

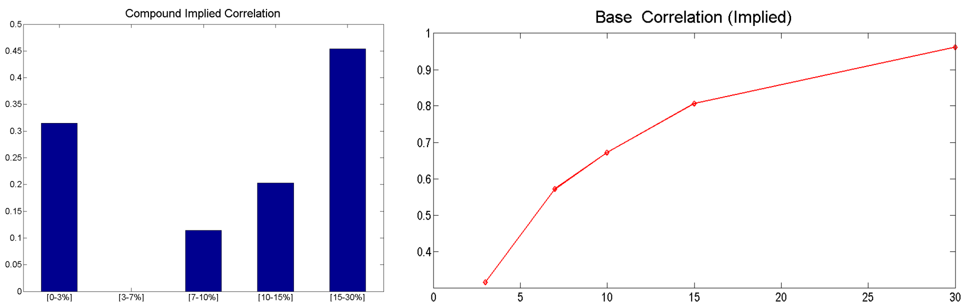

Fig. 1. CDX 17 December 2007 — Left: Compound correlation (undefined on these data for the junior mezzanine tranche 3%–7%). Right: Base correlation (all tranches).

To sum-up, we shall discuss the following notions of deltas of a CDO tranche with respect to individual CDS on all or part of the credit names underlying a CDO tranche, all these deltas being calibrated (in the sense implied by their respective definitions) to the same data set of CDX 17 December 2007:

- *Market compound* (*resp. base*) *spread deltas*: Static Gaussian copula bump-sensitivities for a level of the static Gaussian copula correlation parameter equal to the compound (resp. base) correlation of the tranche;
- *DGC compound spread deltas*: Time-0 values of the dynamic deltas in the sense of (3.7) for a level of the DGC correlation parameter ϱ equal to the compound correlation of the tranche;
- *DGC compound min-variance deltas*: Time-0 values of the dynamic deltas which minimize the risk-neutral variance of the hedging error in a DGC model, for a level of the DGC correlation parameter ϱ equal to the compound correlation of the tranche;
- *DMO min-variance deltas*: Time-0 values of the dynamic deltas which minimize the risk-neutral variance of the hedging error in a DMO model jointly calibrated to all the tranches and to all the CDS which are used as hedging instruments.

These deltas are computed using semi-explicit formulas for the $\partial_{m_k} C_l(s, m, \theta)$ and $\partial_{m_k} C_{a,b}(s, m, \theta)$ which are derived in Secs. A.2 and A.3.

The left panel of Fig. 2 displays the DGC and the market compound spread deltas for all individual CDS (represented by decreasing spread on the x-axis) and CDO tranches, except the junior mezzanine tranche 3%–7% with undefined compound correlation on these data (cf. Fig. 1). For every tranche the DGC and the market compound spread deltas are found very similar numerically (the two delta curves are essentially superposed for each tranche). This means in particular that as also found in [18], the practical notion of market compound spread deltas can be related to a sound dynamic definition, in the sense of the DGC compound spread deltas.

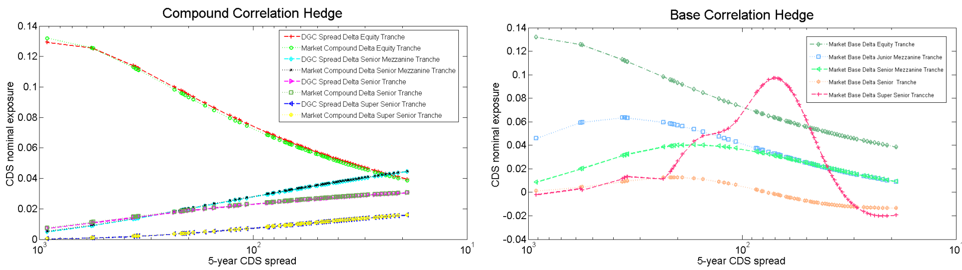

Fig. 2. CDX 17 December 2007 — Left: DGC compound spread deltas versus market compound deltas (undefined on these data for the junior mezzanine tranche 3%–7%). Right: Market base deltas (all tranches).

For comparison the right panel of Fig. 2 also displays the market base spread deltas for all tranches. As opposed to the previous notions of deltas, these are only ad-hoc bump-sensitivities which cannot be related to a sound dynamic approach; but they still provide a possible hedge for the junior-mezzanine tranche 3%–7% with undefined compound correlation. Except for the equity tranche, there is a significant difference between the (DGC or market) compound (left panel) and base (right panel) spread deltas.

Figure 3 displays the DGC compound min-variance deltas for the various tranches other than junior-mezzanine, and for portfolios of hedging CDS comprising

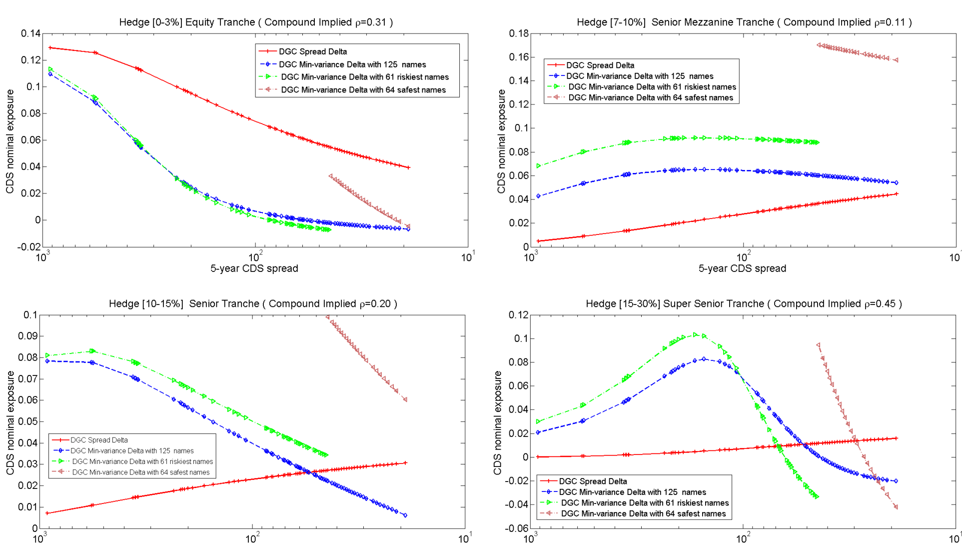

Fig. 3. CDX 17 December 2007 — DGC compound min-variance deltas versus DGC (very close to market) compound spread deltas (all tranches except junior mezzanine for which the compound correlation is undefined on this data set).

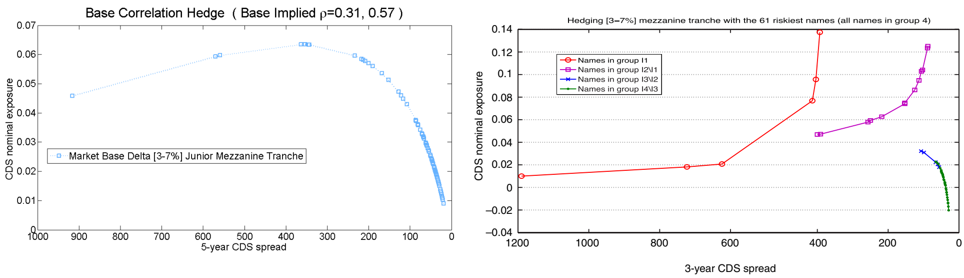

Fig. 4. [Right panel drawn from [5]] CDX 17 December 2007 — Junior mezzanine deltas.

respectively all the 125 underlying names, the 61 riskiest names and the 64 safest names (safest and riskiest in the sense of the corresponding CDS spreads at time 0; in [5] the 61 riskiest names are used for technical reasons which are explained in the paper, which is the reason why we also display the deltas on the 61 riskiest names here and the ones in the complement set of the 64 safest names, for comparison purposes). The corresponding (DGC or market) compound spread deltas are also displayed on the same graphs. For all tranches there are significant numerical differences between these different notions of deltas, as one would expect.

In the case of the junior-mezzanine tranche 3%–7% with undefined compound correlation, there is no consistently calibrated notion of DMO deltas on these data; Fig. 4 displays instead the corresponding static market base correlation spread deltas, as well as the dynamic DMO min-variance deltas based on the portfolio of the 61 riskiest names in the portfolio at time 0, where the DMO model is jointly calibrated to these 61 names and to all the five years CDO tranches at time 0 (see [5]). These two notions of deltas show quite different patterns.

4. Counterparty Risk

We now consider a credit derivative (which could also be a portfolio of contracts) with maturity T between a bank and a defaultable counterparty with default time τ. We denote by $\bar{\tau} = \tau \wedge T$ the effective time horizon of the problem, as there will be no cash flow after it. Our goal is to assess the price-and-hedge correction accounting for counterparty risk with respect to a so-called “clean” price-and-hedge of the contract disregarding this risk.

Remark 4.1 In this paper where our main object is to understand the mathematical structure of the dynamized Gaussian copula, we disregard the default risk of the bank, assuming it risk-free for simplicity. In nowadays market environments banks are of course not default-free and one should take a bilateral counterparty risk perspective. A companion issue is then the cost of funding, as it does not make sense to assume that a default-risky bank can lend and borrow cash at a common and locally risk-free rate r_t (set to zero for even greater simplicity in this paper). The interaction between counterparty risk and funding induces some nonlinearities (see [13, 14]), which in the high-dimensional case of portfolio credit derivatives make it very challenging computationally. Extension of the results of this section to bilateral counterparty risk under funding constraints will be considered in a follow-up work.

In the unilateral counterparty risk setup of this paper, the counterparty risk correction to the contract's value, known as credit valuation adjustment (CVA) (see [4, 9]), boils down to the process Θ given on $[0, \bar{\tau}]$ as

$$\Theta_t = \mathbb{E}[\mathbb{1}_{\tau<T}\xi \,|\, \mathcal{G}_t] \tag{4.1}$$

for a $\mathcal{G}_\tau$-measurable exposure ξ defined as

$$\xi = (1 - R_0)\chi^+ \tag{4.2}$$

in which R_0 is the recovery rate of the counterparty and χ represents the algebraic "debt" χ of the counterparty to the bank at time τ, in the sense of

$$\chi = P_\tau + \Delta_\tau - \Gamma_\tau, \tag{4.3}$$

where:

- P_τ is the clean value of the contract at τ;
- Δ_τ represents a contractual promised cash flow at time τ;
  - — For instance, assuming CDS protection on firm one sold to the bank by the counterparty, this cash flow would correspond to a CDS protection payment due to the bank by the counterparty in case of a joint default of the counterparty and firm one at τ;
- Γ_τ is the value of the margin account at τ, representing the cumulative collateral posted by the counterparty to the bank in order to mitigate counterparty risk.

The reader is referred to [4] for more details, and for the derivation of the CVA process (4.1) as the output of some computation rather than as a definition for simplicity of presentation in this paper. Also note that this definition implicitly assumes that the valuation Q of the contract by the liquidator at the time of default of the counterparty, is a clean valuation $Q = P$ (see [4]).

In view of (4.1)–(4.3), the CVA appears as essentially an option on the clean value of the contract at time τ. In order to compute the CVA one therefore needs a dynamic and tractable model for P_t. Toward this end we use a dynamized Gaussian copula (DGC) model of the default times $(\tau_0 = \tau, \tau_1, \ldots, \tau_n) =: (\tau_l)_{l\in N}$ of the counterparty and of the reference names underlying the credit derivative. Since there are no joint default in this model one can assume here that the contract promises no cash-flow at τ (as opposed to the situation of the dynamized

Marshall-Olkin model of [5] which is used in this paper for comparison purposes in Secs. 3.3 and 4.1.2). So in a DGC setup, one has that $\Delta_\tau = 0$ and ξ in (4.2)–(4.3) reduces to

$$\xi = (1 - R_0)(P_\tau - \Gamma_\tau)^+$$

in which a constant recovery R_0 is postulated for simplicity. The $\mathbb{G}$-Markov property of (t, m_t, θ_t) with generator $\mathcal{A}$ then implies the following pricing equations for the CVA.

Proposition 4.1 (CVA linear BSDE/PDE). *If* $P_t = P(t, m_t, \theta_t)$ *and* $\Gamma_t = \Gamma(t, m_t, \theta_t)$, *then* $\Theta_t = \Theta(t, m_t, \theta_t)$ *where* Θ_t *satisfies the following linear CVA BSDE*: $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$ *and for* $t \in [0, \bar{\tau}]$

$$d\Theta_t = \varsigma(t) \sum_{l \in N} \partial_{m_l} \Theta(t, m_t, \theta_t) dW_t^l + \sum_{j \in J_{t-}} \delta_j \Theta(t, m_t, \theta_{t-}) dM_t^j. \tag{4.4}$$

An equivalent linear CVA PDE with generator $\mathcal{A}$ *holds in* $\Theta = \Theta(s, m, \theta)$.

Note that from the results of Sec. 3, the assumption $P_t = P(t, m_t, \theta_t)$ in this proposition is met for every CDS, CDO tranche, or (by linearity) for every portfolio of CDS and CDOs (in practice the CVA has to be computed at the level of netted portfolios); One also has $\Gamma_t = \Gamma(t, m_t, \theta_t)$ in the "extreme" cases of no ($\Gamma = 0$) or continuous ($\Gamma = P$) collateralization. Regarding the latter, note however that accounting for the delay in setting the collateral in response to market moves, continuous collateralization cannot (and would not even if it could, for operational cost reasons) be implemented in practice. As will be developed in a follow-up work, more realistic cases of path-dependent collateralization can be considered by augmentation of the state space, treating in particular the collateral process Γ as an additional factor.

Equation (4.4) implies that the CVA exposure $\mathbb{1}_{\tau<T}\xi$ at $\bar{\tau}$ can be dynamically replicated by using $2n + 2$ non-redundant hedging instruments, plus a funding riskless (constant) asset. Of course in practice people would more realistically hedge of a selection of risk factors. Specific hedging schemes can be implemented on the basis of the linear BSDE (4.4) or of the equivalent PDE. A CDS on the counterparty can thus be used by the bank for hedging her CVA exposure at time τ (with a hedging CDS clean of counterparty risk and rolled-over in time, as in [4]). However for large n (like with a CDO tranche) the CVA BSDE/PDE are untractable numerically due to the curse of dimensionality. Even the data in these equations become very involved due to the combinatorial structure of the coefficients γ and λ in the generator $\mathcal{A}$. As a consequence, a Monte-Carlo computation of the CVA based on (4.1) (for $t = 0$) seems to be the only feasible computational procedure.

4.1. *Numerics*

To conclude this paper, we provide some results regarding the CVA on a CDS computed in the dynamized Gaussian copula (DGC) model. We shall also give comparative results obtained in the dynamized Marshall-Olkin (DMO) model of [7], which will point out to the issue of CVA model risk.

The following results are derived for τ_0 and τ_1 given as exponential random variables with constant parameters $\bar{\lambda}_0$ and $\bar{\lambda}_1$ (which in the real-life should be calibrated to the related CDS market spreads). Moreover, one uses a function $\varsigma(\cdot)$ in (2.3) constant before T. We shall see below that the number

$$\sqrt{\int_0^T \varsigma^2(u)\,du} = \varsigma(0)\sqrt{T} \in [0,1]$$

(since the L_2-norm of the function $\varsigma(\cdot)$ is one) can then be interpreted as a volatility parameter (also depending on T), which is denoted by $\%(T)$ (to be understood as "the proportion of the volatility of $\int_0^{+\infty} \varsigma(t)dB_t^1$ before T").

4.1.1. *Spread volatilities*

Since the CVA on a CDS is an option on the clean value of the CDS, an important driver of this CVA is the volatility of CDS spreads. We shall now assess this volatility in terms of CDS option implied volatilities. A CDS (call) option with maturity T_a on name one gives the investor the right to enter at T_a a payer CDS on name one with contractual spread κ_1 and termination time $T_b > T_a$. As explained in [8], the corresponding price process is given by

$$O_t = \mathbb{E}[\mathbb{1}_{\{\tau_1 > T_a\}}(C^1_{T_a})^- \,|\, \mathcal{G}_t] \tag{4.5}$$

in which C^1_t stands for the time-t pre-default value of the underlying CDS. A CDS option is typically quoted on the market in terms of its Black implied volatility Σ_t, defined (at time 0) through the following identity, in which F^1 denotes the forward (T_a, T_b)-CDS swap rate process on name one

$$O^1_0 = \left(\int_{T_a}^{T_b} \mathbb{P}(\tau_1 > u)\,du\right)(F^1_0\Phi(d_+) - \kappa_1\Phi(d_-)) \tag{4.6}$$

with

$$d_\pm = \frac{\ln(F^1_0/K)}{\Sigma_0\sqrt{T_a}} \pm \frac{\Sigma_0\sqrt{T_a}}{2}$$

(see Brigo [8] for details regarding the Black model of a CDS forward swap rate). In the DGC model with two names 0 and 1, we compute the price O_0 of the option

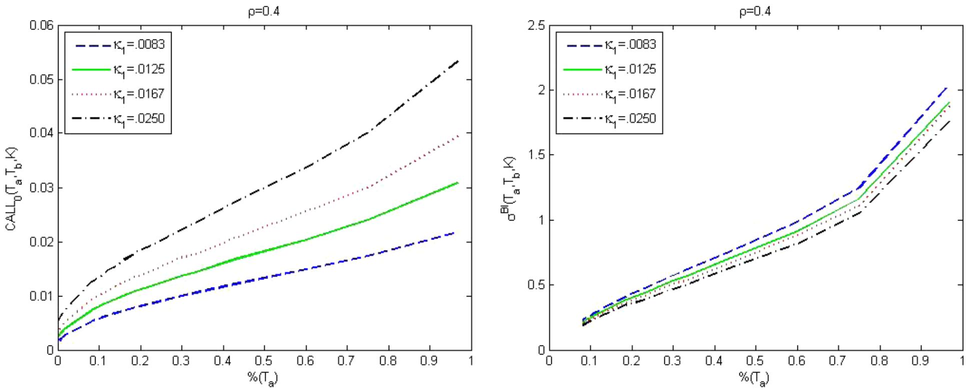

Fig. 5. DGC price (left; Monte Carlo with 5000 scenarios) and corresponding implied volatilities (right) of CDS options on name one for four values of $\bar{\lambda}_1$ as $\%(T_a)$ varies from 0 to 1 (contractual spread of the underlying CDS $\kappa_1 = (1 - R_1)\bar{\lambda}_1$ with $R_1 = 40\%$, correlation parameter $\varrho = 40\%$).

by Monte Carlo simulation based on Formula (4.5) at time 0, in which by (3.1)

$$C^1_{T_a} = \left[(1 - R_1)(1 - G^1_{T_a}(T_b)) - S_1 \int_{T_a}^{T_b} G^1_{T_a}(u)du\right]$$

where G^1 is given by Formula (2.13) (here with two names).

We set $T_a = 3$ years, $T_b = 10$ years, $R_1 = 40\%, \kappa_1 = \bar{\lambda}_1(1 - R_1)$, and use a constant level of risk-free interest rates $r = 5\%$ (with the obvious amendments to all the formulas in case of a non-null but constant funding rate r). For a Gaussian copula correlation parameter $\varrho = 40\%$, Fig. 5 shows the prices (left) and the corresponding implied volatilities (right) of the option as $\%(T_a)$ varies from 0 to 1, and for four values 0.0083, 0.0125, 0.0167 and 0.0250 of the intensities $\bar{\lambda}_1$, corresponding for the chosen recovery of 40% to respective credit spreads of 50, 75, 150 and 200 basis points. The prices O_0 are computed by Monte Carlo simulation based on 5000 scenarios and the implied volatilities are deduced from the prices by numerical solution of (4.6) in Σ_0. The profile of the implied volatility on the right panel justifies the interpretation of the quantity $\%(T_a) = \varsigma(0)\sqrt{T_a}$ as a volatility parameter. The range of implied volatilities obtained as $\%(T_a)$ varies from 0 to 1 is very wide, from a few percents to more than 200% of implied volatility. The implied volatility is slightly decreasing in $\bar{\lambda}_1$, a feature that was already observed in the DMO setup of [7]. As visible on Eq. (3.2), due to absence of immersion in a DGC model, the dynamics of the price of the CDS depend not only on the underlying name 1, but also on name 0 which is present as well in the model. However one expects intuitively that the impact of name 0 should be rather limited quantitatively. This is confirmed on Fig. 6 which shows the prices (left panel) and the corresponding implied volatilities (right panel) of the option as $\%(T_a)$ varies from 0 to 1, for $\bar{\lambda}_1$ fixed to 0.167 (corresponding to a credit spread κ_1 of 100 basis points) and for three

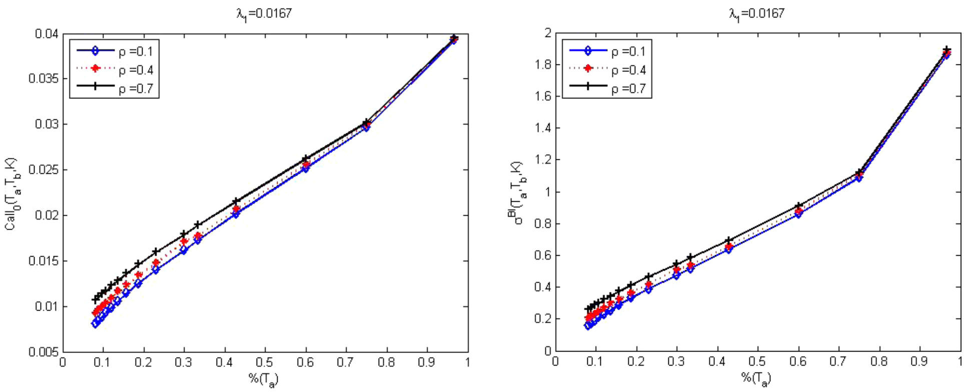

Fig. 6. DGC price (left, Monte Carlo with 5000 scenarios) and corresponding implied volatilities (right) of CDS options on name one for a contractual spread of the underlying CDS for three values of $\varrho\%$ as $\%T_a$ varies from 0 to 1 (contractual spread of the underlying CDS $\kappa_1 = (1 - R_1)\bar{\lambda}_1 = (1 - 40\%) \times 0.167 = 100$ basis points, correlation parameter $\varrho = 40\%$).

values $10\%, 40\%$ and 70% of the Gaussian copula correlation parameter ϱ; The three curves corresponding to the three different values of ϱ are quite close to each other.

4.1.2. *CVA*

Having checked that the model is adequately responsive in terms of volatility of CDS spreads, we now show on the left panel of Fig. 7 the CVA on the CDS computed by Monte Carlo simulation based on Formula (4.1) at time 0 using the same values of the parameters as before and 10^5 scenarios. The CVA at time 0 is shown for a level of the Gaussian correlation parameter ϱ increasing from 0 to 1, for a fixed $\bar{\lambda}_1 = 0.0140$ (corresponding through the assumed recovery $R_1 = 40\%$ to a credit spread κ_1 of 84 basis points). Moreover the right panel of Fig. 7 shows the values of the CVA in the DMO setup of [5, 7] calibrated to the same data. In a DMO setup, dependence between names mostly stems from the possibility of joint defaults. In the context of counterparty risk on credit derivatives, the possibility of joint defaults between the counterparty and the underlying names of a reference contract is a factor of strong wrong-way risk (adverse dependence between the exposure ξ and the default time of a party). Regarding for instance counterparty risk on a CDS, it makes it possible that the default time of the counterparty coincides with that of the name underlying the CDS, which impacts the bank at a high level of exposure (the protection payment non paid by the counterparty). From a mathematical point of view, in a model with joint defaults, the Δ_τ-term can be non-zero and in fact very large in (4.1)–(4.3). Note that even though the two graphs of Fig. 7 represent CVAs computed in models calibrated to the same data, the levels of CVA are different: smaller and small in absolute terms for small ϱ (resp. larger and large in absolute terms for large ϱ) in the DMO setup of [7] than in the DGC model of this paper.

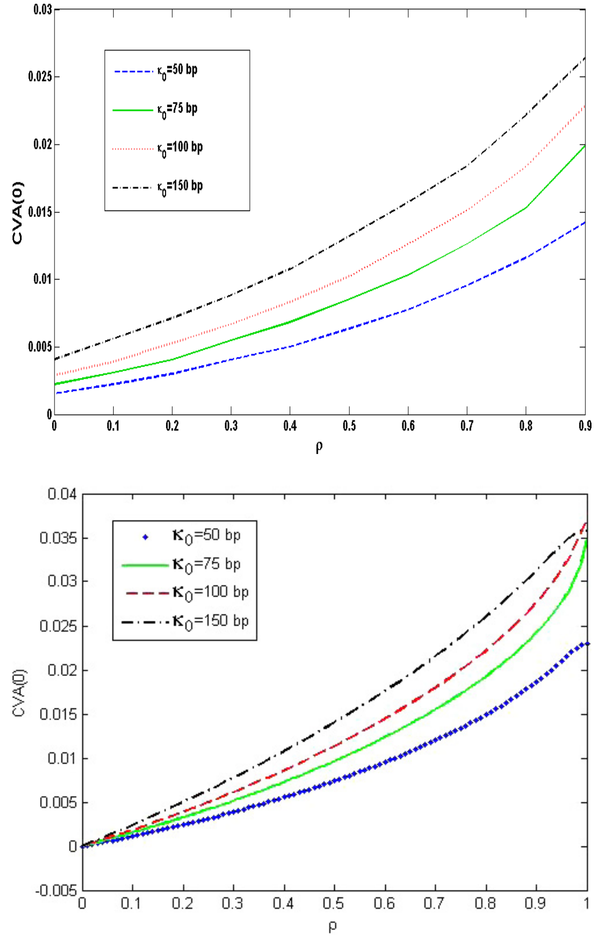

Fig. 7. CVA_0 versus ϱ for $\bar{\lambda}_1 = 0.0140$ (credit spread $\kappa_1 = 84$ basis points) in the DGC (top) vs DMO (bottom) models.

This illustrates the dynamic and optional flavor of the CVA and gives an idea of the related model risk.

Appendix

A.1. *Proof of Lemma* 2.2

Proof. Let in vector form $X_I := (X_i)_{i \in I}$ and $X_J := (X_j)_{j \in J}$ with covariance matrix of $\binom{X_I}{X_J}$ denoted $\begin{pmatrix} R_I & R_{J,I}^T \\ R_{J,I} & R_J \end{pmatrix}$. Setting $Y = X_J - R_{J,I} R_I^{-1} X_I$, one has

$$\begin{pmatrix} X_I \\ Y \end{pmatrix} = \begin{pmatrix} 0 & I_I \\ I_J & -R_{J,I} R_I^{-1} \end{pmatrix} \begin{pmatrix} X_J \\ X_I \end{pmatrix}$$

where

$$\begin{pmatrix} X_I \\ Y \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} R_J - R_{J,I}R_I^{-1}R_{J,I}^T & 0 \\ 0 & R_I \end{pmatrix}\right).$$

Observe that Y is a Gaussian vector independent of X_I. We can thus compute the characteristic function of X_J given X_I as

$$\begin{aligned}\mathbb{E}[e^{iuX_J} \mid X_I] &= \mathbb{E}[e^{iu(Y+R_{J,I}R_I^{-1}X_I)} \mid X_I] \\ &= \mathbb{E}[e^{iuY}]e^{iuR_{J,I}R_I^{-1}X_I} \\ &= e^{-\frac{1}{2}u^T(R_J - R_{J,I}R_I^{-1}R_{J,I}^T)u}e^{iuR_{J,I}R_I^{-1}X_I}.\end{aligned}$$

The conditional distribution of X_J given X_I is therefore $\mathcal{N}(R_{J,I}R_I^{-1}X_I, R_J - R_{J,I}R_I^{-1}R_{J,I}^T)$. We denote by $\mathbf{1}_l$ (resp. $\mathbf{1}_{l\times k}$) the l-dimensional vector (resp. $l \times k$-matrix) whose entries are all equal to 1, and by $R_l(x,y)$ the $l \times l$-matrix with x in diagonal and y outside. One first has for the mean that

$$\begin{aligned}R_{J,I}R_I^{-1}X_I &= \varrho\mathbf{1}_{|J|\times|I|}R_{|I|}^{-1}(1,\varrho)(X_i)_{i\in I} \\ &= \varrho\mathbf{1}_{|J|\times|I|}R_{|I|}(a,b)(X_i)_{i\in I} \\ &= m\mathbf{1}_{|J|}\end{aligned}$$

where $R_{|I|}^{-1}(1,\varrho) = R_{|I|}(a,b)$ with

$$a = -\frac{-(|I|-2)\varrho + 1}{(|I|-1)\varrho^2 + (2-|I|)\varrho - 1}, \quad b = \frac{\varrho}{(|I|-1)\varrho^2 - (|I|-2)\varrho - 1}$$

and

$$m = \frac{(\varrho^2 - \varrho)\sum_{i\in I} X_i}{(|I|-1)\varrho^2 + (2-|I|)\varrho - 1} = \frac{\varrho\sum_{i\in I} X_i}{(|I|-1)\varrho + 1}.$$

For the variance one has likewise that

$$\begin{aligned}R_J - R_{J,I}R_I^{-1}R_{J,I}^T &= R_{|J|}(1,\varrho) - \varrho\mathbf{1}_{|J|\times|I|}R_{|I|}^{-1}(1,\varrho)\varrho\mathbf{1}_{|I|\times|J|} \\ &= R_{|J|}(1,\varrho) - \frac{|I|\varrho^2}{(|I|-1)\varrho+1}\mathbf{1}_{|J|\times|J|} \\ &= \sigma^2 R_{|J|}(1,\rho)\end{aligned}$$

where

$$\sigma^2 = \frac{(|I|-1)\varrho + 1 - \varrho^2|I|}{(|I|-1)\varrho+1}$$

$$\rho = \frac{-\varrho^3 + 2\varrho^2 - \varrho}{(|I|-1)\varrho^2 + (2-|I|)\varrho - 1} = \frac{\varrho}{|I|\varrho+1}. \qquad \square$$

In the remaining sections, we compute the functions $\partial_{m_k}C_l(s,m,\theta)$ and $\partial_{m_k}C_{a,b}(s,m,\theta)$ for any (s,m,θ). We use the notation introduced in (2.11)–(2.13) and (3.3), omitting the argument (s,m,θ) everywhere for the sake of readability. Observe that $Z_t^j(v) = Z_j(v;t,m_t,\theta_t)$, $D_t = D(t,m_t,\theta_t)$, $\Gamma_t^\ell(v) = \Gamma_\ell(v;t,m_t,\theta_t)$,

$G_t^l(v) = G_l(v; t, m_t, \theta_t)$. We also denote

$$\mathcal{Z}_t^l(v) := (Z_t^l(v), (Z_t^j(t))_{j \in J_t \setminus \{l\}}) = \mathcal{Z}_l(v; t, m_t, \theta_t).$$

A.2. *Semi-explicit formula for the gradient of a CDS pricing function*

In the next proposition, we derive a semi-explicit formula for the sensitivities $\partial_{m_k} C_l(s, m, \theta)$ which intervene in the dynamics of the CDS pre-default price in Proposition 3.1(ii).

Proposition A.1 *One has for every $k, l \in N$ and (s, m, θ)*

$$\partial_{m_k} C_l = -\mathbb{1}_{\{\theta_k = 0\}} \left((1 - R_l) L_k(T) + \kappa_l \int_s^T L_k(v) dv \right) - \mathbb{1}_{\{\theta_k \neq 0\}} \left((1 - R_l) L_I(T) + \kappa_l \int_s^T L_I(v) dv \right)$$

where $I = \mathrm{supp}(\theta)$ and for every v, with also $J = N \backslash I$

$$L_k(v) = -\frac{1}{\sigma(s) D^2} (D \partial_{z_k} \Phi_{\rho,\sigma}(\mathcal{Z}_l(v)) - \Phi_{\rho,\sigma}(\mathcal{Z}_l(v)) \partial_{z_j} D)$$

$$L_I(v) = -\frac{\varrho}{\sigma(t) D^2} \frac{1}{(|I| - 1)\varrho + 1} \left(D \sum_{j \in J} \partial_{z_j} \Phi_{\rho,\sigma}(\mathcal{Z}_l(v)) - \Phi_{\rho,\sigma}(\mathcal{Z}_l(v)) \sum_{j \in J} \partial_{z_j} D \right).$$

Proof. Recalling from (3.1) that $C_l = (1 - R_l)(1 - G_l(T)) - \kappa_l (\int_s^T G_l(v) dv)$ with $G_l(v) = \Phi_{\rho,\sigma}(\mathcal{Z}_l(v))/D$, we compute

$$\partial_{m_k} \Phi_{\rho,\sigma}(\mathcal{Z}_l(v)) = -\mathbb{1}_{\{\theta_k = 0\}} \frac{1}{\sigma(s)} \partial_{z_k} \Phi_{\rho,\sigma}(\mathcal{Z}_l(v)) - \mathbb{1}_{\{\theta_k \neq 0\}} \frac{\varrho}{\sigma(s)} \frac{1}{(|I| - 1)\varrho + 1} \sum_{j \in J} \partial_{z_j} \Phi_{\rho,\sigma}(\mathcal{Z}_l(v))$$

$$\partial_{m_k} D = \partial_{m_k} \Phi_{\rho,\sigma}((Z_j(s))_{j \in J}) = -\mathbb{1}_{\{\theta_k = 0\}} \frac{1}{\sigma(s)} \partial_{z_k} \Phi_{\rho,\sigma}((Z_j(s))_{j \in J}) - \mathbb{1}_{\{\theta_k \neq 0\}} \frac{\varrho}{\sigma(s)} \frac{1}{(|I| - 1)\varrho + 1} \sum_{j \in J} \partial_{z_j} \Phi_{\rho,\sigma}((Z_j(s))_{j \in J}), \quad \text{(A.1)}$$

since $\partial_{m_j} Z_l(v) = \frac{-1}{\sigma(s)}$ and $\partial_{m_i} Z_l(s) = -\frac{\varrho}{\sigma(s)} \frac{1}{(|I|-1)\varrho+1}$, for any v, $j \in J$, $i \in I$. The result then follows from $\partial_{m_k} G_l(v) = \mathbb{1}_{\{\theta_k = 0\}} L_k(v) + \mathbb{1}_{\{\theta_k \neq 0\}} L_I(v)$. □

A.3. *Semi-explicit formula for the gradient of a CDO pricing function*

In this section we derive a semi-explicit formula for the sensitivities $\partial_{m_k} C_{a,b}(s,m,\theta)$ which intervene in the dynamics of the CDO price in Proposition 3.2(ii). For $|I| \leq \ell \leq n$, the order $(\ell - |I|)$-coefficient of the polynomial $P^y(x,v)$ in x, denoted by

$$c_\ell^y(v) = \partial_{x^{\ell-|I|}}^{\ell-|I|} P^y(x,v)_{|x=0} / (\ell - |I|)!,$$

is the core term in $C_{a,b}(s,m,\theta)$. In order to compute $\partial_{m_k} c_\ell^y(v)$, we introduce $P^{y,l}(x,v) = \prod_{j \in J \setminus \{l\}} (p_j^y(v)x + q_j^y(v))$ with order $(\ell - 1 - |I|)$-coefficient $c_{\ell-1}^{y,l}(v)$, in which conventionally $c_{|I|-1}^{y,l}(v)$ equals 0. We also denote

$$h_j^y(v) = \frac{h_j(v) - m_j - \sigma(s)\mu - \sigma\sigma(s)\sqrt{\rho}y}{\sigma\sigma(s)\sqrt{1-\rho}}.$$

Lemma A.1 *One has, for any $k \in N$ and $\ell \geq |I|$,*

$$\begin{aligned}
\partial_{m_k} c_\ell^y(v) = {} & \mathbb{1}_{\{k \in J\}} \frac{-1}{\sigma\sigma(s)\sqrt{1-\rho}} \{-c_\ell^{y,k}(v)\phi(h_k^y(v)) \\
& + c_{\ell-1}^{y,k}(v)(\phi(h_k^y(v)) - \phi(h_k^y(s)))\} \\
& + \mathbb{1}_{\{k \in I\}} \frac{-\varrho}{\sigma\sigma(s)\sqrt{1-\rho}((|I|-1)\varrho + 1)} \\
& \times \sum_{j \in J} \{-c_\ell^{y,j}(v)\phi(h_j^y(v)) + c_{\ell-1}^{y,j}(v)(\phi(h_j^y(v)) - \phi(h_j^y(s)))\}.
\end{aligned}$$

Proof. It is straightforward to compute from (3.4)

$$\begin{aligned}
\partial_{m_k} p_j^y(v) = {} & (\phi(h_j^y(v)) - \phi(h_j^y(s))) \\
& \times \left(\mathbb{1}_{\{k=j\}} \frac{-1}{\sigma\sigma(s)\sqrt{1-\rho}} - \mathbb{1}_{\{k \in I\}} \frac{-\varrho}{\sigma\sigma(s)\sqrt{1-\rho}((|I|-1)\varrho+1)} \right), \\
\partial_{m_k} q_j^y(v) = {} & -\phi(h_j^y(v)) \\
& \times \left(\mathbb{1}_{\{k=j\}} \frac{-1}{\sigma\sigma(s)\sqrt{1-\rho}} - \mathbb{1}_{\{k \in I\}} \frac{-\varrho}{\sigma\sigma(s)\sqrt{1-\rho}((|I|-1)\varrho+1)} \right).
\end{aligned} \tag{A.2}$$

Then, by the definition of $P^y(x,v)$, one has that

$$\begin{aligned}
(\ell - |I|)! \partial_{m_k} c_\ell^y(v) &= \partial_{m_k} (\partial_{x^{\ell-|I|}}^{\ell-|I|} P^y(x,v))_{|x=0} \\
&= \partial_{x^{\ell-|I|}}^{\ell-|I|} (\partial_{m_k} P^y(x,v))_{|x=0}
\end{aligned}$$

$$= \partial^{\ell-|I|}_{x^{\ell-|I|}} \left(\partial_{m_k} \left(\prod_{j\in J} (p^y_j(v)x + q^y_j(v)) \right) \right)_{|x=0}$$

$$= \partial^{\ell-|I|}_{x^{\ell-|I|}} \left(\sum_{j\in J} P^{y,j}(x,v)\partial_{m_k}(p^y_j(v)x + q^y_j(v)) \right)_{|x=0}$$

$$= \sum_{j\in J} \partial^{\ell-|I|}_{x^{\ell-|I|}} (P^{y,j}(x,v)\partial_{m_k}(p^y_j(v)x + q^y_j(v)))_{|x=0}$$

$$= (\ell - |I|)! \sum_{j\in J} (c^{y,j}_{\ell-1}(v)\partial_{m_k} p^y_j(v) + c^{y,j}_{\ell}(v)\partial_{m_k} q^y_j(v)).$$

The result follows by plugging (A.2) into the last equality. □

Proposition A.2 *One has for every $k \in N$ and s, m, θ*

$$\partial_{m_k} C_{a,b} = -\mathbb{1}_{\{\theta_k=0\}} \left(\kappa \int_s^T \sum_{\ell=|I|}^n L_{a,b}(\ell) L^\ell_k(v) dv + \sum_{\ell=|I|}^n L_{a,b}(\ell) L^\ell_k(T) \right)$$
$$- \mathbb{1}_{\{\theta_k\neq 0\}} \left(\kappa \int_s^T \sum_{\ell=|I|}^n L_{a,b}(\ell) L^\ell_I(v) dv + \sum_{\ell=|I|}^n L_{a,b}(\ell) L^\ell_I(T) \right)$$

where $I = \operatorname{supp}(\theta)$ and for every l, v, with also $J = N\backslash I$,

$$L^\ell_k(v) = \frac{-1}{D\sigma\sigma(s)\sqrt{1-\rho}} \int_{\mathbb{R}} dy \phi(y)(-c^{y,k}_\ell(v)\phi(h^y_j(v)) + c^{y,k}_{\ell-1}(v)(\phi(h^y_j(v))$$
$$- \phi(h^y_j(s)))) + \frac{1}{D}\Gamma_\ell(v) \frac{-1}{\sigma(s)} \partial_{z_j} \Phi_{\rho,\sigma}((Z_j(v))_{j\in J})$$

$$L^\ell_I(v) = \frac{\varrho}{D\sigma\sigma(s)\sqrt{1-\rho}(\varrho(|I|-1)+1)}$$
$$\times \sum_{j\in J} \int_{\mathbb{R}} \phi(y) dy (-c^{y,j}_\ell(v)\phi(h^y_j(v)) + c^{y,j}_{\ell-1}(v)(\phi(h^y_j(v))$$
$$- \phi(h^y_j(s)))) + \frac{\Gamma_\ell(v)}{D} \left(\frac{\varrho}{\sigma(s)} \frac{1}{(|I|-1)\varrho + 1} \sum_{j\in J} \partial_{z_j} \Phi_{\rho,\sigma}((Z^j_v(v))_{j\in J}) \right).$$

Proof. By the definition of the CDO value process in (3.5), one has

$$\partial_{m_k} C_{a,b} = -\sum_{\ell=|I|}^n L_{a,b}(\ell) \left(\kappa \int_s^T \partial_{m_k} \Gamma_\ell(v) dv + \partial_{m_k} \Gamma_\ell(T) \right).$$

Since $\Gamma_\ell(v) = \int_{\mathbb{R}} c^y_\ell(v)\phi(y)\,dy/D$, therefore $\partial_{m_k}\Gamma_\ell(v) = \frac{1}{D}\partial_{m_k}\int_{\mathbb{R}} c^y_\ell(v)\phi(y)\,dy - \frac{1}{D}\Gamma_\ell(v)\partial_{m_k}D$ where $\partial_{m_k}D$ has been computed in (A.1) and from Lemma A.1

$$\begin{aligned}
\partial_{m_k}\int_{\mathbb{R}} c^y_\ell(v)\phi(y)\,dy &= \int_{\mathbb{R}} \partial_{m_k} c^y_\ell(v)\phi(y)\,dy \\
&= \mathbb{1}_{\{k\in J\}}\frac{-1}{\sigma\sigma(s)\sqrt{1-\rho}}\int_{\mathbb{R}} dy\phi(y) \\
&\quad \times \{-c^{y,k}_\ell(v)\phi(h^y_k(v)) + c^{y,k}_{\ell-1}(v)(\phi(h^y_k(v)) - \phi(h^y_k(s)))\} \\
&\quad + \mathbb{1}_{\{k\in I\}}\frac{-\varrho}{\sigma\sigma(s)\sqrt{1-\rho}((|I|-1)\varrho+1)}\int_{\mathbb{R}} dy\phi(y) \\
&\quad \times \sum_{j\in J}\{-c^{y,j}_\ell(v)\phi(h^y_j(v)) + c^{y,j}_{\ell-1}(v)(\phi(h^y_j(v)) - \phi(h^y_j(s)))\}.
\end{aligned}$$

The results then follows from the fact that $\partial_{m_k}\Gamma_\ell(v) = \mathbb{1}_{\{k\in J\}}L^\ell_k(v) + \mathbb{1}_{\{k\in I\}}L^\ell_I(v)$. □

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