Invariance Times Transfer Properties

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Abstract

Invariance times are stopping times τ such that local martingales with respect to some reduced filtration and an equivalently changed probability measure, stopped before τ, are local martingales with respect to the original model filtration and probability measure. They arise naturally for modeling the default time of a dealer bank, in the mathematical finance context of counterparty credit risk. Assuming an invariance time endowed with an intensity and a positive Azéma supermartingale, this work establishes a dictionary relating the semimartingale calculi in the original and reduced stochastic bases, regarding in particular conditional expectations, martingales, stochastic integrals, random measure stochastic integrals, martingale representation properties, semimartingale characteristics, Markov properties, transition semigroups and infinitesimal generators, and solutions of backward stochastic differential equations.

Keywords: Progressive enlargement of filtration, invariance time, semimartingale calculus, Markov process, backward stochastic differential equation, counterparty risk, credit risk.

Mathematics Subject Classification: 60G07, 60G44.

1 Introduction

Invariance times were introduced in Crépey and Song (2017b) as stopping times τ such that local martingales with respect to a reduced filtration $\mathfrak{F}$ and some equivalently changed probability measure $\mathbb{P}$, stopped before τ, are local martingales with respect to the original model filtration $\mathfrak{G}$ and probability measure $\mathbb{Q}$. Seen from the smaller filtration $\mathfrak{F}$, these are the random times τ for which the enlargement of filtration Jeulin-Yor

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formula can be compensated by the Girsanov formula of an equivalent change of probability measure. Crépey and Song (2017b), which is summarized in Section 2, focused on a characterization of invariance times in terms of the integrability of a tentative $\mathbb{Q}$ to $\mathbb{P}$ measure change density. The present paper establishes a dictionary of transfer properties between the semimartingale calculi in the original and changed stochastic bases, assuming an invariance time τ endowed with an intensity and a positive Azéma supermartingale:

- Theorem 3.1 extends the classical reduced-form credit risk pricing formulas beyond the basic progressive enlargement of filtration setup where the Azéma supermartingale of τ has no martingale component;
- Theorem 4.1 establishes a bijection between the $(\mathfrak{G}, \mathbb{Q})$ (resp. $(\mathfrak{G}, \mathbb{Q})$ continuous / $(\mathfrak{G}, \mathbb{Q})$ purely discontinuous) local martingales stopped before τ and the $(\mathfrak{F}, \mathbb{P})$ (resp. $(\mathfrak{F}, \mathbb{P})$ continuous / $(\mathfrak{F}, \mathbb{P})$ purely discontinuous) local martingales;
- Theorem 5.1 establishes the connection between stochastic integrals in the sense of local martingales in $(\mathfrak{G}, \mathbb{Q})$ and in $(\mathfrak{F}, \mathbb{P})$;
- Theorem 6.1 establishes the connection between the $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$ random measures stochastic integrals;
- Theorem 7.1 establishes the correspondence between (weak or strong) $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$ martingale representation properties;
- Theorem 8.1 yields the relationship between the $(\mathfrak{G}, \mathbb{Q})$ local characteristics of a $\mathfrak{G}$ semimartingale X stopped before τ and the $(\mathfrak{F}, \mathbb{P})$ local characteristics of the $\mathfrak{F}$ semimartingale X', called reduction of X, that coincides with X before τ;
- Theorems 9.1 and 9.2 state conditions under which Markov properties, transition semigroups and infinitesimal generators can be transferred between $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$;
- Theorems 10.1 and 10.2 show the equivalences, within various spaces of solutions, between a nonstandard $(\mathfrak{G}, \mathbb{Q})$ backward SDE (BSDE) stopped before τ and a reduced $(\mathfrak{F}, \mathbb{P})$ BSDE with null terminal condition.

Theoretical interest apart, a concrete motivation for this work is the study of the so called XVA equations, where VA stands for value adjustment and X is a catch-all letter to be replaced by C for credit, F for funding, M for margin, or K for capital. These are the value adjustment equations related to counterparty risk and its capital and funding implications for a dealer bank. Given the misalignment of interest between the shareholders and creditors of a bank, devising financial derivative entry prices from a shareholder indifference point of view leads to XVA BSDEs stopped before the default time τ of the bank, such as the one mentioned in the last bullet point above: see Example 10.1.

For a general reference on the theory of random times and enlargement of filtration with credit risk applications, see Aksamit and Jeanblanc (2017). As discussed in Section A, the notion of invariance time is also related to various approaches that were introduced in the mathematical finance literature for coping with defaultable cash flows based on default intensities.

1.1 Standing Notation and Terminology

The real line and half-line are denoted by $\mathbb{R}$ and $\mathbb{R}_+$; $|\cdot|$ denotes any Euclidean norm (in the dimension of its argument), $\cdot^\top$ means vector transposition; $\mathcal{B}(E)$ denotes the Borel σ algebra on a metrizable space E; $\boldsymbol{\lambda}$ is the Lebesgue measure on $\mathbb{R}_+$; $\boldsymbol{\delta}_a$ denotes a Dirac measure at a point a.

Unless otherwise stated, a function (or process) is real valued; order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense); a time interval is random (in particular, the graph of a random time θ is simply written $[\theta]$). We do not explicitly mention the domain of definition of a function when it is implied by the measurability, e.g. we write "a $\mathcal{B}(\mathbb{R})$ measurable function h (or $h(x)$)" rather than "a $\mathcal{B}(\mathbb{R})$ measurable function h defined on $\mathbb{R}$". For a function $h(\omega, x)$ defined on a product space $\Omega \times E$, we write $h(x)$ (or h_t in the case of a stochastic process), without ω.

We use the terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer (1975) and He, Wang, and Yan (1992). For any semimartingale X and for any predictable X integrable process L, the corresponding stochastic integral is denoted by $\int_0^\cdot L_t dX_t = \int_{(0,\cdot]} L_t dX_t = L \centerdot X$, with the precedence convention $KL \centerdot X = (KL) \centerdot X$ if K is another predictable process such that KL is X integrable. The stochastic exponential of a semimartingale X is denoted by $\mathcal{E}(X)$.

We work with semimartingales on a predictable set of interval type $\mathcal{I}$ as defined in He, Wang, and Yan (1992, Sect. VIII.3). In particular, X is a local martingale on $\mathcal{I}$ (respectively $Y = L \centerdot X$ on $\mathcal{I}$) means that

$$X^{\theta_n} \text{ is a local martingale (respectively } Y^{\theta_n} = L \centerdot (X^{\theta_n} holds)) \tag{1.1}$$

for at least one, or equivalently any, nondecreasing sequence of stopping times $(\theta_n)_{n\geq 0}$ such that $\cup[0, \theta_n] = \mathcal{I}$. The default case where $\mathcal{I} = \mathbb{R}_+$ corresponds to the standard notion of local martingale (respectively stochastic integral).

For any càdlàg process X, for any random time θ, $\Delta_\theta X$ represents the jump of X at θ. We use the convention that $X_{0-} = X_0$ (hence $\Delta_0 X = 0$) and we write X^θ and $X^{\theta-}$ for the processes X stopped at θ and before θ, i.e.

$$X^\theta = X\mathbb{1}_{[0,\theta)} + X_\theta \mathbb{1}_{[\theta,+\infty)},\;\; X^{\theta-} = X\mathbb{1}_{[0,\theta)} + X_{\theta-}\mathbb{1}_{[\theta,+\infty)}. \tag{1.2}$$

The process X is said to be stopped at θ, respectively before θ, if $X = X^\theta$, respectively $X = X^{\theta-}$. We call compensator of a stopping time θ the compensator of $\mathbb{1}_{[\theta,\infty)}$. We say that θ has an intensity γ if θ is positive and if its compensator is given as $\gamma \centerdot \boldsymbol{\lambda}$, for some predictable process γ (vanishing beyond time θ). For any event A, we denote by θ_A the stopping time $\mathbb{1}_A\theta + \mathbb{1}_{A^c}\infty$.

Stochastic integrals of random functions with respect to jump measures and their compensations are meant in the sense of Jacod (1979), to which we also borrow the usage of including the optionality with respect to a reference filtration in the definition of an integer valued random measure. Random measure stochastic integrals and transform of measures by densities are respectively denoted by "$*$" and "$\cdot$".

We denote by $\mathcal{P}(\mathfrak{H})$ and $\mathcal{O}(\mathfrak{H})$ the predictable and optional σ fields with respect to a filtration $\mathfrak{H}$.

2 Invariance Times Revisited

In this section we recall the main results of Crépey and Song (2017b) regarding their conditions (B) and (A) and we present the stronger condition (C), introduced with its first consequences in Crépey, Sabbagh, and Song (2020, Sections 4–6), and which is explored systematically in this work.

We work on a space Ω equipped with a σ field $\mathcal{A}$, a probability measure $\mathbb{Q}$ on $\mathcal{A}$, and a filtration $\mathfrak{G} = (\mathfrak{G}_t)_{t\in\mathbb{R}_+}$ of sub-σ fields of $\mathcal{A}$ satisfying the usual conditions.

2.1 Condition (B)

Let there be given a $\mathfrak{G}$ stopping time τ and a subfiltration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in\mathbb{R}_+}$ of $\mathfrak{G}$ satisfying the usual conditions and the following:

Condition (B) For any $\mathfrak{G}$ predictable process L, there exists an $\mathfrak{F}$ predictable process L', called the $\mathfrak{F}$ predictable reduction of L, such that $\mathbb{1}_{(0,\tau]}L = \mathbb{1}_{(0,\tau]}L'$. ∎

Equivalently (cf. Crépey and Song (2017b, Eq. (2.1))):

$$\forall t \geq 0 \text{ and } B \in \mathfrak{G}_t,\ \ \exists B' \in \mathfrak{F}_t \text{ such that } B \cap \{t < \tau\} = B' \cap \{t < \tau\}. \tag{2.1}$$

This holds in particular (but not only, see Section A) in the classical progressive enlargement of filtration setup, where

$$\mathfrak{G}_t = \mathfrak{F}_t \vee \sigma(\tau \wedge t) \vee \sigma(\{\tau > t\}),\ \ t \in \mathbb{R}_+,$$

i.e. when $\mathfrak{G}$ is the smallest filtration larger than $\mathfrak{F}$ making τ a stopping time.

Let ${}^{o}\cdot$ and ${}^{p}\cdot$ denote the $\mathfrak{F}$ optional and predictable projections. In particular, $\mathsf{S} = {}^{o}(\mathbb{1}_{[0,\tau)})$ is the $\mathfrak{F}$ Azéma supermartingale of τ, with canonical Doob–Meyer decomposition $\mathsf{S} = \mathsf{Q} - \mathsf{D}$, where Q (with $\mathsf{Q}_0 = \mathsf{S}_0$) and D (with $\mathsf{D}_0 = 0$) are the $\mathfrak{F}$ local martingale component and the $\mathfrak{F}$ drift of S. We recall that

$$\mathsf{S}_{\tau-} > 0 \text{ holds on } \{0 < \tau < \infty\} \tag{2.2}$$

(cf. Yor (1978, Lemme 0 page 62)).

Lemma 2.2 in Crépey and Song (2017b) *Under the condition (B):*

1) For any $\mathfrak{G}$ stopping time θ, there exists an $\mathfrak{F}$ stopping time θ', which we call the $\mathfrak{F}$ reduction of θ, such that $\{\theta < \tau\} = \{\theta' < \tau\} \subseteq \{\theta = \theta'\}$.

2) Given a metrizable space E, any $\mathcal{P}(\mathfrak{G}) \times \mathcal{B}(E)$ measurable function $\Psi_t(\omega, x)$ admits a $\mathcal{P}(\mathfrak{F}) \times \mathcal{B}(E)$ measurable function $\Psi'_t(\omega, x)$, called predictable reduction of Ψ, such that $\mathbb{1}_{(0,\tau]}\Psi = \mathbb{1}_{(0,\tau]}\Psi'$ everywhere; Any $\mathcal{O}(\mathfrak{G}) \times \mathcal{B}(E)$ measurable function $\Psi_t(\omega, x)$ admits an $\mathcal{O}(\mathfrak{F}) \times \mathcal{B}(E)$ measurable function $\Psi'_t(\omega, x)$, called optional reduction of Ψ, such that $\mathbb{1}_{[0,\tau)}\Psi = \mathbb{1}_{[0,\tau)}\Psi'$ everywhere.

3) Let M be a $(\mathfrak{G}, \mathbb{Q})$ local martingale stopped before τ. For any $\mathfrak{F}$ optional reduction M' of M, M' is an $\mathfrak{F}$ semimartingale on $\{\mathsf{S}_- > 0\}$ and

$$\mathsf{S}_- \centerdot M' + [\mathsf{S}, M'] \text{ is an } (\mathfrak{F}, \mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_- > 0\}. \tag{2.3}$$

Conversely, for any $\mathfrak{F}$ semimartingale K on $\{\mathsf{S}_- > 0\}$ such that $\mathsf{S}_- \centerdot K + [\mathsf{S}, K]$ is an $(\mathfrak{F}, \mathbb{Q})$ local martingale on $\{\mathsf{S}_- > 0\}$, $K^{\tau-}$ is a $(\mathfrak{G}, \mathbb{Q})$ local martingale on $\mathbb{R}_+$.

4) The Azéma supermartingale S of τ admits the multiplicative decomposition

$$\mathsf{S} = \mathsf{S}_0 \mathcal{Q}\mathcal{D} \text{ on } \{{}^p\mathsf{S} > 0\}, \tag{2.4}$$

where $\mathcal{Q} = \mathcal{E}(\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})$ is an $(\mathfrak{F}, \mathbb{Q})$ local martingale on $\{{}^p\mathsf{S} > 0\}$ and $\mathcal{D} = \mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D})$ is an $\mathfrak{F}$ predictable nonincreasing process on $\{{}^p\mathsf{S} > 0\}$. ∎

Lemma 2.3 in Crépey and Song (2017b) *Under the condition (B), assuming $\mathsf{S}_T > 0$ for some positive constant T, then*

$$\text{two } \mathfrak{F} \text{ optional processes that coincide before } \tau \text{ coincide on } [0, T]. \tag{2.5}$$

In particular, $\mathfrak{F}$ optional (and predictable) reductions are uniquely defined on $[0, T]$. ∎

Lemma A.1 in Crépey and Song (2017b) *If τ has a $(\mathfrak{G}, \mathbb{Q})$ intensity γ, then D is continuous and*

$$\mathcal{E}(\pm\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D}) = e^{\pm\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D}}, \ \ \frac{1}{\mathsf{S}_-}\centerdot\mathsf{D} = \gamma'\centerdot\boldsymbol{\lambda} \tag{2.6}$$

hold on $\{\mathsf{S}_- > 0\}$. ∎

Moreover, supposing $\mathsf{S}_T > 0$ so that reductions are uniquely defined on $[0, T]$, Song (2016, Lemmas 6.4, 6.5 and 6.10) implies that the $\mathfrak{F}$ optional reduction of a càdlàg process is càdlàg on $[0, T]$; the $\mathfrak{F}$ optional reduction of a $\mathfrak{G}$ optional nondecreasing process is an $\mathfrak{F}$ optional nondecreasing process on $[0, T]$; the $\mathfrak{F}$ optional reduction of a $\mathfrak{G}$ semimartingale is an $\mathfrak{F}$ semimartingale on $[0, T]$.

2.2 Condition (A)

In addition to τ, $\mathfrak{F}$, and $\mathfrak{G}$ satisfying the condition (B) as above, let there be given a positive constant T which is fixed throughout the paper. Letters of the families "Q" and "P" are used for $(\mathfrak{F},\mathbb{Q})$ and $(\mathfrak{F},\mathbb{P})$ local martingales, respectively, where $\mathbb{P}$ refers to the following:

Condition (A) There exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathfrak{F}_T$, called invariance probability measure, such that, for any $(\mathfrak{F},\mathbb{P})$ local martingale P, $P^{\tau-}$ is a $(\mathfrak{G},\mathbb{Q})$ local martingale on $[0,T]$. ∎

If so, then we call τ an invariance time and $\mathbb{P}$ an invariance probability measure.

The most standard circumstance ensuring the condition (A) is a basic immersion setup where $(\mathfrak{F},\mathbb{Q})$ local martingales are $(\mathfrak{G},\mathbb{Q})$ local martingales without jump at τ, in which case τ is an invariance time with $\mathbb{P}=\mathbb{Q}$ (i.e. a strict pseudo-stopping time in the terminology of Jeanblanc and Li (2020, Definition 2.1)), for every positive constant T. More generally:

Theorem 3.2 in Crépey and Song (2017b) *The condition (A) holds if and only if $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})$ is a positive $(\mathfrak{F},\mathbb{Q})$ true martingale on $[0,T]$. In this case, a probability measure $\mathbb{P}$ on $\mathcal{A}$ is an invariance probability measure if and only if the $\mathfrak{F}$ density process of $\mathbb{P}$ coincides with*

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})_{\cdot\wedge T} \tag{2.7}$$

on $\{{}^p\mathsf{S}>0\}\cap[0,T]$. In particular, $\mathbb{P}$ defined by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}=\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})_T \text{ on } \mathcal{A}$$

is an invariance probability measure. ∎

Moreover:

Theorem 3.7 in Crépey and Song (2017b) *If τ has a $(\mathfrak{G},\mathbb{Q})$ intensity, then, under the condition (A),*

$$\{\mathsf{S}_->0\}=\{{}^p\mathsf{S}>0\}=\{\mathsf{S}>0\}. \tag{2.8}$$

In addition, for any invariance probability measure $\mathbb{P}$,

$$\begin{gathered}\text{A process } P \text{ is an } (\mathfrak{F},\mathbb{P}) \text{ local martingale on } \{\mathsf{S}_->0\}\cap[0,T]\\ \text{if and only if}\\ \mathsf{S}_-\centerdot P+[\mathsf{S},P] \text{ is an } (\mathfrak{F},\mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_->0\}\cap[0,T].\end{gathered} \tag{2.9}$$ ∎

2.3 Condition (C)

In order to enjoy all of the above properties, we work henceforth under the following **standing assumption** (given the positive constant T already present in the condition (A)):

Condition (C). The condition (A) is satisfied, $\mathsf{S}_T > 0$ holds almost surely, and τ has a $(\mathfrak{G}, \mathbb{Q})$ intensity. ∎

In particular, we then have $\{{}^p\mathsf{S} > 0\} \supseteq [0, T]$, by (2.2). By virtue of the statement encapsulating (2.7), invariance probability measures $\mathbb{P}$ are then uniquely determined on $\mathfrak{F}_T$, on which they only matter anyway (because, in practice, $\mathbb{P}$ is only used for computations in $\mathfrak{F}$ on $[0, T]$). As a consequence, we can then talk of "the invariance probability measure $\mathbb{P}$."

By reduction in our setup, we may and do assume that the $(\mathfrak{G}, \mathbb{Q})$ intensity of τ is of the form $\gamma \mathbb{1}_{(0,\tau]}$, for an $\mathfrak{F}$ predictable process γ uniquely defined on $[0, T]$, and we write $\Gamma = \int_0^{\cdot} \gamma_s ds$, so that Γ^τ is the $(\mathfrak{G}, \mathbb{Q})$ compensator of τ.

Given our focus on the time interval $[0, T]$ hereafter, we may and do assume that optional (respectively predictable) reductions are stopped at T (respectively vanish on (T, ∞)), without loss of generality.

3 Conditional Expectation Transfer Formulas

The $(\mathfrak{G}_t, \mathbb{Q})$ and $(\mathfrak{F}_t, \mathbb{P})$ conditional expectations are denoted by $\mathbb{E}_t$ and $\mathbb{E}'_t$ and we drop the index t at time 0.

The following result, the unconditional version of which corresponds to Theorem 4.1 in Crépey, Sabbagh, and Song (2020), provides an extension of classical results (see e.g. Bielecki, Jeanblanc, and Rutkowski (2009, Chapter 3)) beyond the basic immersion setup where $(\mathfrak{F}, \mathbb{P} = \mathbb{Q})$ local martingales are $(\mathfrak{G}, \mathbb{Q})$ local martingales without jump at τ.

Theorem 3.1 *For any constant $t \in [0, T]$, $[t, T]$ valued $\mathfrak{F}$ stopping time σ, and $\mathfrak{F}_\sigma$ measurable nonnegative random variable χ, for any $\mathfrak{F}$ predictable nonnegative process K, for any $\mathfrak{F}$ optional nondecreasing process A starting from 0, we have, on $\{t < \tau\}$,*

$$\mathbb{E}_t[\chi \mathbb{1}_{\{\sigma < \tau\}}] = \mathbb{E}'_t[\chi e^{-(\Gamma_\sigma - \Gamma_t)}], \tag{3.1}$$

$$\mathbb{E}_t[K_\tau \mathbb{1}_{\{\tau \le T\}}] = \mathbb{E}'_t[\int_t^T K_s e^{-(\Gamma_s - \Gamma_t)} \gamma_s \, ds], \tag{3.2}$$

$$\mathbb{E}_t[A_T^{\tau-} - A_t^{\tau-}] = \mathbb{1}_{\{t<\tau\}} \mathbb{E}'_t[\int_t^T e^{-(\Gamma_s - \Gamma_t)} \, dA_s]. \tag{3.3}$$

Proof. For any $B \in \mathfrak{G}_t$ and B' associated with B as in (2.1), we have by definition of S and $\mathfrak{F}_\sigma$ measurability of χ (using also the tower rule and recalling the assumption $\mathsf{S}_T > 0$ which is part of the condition (C)):

$$\mathbb{E}\left[\mathbb{1}_{\{t<\tau\}}\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\big|\mathfrak{F}_t\big)\mathbb{1}_B\right] = \mathbb{E}\left[\mathsf{S}_t\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\mathbb{1}_{B'}|\mathfrak{F}_t\big)\right] = \mathbb{E}\left[\chi\mathsf{S}_\sigma\mathbb{1}_{B'}\right] = \mathbb{E}\left[\chi\mathbb{1}_{\{\sigma<\tau\}}\mathbb{1}_B\right].$$

Hence

$$\mathbb{1}_{\{t<\tau\}}\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\big|\mathfrak{F}_t\big) = \mathbb{E}\big(\mathbb{1}_{\{\sigma<\tau\}}\chi|\mathfrak{G}_t\big). \tag{3.4}$$

Then (2.4), under the assumption $\mathsf{S}_T > 0$, yields

$$\begin{aligned}\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\big|\mathfrak{F}_t\big) &= \mathbb{E}\Big(\chi\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D})_\sigma\mathcal{E}(\frac{1}{{}^p\mathsf{S}}\boldsymbol{\cdot}\mathsf{Q})_\sigma/\big(\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D})_t\mathcal{E}(\frac{1}{{}^p\mathsf{S}}\boldsymbol{\cdot}\mathsf{Q})_t\big)\Big|\mathfrak{F}_t\Big)\\ &= \mathbb{E}'\Big[\chi\mathcal{E}(-\frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D})_\sigma/\mathcal{E}(-\frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D})_t\Big|\mathfrak{F}_t\Big],\end{aligned} \tag{3.5}$$

by (2.7). In view of (2.6), we obtain (3.1).

For (3.2), we compute, on $\{t < \tau\}$,

$$\begin{aligned}\mathbb{E}_t[K_\tau\mathbb{1}_{\{\tau\le T\}}] &= \mathbb{E}_t[\int_t^T K_s\mathbb{1}_{\{s\le\tau\}}\gamma_s\,ds] = \int_t^T \mathbb{E}_t[K_s\mathbb{1}_{\{s<\tau\}}\gamma_s]\,ds\\ &= \int_t^T \mathbb{E}'_t[K_s e^{-(\Gamma_s-\Gamma_t)}\gamma_s]\,ds = \mathbb{E}'_t[\int_t^T K_s e^{-(\Gamma_s-\Gamma_t)}\gamma_s\,ds],\end{aligned}$$

where (3.1) was used for passing to the second line.

Regarding (3.3), an application of (3.2) yields (still on $\{t < \tau\}$)

$$\begin{aligned}\mathbb{E}_t[(A_{\tau-}-A_t)\mathbb{1}_{\{\tau\le T\}}] &= \mathbb{E}'_t[\int_t^T (A_s-A_t)e^{-(\Gamma_s-\Gamma_t)}\gamma_s\,ds]\\ &= -\mathbb{E}'_t[(A_T-A_t)e^{-(\Gamma_T-\Gamma_t)}] + \mathbb{E}'_t[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\,dA_s].\end{aligned}$$

Using (3.1), we deduce

$$\begin{aligned}\mathbb{E}_t[(A_T^{\tau-}-A_t)] &= \mathbb{E}_t[(A_T-A_t)\mathbb{1}_{\{T<\tau\}}] + \mathbb{E}_t[(A_{\tau-}-A_t)\mathbb{1}_{\{\tau\le T\}}]\\ &= \mathbb{E}'_t[(A_T-A_t)e^{-(\Gamma_T-\Gamma_t)}] - \mathbb{E}'_t[(A_T-A_t)e^{-(\Gamma_T-\Gamma_t)}] + \mathbb{E}'_t[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\,dA_s]\\ &= \mathbb{E}'_t[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\,dA_s].\ \blacksquare\end{aligned}$$

See Section A for the discussion of two alternatives to the formula (3.2) that are known from the mathematical finance literature.

4 Martingale Transfer Formulas

We denote by

- $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$, the set of $(\mathfrak{F},\mathbb{P})$ local martingales stopped at T,
- $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, the set of $(\mathfrak{G},\mathbb{Q})$ local martingales stopped at $\tau-\wedge T$, i.e. before τ and at T,
- $\mathcal{M}^c_T(\mathfrak{F},\mathbb{P})$ and $\mathcal{M}^d_T(\mathfrak{F},\mathbb{P})$, respectively $\mathcal{M}^c_{\tau\wedge T}(\mathfrak{G},\mathbb{Q})$, and $\mathcal{M}^d_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, their respective subsets of continuous local martingales and purely discontinuous local martingales.

Theorem 4.1 *The following bijections hold:*

$$\begin{aligned}
\mathcal{M}_T(\mathfrak{F},\mathbb{P}) &\underset{\cdot'}{\overset{\cdot^{\tau-}}{\rightleftarrows}} \mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q}),\\
\mathcal{M}^c_T(\mathfrak{F},\mathbb{P}) &\underset{\cdot'}{\overset{\cdot^{\tau}}{\rightleftarrows}} \mathcal{M}^c_{\tau\wedge T}(\mathfrak{G},\mathbb{Q}),\\
\mathcal{M}^d_T(\mathfrak{F},\mathbb{P}) &\underset{\cdot'}{\overset{\cdot^{\tau-}}{\rightleftarrows}} \mathcal{M}^d_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q}),
\end{aligned} \tag{4.1}$$

where $\cdot'$ *denotes the* $\mathfrak{F}$ *optional reduction operator.*

***Proof*.** On $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$ the map $\cdot'$ takes its values in the space $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$ because, for any $M\in\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, the process $\mathsf{S}_{-}\centerdot M' + [\mathsf{S}, M']$ is an $(\mathfrak{F},\mathbb{Q})$ local martingale on $\{\mathsf{S}_{-}>0\}$, by (2.3), so that $M'\in\mathcal{M}_T(\mathfrak{F},\mathbb{P})$, by (2.9). Conversely, on $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$ the map $\cdot^{\tau-}$ takes its values in the space $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$ because, for any $P\in\mathcal{M}_T(\mathfrak{F},\mathbb{P})$, $P^{\tau-}\in\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, by the condition (A) which is contained in (C).

To establish the first bijection in (4.1) it remains to show that $(M')^{\tau-}=M$ and $(P^{\tau-})'=P$ in the above. As M is stopped before τ, the first identity is trivially true. Regarding the second one, $(P^{\tau-})'=P$ holds before τ, hence on $[0,T]$, by (2.5), hence on $\mathbb{R}_+$ as both processes $(P^{\tau-})'$ and P are stopped at T.

The second bijection in (4.1) follows by the same steps, noting that the reduction of a continuous process X is continuous on $[0,T]$, by (2.5) applied to the jump process of X.

To prove the third bijection, following He, Wang, and Yan (1992, Theorem 7.34), assuming $M\in\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, we take a $(\mathfrak{G},\mathbb{Q})$ continuous local martingale X and we consider the bracket $[M,X]$. Computing the quadratic variations, we obtain

$$[M,X]=[M,X^{\tau-}]=[M',X']^{\tau-}$$

on $[0,T]$, which shows that $[M',X']$ is the $\mathfrak{F}$ optional reduction of $[M,X]$. Consequently, according to (2.5), $[M,X]=0$ on $[0,\tau\wedge T)$ if and only if $[M',X']=0$ on $[0,T]$. The lemma then follows from the first and second bijections in (4.1). ∎

5 Transfer of Stochastic Integrals in the Sense of Local Martingales

Lemma 5.1 *Let $(\theta_n)_{n\geq 0}$ be a nondecreasing sequence of $\mathfrak{G}$ stopping times tending to infinity. There exists a nondecreasing sequence $(\sigma_n)_{n\geq 0}$ of $\mathfrak{F}$ stopping times such that σ_n tends to infinity and*

$$\theta_n \wedge T \wedge \tau = \sigma_n \wedge T \wedge \tau.$$

Proof. We compute, using (3.1) at $t=0$ for passing to the second line,

$$\begin{aligned}\mathbb{E}'[\mathbb{1}_{\{\theta'_n<T\}}e^{-\Gamma_T}] &\leq \mathbb{E}'[\mathbb{1}_{\{\theta'_n<T\}}e^{-\Gamma_{\theta'_n}}]\\ &= \mathbb{E}[\mathbb{1}_{\{\theta'_n<T\}}\mathbb{1}_{\{\theta'_n<\tau\}}] = \mathbb{E}[\mathbb{1}_{\{\theta_n<T\}}\mathbb{1}_{\{\theta_n<\tau\}}] \to 0 \text{ as } n\to\infty.\end{aligned}$$

This implies that $\mathbb{P}[\theta'_n < T] \to 0$. Hence $\mathbb{Q}[\theta'_n < T] \to 0$, as $\mathbb{P}$ is equivalent to $\mathbb{Q}$ on $\mathfrak{F}_T$. The sequence $\sigma_n = (\theta'_n)_{\{\theta'_n<T\}}, n \geq 0$, satisfies all the desired properties. ∎

Lemma 5.2 *Let A be a $\mathfrak{G}$ adapted nondecreasing càdlàg process. The process $A^{\tau-}$ is $(\mathfrak{G},\mathbb{Q})$ locally integrable on $[0,T]$ if and only if A' is $(\mathfrak{F},\mathbb{P})$ locally integrable on $[0,T]$.*

Proof. Recall that A' is a nondecreasing process (cf. the last paragraph in Section 2.1). Let $(\theta_n)_{n\geq 0}$ be a nondecreasing sequence of $\mathfrak{G}$ stopping times tending to infinity. Let $(\sigma_n)_{n\geq 0}$ be associated with $(\theta_n)_{n\geq 0}$ as in Lemma 5.1. We compute

$$\begin{aligned}\mathbb{E}[\int_0^{\theta_n\wedge T}\mathbb{1}_{\{s<\tau\}}e^{\Gamma_s}dA_s\ ] &= \mathbb{E}[\int_0^{\theta_n\wedge T\wedge\tau}\mathbb{1}_{\{s<\tau\}}e^{\Gamma_s}dA_s\ ]\\ &= \mathbb{E}[\int_0^{\sigma_n\wedge T\wedge\tau}\mathbb{1}_{\{s<\tau\}}e^{\Gamma_s}dA'_s\ ] = \mathbb{E}[\int_0^{\sigma_n\wedge T}\mathbb{1}_{\{s<\tau\}}e^{\Gamma_s}dA'_s\ ] = \mathbb{E}'[A'_{\sigma_n\wedge T}],\end{aligned}$$

by (3.3) (used at $t=0$). This implies the result. ∎

Theorem 5.1 *Let W be a $(\mathfrak{G},\mathbb{Q})$ local martingale stopped before τ and let L be a $\mathfrak{G}$ predictable process. The process L is W integrable in the sense of $(\mathfrak{G},\mathbb{Q})$ local martingales if and only if L' is W' integrable on $[0,T]$ in the sense of $(\mathfrak{F},\mathbb{P})$ local martingales (recall that we assume $L'=0$ on (T,∞)). If so, then*

$$\left(L' \centerdot W' \ \text{ in } (\mathfrak{F},\mathbb{P})\right)^{\tau-} = L \centerdot W \ \text{ in } (\mathfrak{G},\mathbb{Q}) \text{ on } [0,T].$$

Proof. In view of He, Wang, and Yan (1992, Definition 9.1), we only need to check the local integrability of the processes $\sqrt{\int_0^t L_s^2 d[W,W]_s}$ and $\sqrt{\int_0^t (L')_s^2 d[W',W']_s}$ under respectively $(\mathfrak{G},\mathbb{Q})$ and $(\mathfrak{F},\mathbb{P})$. But these local integrabilities are equivalent because of Lemma 5.2.

To prove the identity between the stochastic integrals when they exist, we first note that the identity holds for any L in the class of $\mathfrak{G}$ predictable bounded step processes. By monotone class theorem, this is then extended to the class of $\mathfrak{G}$ predictable bounded processes L. By stochastic dominated convergence, i.e. Theorem 9.30 in He, Wang, and Yan (1992), this is extended further to all $\mathfrak{G}$ predictable processes L which are W integrable under $(\mathfrak{G},\mathbb{Q})$. ∎

6 Transfer of Random Measures Stochastic Integrals

Given a Polish space E endowed with its Borel σ algebra $\mathcal{B}(E)$, we recall from He, Wang, and Yan (1992, Theorem 11.13) that, for any (optional) integer valued random measure π, there exists an E valued optional process β and an optional thin set, of the form $\cup_{n\in\mathbb{N}}[\theta_n]$ for some sequence of stopping times $(\theta_n)_{n\geq 0}$, such that

$$\pi = \sum_{s} \boldsymbol{\delta}_{(s,\beta_s)} \mathbb{1}_{\{s\in\cup_{n\in\mathbb{N}}[\theta_n]\}}. \tag{6.1}$$

Hence, for any nonnegative $\mathcal{A}\times\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(E)$ measurable function Ψ,

$$\Psi * \pi = \sum_{s<\cdot} \Psi_s(\beta_s)\mathbb{1}_{\{s\in\cup_{n\in\mathbb{N}}[\theta_n]\}} = \sum_{\theta_n<\cdot} \Psi(\theta_n,\beta_{\theta_n})\mathbb{1}_{\{\theta_n<\infty\}}. \tag{6.2}$$

Lemma 6.1 *The $\mathfrak{G}$ optional integer valued random measure π on $\mathbb{R}_+\times E$ admits an $\mathfrak{F}$ optional reduction, i.e. an $\mathfrak{F}$ optional integer valued random measure π' on $\mathbb{R}_+\times E$ such that $\mathbb{1}_{[0,\tau)}.\pi = \mathbb{1}_{[0,\tau)}\cdot\pi'$.*

Proof. We have, for any nonnegative $\mathcal{A}\times\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(E)$ measurable function Ψ,

$$\begin{aligned}\Psi * (\mathbb{1}_{[0,\tau)}.\pi) &= \sum_{s<\cdot} \mathbb{1}_{\{s<\tau\}}\Psi_s(\beta'_s)\mathbb{1}_{\{s\in\cup_{n\in\mathbb{N}}[\theta_n]\}}\\ &= \sum_{s<\cdot} \mathbb{1}_{\{s<\tau\}}\Psi_s(\beta'_s)\mathbb{1}_{\{s\in\cup_n[\theta'_n]\}} = \Psi * (\mathbb{1}_{[0,\tau)}\cdot\pi'),\end{aligned}$$

where $\pi' = \sum_{s<\cdot} \boldsymbol{\delta}_{(s,\beta'_s)}\mathbb{1}_{\{s\in\cup_n[\theta'_n]\}}$ defines an $\mathfrak{F}$ optional integer valued random measure, by He, Wang, and Yan (1992, Theorem 11.13). ∎

In the remainder of the paper, we fix the space E, a $\mathfrak{G}$ optional integer valued random measure π, and the related notation as in the above. We introduce the spaces of random functions $\widehat{\mathcal{P}}(\mathfrak{F}) = \mathcal{P}(\mathfrak{F})\times\mathcal{B}(E)$ and $\widehat{\mathcal{P}}(\mathfrak{G}) = \mathcal{P}(\mathfrak{G})\times\mathcal{B}(E)$. We denote the $(\mathfrak{F},\mathbb{P})$ compensator of $\mu = \pi'$ by ν.

Lemma 6.2 *The $(\mathfrak{G},\mathbb{Q})$ compensator of $\mathbb{1}_{[0,\tau)}\cdot\mu$ is $\mathbb{1}_{[0,\tau]}\cdot\nu$ on $[0,T]$.*

Proof. By Lemma 5.2, for any $\Psi\in\widehat{\mathcal{P}}(\mathfrak{G})$ such that the process $|\Psi| * \pi$ is $(\mathfrak{G},\mathbb{Q})$ integrable, the processes $|\Psi'| * \mu$ and $|\Psi'| * \nu$ are $(\mathfrak{F},\mathbb{P})$ locally integrable (recalling that $\Psi' = 0$ on (T,∞) by assumption). It follows that the process

$$P = \Psi' * \mu - \Psi' * \nu$$

is an $(\mathfrak{F},\mathbb{P})$ local martingale (cf. He, Wang, and Yan (1992, p. 301)). By the condition (A), the stopped process

$$P^{\tau-} = \mathbb{1}_{[0,\tau)}\Psi' * \mu - \mathbb{1}_{[0,\tau)}\Psi' * \nu = \mathbb{1}_{[0,\tau)}\Psi * \mu - \mathbb{1}_{[0,\tau]}\Psi * \nu$$

is a $(\mathfrak{G},\mathbb{Q})$ local martingale, where $\mathbb{1}_{[0,\tau)}\Psi * \nu = \mathbb{1}_{[0,\tau]}\Psi * \nu$ because τ avoids the predictable stopping times. As $\mathbb{1}_{[0,\tau]}.\nu$ is a $\mathfrak{G}$ predictable random measure, this proves the lemma. ∎

Theorem 6.1 *For any $\Psi \in \widehat{\mathcal{P}}(\mathfrak{G})$, Ψ is $(\mathbb{1}_{[0,\tau)}\boldsymbol{.}\mu - \mathbb{1}_{[0,\tau]}\boldsymbol{.}\nu)$ stochastically integrable in $(\mathfrak{G}, \mathbb{Q})$ on $[0, T]$ if and only if Ψ' is $(\mu - \nu)$ stochastically integrable in $(\mathfrak{F}, \mathbb{P})$ on $[0, T]$. If so, then*

$$\big(\Psi' * (\mu - \nu) \quad in\ (\mathfrak{F}, \mathbb{P})\big)^{\tau-} = \Psi * (\mathbb{1}_{[0,\tau)}.\mu - \mathbb{1}_{[0,\tau]}.\nu) \quad in\ (\mathfrak{G}, \mathbb{Q})\ on\ [0, T].$$

Proof. In view of He, Wang, and Yan (1992, Definition 11.16), the integrability relationship between Ψ and Ψ' is the consequence of Lemma 5.2.

To prove the identity between the corresponding integrals when they exist, we note that

$$(\Psi' * (\mu - \nu))^{\tau-} \text{ and } \Psi * (\mathbb{1}_{[0,\tau)}.\mu - \mathbb{1}_{[0,\tau]}.\nu)$$

are $(\mathfrak{G}, \mathbb{Q})$ purely discontinous local martingales. By virtue of He, Wang, and Yan (1992, Theorem 7.42 and Definition 11.16), they are then equal because they have the same jumps, namely

$$\begin{aligned}
\Delta_t(\Psi' * (\mu - \nu))^{\tau-} &= \big(\Psi'(t, \beta'_t)\mathbb{1}_{\{t \in \cup_{n\in\mathbb{N}}[\theta'_n]\}} - \int_{\{t\}\times E} \Psi'_s(e)\nu(ds, de)\big)\mathbb{1}_{\{t<\tau\}} \\
&= \big(\Psi_t(\beta_t)\mathbb{1}_{\{t \in \cup_{n\in\mathbb{N}}[\theta_n]\}} - \int_{\{t\}\times E} \Psi_s(e)\nu(ds, de)\big)\mathbb{1}_{\{t<\tau\}} \\
&= \Delta_t\big(\Psi * (\mathbb{1}_{[0,\tau)}.\mu - \mathbb{1}_{[0,\tau]}.\nu)\big),
\end{aligned}$$

as $\mathbb{1}_{[0,\tau)}.\nu = \mathbb{1}_{[0,\tau]}.\nu$ (because τ avoids the $\mathfrak{G}$ predictable stopping times). ∎

7 Transfer of Martingale Representation Properties

We consider martingale representations with respect to martingales and compensated jump measures as in Jacod (1979), which corresponds to the notion of weak representation in He, Wang, and Yan (1992). As in He, Wang, and Yan (1992), when no jump measure is involved, we talk of strong representation.

Let W be a d variate $(\mathfrak{G}, \mathbb{Q})$ local martingale stopped before τ. We assume the random measure π stopped before τ, in the sense that $\cup_{n\in\mathbb{N}}[\theta_n] \subseteq (0, \tau)$. We write $B = W'$, $\mu = \pi'$. Let ρ and ν denote the $(\mathfrak{G}, \mathbb{Q})$ compensator of π and the $(\mathfrak{F}, \mathbb{P})$ compensator of μ, so that $\rho = \mathbb{1}_{[0,\tau]}\cdot\nu$, by Lemma 6.2.

Lemma 7.1 *Given $(\mathcal{P}(\mathfrak{G}))^{\times d}$ and $\widehat{\mathcal{P}}(\mathfrak{G})$ measurable integrands L and Ψ, if*

$$M = L \boldsymbol{.} W + \Psi * (\pi - \rho) \tag{7.1}$$

holds in $(\mathfrak{G}, \mathbb{Q})$ on $[0, T]$, then $M' = L' \boldsymbol{.} B + \Psi' * (\mu - \nu)$ holds in $(\mathfrak{F}, \mathbb{P})$ on $[0, T]$.

Conversely, given $(\mathcal{P}(\mathfrak{F}))^{\times d}$ and $\widehat{\mathcal{P}}(\mathfrak{F})$ measurable integrands K and Φ, if

$$P = K \boldsymbol{.} B + \Phi * (\mu - \nu) \tag{7.2}$$

holds in $(\mathfrak{F}, \mathbb{P})$ on $[0, T]$, then $P^{\tau-} = K \boldsymbol{.} B^{\tau-} + \Phi * (\mathbb{1}_{[0,\tau)}\cdot\mu - \mathbb{1}_{[0,\tau]}\cdot\nu)$ holds in $(\mathfrak{G}, \mathbb{Q})$ on $[0, T]$.

Proof. This is the consequence of Theorems 5.1 and 6.1. ∎

Remark 7.1 In the representation (7.1), the integrands L and Ψ corresponding to a given process M are unique modulo $d[W,W]$ (with $d[W,W]_s$-*a.e.* in the multivariate sense of Jacod and Shiryaev (2003)) and ρ negligible sets, respectively. Likewise, in the representation (7.2), the integrands K and Φ corresponding to a given process P are unique modulo $d[B,B]$ (with $d[B,B]_s$-*a.e.* in the multivariate sense of Jacod and Shiryaev (2003)) and ν negligible sets. ∎

As an immediate consequence of Lemma 7.1:

Theorem 7.1 *The space $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$ admits a weak representation by W and π if and only if the space $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$ admits a weak representation by $B = W'$ and $\mu = \pi'$.* ∎

Applying Theorem 7.1 with $\mu \equiv 0$, one obtains the strong martingale representation transfer property.

See Gapeev, Jeanblanc, and Wu (2021, 2022) for other transfers of martingale representation properties, in respective Brownian and marked point process enlargement of filtration setups (progressive but also initial as already before in Fontana (2018)) satisfying Jacod's equivalence hypothesis, i.e. the existence of positive conditional density for τ with respect to $\mathfrak{F}$, as opposed to a semimartingale progressive enlargement of filtration setup under the condition (C) in this work. See also Jeanblanc and Song (2015) or (until τ) Choulli, Daveloose, and Vanmaele (2020) and Choulli and Alharbi (2022) (also after τ) for rather general transfers of martingale representation properties in a progressive enlargement of filtration setup. From a technical viewpoint our setup stopped before τ (as dictated by the motivating application of Example 10.1) is elementary once the underlying Theorems 5.1 and 6.1 are in place.

Remark 7.2 Of course we cannot say anything beyond τ, but the motivating application of Example 10.1 never requires to go beyond τ.

8 Semimartingale Characteristic Triplets Transfer Formula

Let there be given a semimartingale X stopped before τ (i.e. such that $X = X^{\tau-}$) in some filtration $\mathfrak{H}$ under a probability measure $\mathbb{M}$, with jump measure π^X. The characteristic triplet of X is composed of:

$b^{X,\mathfrak{H},\mathbb{M}}$, the drift part of the truncated semimartingale $X - (x\mathbb{1}_{\{|x|>1\}})_*\pi^X$;

$a^{X,\mathfrak{H},\mathbb{M}} = \langle X^c, X^c\rangle$, the angle bracket of the continuous martingale part of X (the diffusion part of X);

$c^{X,\mathfrak{H},\mathbb{M}} = (\pi^X)^{p,\mathfrak{H},\mathbb{M}}$, the predictable dual projection of π^X (called in He, Wang, and Yan (1992) the Lévy system of X, i.e. the extension to a semimartingale setup of the notion of a Lévy measure).

The following results show that the $(\mathfrak{F},\mathbb{P})$ characteristic triplet of the optional reduction X' of a $(\mathfrak{G},\mathbb{Q})$ semimartingale stopped before τ, X, is the predictable reduction of the $(\mathfrak{G},\mathbb{Q})$ characteristic triplet of X. Moreover, if the $(\mathfrak{G},\mathbb{Q})$ semimartingale X is special, then so is X' and the $(\mathfrak{F},\mathbb{P})$ drift of X' is the predictable reduction of the $(\mathfrak{G},\mathbb{Q})$ drift of X.

Theorem 8.1 *Let X be a $(\mathfrak{G},\mathbb{Q})$ semimartingale stopped before τ. Let X' be the optional reduction of X, an $(\mathfrak{F},\mathbb{Q})$ (hence $(\mathfrak{F},\mathbb{P})$) semimartingale as recalled in the last paragraph of Section 2.1, with $(\mathfrak{F},\mathbb{P})$ jump measure denoted by $\pi^{X'}$. We have*

$$\left(b^{X,\mathfrak{G},\mathbb{Q}}, a^{X,\mathfrak{G},\mathbb{Q}}, c^{X,\mathfrak{G},\mathbb{Q}}\right) = \left((b^{X',\mathfrak{F},\mathbb{P}})^{\tau}, (a^{X',\mathfrak{F},\mathbb{P}})^{\tau}, \mathbb{1}_{[0,\tau]}.c^{X',\mathfrak{F},\mathbb{P}}\right) \text{ on } [0,T]. \quad (8.1)$$

Proof. We have the identity $\mathbb{1}_{[0,\tau)}.\pi^{X} = \mathbb{1}_{[0,\tau)}.\pi^{X'}$ on $[0,T]$. So,

$$X-(x\mathbb{1}_{\{|x|>1\}})_{*}\pi^{X} = \left(X' - (x\mathbb{1}_{\{|x|>1\}})_{*}\pi^{X'}\right)^{\tau-} = (P^c)^{\tau-}+(P^d)^{\tau-}+(b^{X',\mathfrak{F},\mathbb{P}})^{\tau-} \quad (8.2)$$

on $[0,T]$, where P is the $(\mathfrak{F},\mathbb{P})$ canonical Doob–Meyer martingale component of the $(\mathfrak{F},\mathbb{P})$ special semimartingale $X'-(x\mathbb{1}_{\{|x|>1\}})_{*}\pi^{X'}$ on $[0,T]$, with continuous and purely discontinuous parts P^c and P^d. By the condition (A), $P^{\tau-}$ is a $(\mathfrak{G},\mathbb{Q})$ local martingale on $[0,T]$. Therefore, from the formula (8.2), we conclude that

$$b^{X,\mathfrak{G},\mathbb{Q}} = (b^{X',\mathfrak{F},\mathbb{P}})^{\tau-} = (b^{X',\mathfrak{F},\mathbb{P}})^{\tau} \text{ on } [0,T]$$

(as $\Delta_\tau b^{X',\mathfrak{F},\mathbb{P}} = 0$, because τ is totally inaccessible.) Now, applying Lemma 6.2 with $E=\mathbb{R}$, we also conclude

$$c^{X,\mathfrak{G},\mathbb{Q}} = \pi^{X\,p,\mathfrak{G},\mathbb{Q}} = (\mathbb{1}_{[0,\tau)}.\pi^{X})^{p,\mathfrak{G},\mathbb{Q}} = \mathbb{1}_{[0,\tau]}.(\pi^{X'})^{p,\mathfrak{F},\mathbb{P}} = \mathbb{1}_{[0,\tau]}.c^{X',\mathfrak{F},\mathbb{P}}.$$

Finally, according to the second and third bijections in (4.1), we have

$$(P^c)^{\tau-} \in \mathcal{M}^c_{\tau\wedge T}(\mathfrak{G},\mathbb{Q}), \;\; (P^d)^{\tau-} \in \mathcal{M}^d_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q}).$$

Hence we conclude from (8.2) that $X^c = (P^c)^{\tau-}$ is the continuous martingale part of X in $(\mathfrak{G},\mathbb{Q})$ and therefore

$$a^{X,\mathfrak{G},\mathbb{Q}} = [X^c,X^c] = [(P^c)^{\tau-},(P^c)^{\tau-}] = [P^c,P^c]^{\tau} = (a^{X',\mathfrak{F},\mathbb{P}})^{\tau}. \blacksquare$$

Corollary 8.1 *Suppose that a $(\mathfrak{G},\mathbb{Q})$ semimartingale $X = X^{\tau-}$ is special on $[0,T]$. Then X' is an $(\mathfrak{F},\mathbb{P})$ special semimartingale on $[0,T]$. Denoting by $\beta^{X,\mathfrak{G},\mathbb{Q}}$ and $\beta^{X',\mathfrak{F},\mathbb{P}}$ the $(\mathfrak{G},\mathbb{Q})$ drift of X and the $(\mathfrak{F},\mathbb{P})$ drift of X', we have*

$$\beta^{X,\mathfrak{G},\mathbb{Q}} = (\beta^{X',\mathfrak{F},\mathbb{P}})^{\tau} \text{ on } [0,T]. \quad (8.3)$$

Proof. As X' is already known to be an $(\mathfrak{F},\mathbb{P})$ semimartingale and because special semimartingale means one with locally integrable jumps, the special feature of X' follows from Lemma 5.2. Note that, by He, Wang, and Yan (1992, Lemma 7.16 and Theorem 11.24), the function $|x|\mathbb{1}_{\{|x|>1\}}$ is $c^{X',\mathfrak{F},\mathbb{P}}$ integrable on $[0,T]$. Consequently

$$\beta^{X',\mathfrak{F},\mathbb{P}} = b^{X',\mathfrak{F},\mathbb{P}} + (x\mathbb{1}_{\{|x|>1\}})_{*}c^{X',\mathfrak{F},\mathbb{P}} \text{ on } [0,T].$$

The analogous $(\mathfrak{G},\mathbb{Q})$ relationship holds for X. Hence (8.3) follows from (8.1). $\blacksquare$

9 Markov Transfer Formulas

In this section we study the transfer of Markov properties between $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$. The reader is referred to Sharpe (1988, proof of Proposition (60.2)) regarding the definition of the semigroup generated by a Markov family.

We suppose that the filtration $\mathfrak{G}$ is generated by a $(\mathfrak{G}, \mathbb{Q})$ quasi-left continuous strong Markov semimartingale X with state space $\mathbb{R}^d$. We denote by an index $\cdot^t$ everything related to the Markov process X translated by time t. We assume that τ is a terminal time of X, i.e. (see e.g. Blumenthal and Getoor (2007, (3.7) Remark p.108))

$$\tau = \tau^t + t \text{ if } \tau > t. \tag{9.1}$$

We assume further that the $(\mathfrak{G}, \mathbb{Q})$ intensity process of τ takes the form $\gamma(X_\cdot)\mathbb{1}_{(0,\tau]}$, for some $\mathcal{B}(\mathbb{R}^d)$ measurable function γ. Let

$$\mathsf{M}_s = e^{\Gamma_s}\mathbb{1}_{\{s<\tau\}} = e^{\int_0^s \gamma(X_u)du}\mathbb{1}_{\{s<\tau\}}.$$

Lemma 9.1 *M is a multiplicative functional of X, i.e., $\mathsf{M}_t = \mathsf{M}_s\mathsf{M}^s_{t-s}$, and is a $(\mathfrak{G}, \mathbb{Q})$ local martingale. The multiplicative functional M defines a probability transition function $(\mathcal{T}_t)_{t\in\mathbb{R}_+}$.*

Proof. The first part can be checked by definition of M. For the second part, we check by the Doléans-Dade exponential formula that

$$\mathsf{M} = \mathcal{E}(-\mathbb{1}_{[\tau,\infty)} + \Gamma^\tau).$$

The last part follows from Sharpe (1988, (65.3), proof of Proposition(56.5)), Sharpe (1988, Hypothesis (62.9)) and Sharpe (1988, Theorem (62.19)). ∎

Theorem 9.1 *The reduction X' of X is an $(\mathfrak{F}, \mathbb{P})$ strong Markov process with the transition semigroup $(\mathcal{T}_t)_{t\in[0,T]}$.*

Proof. For $A \in \mathfrak{F}_s$, h Borel bounded and $0 < s < s+t \le T$, we have

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_A h(X_{t+s})e^{\int_0^{t+s}\gamma(X_u)du}\mathbb{1}_{\{t+s<\tau\}}] \\
=\ &\mathbb{E}[\mathbb{1}_A e^{\int_0^s \gamma(X_u)du}\mathbb{1}_{\{s<\tau\}} h(X^s_t) e^{\int_0^t \gamma(X^s_u)du}\mathbb{1}_{\{t<\tau^s\}}] \\
=\ &\mathbb{E}[\mathbb{1}_A e^{\int_0^s \gamma(X_u)du}\mathbb{1}_{\{s<\tau\}}\mathcal{T}_t h(X_s)],
\end{aligned}$$

which is rewritten in terms of X' through the first expectation transfer formula in (3.1) as

$$\mathbb{E}'[\mathbb{1}_A h(X'_{t+s})] = \mathbb{E}'[\mathbb{1}_A \mathcal{T}_t h(X'_s)].$$

This proves that X' is an $(\mathfrak{F}, \mathbb{P})$ Markov process with the transition semigroup $(\mathcal{T}_t)_{t\in\mathbb{R}_+}$. If we rewrite the above computation for s replaced by an $\mathfrak{F}$ stopping time σ, we prove that X' is an $(\mathfrak{F}, \mathbb{P})$ strong Markov process on $[0, T]$. ∎

The next question is how to determine the generator of the semigroup $(\mathcal{T}_t)_{t\in\mathbb{R}_+}$. We suppose that the Markov process X is of the form $X = (Y, Z)$, with Y stopped before τ and Z constant (0, say) before τ.

Example 9.1 We may consider for X the following dynamic copula models of portfolio credit risk, with any of the modeled default times in the role of τ in this paper:

- The dynamic Marshall-Olkin copula (DMO) model, shown in Crépey and Song (2016, Theorem 9.2) to satisfy the condition (C), for $\mathbb{P} = \mathbb{Q}$ in the embedded condition (A) (case of a strict pseudo-stopping time in the terminology of Jeanblanc and Li (2020, Definition 2.1));
- The dynamic Gaussian copula (DGC) model, shown in Crépey and Song (2017a) to satisfy the condition (C) with $\mathbb{P} \neq \mathbb{Q}$, provided the correlation coefficient $\varrho > 0$ in the model is small enough. In particular, the condition (C) holds in the univariate DGC model (there is then no correlation ϱ involved) of Section A.

Suppose X is a $(\mathfrak{G}, \mathbb{Q})$ Markov process which is the solution of the following martingale problem:

$$v(X_t) - \int_0^t \mathcal{L}v(X_s)ds \text{ is a } (\mathfrak{G}, \mathbb{Q}) \text{ local martingale for all } v \in \mathcal{D}(\mathcal{L}),$$

where $\mathcal{L}$ is the generator of X, with domain $\mathcal{D}(\mathcal{L}) \subseteq$ the set of the $\mathcal{B}(\mathbb{R}^d)$ measurable bounded functions. Let $\mathcal{D}' = \{u \equiv u(y); \widehat{u}(y,z) := u(y) \text{ is in } \mathcal{D}(\mathcal{L})\}$ and let $\mathcal{L}'$ be the operator on $\mathcal{D}'$ defined by, for $u \in \mathcal{D}'$,

$$\mathcal{L}'u(y) = \mathcal{L}\widehat{u}(y, 0). \tag{9.2}$$

Theorem 9.2 *We suppose that $(\mathcal{D}', \mathcal{L}')$ satisfies the conditions of Ethier and Kurtz (1986, Theorem 4.1 of Chapter 4, p.182). Then $X' = (Y', 0)$, Y' is an $(\mathfrak{F}, \mathbb{P})$ strong Markov process on $[0, T]$, and the generator of Y' is an extension of $(\mathcal{D}', \mathcal{L}')$.*

Proof. Clearly, $X' = (Y', 0)$. Hence, Theorem 9.1 implies that Y' is an $(\mathfrak{F}, \mathbb{P})$ strong Markov process on $[0, T]$. For $u \in \mathcal{D}'$,

$$\widehat{u}(X_t) - \textstyle\int_0^t \mathcal{L}\widehat{u}(X_s)ds = u(Y_t) - \int_0^t \mathcal{L}\widehat{u}(X_s)ds$$

is a $(\mathfrak{G}, \mathbb{Q})$ local martingale. As Y is stopped before τ,

$$u(Y_t)^{\tau-} - \int_0^{t\wedge\tau} \mathcal{L}\widehat{u}(X_s)ds \text{ is a } (\mathfrak{G}, \mathbb{Q}) \text{ local martingale stopped before } \tau.$$

On $[0, \tau)$, $\mathcal{L}\widehat{u}(X_s) = \mathcal{L}\widehat{u}(Y_s, 0) = \mathcal{L}'u(Y_s)$, by (9.2). Hence,

$$u(Y_t)^{\tau-} - \int_0^{t\wedge\tau-} \mathcal{L}'u(Y_s)ds \text{ is a } (\mathfrak{G}, \mathbb{Q}) \text{ local martingale stopped before } \tau.$$

Passing to the reduction, we obtain that

$$u(Y_t') - \int_0^t \mathcal{L}'u(Y_s')ds \text{ is an } (\mathfrak{F}, \mathbb{P}) \text{ local martingale on } [0, T].$$

Therefore, Y' is the solution of the martingale problem of $\mathcal{L}'$. The result then follows from Ethier and Kurtz (1986, Theorem 4.1 of Chapter 4, p.182). ∎

10 BSDE Transfer Properties

In this section, τ satisfying the condition (C) on $[0,T]$ as before, we reduce a $(\mathfrak{G},\mathbb{Q})$ backward stochastic differential equation (BSDE) stopped before τ and at T to a simpler $(\mathfrak{F},\mathbb{P})$ BSDE stopped at T.

We suppose E Euclidean and $(E,\mathcal{B}(E))$ endowed with a σ finite measure m integrating $(1\wedge|e|^2)$ on E. We consider the space $\mathbb{L}_0$ of the $\mathcal{B}(E)$ measurable functions u endowed with the topology of convergence in measure induced by m.

Given a $\mathcal{P}(\mathfrak{G})\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{L}_0)$ measurable function $g=g_t(z,l,\psi)$, we can define, by monotone class theorem, a $\mathcal{P}(\mathfrak{F})\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{L}_0)$ reduction $g'=g'_t(z,l,\psi)$ of g such that $\mathbb{1}_{(0,\tau]}g=\mathbb{1}_{(0,\tau]}g'$. Let A be a $\mathfrak{G}$ finite variation (càdlàg) process.

Adopting the setup of Section 7, we consider the $(\mathfrak{G},\mathbb{Q})$ BSDE for a $\mathfrak{G}$ adapted process Z, a $(\mathcal{P}(\mathfrak{G}))^{\times d}$ measurable process L integrable against $B^{\tau-}$ in $(\mathfrak{G},\mathbb{Q})$ on $[0,T]$, and a $\widehat{\mathcal{P}}(\mathfrak{G})$ measurable function Ψ stochastically integrable against $(\mathbb{1}_{[0,\tau)}\cdot\mu-\mathbb{1}_{[0,\tau]}\cdot\nu)$ in $(\mathfrak{G},\mathbb{Q})$ on $[0,T]$, such that, in $(\mathfrak{G},\mathbb{Q})$,

$$\left\{\begin{array}{l}\int_0^{\tau\wedge T}|g_s(Z_{s-},L_s,\Psi_s)|ds<\infty \text{ and}\\ \qquad\qquad \int_0^{\cdot}\mathbb{1}_{\{s<\tau\}}|dA_s| \text{ is } (\mathfrak{G},\mathbb{Q}) \text{ locally integrable on } [0,T],\\ \\ Z_t^{\tau-\wedge T}+\int_0^{t\wedge\tau\wedge T}\left(g_s(Z_{s-},L_s,\Psi_s)ds+dA_s^{\tau-}\right)\\ \qquad\qquad =L\centerdot B_t^{\tau-}+\Psi*(\mathbb{1}_{[0,\tau)}\cdot\mu-\mathbb{1}_{[0,\tau]}\cdot\nu)_t,\ \ t\in\mathbb{R}_+,\\ \\ Z \text{ vanishes on } [\tau\wedge T,+\infty).\end{array}\right. \tag{10.1}$$

We also consider the $(\mathfrak{F},\mathbb{P})$ BSDE for an $\mathfrak{F}$ adapted process U, a $(\mathcal{P}(\mathfrak{F}))^{\times d}$ measurable process K integrable against B in $(\mathfrak{F},\mathbb{P})$ on $[0,T]$, and a $\widehat{\mathcal{P}}(\mathfrak{F})$ measurable function Φ stochastically integrable against $(\mu-\nu)$ in $(\mathfrak{F},\mathbb{P})$ on $[0,T]$, such that, in $(\mathfrak{F},\mathbb{P})$,

$$\left\{\begin{array}{l}\int_0^{T}|g'_s(U_{s-},K_s,\Phi_s)|ds<\infty \text{ and}\\ \qquad\qquad \int_0^{\cdot}|dA'_s| \text{ is } (\mathfrak{F},\mathbb{P}) \text{ locally integrable on } [0,T],\\ \\ U_t^{T}+\int_0^{t\wedge T}\left(g'_s(U_{s-},K_s,\Phi_s)ds+dA'_s\right)=K\centerdot B_t+\Phi*(\mu-\nu)_t,\ \ t\in\mathbb{R}_+,\\ \\ U \text{ vanishes on } [T,+\infty).\end{array}\right. \tag{10.2}$$

Note that the $(\mathfrak{G},\mathbb{Q})$ BSDE (10.1) is stopped at $\tau-\wedge T$, whereas the $(\mathfrak{F},\mathbb{P})$ BSDE (10.2) is stopped at T.

Example 10.1 Given a bank with default time τ, a $\mathfrak{G}$ stopping time θ representing the default time of a client of the bank, and a nonnegative $\mathfrak{G}$ optional process G representing the liability of the client to the bank, then the process $A=\int_0^{\cdot}G_s\boldsymbol{\delta}_\theta(ds)$ represents the counterparty credit exposure of the bank to its client. In this case

$$|dA_s|=G_s\boldsymbol{\delta}_\theta(ds),\ \ A^{\tau-}=\int_0^{\cdot}\mathbb{1}_{\{s<\tau\}}G_s\boldsymbol{\delta}_\theta(ds),\ \ A'=\int_0^{\cdot}G'_s\boldsymbol{\delta}_{\theta'}(ds).$$

The coefficient g represents the risky funding costs of the bank entailed by its credit riskiness. For the reason explained in the next-to-last paragraph of Section 1, all cash flows are stopped before the bank default time τ. This results in a BSDE of the form (10.1) for the valuation of counterparty risk (CVA) and of its funding implications to the bank (FVA). The cost of capital (KVA) also obeys an equation of the form (10.1): see Crépey (2022, Eqns (2.12), (2.13), and (2.17)).

10.1 Transfer of Local Martingale Solutions

The result that follows states the equivalence between the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) considered within the above-introduced spaces of solutions for the triples (Z, L, Ψ) and (U, K, Φ), called local martingale solutions henceforth (as the right-hand sides in the second lines of (10.1) and (10.2) are then respectively $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$ local martingales).

Theorem 10.1 *The* $(\mathfrak{G}, \mathbb{Q})$ *BSDE* (10.1) *and the* $(\mathfrak{F}, \mathbb{P})$ *BSDE* (10.2) *are equivalent in their respective spaces of local martingale solutions. Specifically, if* (Z, L, Ψ) *solves (10.1), then* $(U, K, \Phi) = (Z, L, \Psi)'$ *solves* (10.2)*. Conversely, if* (U, K, Φ) *solves* (10.2)*, then* $(Z, L, \Psi) = (\mathbb{1}_{[0,\tau)} U, \mathbb{1}_{[0,\tau]} K, \mathbb{1}_{[0,\tau]} \Phi)$ *solves* (10.1)*.*

Proof. Through the correspondence stated in the theorem between the involved processes:

- The equivalence between the Lebesgue integrability conditions (first lines) in (10.1) and (10.2) follows from Lemma 5.2;
- The equivalence between the martingale conditions (second lines) in (10.1) and (10.2) follows from Theorems 5.1 and 6.1;
- The terminal condition for U in (10.2) obviously implies the one for $Z = \mathbb{1}_{[0,\tau)} U$ in (10.1), whereas the terminal condition in (10.1) implies $Z_T \mathbb{1}_{\{T<\tau\}} = 0$, hence by taking the $\mathfrak{F}_T$ conditional expectation:

  $$0 = \mathbb{E}[Z_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = \mathbb{E}[Z'_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = Z'_T \mathsf{S}_T,$$

  yielding $U_T = Z'_T = 0$ (as S_T is positive under the condition (C)). ∎

10.2 Transfer of Square Integrable Solutions

We now consider the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) within suitable spaces of square integrable solutions.

We assume that the compensator ν of $\mu = \pi'$ is given as $\zeta_t(e) m(de) dt$, where ζ is a nonnegative and bounded integrand in $\mathcal{P}(\mathfrak{F})$. We write, for any $t \geq 0$ and $\mathcal{B}(E)$ measurable function u,

$$|u|_t^2 = \int_E u(e)^2 \zeta_t(e) m(de).$$

We write $Y_t^* = \sup_{s \in [0,t]} |Y_s|$.

Lemma 10.1 *For any real valued càdlàg $\mathfrak{F}$ adapted process V, respectively nonnegative $\mathfrak{F}$ predictable process X, we have*

$$\mathbb{E}\Big|V_0^2 + \int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} d(V^*)_s^2\Big] = \mathbb{E}'[(V^*)_T^2]; \quad (10.3)$$

$$\mathbb{E}\Big[\int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} X_s ds\Big] = \mathbb{E}'\Big[\int_0^T X_s\ ds\Big]. \quad (10.4)$$

Proof. The formula (3.3) used at $t = 0$ yields:

- For $A = \int_0^\cdot e^{\int_0^s \gamma_u du} d(V^*)_s^2$,

$$\mathbb{E}\Big[\int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} d(V^*)_s^2\Big] = \mathbb{E}'[(V^*)_T^2] - \mathbb{E}'[V_0^2];$$

- For $A = \int_0^\cdot e^{\int_0^s \gamma_u du} X_s\ ds$,

$$\mathbb{E}\Big[\int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} X_s ds\Big] = \mathbb{E}'\Big[\int_0^T X_s\ ds\Big]. \ \blacksquare$$

Considering the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) for (Z, L, Ψ) and the reduced $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) for (U, K, Φ), with local martingale solutions (if any) such that

$$(U, K, \Phi) = (Z, L, \Psi)',\ \ (Z, L, \Psi) = (\mathbb{1}_{[0,\tau)} U, \mathbb{1}_{[0,\tau]} K, \mathbb{1}_{[0,\tau]} \Phi) \quad (10.5)$$

(cf. Theorem 10.1), we define

$$\begin{aligned}
\|(Z, L, \Psi)\|_2^2 &= \mathbb{E}\Big[|Z_0|^2 + \int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} d(Z^*)_s^2\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} \big(|L_s|^2 + |\Psi_s|_s^2\big) ds\Big], \\
(\|(U, K, \Phi)\|_2')^2 &= \mathbb{E}'[(U^*)_T^2] + \mathbb{E}'\Big[\int_0^T \big(|K_s|^2 + |\Phi_s|_s^2\big)\ ds\Big].
\end{aligned}$$

We consider the respective subspaces of square integrable solutions of the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and of the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) defined by $\|\cdot\|_2 < +\infty$ and $Z = 0$ on $[\tau \wedge T, +\infty)$, respectively $\|\cdot\|_2' < +\infty$ and $U = 0$ on $[T, +\infty)$, dubbed $\|\cdot\|_2$ and $\|\cdot\|_2'$ solutions hereafter.

Theorem 10.2 *Given local martingale solutions (Z, L, Ψ) to the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and (U, K, Φ) to the reduced $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2), we have*

$$\|(Z, L, \Psi)\|_2 = \|(U, K, \Phi)\|_2'. \quad (10.6)$$

The $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) considered in terms of $\|\cdot\|_2$ solutions and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) considered in terms of $\|\cdot\|_2'$ solutions are equivalent through the correspondence (10.5).

Proof. Given respective local martingale solutions (Z, L, Ψ) and (U, K, Φ) to (10.1) and (10.2), then related through (10.5) as seen in Theorem 10.1, Lemma 10.1 applied to $V = U$ and $X = |K_\cdot|^2 + |\Phi_\cdot|_\cdot^2$ proves (10.6).

Given the equivalence of Theorem 10.1 between (10.1) and (10.2) in the sense of local martingale solutions, their equivalence in the sense of square integrable solutions follows from the transfer of norms formula (10.6). ∎

10.3 Application

Assuming $\int_0^T |dA'_s|$ integrable under $\mathbb{P}$ and a (weak) martingale representation of the form studied in Theorem 7.1, we define the process R and its $(\mathfrak{F}, \mathbb{P})$ martingale part P given as

$$R_t = \mathbb{E}'\big[\int_t^{T\wedge t} dA'_s \,|\, \mathfrak{F}_t\big] \text{ and } P_t = \mathbb{E}'\big[\int_0^T dA'_s \,|\, \mathfrak{F}_t\big],\ \ t \in \mathbb{R}_+.$$

Let $f_s(u, k, \phi) = g'_s(R_{s-} + u, K_s^P + k, \Phi_s^P + \phi)$, where K^P and Φ^P are the integrands in the representation (7.2) of P (cf. Remark 7.1).

Proposition 10.1 *Suppose that $\int_0^T |dA'_s|$ is $\mathbb{P}$ square integrable and*

(i) the functions $u \mapsto f_t(u, k, \phi)$ are continuous. Moreover, f is monotonous with respect to u, i.e.

$$(f_t(u_1, k, \phi) - f_t(u_2, k, \phi))(u_1 - u_2) \leq C(u_1 - u_2)^2;$$

(ii) $\mathbb{E}' \int_0^T \sup_{|u|\leq c} |f_t(u, 0, 0) - f_t(0, 0, 0)| dt < \infty$ holds for every positive c;

(iii) f is Lipschitz continuous with respect to k and ϕ, i.e.

$$|f_t(u, k_1, \phi_1) - f_t(u, k_2, \phi_2)| \leq C(|k_1 - k_2| + |\phi_1 - \phi_2|_t);$$

(iv) $\mathbb{E}' \int_0^T |f_t(0, 0, 0)|^2 dt < +\infty$.

Then the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) have unique $\|\cdot\|_2$ and $\|\cdot\|'_2$ solutions, respectively, and these solutions are related through (10.5).

Proof. Note that $\int_0^T |dA'_s|$ being $\mathbb{P}$ square integrable implies that $\mathbb{E}'\big[(R^*)_T^2\big] < \infty$. Through the correspondence

$$U = R + V,\ \ K^U = K^P + K^V,\ \ \Phi^U = \Phi^P + \Phi^V,$$

the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) for (U, K^U, Φ^U) is equivalent (in both senses of $(\mathfrak{F}, \mathbb{P})$ local martingale solutions and $\|\cdot\|'_2$ solutions) to the following $(\mathfrak{F}, \mathbb{P})$ BSDE for (V, K^V, Φ^V):

$$\begin{cases} \int_0^T |f_s(V_{s-}, K_s^V, \Phi_s^V)| ds < \infty, \\ V_t^T + \int_0^{t\wedge T} f_s(V_{s-}, K_s^V, \Phi_s^V) ds = K^V \centerdot B_t + \Phi^V * (\mu - \nu)_t, \\ V_T = 0. \end{cases} \tag{10.7}$$

Under the assumptions of the proposition, the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.7) for (V, K^V, Φ^V) satisfies the assumptions of Kruse and Popier (2016, Theorem 1). Hence it has a unique $\|\cdot\|_2'$ solution. So has in turn the BSDE (10.2). The result then follows by an application of Theorem 10.2. ∎

Remark 10.1 Kruse and Popier (2016, Theorem 1) is only derived in the case a Poisson measure π, but one can readily check that all their computations performed under square integrable assumptions are still valid in our more general integer valued random measure setup. Also, in view of Bouchard, Possamaï, Tan, and Zhou (2018), Kruse and Popier (2016)'s condition of a quasi-left continuous filtration is in fact not needed.

The reader is referred to Crépey (2022, Lemma B.1, Proposition B.1 and Theorem 6.1) and Crépey, Sabbagh, and Song (2020, Section 6) for variations on the above results, in the respective cases where f only depends on u (in the notation of Proposition 10.1 above) and no martingale representation property needs to be assumed, or where f is assumed to be Lipschitz but also exhibits a dependence on a conditional expected shortfall of a future increment of the martingale part of the solution.

Remark 10.2 Earlier occurrences of such results are Crépey and Song (2015, 2016), with the difference that these earlier works were about BSDEs stopped at a random time. The more recent papers, instead, with the motivation recalled in the next-to-last paragraph of Section 1 of the present paper, are about BSDEs stopped before a random time: compare e.g. stopping at ϑ in the second line (Crépey and Song, 2015, Eqn (2.1)) versus stopping before τ in the second part of (10.1).

Analogous techniques could be used to simplify $(\mathfrak{G}, \mathbb{Q})$ optimal stopping or stochastic control problems into reduced $(\mathfrak{F}, \mathbb{P})$ reformulations: cf., in the case of BSDEs or control problems stopped at time τ (as opposed to stopped before τ in our setup), Kharroubi and Lim (2014) and Jiao, Kharroubi, and Pham (2013) (assuming that a driving $(\mathfrak{F}, \mathbb{Q})$ Brownian motion, stopped at τ, is a $(\mathfrak{G}, \mathbb{Q})$ martingale), Aksamit, Li, and Rutkowski (2021) (who provide some comparative comments with our approach in their Remark 8.2), or Alsheyab and Choulli (2021).

A Intensity Based Pricing Formulas, Survival Measure and Invariance Times

This section puts Theorem 3.1 (specifically, the formula (3.2)) in perspective with Duffie, Schroder, and Skiadas (1996, Proposition 1) and Collin-Dufresne, Goldstein, and Hugonnier (2004, Theorem 1). This is done in the setup of a specific Markov model where the issues at stake can be understood based on Feynman-Kac representations. For other renewed views on the seminal formulas of Duffie, Schroder, and Skiadas (1996), see Jeanblanc and Li (2020).

A.1 The Univariate Dynamic Gaussian Copula Model

We consider the following single-name version of the dynamic Gaussian copula model of Crépey and Song (2017a). Let

$$\tau = \Psi\big(\int_0^{+\infty} \varsigma(s) dB_s\big), \tag{A.1}$$

where Ψ is a continuously differentiable increasing function from $\mathbb{R}$ to $(0, +\infty)$, ς is a Borel function on $\mathbb{R}_+$ such that $\int_0^{+\infty} \varsigma^2(u)du = 1$, and B is an $(\mathfrak{F}, \mathbb{Q})$ Brownian motion, with $\mathfrak{F}$ taken as the augmented natural filtration of B. The full model filtration $\mathfrak{G}$ is given as the augmented filtration of the progressive enlargement of $\mathfrak{F}$ by τ. Hence, the random time τ is an $\mathfrak{F}_\infty$ measurable $\mathfrak{G}$ stopping time τ.

By Theorem 2.2, Lemma 3.2, and Remark 4.1 in Crépey and Song (2017a), the condition (C) holds in this setup, for some probability measure $\mathbb{P}$ distinct from $\mathbb{Q}$ but equivalent to it on $\mathfrak{F}_T$, on which $\mathbb{P}$ is uniquely determined through (2.7). Let

$$h_t = \mathbb{1}_{\{\tau \le t\}}, \;\; m_t = \int_0^t \varsigma(s)dB_s, \;\; k_t = (h_t, \tau \wedge t), \;\; \nu^2(t) = \int_t^{+\infty} \varsigma^2(s)ds, \tag{A.2}$$

and assume ν positive for all t. By application of results in Ethier and Kurtz (1986), one can show that the process (m, k) is $(\mathfrak{G}, \mathbb{Q})$ Markov.

Remark A.1 The reason why we introduce $(\tau \wedge t)$ on top of the indicator h_t in k_t is because of a dependence of the post-τ behavior of the model on the value of τ itself. The state augmentation by $(\tau \wedge t)$ takes care of this path-dependence. ∎

By definition (A.1) of τ, we have

$$\mathbb{Q}(\tau > t \,|\, \mathfrak{F}_t) = \Phi\big(\frac{\Psi^{-1}(t) - m_t}{\nu(t)}\big), \;\; t \in \mathbb{R}_+, \tag{A.3}$$

where Φ denotes the standard normal cdf. The process on the right hand side of (A.3) has infinite variation. This shows that the reference filtration $\mathfrak{F}$ is not immersed into the full model filtration $\mathfrak{G}$. This lack of immersion makes it more interesting from the point of view of the different approaches that we want to compare. This is our motivation for working in this particular model in this section.

Theorems 2.2 and 2.4 in Crépey and Song (2017a) show the existence of processes of the form

$$\beta_t = \beta(t, m_t, k_t) \text{ and } \gamma_t = \gamma\,(t, m_t, k_t) = \gamma_t \mathbb{1}_{(0,\tau]}, \;\; t \in \mathbb{R}_+, \tag{A.4}$$

for continuous functions β and γ with linear growth in m, such that

$$\begin{aligned} &dW_t = dB_t - \beta_t dt \text{ is a } (\mathfrak{G}, \mathbb{Q}) \text{ Brownian motion and} \\ &\text{the process } \gamma \text{ is the } (\mathfrak{G}, \mathbb{Q}) \text{ intensity of } \tau. \end{aligned} \tag{A.5}$$

The proof of the following result is deferred to Section A.3.

Proposition A.1 *Let a process $m^\star$ satisfy*

$$dm^\star_t = \varsigma(t)\big(dW^\star_t + \beta(t, m^\star_t, (0,t))dt\big), \;\; 0 \le t \le T, \tag{A.6}$$

starting from $m^\star_0 = 0$, for some Brownian motion $W^\star$ with respect to some stochastic basis $(\mathfrak{G}^\star, \mathbb{Q}^\star)$. Denoting the $\mathbb{Q}^\star$ expectation by $\mathbb{E}^\star$, we have, for any bounded Borel function $G(t,m)$,

$$\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}} G(\tau, m_\tau)\big] = \mathbb{E}^\star\Big[\int_0^T e^{-\int_0^t \gamma(s, m^\star_s, (0,s))ds}\gamma(t, m^\star_t, (0,t))G(t, m^\star_t)dt\Big]. \tag{A.7}$$

A.2 Discussion

From (A.2) and (A.4)–(A.5), it holds that

$$dm_t = \varsigma(t)\big(dW_t + \beta(t, m_t, k_t)dt\big), \;\; t \in \mathbb{R}_+, \tag{A.8}$$

which, for $t \ge \tau$ (so that $k_t = (1, \tau)$), diverges from the specification (A.6). Hence the pair (m, W) is not an eligible choice for $(m^\star, W^\star)$ in Proposition A.1. As a consequence, we expect a contrario from Proposition A.1 that

$$\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}} G(\tau, m_\tau)\big] = \mathbb{E}\Big[\int_0^T e^{-\int_0^t \gamma(s, m_s, (0,s))ds}\gamma(t, m_t, (0,t))G(t, m_t)dt\Big] \tag{A.9}$$

does *not* hold in general (but only in special cases, including obviously $G = 0$).

Indeed, let

$$V_t = \mathbb{E}\Big[\int_t^T e^{-\int_t^s \gamma(u, m_u, (0,u))du}\gamma(s, m_s, (0,s))G(s, m_s)ds \,\Big|\, \mathfrak{G}_t\Big], \;\; t \in \mathbb{R}_+,$$

so that V_0 is equal to the right hand side in (A.9). By an application of Duffie, Schroder, and Skiadas (1996, Proposition 1) with $X = r = 0$ and $h_\cdot = \gamma(\cdot, m_\cdot, (0, \cdot))$ on $[0, T]$ there (noting that any process coinciding with the $(\mathfrak{G}, \mathbb{Q})$ intensity of τ before τ can be used as a process h in their setup), we have

$$\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}} G(\tau, m_\tau)\big] = V_0 - \mathbb{E}(V_\tau - V_{\tau-}). \tag{A.10}$$

In a basic immersed setup, $\mathbb{E}(V_\tau - V_{\tau-})$ vanishes and (A.9) actually holds: See the comments before Section 3 in Duffie, Schroder, and Skiadas (1996), page 1379 in Collin-Dufresne, Goldstein, and Hugonnier (2004), or following (3.22), (H.3) and Proposition 6.1 in Bielecki and Rutkowski (2001)). But, in general, $\mathbb{E}(V_\tau - V_{\tau-})$ is nonnull and intractable.

Instead, an eligible choice in Proposition A.1 consists in using $m^* = m$, $W^* = W$, and $\mathbb{Q}^* =$ the survival measure $\mathbb{S}$ with $(\mathfrak{G}, \mathbb{Q})$ density process

$$e^{\int_0^{\cdot\wedge T} \gamma(u, m_u, (0,u))du}\mathbb{1}_{\{\tau > \cdot\wedge T\}}$$

(assuming $e^{\int_0^{\tau\wedge T}\gamma(u,m_u,(0,u))du}$ integrable under $\mathbb{Q}$). As noted in Collin-Dufresne, Goldstein, and Hugonnier (2004, Lemma 1(i)), $\mathbb{1}_{[\tau,+\infty)}=0$ holds $\mathbb{S}$ almost surely on $[0,T]$. The "survival measure" idea and terminology were first introduced in Schönbucher (1999, 2004). One can then fix the discrepancy in (A.9) (in a progressive enlargement of filtration setup without immersion) by singularly changing the probability measure $\mathbb{Q}$ to $\mathbb{Q}^*=\mathbb{S}$, while sticking to the original model filtration $\mathfrak{G}$ (or, more precisely, resorting under $\mathbb{Q}^*=\mathbb{S}$ to the $\mathbb{S}$ augmentation $\bar{\mathfrak{G}}$ of $\mathfrak{G}$, obtained by adding to $\mathfrak{G}_0$ all the $\mathbb{S}$ null sets $A\in\mathcal{A}$ such that $A\subseteq\{\tau\le T\}$). This specification of the formula (A.7) corresponds to Collin-Dufresne, Goldstein, and Hugonnier (2004, Theorem 1).

Another eligible choice for $(m^\star,W^\star)$ in Proposition A.1 consists in using

$$m^*=m \text{ and } dW^*_t=dB_t-\beta(t,m_t,(0,t))dt.$$

As it follows from Lemma 3.5 and Section 4.4 in Crépey and Song (2017a), this process W^* is a $(\mathfrak{G}^*=\mathfrak{F},\mathbb{Q}^*=\mathbb{P})$ Brownian motion. The corresponding formula (A.7) is then none other than our formula (3.2) (for $t=0$). This approach fixes the discrepancy in (A.9) (in a non-immersed setup) by reducing the filtration from $\mathfrak{G}$ to a smaller $\mathfrak{F}$, while changing the probability measure "as little as possible", i.e. equivalently on $\mathfrak{F}_T$ (in a basic immersive setup, an invariance time approach would not change $\mathbb{Q}$ at all, whereas Collin-Dufresne, Goldstein, and Hugonnier (2004)'s measure change would still be singular).

Note that Collin-Dufresne, Goldstein, and Hugonnier (2004)'s $(m^*,W^*,\mathfrak{G}^*,\mathbb{Q}^*)=(m,W,\bar{\mathfrak{G}},\mathbb{S})$ approach only provides a transfer of conditional expectation formulas (even the semimartingale property may be lost after their singular measure change). By contrast, the approach $(m^*,W^*,\mathfrak{G}^*,\mathbb{Q}^*)=(m,B-\beta(\cdot,m_\cdot,(0,\cdot)\centerdot\boldsymbol{\lambda}),\mathfrak{F},\mathbb{P})$ of this paper results in a transfer of semimartingale calculus as a whole. One concrete motivation for this work is the solution of BSDEs stopped before their terminal time. As Section 10 illustrates, in order to deal with these, conditional expectation formulas are not enough: the entire semimartingale calculus of this paper is required.

A.3 Proof of Proposition A.1

By the $(\mathfrak{G},\mathbb{Q})$ Markov property of the process (m,k), noting that τ is the hitting time of 1 by the h component of the process k, we have

$$\begin{aligned}\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}}G(\tau,m_\tau)\,|\,\mathfrak{G}_t\big]=\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}}G(\tau,m_\tau)\,|\,(m_t,k_t)\big]=\\ v(t,m_t,k_t)=u_{h_t}(t,m_t),\ \ t\in[0,\tau\wedge T],\end{aligned} \tag{A.11}$$

for suitable Borel bounded functions $v(t,m,k)$ and $u_h(t,m)=v(t,m,k=(h,t))$. As a martingale, the process $u_{h_t}(t,m_t)$, $t\in[0,\bar{\tau}=\tau\wedge T]$, has a vanishing drift. Hence, by an application of the Itô formula to this process, using (A.5), the pair function

$u = (u_0(t,m), u_1(t,m))$ formally solves

$$\begin{cases} u_0(T,m) = u_1(T,m) = 0, \;\; m \in \mathbb{R}, \\ u_1(t,m) = G(t,m), \;\; t < T, m \in \mathbb{R}, \\ \partial_t u_0(t,m) + \varsigma(t)\beta(t,m,(0,t))\partial_m u_0(t,m) + \dfrac{\varsigma(t)^2}{2}\partial^2_{m^2} u_0(t,m) \\ \qquad\qquad + \gamma(t,m,0)\big[u_1(t,m) - u_0(t,m)\big] = 0, \;\; t < T, m \in \mathbb{R}, \end{cases} \tag{A.12}$$

Remark A.2 At least, the above holds assuming u regular enough for applicability of the Itô formula. Given the discussion format of this appendix, we content ourselves with the above formal argument, without introducing weak solutions of (A.12).

Note that the system of equations (A.12) reduces to $u_1 = \mathbb{1}_{[0,T)}G$ and to the following equation for u_0:

$$\begin{cases} u_0(T,m) = 0, \;\; m \in \mathbb{R}, \\ \partial_t u_0(t,m) + \varsigma(t)\beta(t,m,0)\partial_m u_0(t,m) + \dfrac{\varsigma(t)^2}{2}\partial^2_{m^2} u_0(t,m) \\ \qquad\qquad - \gamma(t,m,0)u_0(t,m) + \gamma(t,m,0)G(t,m) = 0, \;\; t < T, m \in \mathbb{R}. \end{cases} \tag{A.13}$$

Putting together (A.11) and the Feynman-Kac representation of the solution u_0 of (A.13) at the origin yields

$$\begin{aligned} \mathbb{E}\big[\mathbb{1}_{\{\tau<T\}}G(\tau, m_\tau)\big] &= u_0(0,0) \\ &= \mathbb{E}^\star\Big[\int_0^T e^{-\int_0^t \gamma(s,m^\star_s,(0,s))ds}\gamma(t, m^\star_t, (0,t))G(t, m^\star_t)dt\Big], \end{aligned}$$

for any process $m^\star$ as stated in the proposition, which is therefore proven.

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