Percolation on hyperbolic groups

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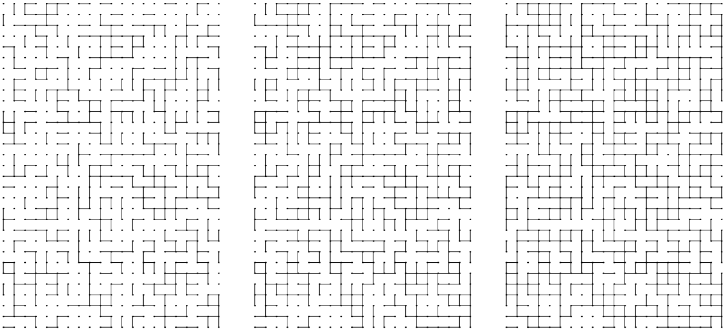
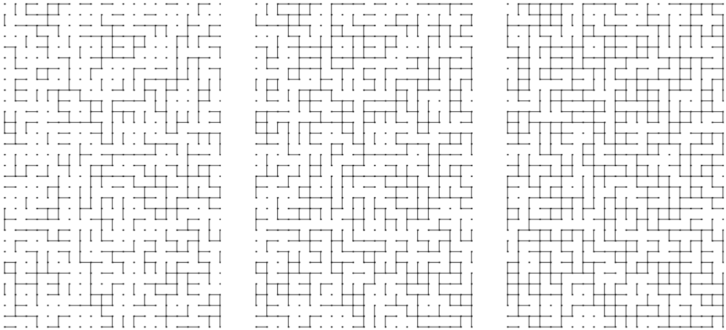
Percolation basics

◇ Every edge of a graph G is deleted or retained independently with retention probability p.



◇ Retained edges are **open**, deleted edges are **closed**. Open connected components are **clusters**.

◇ $\theta(p)$ probability that the origin is in an infinite cluster. **Critical probability** $p_c(G) = \inf\{p \in [0,1] : \theta(p) > 0\}$.



◇ **There is a phase transition:** Typically $0 < p_c < 1$.

◇ For the square grid $p_c = 1/2$ (Kesten, 1980's). Usually we don't expect p_c to have a nice or interesting value.

Two big questions

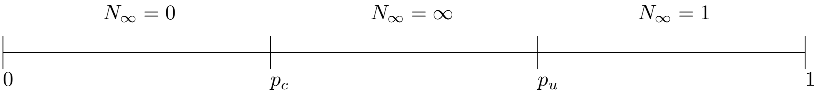
- Question 1: What is the nature of the phase transition? Is there percolation at p_c? What are the critical exponents? etc.
- Question 2: When is there a *unique* infinite cluster?

We'll restrict attention to **transitive** graphs, where for every two vertices u, v there is an automorphism (symmetry) of the graph sending u to v.

The uniqueness problem

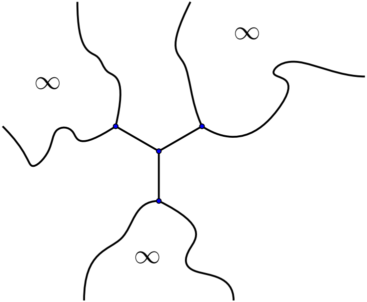
We define the **uniqueness threshold** to be

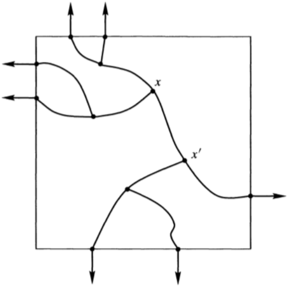
$$p_u(G) = \inf\{p \in [0,1] : \exists \text{ a } \textit{unique} \text{ infinite cluster a.s.}\}$$



- Newman and Schulman 1981: If G is transitive then $G[p]$ has either $0, 1$ or infinitely many infinite clusters.
- Häggströmm, Peres, and Schonmann 1999: If $p \in (p_u, 1]$ then $G[p_u]$ has a unique infinite cluster a.s.

◇ Aizenman, Kesten, and Newman 1987, Burton and Keane 1989: For each $p \in [0,1]$, $\mathbb{Z}^d[p]$ either has one infinite cluster or no infinite clusters almost surely. In particular $p_c = p_u$.





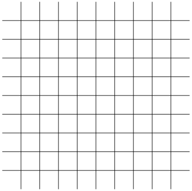
Amenability and nonamenability

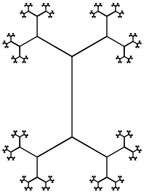
These proofs readily generalize to any **amenable** transitive graph.

Here, a locally finite graph $G = (V, E)$ is said to be **nonamenable** if its **Cheeger constant**

$$h(G) = \inf \left\{ \frac{\#\{\text{edges in the boundary of } W\}}{\#\{\text{edges with both endpoints in } W\}} : W \subset V \text{ finite} \right\}$$

is positive.

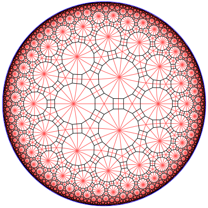


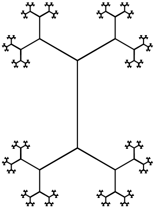


Conjecture (Benjamini and Schramm 1996)

For a transitive graph G, $p_c(G) = p_u(G)$ if and only if G is amenable.

Aizenman-Kesten-Newman/Burton-Keane proofs extend to any amenable transitive graph, so the problem is to prove the 'only if' direction.





Note that for the k-regular tree we have that $p_c = 1/(k-1)$, but p_u is trivially 1, so $p_c < p_u$. What about the graph on the left?

Old results

- Grimmett and Newman 1990: $0 < p_c < p_u < 1$ for $T_k \times \mathbb{Z}^d$ if k large.
- Perturbative criteria for $p_c < p_u$ and $\nabla_{p_c} < \infty$. (Benjamini & Schramm, Pak & Smirnova-Nagnibeda)
- Lalley 1998, Benjamini and Schramm 2001: $p_c < p_u$ for transitive nonamenable **planar** graphs.
- Gaboriau 2005: "cost > 1" implies $p_c < p_u$.

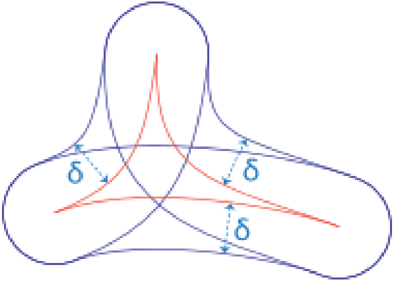
In particular, any transitive graph that looks like the hyperbolic plane has $p_c < p_u$.

New results

Theorem (H. 2018)

Let G be a nonamenable, transitive, **Gromov hyperbolic** *graph. Then $p_c(G) < p_u(G)$.*

E.g. graphs that look like d-dimensional hyperbolic space $\mathbb{H}^d$.

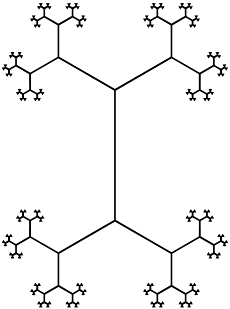


Critical behaviour

- The p_c vs p_u question is very closely connected to understanding percolation at p_c.
- Question 1: Are there infinite clusters at p_c?
- Question 2: Understand **critical exponents**, e.g., we expect that there exist β, δ, γ such that

$$
\begin{aligned}
\mathbb{P}_{p_c+\varepsilon}(|C| = \infty) &\approx \varepsilon^{\beta} \\
\mathbb{P}_{p_c}(|C| \geq n) &\approx n^{-\delta} \\
\mathbb{E}_{p_c-\varepsilon}|C| &\approx \varepsilon^{-\gamma}
\end{aligned}
$$

Percolation on trees



- Percolation on a d-regular tree is essentially just a a Galton-Watson branching process with offspring distribution $\text{Binomial}(d-1, p)$.
- We can compute everything explicitly, and find that e.g.:

$$p_c = 1/(d-1)$$
$$\theta(p_c) = 0$$
$$\left.\begin{aligned} \mathbb{P}_{p_c+\varepsilon}(|C| = \infty) &\approx \varepsilon^1 \\ \mathbb{P}_{p_c}(|C| \geq n) &\approx n^{-1/2} \\ \mathbb{E}_{p_c-\varepsilon}|C| &\approx \varepsilon^{-1} \end{aligned}\right\} \textbf{critical exponents}$$

When $p_c < p < 1$ there are **infinitely many** infinite clusters.

Conjectures for percolation on Euclidean lattices

No percolation at p_c in any dimension above 1. Totally open for $3 \le d \le 6$.

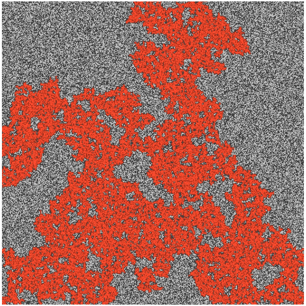
$$d = 2 \qquad \left.\begin{aligned} \mathbb{P}_{p_c+\varepsilon}(|C| = \infty) &\approx \varepsilon^{5/36} \\ \mathbb{P}_{p_c}(|C| \ge n) &\approx n^{-5/91} \\ \mathbb{E}_{p_c-\varepsilon}|C| &\approx \varepsilon^{-43/18} \end{aligned}\right\} \text{weird rational exponents}$$

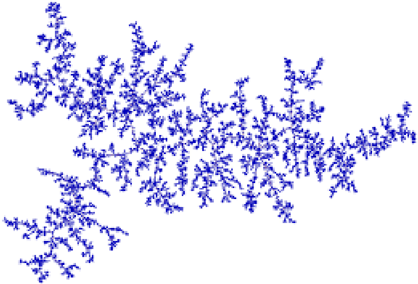
Proven for *site percolation on the triangular lattice* by Kesten, Smirnov, and Lawler, Schramm, and Werner.

$$d \ge 7 \qquad \left.\begin{aligned} \mathbb{P}_{p_c+\varepsilon}(|C| = \infty) &\approx \varepsilon^{1} \\ \mathbb{P}_{p_c}(|C| \ge n) &\approx n^{-1/2} \\ \mathbb{E}_{p_c-\varepsilon}|C| &\approx \varepsilon^{-1} \end{aligned}\right\} \text{same as on tree, a.k.a. "mean-field"}$$

Proven for *large d* by Hara and Slade (1990); **lace expansion**.

Scaling limits





In low dimensions macroscopic cycles play a significant role; in high dimensions they do not.

Critical behaviour

For nonamenable graphs, the most basic question of whether there are infinite clusters at p_c or not is already understood:

- Benjamini, Lyons, Peres, and Schramm 1999: No infinite clusters at p_c for **unimodular** nonamenable transitive graphs.
- Timár 2006: At most one infinite cluster at p_c on **nonunimodular** amenable transitive graphs.
- H. 2016: No infinite clusters at p_c on transitive graphs of **exponential volume growth**.
- Hermon and H. 2018: No infinite clusters at p_c on certain transitive graphs of intermediate growth.

Critical behaviour

We expect that on any nonamenable transitive graph, we have mean-field critical behaviour

$$\left.\begin{aligned} \mathbb{P}_{p_c+\varepsilon}(|C|=\infty) &\approx \varepsilon^1 \\ \mathbb{P}_{p_c}(|C|\geq n) &\approx n^{-1/2} \\ \mathbb{E}_{p_c-\varepsilon}|C| &\approx \varepsilon^{-1} \end{aligned}\right\}$$

same as on tree, a.k.a. "mean-field"

Theorem (H. 2018)

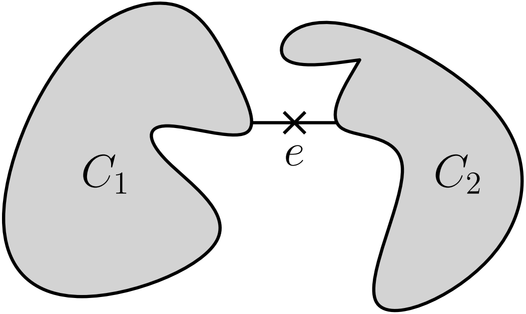
Let G be a transitive graph of exponential growth. Then there exist positive constants c, C such that

$$\begin{aligned} \mathbb{P}_{p_c+\varepsilon}(|C|=\infty) &\leq \varepsilon^c \\ \mathbb{P}_{p_c}(|C|\geq n) &\leq n^{-c} \\ \mathbb{E}_{p_c-\varepsilon}|C| &\leq \varepsilon^{-C} \end{aligned}$$

Differential inequalities

To get information near criticality, where combinatorial methods fail, **differential inequalities** play a crucial role.

$$\chi_p := \mathbb{E}_p|C| \qquad \frac{\mathrm{d}}{\mathrm{d}p}\chi_p \asymp \mathbb{E}_p|C_1||C_2|\mathbb{1}(C_1 \neq C_2)$$



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We always have

$$\frac{\mathrm{d}}{\mathrm{d}p}\chi_p \leq C\chi_p^2 \qquad \text{which implies that} \qquad \chi_{p_c-\varepsilon} \geq c\varepsilon^{-1}.$$

To establish mean-field behaviour, key step is to prove

$$\frac{\mathrm{d}}{\mathrm{d}p}\chi_p \geq c\chi_p^2 \qquad \text{which implies that} \qquad \chi_{p_c-\varepsilon} \leq C\varepsilon^{-1}.$$

Critical behaviour

- The **triangle** diagram is defined to be

$$\nabla_p = \sum_{y,z} \mathbb{P}_p(x \leftrightarrow y)\mathbb{P}_p(y \leftrightarrow z)\mathbb{P}_p(z \leftrightarrow x) \qquad x \text{ fixed}$$

- Aizenman and Newman 1984: If $\nabla_{p_c} < \infty$ then $\frac{\mathrm{d}}{\mathrm{d}p}\chi_p \geq c\chi_p^2$.
- $\nabla_{p_c} < \infty$ shown to hold on $\mathbb{Z}^d$ for large d by Hara and Slade. Expected to hold for all $d > 6$.

Conjecture

Every transitive nonamenable graph has $\nabla_{p_c} < \infty$.

A new conjecture

- Let $T_p(u,v) = \mathbb{P}_p(u \leftrightarrow v) \in [0,1]^{V^2}$ be the *matrix* of connection probabilities.
- $p_{2\to 2} = \sup\{p \in [0,1] : T_p \text{ is a bounded operator on } L^2(V)\}$.

Conjecture (H. 2018)

Let G be a transitive nonamenable graph. Then $p_c < p_{2\to 2}$.

$p_c < p_{2\to 2}$ implies $p_c < p_u$ and $\nabla_{p_c} < \infty$.

Also lets one compute some exponents that aren't known to follow from triangle condition.

A strategy

Lemma

To prove $p_c < p_{2\to 2}$, it suffices to prove that

- $\|T_p\|_{1\to 1} = \chi_p \asymp (p_c - p)^{-1}$ *as* $p \uparrow p_c$.
- *If* $S_p = T_p / \|T_p\|_{1\to 1}$ *then* $\|S_p\|_{2\to 2} \to 0$ *as* $p \uparrow p_c$.

S_p is the transition matrix of a 'highly spread out' random walk, so it's highly plausible that both conditions are true in the nonamenable context.

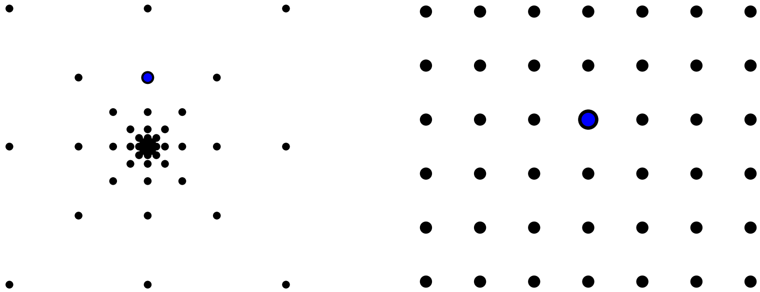
Theorem (H. 2018)

Let G be a nonamenable, transitive, **Gromov hyperbolic** *graph. Then $p_c(G) < p_{2\to2}(G)$.*

The Magic Lemma

Theorem (Benjamini-Schramm Magic Lemma 2001)

Viewed from a random point, any finite set in $\mathbb{R}^d$ looks like it has at most two accumulation points, one of which is at infinity.

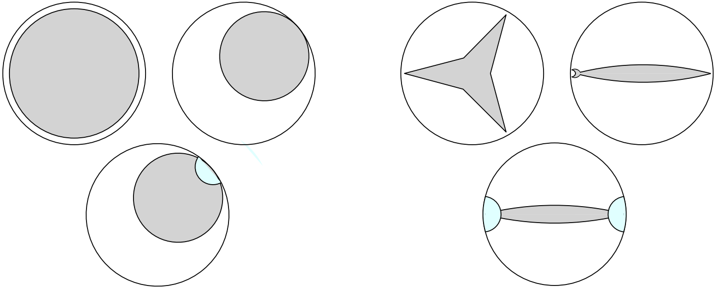


The Hyperbolic Magic Lemma

Say a set in $\mathbb{H}^d$ is **separated** if all its points have distance at least c from each other.

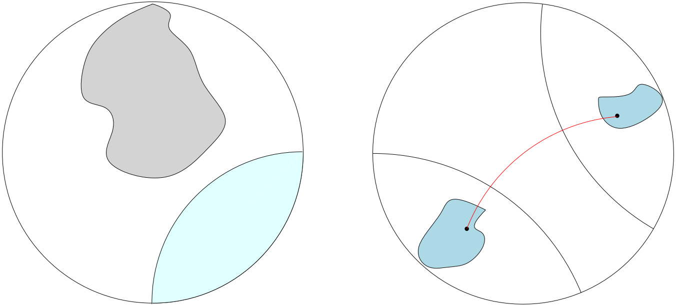
Corollary

Viewed from a random point, any finite, separated set has most of its volume in either one or two distant half-spaces.



Corollary of the Magic Lemma: *For any finite set of vertices A in a nonamenable Gromov hyperbolic graph, most points of A are near the boundary of the convex hull of A.*

Similar observations proven previous by Benjamini and Eldan.



We use this fact to prove the differential inequality $\frac{\mathrm{d}}{\mathrm{d}p}\chi_p \geq c\chi_p^2$.

We also use the Magic Lemma to prove that $\|S_p\|_{2\to 2} \to 0$ as $p \uparrow p_c$ which, by the lemma, completes the proof that $p_c < p_{2\to 2}$.

- Roughly speaking, the Magic Lemma implies that the only way for a spread out random walk not to have small spectral radius is for it to be concentrated on a line-like set.
- This can then be ruled out for percolation. This is rather technical, but relies on the abudance of free subgroups in hyperbolic groups.

Problems

- Which Cayley graphs have uniqueness at p_u? Not understood for graphs that look like $\mathbb{H}^d$ for $d \geq 3$ or for non-planar graphs that look like $\mathbb{H}^2$.
- When is $p_{2\to 2} = p_u$? There is always non-uniqueness at $p_{2\to 2}$.
- Are there interesting/useful analogues of the Magic Lemma in other geometric contexts?
- Sufficient conditions for sequences of increasingly spread-out random walks to have spectral radius tending to zero?