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The mathematics of the past: distinguishing its history from our heritage

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For the centenary of Jean Cavaillès (1903–1944), historian and philosopher of mathematics, and self-chosen victim of Nazi millenarianism

Abstract

Mathematics shows much more durability in its attention to concepts and theories than do other sciences: for example, Galen may not be of much use to modern medicine, but one can still read and use Euclid. One might expect that this situation would make mathematicians sympathetic to history, but quite the opposite is the case. Their normal attention to history is concerned with heritage: that is, how did we get here? Old results are modernized in order to show their current place; but the historical context is ignored and thereby often distorted. By contrast, the historian is concerned with what happened in the past, whatever be the modern situation. Each approach is perfectly legitimate, but they are often confused. The difference between them is discussed, with examples exhibited; these will include Euclid, set theory, limits, and applied mathematics in general.

Sumário

Nota-se na Matemática uma muito maior durabilidade em relação a conceitos e teorias do que nas outras ciências: por exemplo, Galeno não será muito útil para a Medicina moderna, mas Euclides ainda pode ser lido e utilizado. Poder-se-ia esperar que esta situação levasse os matemáticos a simpatizarem com a história, mas acontece precisamente o oposto. Normalmente a sua perspectiva da história é de *herança*; isto é, "como chegamos aqui?". Os resultados antigos são modernizados para que se possa ver o seu lugar actual, mas o contexto histórico é ignorado e portanto frequentemente distorcido. Por contraste, o historia dor preocupa-se com o que aconteceu no passado independentemente da situação moderna. Ambas as abordagens são perfeitamente válidas, mas são frequentemente confundidas. A diferença entre elas é discutida, e apresentam-se exemplos: estes incluem Euclides, teoria de conjuntos, limites e Matemática Aplicada em geral.

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However eager to tell us how scientists of the seventeenth century used their inheritance from the sixteenth, the scholars seem to regard as irrelevant anything a scientist today might think about any aspects of science, including his own debt to the past or reaction against it. —C.A. Truesdell III, Essays in the History of Mechanics (Foreword) [1968]

You think that the world is what it looks like in fine weather at noonday; I think that it seems like in the early morning when one first wakes from deep sleep. —A.N. Whitehead to B. Russell in B. Russell, Portraits from Memory and Other Essays (p. 41) [1956]

As all historians know, the past is a great darkness, and filled with echoes. Voices may reach us from it; but [...] try as we may, we cannot always decipher them in the clearer light of our own day. —Margaret Atwood, end of The Handmaid's Tale [1985]

1. The pasts and the futures

1.1. The basic distinction

The growth of interest and work in the history of mathematics during the past three decades or so has led to reactions among mathematicians. Some of them have been welcoming, and indeed have contributed their own historical research; but many others have been cautious, even contemptuous, about the work produced by practicing historians, especially on account of the historians' apparently limited knowledge of mathematics.[1] By the latter they usually mean some modern version of the mathematics in question, and the failure of historians to take due note of it.

There is a deep and general distinction involved here, locatable in any branch of mathematics, any period, any culture, and possibly involving teaching or popularization of mathematics as well as its research. It seems to be sensed by people working in history, whether they come to the subject with mainly a historical or a mathematical motivation. However, it has not been much discussed in the literature; even the survey [May, 1976] of historiography jumps across it.

I use the words "history" and "heritage" to name two interpretations of a mathematical theory; the corresponding actors are "historians" and "inheritors" (or "heirs"), respectively. The word "notion" serves as the umbrella term to cover a theory (or definition, proof-method, technique, algorithm, notation(s), whole branch of mathematics, ...), and the letter "N" to symbolize it. A sequence of notions in recognized order in the development of a mathematical theory is notated '$N_0, N_1, N_2, \ldots$.'

By "history" I refer to the details of the development of N: its prehistory and concurrent developments; the chronology of progress, as far as it can be determined; and maybe also the impact in the immediately following years and decades. History addresses the question "what happened in the past?" and gives descriptions; maybe it also attempts explanations of some kinds, in order to answer the corresponding "why?" question (Section 3.10). History should also address the dual questions "what did not happen in the past?" and "why not?"; false starts, missed opportunities [Dyson, 1972], sleepers, and repeats are noted and maybe explained. The (near-)absence of later notions from N is registered, as well as their eventual arrival; *differences* between N and seemingly similar more modern notions are likely to be emphasized.

[1] Another point of division between the two disciplines is techniques and practices specific to historical work, such as the finding, examination, and deployment of manuscript sources and of large-scale bibliographies. The latter are rehearsed, for the pre-computer age, in May [1973, 3–41]. They are not directly relevant to this paper.

By "heritage" I refer to the impact of N upon later work, both at the time and afterward, especially the forms which it may take, or be embodied, in later contexts.[2] Some modern form of N is usually the main focus, with attention paid to the course of its development. Here the mathematical relationships will be noted, but historical ones in the above sense will hold much less interest. Heritage addresses the question "how did we get here?," and often the answer reads like "the royal road to me." The modern notions are inserted into N when appropriate, and thereby N is unveiled (a nice word proposed to me by Henk Bos): *similarities* between N and its more modern notions are likely to be emphasized; the present is *photocopied* onto the past.

Both kinds of activity are quite legitimate, and indeed important in their own right; in particular, mathematical research often seems to be conducted in a heritage-like way (Section 3.1), whether the predecessors produced their work long ago or very recently. *The confusion of the two kinds of activity is not legitimate*, either taking heritage to be history (frequently the mathematicians' view—and historians' sometimes!) or taking history to be heritage (the occasional burst of excess enthusiasm by a historian); indeed, such conflations may well mess up both categories, especially the historical record. In the case of sequences of notions, a pernicious case arises when N_1 is a logical consequence or a generalization of N_0, and the claim is made that a knower of N_0 knew N_1 also [May, 1975a]; an example is given in Section 3.5.

A philosophical difference is that inheritors tend to focus upon knowledge alone (theorems as such, and so on), while historians also seek motivations, causes, and understanding in a more general sense. The distinction sometimes made by historians of science between "internal" and "external" history forms part of this difference. Each category is explicitly metatheoretical, though history may demand the greater finesse in the handling of different levels of theory.

A third category of writing is when a theory is laid out completely time-free with all developments omitted, historical or otherwise; for example, as a strictly axiomatized theory. This kind of writing is also quite legitimate; it tells us that "we are here." A similar fourth category is large-scale bibliographies, including classifications and indexing by topic. These categories are neither history nor heritage, although they may well involve both.[3] Apart from noting that they too will be influenced by history though probably without the knowledge of the practitioners (Section 5.4), I shall not consider them further here.

1.2. Some literature

Two prominent types of writing in which heritage is the main guide are review articles and lengthy reports. Names, dates, and references are given frequently, and chronology (of publication) may well be checked quite scrupulously; but motivations, cultural background, processes of genesis, and historical complications are usually left out. A golden period in report writing was at the turn of the 19th and 20th centuries, especially in German, with two main locations: the reports, often lengthy, in the early volumes of the *Jahresberichte* of the *Deutsche Mathematiker-Vereinigung* (1892–); and the articles composing the

[2] In my first lectures on this topic I used the word "genealogy" to name this concept. I now prefer "heritage," partly on semantic grounds and partly for its attractive similarity to "history" in English as another three-syllable word in English beginning with "h."

[3] A current project to classify the primary literature as reviewed in the *Jahrbuch ueber die Fortschritte der Mathematik* (1867–1942) imposes a modern division into topics and subtopics. My efforts to handle the early articles on mechanics were quite unsatisfying: heritage dominated a task intrinsically historical, at least for the early decades of that period.

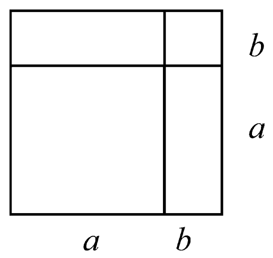
Encyklopädie der mathematischen Wissenschaften (1898–1935) with its unfinished extension into the French *Encyclopédie des sciences mathématiques* (1904–1916?) [Gispert, 1999]. Some of these texts are quite historical.[4]

Among modern examples of heritage-oriented historical writings, Jean Dieudonné's lengthy account of algebraic and differential topology in the 20th century is (impressively) typical [Dieudonné, 1989], and several of the essays in the Bourbaki history have the same character [Bourbaki, 1974]. André Weil's widely read advice [1980] on how to do history is largely driven by needs of heritage and even dismissive of history, especially concerning the relative importance of judgements of the mathematics of the past (Section 2). An interesting slip is his use of "history of mathematics" and "mathematical history" as synonyms, whereas the expressions denote quite different subjects [Grattan-Guinness, 1997, 759–761].

2. An example

The distinction between history and heritage has been cast above in as general a manner as possible; any piece of mathematics from any culture will be susceptible to it. Here is an example, mathematically simple but historically very important (a contrast which itself manifests the distinction).

Book 2, Proposition 4 of Euclid's *Elements* comprises this theorem about "completing the square":



From the late 19th century onwards an influential historical interpretation developed, in which Euclid was taken to be a "geometric algebraist," handling geometrical notions and configurations but actually practicing common algebra. (Compare the remarks in Section 1.2 on history and heritage at that time.) Under this interpretation the diagram is rendered as

$$(a+b)^2 = a^2 + 2ab + b^2. \tag{1}$$

However, historical disquiet should rise.

First, (1) is a piece of algebra, which Euclid did not use, even covertly: his diagram does not carry the letters a and b.[5] His theorem concerned geometry, about the large square being composed of four parts, with rectangles to the right and above the smaller square and a little square off in the north-east corner; indeed, he specifically defined as "the gnomon," the L-shape formed by the three small regions

[4] See Dauben [1999] on the journals for the history of mathematics at that time.

[5] A characterization of algebra is needed. 'The determination of unknowns' is a necessary but not a sufficient condition; for under it most mathematics is algebra! I would also require the *explicit* representation of knowns and unknowns by special words and/or symbols, and articulation of operations upon them (such as addition or concatenation), relationships between them (such as inequalities and expansions), and their basic laws. On the specification of ancient "algebra" see Høyrup [2002, Chapter 7].

[*Elements*, Book 2, Definition 2], known also for its use in sundials and the measurement of time. All these geometrical relationships, essential to the theorem, are lost in the single sign '+' in (1).

Further, a and b are associated with numbers, and thereby with lengths and their multiplication. But Euclid worked with lines, regions, solids, and angles, not any arithmeticized analogues such as lengths, areas, volumes, or degrees; he never multiplied geometrical magnitudes of any kind (in important contrast to his arithmetic in Books 7–9, where he multiplied integers in the usual way). Hence 'a^2' is already a historical distortion [Grattan-Guinness, 1996].[6]

For reasons such as this the algebraic reading of Euclid has been discredited by specialist historians in recent decades. By contrast, it is still advocated by mathematicians, such as Weil [1980], who even claimed that group theory is *necessary* to understand Book 5 (introducing ratios, and forming propositions and other theorems involving geometrical magnitudes) and Book 7 (introducing basic properties of positive integers)! An interesting practitioner of the reading of Euclid as a geometric algebra was T.L. Heath, whose translation and edition of Euclid, first published in the 1900s, is still the major source in English [Euclid, 1926]. I am assured by Greek specialists that his translation is generally faithful to the original. To take an important example, he writes "square on the side," not "square of the side," which can easily be confused with "side squared" and thus lead to the algebra of (1); even Heath's distinguished predecessor Robert Simson had used it in his influential edition [Euclid, 1756—for example, p. 51 for Book 2, Proposition 4 and (1)].[7] Yet in his commentaries Heath rewrote many of Euclid's propositions in common algebra without seeming to notice the variance from his own translation that inevitably follows (see his summary of geometric algebra in Euclid [1926, Vol. 1, 372–374]): in a few cases his algebraic proofs differ from Euclid's originals (for example, Book 6, Proposition 28).

It is now much better understood that identity (1) belongs to the heritage from Euclid, especially among some Arabs with their word-based algebra (the phrase "completing the square" is Arabic in origin), and then in European mathematics, when symbols for quantities and operations were gradually introduced.[8] The actual version used in (1) corresponds more or less to the early 17th century, with figures such as Thomas Harriot and René Descartes; Euclid and the relevant Arabs are part of their history, they are part of the heritage from Euclid and those Arabs, and our use of (1) forms part of our heritage from both of them.[9] Here we have various history and heritage statements, all in one sentence: fine, but do not muddle them up!

This advice seems to have been offered by E.J. Dijksterhuis (1892–1965) in his inaugural lecture as Professor of the History of Exact Sciences at Utrecht University in 1953. He used the adjectives "genetic" or "evolutionary" to characterize heritage and "phenomenological" for history [Struik, 1980, 12–13: the last adjective was perhaps not well chosen]. Not coincidentally, his edition of Euclid was much more

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[6] Again, Euclid defined lines as "breadthless" (Book 1, Definition 2); often criticized by inheritors, he made clear an aspect of his own history, in replacing the Babylonian use of "lines" *as objects with width* [Høyrup, 1995; and 2002, *passim*].

[7] Translations of mathematical texts often entail tricky questions of history and heritage, along with semantic and syntactic issues. These latter are especially marked when the languages involved belong to different families; in particular, Hoe [1978] translates Chinese into English or French character by character rather than by the word structure of the final language. See also Section 4.6 on general words.

[8] There is of course another large history and heritage from Euclid, inspired by the alleged rigour of this proofs. It links in part to the modernization of his geometry, but I shall not discuss it here.

[9] This last feature applies also, regrettably, to the supposed history [Rashed, 1994] of Arabic algebra, where the Arabs seem already to have read Descartes.

historically sensitive than Heath's, with the notations of geometric algebra avoided; for example, the square on side a was denoted $T(a)$, with T for "tetragon" [Euclid, 1929–1930].

In the rest of this paper I shall concentrate upon general historical and historiographical issues. In so doing no claim is made that history is superior to heritage, or superordinate upon it. A companion essay to this one dealing with good and bad practices in the prosecution of heritage is very desirable. History and heritage are twins, each profiting from practices used in the other. I only claim, *outside* of the discussion to follow, that it is often worthwhile to have some knowledge of the history of any context or subject in which one is interested.

3. Some attendant distinctions

3.1. Pre- and posthistory

The distinction between history and the heritage of N clearly involves its relationship to its prehistory and to its posthistory. The historian may well try to spot the historical *foresight*—or maybe lack of foresight—of his historical figures, the ways in which they thought or hoped that the notions at hand may be developed. He should be aware of the merit as well as the difficulties of "not being wise after the event" [Agassi, 1963, 48–67]. By contrast, the inheritor may seek historical *perspective* and hindsight about the ways the notions actually seemed to have developed. This distinction, quite subtle, is often overlooked.

The distinction is emphatically *not* that between success and failure; history also records successes, but with the slips and delays also exposed. A nice example is Hawkins [1970], a fine history of the application of point set topology to refine the integral from the Cauchy–Riemann version through content in the sense of Camille Jordan and Georg Cantor to the measure theory of Emile Borel and Henri Lebesgue. Hawkins not only records the progress achieved but also carefully recounts conceptual slips made en route: for example, the belief until its refutation that denumerable set, set of measure zero, and nowhere dense set were coextensive concepts.

The general situation may be expressed as follows. Let N_0, N_1, and N_2 form a sequence of (say) three notions holding some contextual (not necessarily logical) relationship, and lying in forward chronological order; then the heritage of N_1 for N_2 belongs also to the history of N_2 relative to N_0 and N_1. In both history and heritage it is worth finding out whether or not N_0 played an active role in the creation of $N_1, N_2, \ldots$ (as with the Euclid example for some Arabs), or if it is simply being used as a test case for them. However, more is involved than the difference between pre- and posthistory; for *both* categories use posthistory, though in quite different ways. In the elaboration below some further examples will be used below, though for reasons of space they are treated rather briefly; fuller historical accounts would take note of interactions with the development of other relevant notions.

3.2. History is usually a story of heritages

The historian records developments and events where normally an historical figure inherited knowledge from the past in order to make his own contributions heritage style. Conversely, heritage unavoidably involves various histories. Some attention to the broad features of history may well enrich the inheritance, and perhaps even suggest a research topic.

Sometimes tiers of history may be exposed. Work produced in, say, 1700 was historical in 1800 and in 1900 as well as in 2000. Thus the historian in 2000 may have needed to note how it was (mis-)understood by later figures, including historians as well as mathematicians, when it formed parts of their heritages. If a mathematician really did treat a predecessor in an historical spirit, at least as he (mis-)understood it, then the (now meta-)historian should record accordingly (see, for example, Stedall [2001] on John Wallis's partly and dubiously historical *Algebra* of 1685).

3.3. Types of influence

Types of influence raise important issues. Heritage is likely to focus only upon positive influence, whereas history needs to take note also of negative influences, especially of a general kind, such as reaction against some notion or the practice of it or importance accorded some context. For example, one motive for A.L. Cauchy to found mathematical analysis in the 1820s upon a theory of limits (Section 4.1) was his rejection of J.L. Lagrange's approach to the calculus using only notions from algebra. Further, as part of his new regime Cauchy stipulated that "a divergent series has no sum," regarding as illegitimate the results obtained by Leonhard Euler [Hofmann, 1959] and various other contemporaries and successors; but in the 1890s Borel reacted against precisely this decree and became a major figure in the development of summability and formal power series [Tucciarone, 1973]. Thus we have some heritage from Euler and from Cauchy and some history of Borel at the same time.

3.4. The role of chronology

The role of chronology differs greatly. In history it can form a major issue; for example, possible differences between the creations of a sequence of notions and those of their publication. Further, the details available may only give a crude or inexact time course, so that some questions of chronology remain unanswerable. It is particularly difficult or even impossible to determine for ancient mathematics and for ethnomathematics. In heritage chronology is much less significant: however, mathematicians often regard questions of the type "Who was the first mathematician to ...?" as the prime type of historical question to pose [May, 1975b], whereas historians recognize them as often close to meaninglessness when the notion involved is very general or basic. For example, "... to use a function?" could excite a large collection of candidates according to the state, generality or abstractness of the function theory involved ([Thiele, 2000]; compare Section 4.6). The only questions of this kind of genuine historical interest concerns priority disputes, when intense parallel developments among rivals are under investigation, and chronology is tight—and where again maybe no answer can be found.

3.5. Use of notions later than N

This is a major matter. Later notions are *not* to be ignored; the idea of forgetting the later past of an historical episode, often put forward as desirable historiography, is impossible to achieve, since the historian has to know which notions *are* later, and this requires the historical task already to have been accomplished (Section 5.1). Instead, when studying the history of N_0, by all means *recognize* the place of later notions N_1, $N_2, \dots,$ but *avoid* feeding them back into N_0 itself. For if that does happen, the novelties that attended the emergence of N_1, $N_2, \dots,$ will not be registered. Instead time loops are created, with cause and effect over time becoming reversed: when N_2 and N_1 are shoved into N_0, then they seem to be involved in its creation, whereas the *converse* is (or may be) the case. In such situations not only is the history of N_0 messed up but also that of the intruding successors, since their *absence* before introduction is not registered.

For example, in the late 18th century Lagrange realized that the solvability of polynomial equations by algebraic operations alone was connected to properties of certain functions of their roots when the latter were permuted; and this achievement played a role in the development of group theory during the 19th century [Wussing, 1984, 70–84]. Now to describe his work *in terms of* group theory not only distorts Lagrange but also muddies the (later) emergence of that theory itself by failing to note its absence in him. Sometimes such modernizations are useful to save space on notations, say, or to summarize mathematical relationships, but the ahistorical character should be stressed: “in terms of group theory (which Lagrange did not have), his theorem on roots may be stated thus: . . . .”

A valuable use of later notions when studying the history of N_0 is as sources for questions to ask about N_0 itself—but do not expect positive answers! However, negative answers need to be examined carefully; lack of evidence does not provide evidence of lack.

By contrast, when studying the heritage of N_0, by all means feed back $N_1, N_2, \ldots,$ to create new versions; it may be clarified by such procedures. The chaos in the resulting history is not significant; maybe even a topic for mathematical research will emerge. But it is only negative feedback, unhelpful for both history and heritage, to attack a historical figure for having found only naïve or limited versions of a theory that, as his innovations, helped to lead to the later versions upon which the attack is based. To resume the case of summability from Section 3.3, it is not informative to regard Euler on that topic as an idiot; but also he did not foresee the rich panoply of uses to which “divergent series” are now put.

3.6. A schematic representation of the distinction

The difference is shown in Fig. 1, where time runs from left to right. For history the horizontal arrows do not impinge positively upon the preceding notions whereas those for heritage do. That is, in history one should avoid feeding later notions back into N if they did not play roles there; by contrast, such practices are fine for the purposes of heritage and indeed constitute a common and fruitful way of conducting research (Section 3.1).

Each N may be a collection of notions, with some or maybe all some playing roles in the creation of successors in the next collection. Arrows pointing forwards in time could be drawn, to represent foresight, hopes for further progress.

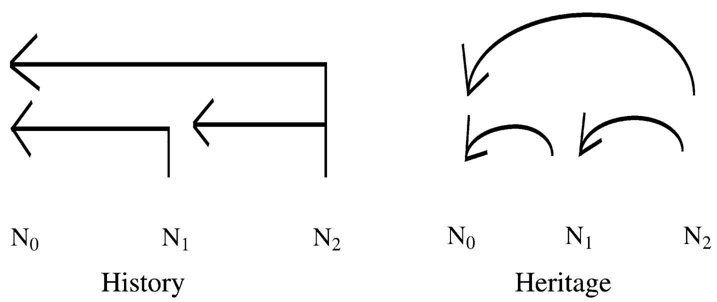

Fig. 1.

3.7. Foundations up or down?

The distinction can be extended when N is an axiomatized theory, which proceeds logically through concepts $C_1, C_2, C_3, \ldots$; for to some extent the respective historical origins move *backward* in time, thus broadly the reverse of the historical record. A related difference is thereby exposed: heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows that foundations are dug down, and not necessarily onto firm territory. For example, the foundations of arithmetic may start with mathematical logic in a version of the 1900s (hopefully free from paradoxes!), use set theory as established mainly by Cantor in the 1880s and 1890s, define progressions via the Peano axioms of the later 1880s, and then lay out the main properties of integers as established long before that.

A figure important in that story is Richard Dedekind, with his book of 1888 on the foundations of arithmetic. The danger of making historical nonsense out of heritage is well shown in a supposed new translation. A typical example of the text is the following passage, where Dedekind's statement that (in literal translation) "All simply infinite systems are similar to the number-series *N* and consequently by (33) also to one another" comes out as "*All unary spaces are bijective* [1] *to the unary space* [2]*N and consequently, by §33,* [3] *also to one another*"; moreover, of the three editorial notes, the first one admits that "isomorphic" would be more appropriate for Dedekind but the second one informs that "*unary space* [...] is what he means" ... [Dedekind, 1995, 63].

3.8. Indeterminism or determinism?

Especially if history properly records missed opportunities and delayed and late arrivals of conception and/or publication, it will carry an indeterministic character: the history did indeed pass through the sequence of notions $N_0, N_1, N_2, \ldots$, but it might have been otherwise. Everything in this paper is proposed in an explicitly indeterministic spirit. The inheritor can take a hint from the historian here: in the past, many theories have developed slowly and/or fitfully, with long periods of sleep; so which theories are sleeping today?

By contrast, even if not explicitly stressed, a deterministic impression is likely to be conveyed by heritage: N_0 *had* to lead to N_1. Appraisal of historical figures as "progressive" or "modern," in any context, is normally of this kind: the appropriate features of their work are stressed, the others ignored. In this respect, and in some others such as the stress on hindsight and the flavor of determinism, heritage resembles Whig history, the seemingly inevitable success of the actual victors, with predecessors assessed primarily in terms of similarities with the dominant position. For scientists Isaac Newton as a modern scientist gains a "yes," but Isaac Newton the major alchemist is a "no."[10] Again, the inheritor may read something by, say, Lagrange and exclaims: "My word, Lagrange here is very modern!"; but the historian should reply: "No, we are very Lagrangian."

A fine example of indeterminism is provided by the death of Bernhard Riemann in 1866. The world lost a very great mathematician, and early; had he lived longer, new theories might have come from him that arrived only later or maybe not at all. On the other hand, his friend Dedekind published in

[10] [Arnol'd, 1990] is a supposedly historical assessment of Isaac Newton's remarkable theorem in the *Principia* on the class of closed convex curves expressible by algebraic formulae; apparently it was a theorem about the topology of Abelian integrals (Chapter 5, including a fantasy on p. 85 about Cauchy's motivation to complex-variable analysis).

1867 two manuscripts that Riemann had prepared in 1854 for his *Habilitation* but had left unpublished, seemingly indefinitely. While both manuscripts contained notions already present in the work of some other mathematicians, they made rapid and considerable impacts on their appearance. Had the one on mathematical analysis and especially trigonometric series not appeared then, there is no reason to assume that Cantor, a young number theorist in the early 1870s, would have tackled the problem of exceptional sets for Fourier series (to use the later name) which Riemann exposed, and thereby invented the first elements of his set theory [Dauben, 1979, Chapters 1–2]; but then many parts of mathematical analysis would have developed differently. The other manuscript, on the foundations of geometry, is noted at the end of the next section.

3.9. Revolutions or convolutions?

In appraising heritage, interest lies mainly in notions in (fairly) finished form without special concern about the dynamics of their production. A deterministically construed heritage can convey the impression that the apparently inevitable progress shows mathematics to be a *cumulative* discipline.

But history suggests otherwise; some theories die away, or at least their status reduced. The status or even occurrence of revolutions in mathematics is historically quite controversial [Gillies, 1992]; I have proposed the meta-notion of convolution, where new and old notions wind around each other as a (partly) new theory is created [Grattan-Guinness, 1992]. Convolution lies between, and can mix, three standard categories: revolution, in the sense of strict *replacement* of theory; innovation, where replacement is absent or plays a minor role (I do not know of a case where even a remarkably novel notion came from literally *no* predecessors); and evolution, similar to convolution in itself but carrying many specific connotations in the life sciences that are not necessarily relevant here.

One of the most common ways in which old and new mix is when a new notion is created by connecting two or more old notions in a novel way. Among very many cases, in 1593 François Viète connected Archimedes's algorithmic exhaustion of the circle using the square, regular octagon, ... with the trigonometry of the associated angles and obtained this beautiful infinite product

$$2/\pi = \sqrt{1/2}\sqrt{1/2 + 1/2\sqrt{\sqrt{1/2}}}\sqrt{1/2 + 1/2\sqrt{1/2 + 1/2\sqrt{1/2}}}\sqrt{\ldots}. \tag{2}$$

Again, in the 1820s Niels Henrik Abel and Carl Jacobi independently linked the notion of the inverse of a mathematical function with Adrian-Marie Legendre's theory of "elliptic functions" (to us, elliptic integrals) to produce their definitive theories of elliptic functions. Heritage may also lead to new connections being effected.

Sometimes convolutions, revolutions, and traditions can be evident together. A very nice case is found in the work of Joseph Fourier in the 1800s on heat diffusion [Grattan-Guinness and Ravetz, 1972]:

(1) Apart from an unclear and limited anticipation by J.-B. Biot, he innovated the differential equation to represent the phenomenon.
(2) The method that he used to obtain it was traditional, namely Euler's version of the Leibnizian differential and integral calculus (which is noted in Section 4.1).
(3) He refined the use of boundary conditions to adjoin to the internal diffusion equation for solid bodies.
(4) He revolutionized understanding of the solution of the diffusion equation for finite bodies by using infinite trigonometric series; the solution had been known before him but was importantly

misunderstood, especially about the manner in which a periodic series could represent a general function at all.

(5) He innovated the Fourier integral solution for infinite bodies.

Delays often arise from connections *not* being made. A well-known puzzle is the slowness to recognize non-Euclidean geometries when there was a long history of mapmaking which surely exhibits one kind of such a geometry. J.H. Lambert is an especially striking figure, as he worked with some luster in both areas in the later 18th century. The answer seems to be that, like his predecessors and several successors, he understood the geometry problem as being just the status, especially provability, of the parallel axiom *within the Euclidean framework* rather than the more general issue of alternative geometr*ies*, which was fully grasped only by Riemann in his 1854/1867 manuscript [Gray, 1989]. Thus the link, which seems so clear in our heritage, was not obvious in the earlier times.

3.10. Description or explanation?

Both history and heritage are concerned with description; but, as was mentioned in Section 1.1, history should also attempt explanations of the developments found, and also of the delays and missed opportunities that are noticed. These explanations can be of various kinds; not just of the technical insights that were gained but also of the social background, such as the (lack of) educational opportunities for mathematics in the community or country involved. Especially in ancient and medieval times, and not only in the West, prevalent philosophical and/or religious stances could play important roles. One feature especially of the 19th century which needs explanation is the differences between nations in the (*un*)*popularity* of topics or branches of mathematics (France doing loads of mathematical analysis, England and Ireland rather little of it but working hard at several new algebras, and so on).

Heritage studies need to consider explanation only from a formal or epistemological point of view. For example, it would explain the mystery of having to use complex numbers when finding the real roots of polynomials with real coefficients in terms of closure of operations over sets, an insight which has its own history [Sinaceur, 1991, pt. 2].

3.11. Levels of (un)importance

This last task relates to another difference; that a notion rises and/or falls in importance. Heritage does not need to give such changes much attention; the modern level of importance is taken for granted. But history should watch and ponder upon the changes carefully. A general class of cases is considered in Section 4.4.

A fine example is provided by trigonometry. For a long time it has been an obviously useful but rather minor topic in a course in algebra—and, correspondingly, there has been no detailed general history of it since von Braunmühl [1900, 1903]. By contrast, in the late Middle Ages it was a major branch of mathematics; and handled geometrically, so that, for example, the sine was a length measured against the hypotenuse as unit, not as a ratio of lengths. In further contrast, spherical trigonometry was more important than planar trigonometry because of its use in astronomy and navigation.

As a converse example, probability theory and especially mathematical statistics had long and slow geneses; most of the principal notions in statistics are less than then two centuries old, and the cluster of them which is associated with Karl Pearson and his school has celebrated its centenary only recently. The slowness of the arrival of this discipline, now one of the most massive parts of mathematics while often functioning separate from it, is one of the great mysteries of the history of mathematics; its modest place during most of the 19th century is especially astonishing. But this tardiness need not disturb a seeker of heritage within it.

3.12. Handling muddles

One way in which knowledge of all kinds increases, especially the mathematical, is by the cleaning up of unclarities and ambiguities by bringing in new distinctions of sense; for example, the convergence of series of functions was split, largely by Karl Weierstrass and his followers from the 1870s onwards, into various modes of uniform, nonuniform, and quasi-uniform convergence [Hardy, 1918]. Such housework forms part of the heritage that the mathematician will deploy (unless he has reason to question it historically). The historian will also note the modern presence of such distinctions, but he should try to *reconstruct* the old unclarities, as clearly as possible, so that the history of the distinctions is itself studied. Section 4.1 includes an important example.

This historical procedure seems to contradict the claim of Section 3.5 that history usually stresses differences between notions while heritage highlights similarities; for preserving muddles keeps things the same while cleaning them up brings out differences. However, there is no difficulty; to continue with the example of the various modes of convergence before the Weierstrassians, the historian will stress the difference between the ignorance of them among predecessors and our knowledge of them while the inheritor will insert them into that earlier work and so make it more similar to the later version.

3.13. On some consequences for mathematics education

The issue of heuristics in mathematics, and the discovery and later justification of mathematical notions, is strongly present in this discussion, with obvious bearing upon mathematics education. The tradition there, especially at university level or equivalent, is to teach a mathematical theory in a manner very much guided by heritage. But reactions of students—including myself, as I still vividly recall—are often distaste and bewilderment; not particularly that mathematics is very hard to understand and even to learn but mainly that it turns up in “perfect” dried-out forms, so that if there are any mistakes, then necessarily the student made them. Mathematical theories come over as all answers but no questions, all solutions but no problems—and only the cleverest students possess enough intelligence to understand it.

A significant part of the growth in interest in the history of mathematics has been inspired by the negative influence (Section 3.3) of such situations, and there is now a strong international movement for making use of history in the teaching of mathematics, at all levels [Fauvel and van Mannen, 2000]. In a companion paper [Grattan-Guinness, 2004] I consider the bearing of the distinction between history and heritage upon mathematics education in some detail; the main points are rehearsed here, and another one in Section 5.4.

Long ago I proposed the metatheoretical notion of “history–satire,” where the broad historical record is respected but many of the complications often contained in the messy details are omitted or elided [Grattan-Guinness, 1973]: if one stays solely within, say, Newton’s historical context all the time, then one will stop where Newton stopped. Otto Toeplitz’s “genetic approach” to the calculus [Toeplitz, 1963] is close to a special case of this approach [Schubring, 1978]. (Note from Section 1.2 the use of “genetic” by Dijksterhuis to characterize heritage.) It is also very well deployed in Bressoud [1994], a textbook on real-variable mathematical analysis.

Where does mathematical education lie in between history and heritage? My answer is: exactly there, and a very nice place it is. Educators can profitably use both history *and* heritage for their purposes. For example, the algebraic version of Euclid, so important in its heritage, is often and well used in this kind of teaching. But also available is the real Euclid of arithmetic and geometry, including the beautiful theory of ratios, for me the mathematical jewel of the work, both fine mathematics in its own right and an excellent route in to the notoriously difficult task of teaching (the different topic of) rational numbers. (To make another contrast between history and heritage, Euclid used only the reciprocals $1/m$ among the rational numbers, and no irrational numbers at all.) Following history–satire, the differences between the two Euclids should be stressed; indeed, they could start off lots of nice points about the relationships between these three branches of mathematics in elementary contexts, such as the difference between lines (geometry without arithmetic) and lengths (geometry with arithmetic). A recent attractive study of the history of algebra, including the role of Euclid, is provided by Bashmakova and Smirnova [2000], though in my view the authors conflate history and heritage statements throughout [Grattan-Guinness, 2004, Section 8].

4. Prevailing habits: six cases

> Anything that has become background, or context, or tradition is no longer salient, sometimes no longer represented symbolically at all. —James Franklin, The Science of Conjecture: Evidence and Probability before Pascal (p. 344) [2001]

I consider six special cases of aspects of mathematics where the conflation of history and heritage seems to be especially acute, including among historians. The cause seems to be habitual use of the notions involved, so commonplace as not to be questioned. The examples come mostly from the 19th and early 20th centuries, which not accidentally is my own main period of research; thus no claim of optimal importance or variety is made for them. Examples of the distinctions made in Section 3 are also included.

4.1. The calculus and the theory of limits

There have been four main ways of developing the calculus [Grattan-Guinness, 1987]: in chronological order:

(1) Newton’s “fluxions” and “fluents” (1660s onwards), for theory of limits deployed, though not convincingly;
(2) G.W. Leibniz’s “differential” and “integral” calculus, based upon dx and $\int x$ (1670s onwards), with infinitesimals central to and limits absent from all the basic concepts: reformulated by Euler in the mid-1750s by adding in the “differential coefficient,” the forerunner of the derivative;
(3) Lagrange’s algebraization of the theory, in an attempt to avoid both limits and infinitesimals, with a new basis sought in Taylor’s power-series expansion (1770s onwards), and the successive differential coefficients reconceived in terms of the coefficients of the series as the “derived functions”; and
(4) Cauchy’s approach based upon a firm *theory* (and not just intuition) of limits (1810s onwards); from it he defined the basic notions of the calculus (including the derivative as the limiting value of the

difference quotient) and also of the theories of functions and of infinite series, to create "mathematical analysis."

Gradually the last tradition gained wide acceptance, with major refinements brought in with Karl Weierstrass and followers from the mid-century onwards. In particular, they honed Cauchy's basically single-limit theory into one of multiple limits with a plethora of new distinctions (including the modes of convergence noted in Section 3.12). Thus it has long been the standard way of teaching the calculus; but historians should beware using it to rewrite the history of the calculus where any of the other three traditions, *especially* Newton and Cauchy's, are being studied. It also contains an internal danger. The (post-)Weierstrassian refinements have become standard fare, and are incorporated into the heritage of Cauchy; but it is mere feedback-style ahistory to read Cauchy (and contemporaries such as Bernard Bolzano) as if they had read Weierstrass already [Freudenthal, 1971]. On the contrary, their own pre-Weierstrassian muddles need historical reconstruction, and clearly (Section 3.12). Again by contrast, inheritors can acknowledge such anachronisms but ignore them, and just see whether or not the mathematics produced is interesting.

4.2. Part-whole theory and set theory

An important part of Cauchy's tradition by (some of) the Weierstrassians was the introduction from the early 1870s of set theory, principally by Cantor (Section 3.8). Gradually it too gained a prominent place in mathematics and then in mathematics education; so again conflations lurk around its history. They can occur not only in putting set-theoretical notions into the prehistory, but also, in particular, in confusing that theory with the traditional way of handling collections from antiquity: namely, the theory of whole and parts, where a class of objects contains only parts (such as the class of Australian cathedrals as a part of the class of cathedrals), and membership was not distinguished from inclusion. Relative to set theory parthood corresponds to improper inclusion, but the theory can differ philosophically from Cantor's doctrine, on matters such as the status of the empty class/set, and the class/set as one and as many; so care is needed. An interesting example occurs in avoiding the algebraization of Euclid mentioned in Section 2: Mueller [1981] proposed an algebra alternative to that in (1) in Section 2 above, but he deployed set theory in it, whereas Euclid had followed the traditional theory, so that a different distortion arises. As in earlier points, inheritors need feel no discomfort.

4.3. Vectors and matrices

In a somewhat disjointed way vector and matrix algebras and vector analysis gradually developed during the 19th century, and slowly became staple techniques during the 20th century, including in mathematics education [Grattan-Guinness, 1994, articles 6.2, 6.7, 6.8, 7.12]. But then the danger just highlighted arises again; for earlier work was not thought out that way. The issue is *not* just one of notation; the key lies in the associated notions, especially the concept of laying out a vector as a row or column of quantities and a matrix as a square or rectangular array, and manipulating them separately or together according to stipulated rules and definitions. Similar remarks can be applied to tensor analysis.

A particularly influential example of these anachronisms is Truesdell; in very important pioneering historical work of the 1950s he expounded achievements by especially Euler in continuum mathematics that previously had been largely ignored (see, for example, Truesdell [1954]). However, in the spirit of heritage in his remark quoted at the head of this paper, he treated Euler as already familiar with vector analysis and some matrix theory, and also using derivatives as defined via the theory of limits, whereas in fact Euler had actually used his own elaboration of Leibniz's version of the calculus mentioned in Section 4.1. Therefore Truesdell's Euler was out of chronological location by at least a century. It is quite amusing to read Truesdell's editorial commentaries and then Euler's original texts in the same volumes (11 and 12 of the second series) of the *Opera omnia*. Much historical reworking of Euler's mechanics is needed, not only to clarify what and how he had actually done and not done but also to eliminate the mess-ups of feedback. The history of vectors and matrices needs to be clarified by noting the absence of these notions in Euler.

4.4. The status of applied mathematics

This case exemplifies the variation of levels of importance raised in Section 3.11, in a case where certain features of heritage have affected levels of historical interest. During the middle of the 19th century the professionalization of mathematics increased quite notably in Europe; many more universities and other institutions of higher education were created or expanded, so that the number of jobs increased. During that period, a rather snobbish preference for pure over applied or even applicable mathematics began to develop in the German states and then Germany, and later internationally. Again this change has affected mathematics education, for the worse.[11]

The tendency has also influenced historical work in that the history of pur(ish) topics has been studied far more than that of applications; the history of military mathematics is especially ignored. But a mismatch of levels of importance arises; for prior to the change applications and applicability were very much the governing motivation for mathematics, and the balance of historical research should better reflect it. Euler is a very good case; studies of his contributions to purish mathematics far exceed those of his applied mathematics (hence the importance of Truesdell's initiative in looking in detail at his mechanics). Some negative influence from current practice is required of historians to correct this imbalance.

4.5. The place of axiomatization

From the late 19th century onwards David Hilbert encouraged the axiomatization of mathematical theories, in order to make clearer the assumptions made and also to study metaproperties of consistency, completeness, and independence. His advocacy, supported by various followers, has given axiomatization a high status in mathematics, and thence in mathematics education. But once again dangers of distortion of earlier work attend, for Hilbert's initiative was then part of a *new* level of concern with axiomatization [Cavaillès, 1938]; earlier work was rarely so preoccupied, although the desire to make clear basic assumptions was frequently evident (for example, in the calculus, as reviewed in Section 4.1). Apart from Euclid, of the other figures named above only Dedekind can be regarded as an axiomatizer; it is out of line so to characterize the others, even Lagrange, Cauchy, Weierstrass, or Cantor.

[11] Both history and heritage attach to the words "pure" and "applied" mathematics, and to cousins such as "mixed." The history of these adjectives is itself worth study.

4.6. *Words of general import*

One aim of many mathematical theories is generality; and attendant to this aspiration is the use of correspondingly wide-ranging words or phrases, such as “arbitrary” or “in any manner,” to characterize pertinent notions. The expressions may well still be used in many modern contexts; so again the danger of identification with their past manifestations needs to be watched.

A good example is the phrase “any function” in the calculus and the related theory of functions; it or some cognate (such as “functio quomodocumque”) will be found with (at least) John Bernoulli in the early 18th century, Euler about 40 years later, Lagrange and S.-F. Lacroix around 1800, J.P.G. Dirichlet in the late 1820s, and Lebesgue and the French school of analysts in the early 20th century. Nowadays it is usually taken to refer to a mapping (maybe with special conditions such as isomorphism), with set theory used to specify range and domain and no other details or conditions. But the universe of functions has not always been so vast; generality has always belonged to its period of assertion. In particular, Dirichlet [1829] mentioned the characteristic function of the irrational numbers (to use the modern name); but he quite clearly regarded it as a pathological case, for it did not possess an integral. The difference is great between his situation and that of Lebesgue’s time, for the integrability of such a function was a good test case of the new theory of measure to which he was a major contributor; indeed, this detail is part of the heritage from Dirichlet.

5. History and heritage as metatheories

So far the concerns and examples treated in this paper have centered on mathematics alone; but clearly the issue of history and heritage is more general. One can see the same kinds of issue arising in the histories of the other sciences and of technology [Pickstone, 1995] and indeed outside the sciences altogether; for example, some nice examples arise in music, in connection with preferred practices in the execution of “authentic performance” of older works. Thus, while mathematics seems to provide by far the richest context and examples (at least to my knowledge), the issues themselves have a broader remit. In this section I state the four principles that inform the discussion above.

5.1. *History is unavoidable*

We work out in the present from the past, whether we like it or not. Thus ignorance of history does not produce immunity from it any more than ignorance of food poisoning saves one from attacks of it. On the contrary, influence is all the more likely to be exerted.

This principle brings into question a basic issue in mathematics (and other sciences) and its teaching: namely, should one bother with the history or ignore it completely? Recognition of its unavoidability shows that *the question itself is falsely posed*: the issue is *not* history yes or no, but history how? A dried-out formulation of a theory of the kind mentioned at the end of Section 1.1, denuded of human names, background, or heuristic, is still not immune from history; for example, it continues a historical tradition of presenting mathematical theories in a dried-out formulation, denuded of human names, background, and heuristic. For the same reasons, heritage also is unavoidable. So it is better to be aware of both of them, and the relationships that they excite and unavoidably impose.

5.2. The stratification and self-reference of knowledge

If history is unavoidable, then it has to be addressed somehow. We have some historical text before us; say Euclid's *Elements.* How can we read it in a historical spirit? A popular answer, put forward for all kinds of history, goes as follows. When reading Euclid's work, forget all theories in the field involved that have been developed since; step into his shoes (more likely sandals, in this case) and read his work with his eyes.

Unfortunately, as was noted briefly in Section 3.5, this method suffers from a difficulty; namely, it is *completely useless.* For in order to ignore all knowledge produced since Euclid one needs to know what that knowledge is in the first place. But in order to know that we must be able distinguish it from the knowledge produced before and during Euclid's time. But in order to know that we need to know the history of Euclid's work—before studying the history of Euclid's work! Q.E.D.[12]

To avoid this contradiction it is necessary to realize that when the historian studies his historical figures he has to realize that he is thinking *about them,* not *with* them. It is claimed that the distinction between theory and metatheory is of *central* importance for knowledge, whether mathematical or of any other kind. The position of the horizontal arrows *above* the notions in the history part of Fig. 1 forms an image of this situation, in contrast to the feedback imaged in the heritage part of Fig. 1.

The importance of this distinction lies in its generality.[13] This emerged from the 1930s onwards, inspired principally by the logicians Kurt Gödel and Alfred Tarski after several partial anticipations, of which David Hilbert's program of metamathematics as practiced during the 1930 was the most notable [Grattan-Guinness, 2000, Chapters 8–9]. In most other disciplines the distinction is too obvious to require special emphasis; clearly a difference of category exists between, say, properties of light and laws of optics, or between a move in chess and a rule of chess. By contrast, in logic, a very general branch of knowledge, the distinction is uniquely subtle (and therefore desirable); for example, "and" features in both logic and metalogic, and failure to register the distinction led to much incoherence and even to paradoxes such as one arising from "this proposition is false." Its importance and generality can be seen in Tarski's theory of truth (his own main way to this distinction): "snow is white" (in the "metalanguage," Tarski's word) if and only if snow is white (in the language). His theory is neutral with respect to most philosophies and sidesteps generations of philosophical anxiety about making true (or false) judgements or holding such beliefs.

Consider now a mathematical theory M. Its history is one kind of metatheory of it, its heritage is another, Hilbert-style metamathematics is a third if M is suitably axiomatized, questions about how to teach it are a fourth, and there may well be others.

As with theory itself, metatheory requires its own metametatheory, and so on up as far as may be needed; thus theory becomes stratified. An example of metametatheory is the history of the history of mathematics, upon which a comprehensive book has recently been published [Dauben and Scriba, 2002];

[12] In a posthumously published consideration of "History as re-enactment of past experience," which has been much discussed by philosophers of history, Collingwood [1946, 282–289] took Book 1, Proposition 5 of Euclid's *Elements*, that "In isosceles triangles the angles at the base are equal to one another," and contrasted Euclid's own thoughts about the theorem from the thoughts about it made by a later historian. However, he tended to stress the similarities of the thoughts rather than the differences, and did not explicate metatheory in the way advocated in this paper.

[13] Generality is not a necessary virtue. I agree with the maxim attributed to Saunders Mac Lane: "We do not need the greatest generality, but the right generality."

the comments on Heath's translation of Euclid and his own algebraic rewritings (Section 2), and the example of Wallis (Section 3.2), also belong to the history of history. Another example, indeed a self-referring one (Section 6), is this paper; it belongs to the history of history (of mathematics), although whether it also will enjoy a heritage is another matter! If such a miracle were to occur, then the paper would belong to another third-order theory: the history of philosophy of history, an interesting subject for which a good sourcebook has recently been published [Burns and Rayment-Pickard, 2000].

One great advantage of adopting stratification is that the assumptions chosen to underlie the theory do not have to be adopted also for its metatheory. An interesting and explicit case lies in L.E.J. Brouwer. Especially from the 1920s, he put forward a constructivist approach to mathematics, called "intuitionism," in which he rejected the law of excluded middle [van Dalen, 1999]. However, his metamathematics, which he called "mathematics of the second order," was classical, with that law in place; a proof was intuitionistically correct or not. No contradiction arises, since the levels are different.

The same freedom attends the historian when he sees himself as metatheorist. For he does *not* have to defend or even like what he tries to describe or to explain. Why should he? After all, he was not there. This point aligns with commonplace understandings; that a historian of, say, Hinduism does not have to be a Hindu, although he might be. Similarly, the inheritor has to take what he can find, maybe without enthusiasm. Stratification also sidesteps the fashionable modern chatter about narratives and discourses [Windschuttle, 1997] and the relativism and just waffle that often accompanies it.

5.3. Knowledge is based upon ignorance

This is true in the important sense that theories explain knowns in terms of unknowns. To take Euclid again, the primitives in his geometry include the "common notions" and axioms given in Book 1 (and indeed more axioms than he realized, as has been understood for over a century); but these primitives cannot be known in terms of other notions, for then they would no longer be primitive. To take another case, one of the bases of parts of Newton's mechanics is his inverse-square law of central attraction, which is unknown, maybe unknowable, and certainly mysterious!

This principle is worth stressing partly because it is often confused with an important but quite different way in which theories develop; namely, having being created in one context, they are then applied to new ones to see how they fare. To continue with the Newton case, Euler and others applied his theory to areas of continuum mechanics such as elasticity theory and fluid mechanics, where Newton had not said a great deal. Euler also took the second law of motion to apply in *any* direction whereas Newton himself had restricted it to special directions such as tangents and normals to given curves [Truesdell, 1968, Chapters 3 and 5]. Such developments tempt one to say that Newton's theory explained the unknown in terms of the known; but such claims are *methodological*, concerning the important process of changing from contexts already known to contexts currently unknown. But the principle put forward in this section is *epistemological*, concerning the structure of theories as such.

5.4. Knowledge and ignorance go together

This is true at the metatheoretic level in a profound way. For we have knowledge (of a fact, or theorem, or whatever); and maybe also knowledge of that knowledge, such as a proof of a theorem. But we also have *knowledge of ignorance*, especially when forming a problem or conjecture: when asking whether or not some property does or does not obtain in a theory, the poser *knows that he does not know* the answer. There is also *unawareness, that is, ignorance of ignorance*, when one can see that some workers on a topic did not know that they did not know the substance of the problem because the properties and connections were not known at the time in question. The emphasis upon ignorance, especially the granting to it of a status metatheoretically equal to that of knowledge, is the principal novelty of the approach advocated here.

Posing of problems enjoys high prestige in mathematics. To recall a historically famous case, Hilbert [1901] posed in 1900 a string of them (some in rather sketched form) for mathematicians to tackle. In each case he *knew* that answers were not yet known. One of them was Cantor's continuum hypothesis about the number of real numbers, which claimed that two particular infinite numbers were equal to each other. Speculations on the infinite go back earlier than Cantor, but his predecessors did not know that they did not know whether or not his continuum hypothesis was true or false because none of them knew of two different ways of constructing infinite numbers in the first place.

The same kind of relationship obtains also in history (of mathematics as just one special case). The historian knows various facts, say, and can even prove some, for example, by finding authoritative documents. But he too can pose problems, concerning matters that he knew that he does not know; and he can be unaware of other problems until new connections come to light. Layers of history as exemplified in Section 3.2 concerning the history of history can also involve knowledge and ignorance; what historians did (not) know at intermediate periods.

This scheme works also for (mathematics) education. One important task there is laying out a syllabus, and the planning could focus much on deciding how long the students will be kept unaware of some topic or theorem, when an associated problem should be posed, and when solutions be given to it. To use common algebra again, a school course hoping to advance as far as the formula for the root of a cubic equation will surely spare the youngsters knowledge of the horrible cube roots to come, but the problem could be posed when the roots of the quadratic equation have been dealt with; and when the formula has been obtained, further new questions posed, such as formulae for all three roots, and the possibility of going further with formulae for the roots of the quartic, the quintic (big shock to come!), the sextic, ... At every main stage in the teaching the interplay of knowledge and ignorance could play a major role in the teaching, though preferably not muddled together (Section 3.13).

6. Philosophical prospects

As logicians have long known, generality skirts self-reference, which sometimes generates paradoxes. Here is a hopefully virtuous example. The discussion in the last section was a sketch of a theory of relationships between knowledge and ignorance. I know that it constitutes a problem (and, I believe, an important one); but I am ignorant of a general solution of it, which would be a detailed account of the main relationships and their own metarelations. Elaboration could be well guided by consideration of the many ways in which changes take place in notions, especially in theorems and theories. These include extending known notions, generalizing them, and/or abstracting them; making new classifications of the mathematical objects involved; reacting to counterexamples by seeking the defective components of the refuted theory; exposing hitherto unnoticed assumptions; in cases where foundations are significant, interchanging some theorems, axioms, and/or definitions; devising new algorithms, or modifying old ones; making new applications or extending known ones, both within mathematics and to other disciplines; and forging new connections between branches of mathematics, or eliminating or avoiding some (recall from Section 2, for example, that in his geometry Euclid did not arithmeticize his magnitudes).

A general theory of history and heritage would not be restricted to mathematics, which, however, is a particular rich source of examples and issues. What are the prospects for further philosophical progress?

There are other intellectual contexts in which ignorance plays an active role. In mathematics it can come into economics, where the actors in an economic situation are ignorant of the intentions of the other actors. In probability theory, values are sometimes interpreted in terms of degrees of ignorance. Some nonclassical logics are relevant, such as the logic of asking questions [Wisniewski, 1995]. Ignorance has been aired occasionally in science; in particular, much interest was aroused by two "encyclopedias" on it for science and for medicine, collections of articles in which specialists posed then unsolved problems for their fields [Duncan and Weston-Smith, 1977, 1984].

All these cases are perfectly respectable, though inevitably oriented to specific contexts. To find the generality that informs ours, we must move to philosophy proper, especially theories of knowledge. However, there a kind of converse scenario emerges: the generality is indeed present, but ignorance is treated like a disease, to be cured by the acquisition of knowledge, however, the philosophy at hand claims that this is to be done. The same attitude seems to inform those philosophies of history that address ignorance at all.[14]

The tradition in which this approach has been developed most systematically is scepticism, in which Descartes was a major figure. It is a highly dystopian philosophy, a disenchantment inspired by the fact that one does not know things for certain.[15] Well, that is true, and for certain (note the use of metatheory here); but scepticism can degenerate into unwelcomely negative positions, such as pure relativism or nihilism (nobody can know anything, at least not better than anyone else).

The insight lacking is the positive one that *it is nice to be ignorant, for that is where the problems come from*. The only philosophy of which I am aware which both exhibits this insight and also carries the required generality is to be found in some writings of a philosopher who was deeply influenced by Tarski from an early stage in his career: Karl Popper. I have in mind his concern with "the sources of knowledge and of ignorance" [Popper, 1963, Intro.] and with the tricky self-referential problem of rationally criticizing rationality itself [Watkins, 1969]. Also relevant are his detailed metaphilosophical arguments for indeterminism and against determinism [Popper, 1982], which he also applied to historiography itself.[16]

But even from Popper the hints are limited. Like most philosophers he said little about the formation of scientific (including mathematical) theories; he was mainly concerned instead with the ways in which

[14] A wide-ranging survey of other philosophical approaches to history in general is provided by Stanford [1997]. There is no explicit discussion of the historiography of mathematics, though some space is given to that for science: his references to mathematics concern either its place in the history of science or its use in mathematical history.

[15] See Unger [1975] for a nice elaboration of sceptical positions, some linked to (lack of) facts and others related to (possibly false) beliefs. However, Chapter 7 on "the impossibility of truth" is a very disappointing monistic treatment based for some reason on relating proposed truth to "the whole truth about the world," a notion that is indeed impossible to handle (but then why invoke it?), so that truth as a notion is rejected. No use is made of stratification, not even in the (brief) treatment of Tarski's theory.

[16] See Popper [1945, Chapter 25; 1957]. Note carefully his rather nonstandard use of the word "historicism." On the historiography of science, in a somewhat Popperian spirit, see also Agassi [1963]. Stanford [1997] considers other aspects of Popper's philosophy than those mooted in this paper, and seriously misdescribes him as a philosopher "of positivist inclinations" (p. 39). The excellent index does not have entries for "ignorance," "(in)determinism," or "self-reference."

they could be tested. Further, despite his strong emphasis on theories, he was dismissive of questions of ontology, that is, doctrines concerning existence and in being both the physical world and in commitments of these kinds made in theories [Grattan-Guinness, 1986].[17] He also did not write much on the philosophy of mathematics and was disinclined to enter into discussion of it (personal experience, on several occasions); ironic, then, that mathematics is such a rich source! However, maybe his insights can be elaborated; if so, the outcome would corroborate one of his maxims: “all life is problem-solving.”

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References

Agassi, J., 1963. Towards a Historiography of Science. Mouton, The Hague.
Arnol’d, V.I., 1990. Huygens and Barrow, Newton and Hooke. Pioneers in Mathematical Analysis and Catastrophe Theory from Evoluents to Quasi-crystals. Birkhäuser, Basel.
Atwood, M., 1985. The Handmaid’s Tale. Bloomsbury, London.
Bashmakova, I.G., Smirnova, G., 2000. The Beginning and Evolution of Algebra. Mathematical Association of America, Washington, DC. Transl. A. Schenitzer.
Bourbaki, N., 1974. Eléments d’histoire des mathématiques, second ed. Hermann, Paris.
Bressoud, R., 1994. A Radical Approach to Real Analysis. Mathematical Association of America, Washington, DC.
Brink, C., Rewitzky, I., 2002. Three dual ontologies. J. Philos. Logic 31, 543–568.
Burns, R.M., Rayment-Pickard, H. (Eds.), 2000. Philosophies of History. From Enlightenment to Postmodernity. Blackwells, Oxford.
Cavaillès, J., 1938. Méthode axiomatique et formalisme. 3 pts. Hermann, Paris;
Also in: Oeuvres complètes de philosophie des sciences. Hermann, Paris, 1994, pp. 221–374.
Collingwood, R.G., 1946. The Idea of History. Oxford Univ. Press, Oxford.

[17] The place of ignorance in this scheme makes questions of ontology L and ontological commitments particularly important but tricky to handle, especially as we have to consider mathematics with science, philosophy with logics, and history with its methods. Ontology has to cover not only the actual physical world but also possible ones asserted in theories [Jacquette, 2002], including reconstructions of the past, where the issues are extremely perplexing [Hacking, 2002, Chapter 1]. For me, a key notion is reification, an as-if approach in which the referents of a theory (of any kind) are assumed to exist in a liberal spirit, but where any of them are readily thrown away or reduced to others if deemed necessary, worthwhile, or just convenient [Grattan-Guinness, 1987]. Preferences over ontology are often made between objects, properties, and propositions as candidate categories for existence; but in many contexts this may not be necessary, since the associated ontologies are intertranslatable [Brink and Rewitzky, 2002].

Dauben, J.W., 1979. Georg Cantor. Harvard Univ. Press, Cambridge, MA. [Reprint 1990, Princeton Univ. Press, Princeton, NJ.]
Dauben, J.W., 1999. Historia mathematica: 25 years/context and content. Historia Math. 26, 1–28.
Dauben, J.W., Scriba, C.J. (Eds.), 2002. Writing the History of Mathematics: Its Historical Development. Birkhäuser, Basel.
Dedekind, J.W.R., 1995. What Are Numbers and What Should They Be? RIM Press, Orono, ME. [German original 1888.]
Dieudonné, J., 1989. A History of Algebraic and Differential Topology 1900–1960. Birkhäuser, Basel.
Dirichlet, J.P.G. Lejeune, 1829. Sur la convergence des séries trigonométriques. J. Reine Angew. Math. 4, 157–169; Also in: Gesammelte Werke, vol. 1. Reimer, Berlin, 1889. [Reprint 1969, Chelsea, New York, pp. 117–132.]
Duncan, R., Weston-Smith, M. (Eds.), 1977. The Encyclopaedia of Ignorance. Pergamon, Oxford.
Duncan, R., Weston-Smith, M. (Eds.), 1984. The Encyclopaedia of Medical Ignorance. Pergamon, Oxford.
Dyson, F., 1972. Missed opportunities. Bull. Amer. Math. Soc. 78, 635–652.
Euclid, 1756. The Elements of Euclid. (Ed. and notes by R. Simson). Foulis, Edinburgh. [Contains Books 1–6, 11, 12.]
Euclid, 1926. The Elements. In: T.L. Heath (Ed. Transl.) The Thirteen Books of Euclid's Elements, 3 vols, second ed. Cambridge Univ. Press, Cambridge, UK. [Reprint 1956; Dover, New York. First ed. 1908.]
Euclid, 1929–1930. In: Dijksterhuis, E.J. (Ed.), De Elementen Van Euclides, 2 vols. Noordhoff, Groningen.
Fauvel, J., van Mannen, J. (Eds.), 2000. History in Mathematics Education. The ICME Study. Kluwer Academic, Dordrecht.
Franklin, J., 2001. The Science of Conjecture. Evidence and Probability before Pascal. Johns Hopkins Univ. Press, Baltimore.
Freudenthal, H., 1971. Did Cauchy plagiarise Bolzano? Arch. Hist. Exact Sci. 7, 375–392.
Gillies, D. (Ed.), 1992. Revolutions in Mathematics. Clarendon, Oxford.
Gispert, H., 1999. Les débuts de d'histoire des mathématiques sur les scènes internationales et le cas de l'entreprise encyclopédique de Felix Klein et Jules Molk. Historia Math. 26, 344–360.
Grattan-Guinness, I., 1973. Not from nowhere. History and philosophy behind mathematical education. Int. J. Math. Ed. Sci. Tech. 4, 421–453.
Grattan-Guinness, I., 1986. What do theories talk about? A critique of Popperian fallibilism, with especial reference to ontology. Fundamenta Scientiae 7, 177–221.
Grattan-Guinness, I., 1987. What was and what should be the calculus? In: Grattan-Guinness, I. (Ed.), History in Mathematics Education. Belin, Paris, pp. 116–135.
Grattan-Guinness, I., 1992. Scientific revolutions as convolutions? A sceptical enquiry. In: Demidov, S.S., Folkerts, M., Rowe, D.E., Scriba, C.J. (Eds.), Amphora. Festschrift für Hans Wussing zu seinem 65. Geburtstag. Birkhäuser, Basel, pp. 279–287.
Grattan-Guinness, I. (Ed.), 1994. Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences. Routledge, London.
Grattan-Guinness, I., 1996. Numbers, magnitudes, ratios and proportions in Euclid's *Elements*: how did he handle them? Historia Math. 23, 355–375. [Printing correction: 24 (1997) 213.]
Grattan-Guinness, I., 1997. The Fontana History of the Mathematical Sciences. The Rainbow of Mathematics. Fontana, London; Reprinted as The Norton History of the Mathematical Sciences. The Rainbow of Mathematics. Norton, New York, 1998.
Grattan-Guinness, I., 2000. The Search for Mathematical Roots, 1870–1940. Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel. Princeton Univ. Press, Princeton, NJ.
Grattan-Guinness, I., 2004. History or heritage? An important distinction in mathematics and for mathematics education. American Mathematical Monthly, in press.
Grattan-Guinness, I., in collaboration with Ravetz, J.R. 1972. Joseph Fourier 1768–1830. A Survey of His Life and Work, Based on a Critical Edition of His Monograph on the Propagation of Heat, Presented to the Institut de France in 1807. MIT Press, Cambridge, MA.
Gray, J.J., 1989. Ideas of Space, second ed. Clarendon, Oxford/New York.
Hacking, I., 2002. Historical Ontology. Harvard Univ. Press, Cambridge, MA.
Hardy, G.H., 1918. Sir George Stokes and the concept of uniform convergence. Proc. Cambridge Philos. Soc. 19, 148–156; Also in: Collected Papers, vol. 7, Oxford, 1979, pp. 505–513.
Hawkins, T.W., 1970. Lebesgue's Theory of Integration. Univ. of Wisconsin Press, Madison. [Reprint 1975, Chelsea, New York.]
Hilbert, D., 1901. Mathematische Probleme. Arch. Math. Phys. 1 (3), 44–63, 213–237. [Various reprints. English transl.: Mathematical problems. Bull. Amer. Math. Soc. 8 (1902) 437–479; itself variously reprinted.]
Hoe, J., 1978. The Jade Mirror of the four unknowns—some reflections. New Zealand Math. Chronicle 7, 125–156.
Hofmann, J.E., 1959. Um Eulers erste Reihestudien. In: Schröder, K. (Ed.), Sammelband der zu Ehren des 250. Geburtstages Leonhard Eulers. Akademie Verlag, Berlin, pp. 139–208.

Høyrup, J., 1995. Linee larghe. Un'ambiguità geometrica dimenticata. Boll. Storia Sci. Mat. 15, 3–14.
Høyrup, J., 2002. Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and Its Kin. Springer-Verlag, New York.
Jacquette, D., 2002. Ontology. Acumen, Chesham.
May, K.O., 1973. Bibliography and Research Manual in the History of Mathematics. Univ. of Toronto Press, Toronto.
May, K.O., 1975a. Historiographic vices. I. Logical attribution. Historia Math. 2, 185–187.
May, K.O., 1975b. Historiographic vices. II. Priority chasing. Historia Math. 2, 315–317.
May, K.O., 1976. What is good history and who should do it? Historia Math. 3, 449–455.
Mueller, I., 1981. Philosophy of Mathematics and Deductive Structure in Euclid's Elements. MIT Press, Cambridge, MA.
Pickstone, J.V., 1995. Past and present knowledge in the practice of the history of science. Hist. Sci. 33, 203–224.
Popper, K.R., 1945. The Open Society and Its Enemies. Routledge and Kegan Paul, London.
Popper, K.R., 1957. The Poverty of Historicism. Routledge and Kegan Paul, London.
Popper, K.R., 1963. Conjectures and Refutations. Routledge and Kegan Paul, London.
Popper, K.R., 1982. The Open Universe. An Argument for Indeterminism. Hutchinson, London.
Rashed, R., 1994. The Development of Arabic Mathematics: Between Arithmetic and Algebra. Kluwer Academic, Boston/London/Dordrecht.
Russell, B.A.W., 1956. Portraits from Memory and Other Essays. Allen & Unwin, London.
Schubring, G., 1978. Das genetische Prinzip in der Mathematik-Didaktik. Klett-Cotta, Stuttgart.
Sinaceur, H., 1991. Corps et modèles. Vrin, Paris.
Stanford, M., 1997. An Introduction to the Philosophy of History. Blackwells, Oxford.
Stedall, J.A., 2001. Of our own nation: John Wallis's account of mathematical learning in medieval England. Historia Math. 28, 73–122.
Struik, D.J., 1980. Why study the history of mathematics? UMAP J. 1, 3–28.
Thiele, R., 2000. Frühe Variationsrechnung und Funktionsbegriff. In: Thiele, R. (Ed.), Mathesis. Festschrift zum siebzigsten Geburtstag von Matthias Schramm. Diepholz, Berlin, pp. 128–181.
Toeplitz, O., 1963. The Calculus. A Genetic Approach. Univ. of Chicago Press, Chicago.
Truesdell III, C.A., 1954. Prologue. In: L. Euler Opera omnia, Ser. 2, vol. 12. Orell Füssli, Basel, pp. ix–cxxv. [On fluid mechanics.]
Truesdell III, C.A., 1968. Essays in the History of Mechanics. Springer-Verlag, Berlin.
Tucciarone, J., 1973. The development of the theory of summable divergent series from 1880 to 1925. Arch. Hist. Exact Sci. 10, 1–40.
Unger, P., 1975. Ignorance. A Case for Scepticism. Clarendon, Oxford.
van Dalen, D., 1999. Mystic, Geometer, and Intuitionist. The Life of L.E.J. Brouwer, vol. 1. In: The Dawning Revolution. Clarendon, Oxford.
von Braunmühl, A., 1900, 1903. Vorlesungen über Geschichte der Trigonometrie, 2 vols. Teubner, Leipzig.
Watkins, J.M.W., 1969. Comprehensively critical rationalism. Philosophy 44, 57–62. [See also the ensuing discussion by various authors in Philosophy 46 (1971) 43–61.]
Weil, A., 1980. History of mathematics: why and how. In: Lehto, O. (Ed.), Proc. International Congress of Mathematicians, Helsinki 1978, vol. 1. Academia Scientarum Fennica, Helsinki, pp. 227–236;
Also in: Collected Papers, vol. 3. Springer-Verlag, New York, 1980, pp. 434–443.
Windschuttle, K., 1997. The Killing of History. How Literary Critics and Social Theorists Are Murdering Our Past. The Free Press, New York.
Wisniewski, A., 1995. The Posing of Questions. Logical Foundations of Erotetic Sciences. Kluwer Academic, Dordrecht.
Wussing, H., 1984. The Genesis of the Abstract Group Concept. MIT Press, Cambridge, MA.