AN INTRODUCTION TO PROBABILISTIC METHODS IN STOCHASTIC CONTROL

Lectures at Pescara University, April 96

Laurent MAZLIAK
Laboratoire de Probabilités, Université Paris VI
75252 PARIS CEDEX 05, FRANCE

In these lectures we want to present a survey of elementary probabilistic theory for the study of a stochastic control problem. We have tried to make a self-content document, requiring as little prerequisite as possible. Of course, in order to keep to this course a reasonable size with so ambitious a program, many proofs are omitted, mostly those of classical results of the theory; we have given references for them in important textbooks.
The course is divided in five chapters. The first three are general and basic while the last two chapters go more deeply in specificaly probabilistic techniques.
We begin by a chapter introducing basic stochastic calculus. The second chapter is devoted to a general presentation of deterministic and stochastic control problems. Then we look for a general existence theorem for an optimal control using probabilistic means. This is the aim of Chapters 3 to 5.

Chapter 1: A quick survey of Stochastic Calculus

In this chapter, we are going to present the main probabilistic tools used to formulate the control problems we shall study in the following chapters. Of course, as it is not our aim to expose this theory thoroughly, we will avoid a great number of technicalities and try to emphasize the main results and formulae that are useful for our purpose.
The initial point of stochastic calculus is the construction of the stochastic integral- or Ito integral- based on the properties of brownian motion. We begin by presenting this construction in the most simple way and exposing the most important formula that allows calculations, namely Itô's formula.
At the end of this chapter, we say some words about stochastic differential equations which are the natural framework of the stochastic control problems studied later.

1- Brownian motion

In all this chapter, $(\Omega, \mathcal{F}, P)$ is a fixed probability space. We consider on this space a stochastic process $(B_t)_{t\geq 0}$ which is to say that for each $t \geq 0$ one considers a real random variable B_t.

Definition 1.1: *The stochastic process $(B_t)_{t\geq 0}$ is said to be a brownian motion if*

(i) $B_0 = 0$ a.s.

(ii) $(B_t)_{t\geq 0}$ is a process with independent increments

(iii) B_t is a centered gaussian random variable with variance $t, \forall t$

(iv) $t \to B_t(\omega)$ is continuous, a.s.

We will not talk about the construction of such a process which can be studied in most standard reference books (see for example [3]).
Observe that with the above definition, one gets that for $s < t$, $B_t - B_s$ is gaussian, centered with variance $t - s$, which says that brownian motion is stationary. Indeed, one has

$$B_t = B_s + (B_t - B_s)$$

Therefore, using characteristic functions and the independence of B_s and $B_t - B_s$, one has

$$E(e^{ir[B_t - B_s]}) = e^{-\frac{r^2}{2}(t-s)}$$

One can also compute the covariance function of brownian motion: for all $(s,t) \in (\mathbb{R}^+)^2$,

$$E(B_s B_t) = s \wedge t$$

Suppose indeed that $s < t$. Then,

$$E(B_sB_t) = E(B_s(B_s + (B_t - B_s)) = E(B_s^2) + E(B_s(B_t - B_s)) =$$

$$= E(B_s^2) = \mathrm{Var}(B_s) = s$$

Let now $t \geq 0$. If $\Delta = \{t_0 = 0 < t_1 < \ldots < t_n = t\}$ is a subdivision of $[0,t]$, one denotes by T_t^Δ the random variable

$$\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2$$

Let $\mid \Delta \mid = \sup_{0 \leq k \leq n-1} \mid t_{k+1} - t_k \mid$ be the step of the subdivision. One then has the following result.

Theorem 1.2: *If (Δ_n) is a sequence of subdivisions of $[0,t]$ such that $\mid \Delta_n \mid \to 0$, then $T_t^{\Delta_n}$ converges in probability towards t*

Proof: We in fact prove the convergence in L^2.
Denote $\Delta_n = \{t_0^n = 0 < t_1^n < \ldots < t_k^n = t\}$

$$E([T_t^{\Delta_n} - t]^2) = \sum_{j=0}^{k-1} E([(B_{t_{j+1}^n} - B_{t_j^n})^2 - (t_{j+1}^n - t_j^n)]^2)$$

because the random variables in the last sum are independent and centered.
Now, if Y is a centered gaussain random variable,

$$E(Y^4) = 3E(Y^2)^2$$

and so

$$E([(B_{t_{j+1}^n} - B_{t_j^n})^2 - (t_{j+1}^n - t_j^n)]^2) =$$

$$E([(B_{t_{j+1}^n} - B_{t_j^n})^4 - 2((t_{j+1}^n - t_j^n)(B_{t_{j+1}^n} - B_{t_j^n})^2 + (t_{j+1}^n - t_j^n)]^2) = 2(t_{j+1}^n - t_j^n)^2 \leq$$

$$\leq 2(t_{j+1}^n - t_j^n) \mid \Delta_n \mid$$

and so

$$E([T_t^{\Delta_n} - t]^2) \leq 2t \mid \Delta_n \mid \to 0 \quad \square$$

One deduces the following corollary that proves that one cannot define a Riemann-Stieltjes integration with respect to Brownian motion.

Corollary 1.3: *The brownian paths are a.s. of infinite variation on any interval*

Proof: It naturally suffices to prove the result for an interval $[0, \alpha] \subset \mathbb{R}^+$.

From the previous result, one finds a sequence (Δ_n) of subdivisions of $[0,\alpha]$ such that $\sum_{t_i \in \Delta_n} (B_{t_{i+1}} - B_{t_i})^2 \to \alpha$ a.s.

Now,

$$\sum_{t_i \in \Delta_n} (B_{t_{i+1}} - B_{t_i})^2 \leq V. \sup_{t_i \in \Delta_n} \mid B_{t_{i+1}} - B_{t_i} \mid$$

where $V = \sup_{\Delta \subset [0,\alpha]} \sum_{t_i \in \Delta} \mid B_{t_{i+1}} - B_{t_i} \mid$ is the total variation of B over $[0,\alpha]$: the supremum is taken here over all the subdivisions of $[0,\alpha]$. The previous inequality is impossible as the left hand side tends to α and the right hand side tends to 0 because (B_t) is continuous.

□

We then have to construct a new notion of integration.

2- Martingales

We begin by introducing a fundamental technical notion. We now consider a filtration $(\mathcal{F}_t)$ on Ω i.e. an increasing family of σ-algebras.

Definition 2.1: *An $(\mathcal{F}_t)$-martingale is a stochastic process $(M_t)_{t\geq 0}$ such that*

(i) M_t is $\mathcal{F}_t$ measurable, $\forall t \geq 0$

(ii) $E(\mid M_t \mid) < \infty$, $\forall t \geq 0$

(iii) $\forall 0 \leq s \leq t$,

$$E(M_t/\mathcal{F}_s) = M_s$$

Remark 2.2: An important consequence of Definition 2.1 is that $\forall t \geq 0, E(M_t) = E(M_0)$

Example 2.3: It is an easy exercise to prove that if (B_t) is a brownian motion and if $\mathcal{F}_t = \sigma(B_s, s \leq t)$ is its natural filtration, then (B_t) and $(B_t^2 - t)$ are both $\mathcal{F}_t$-martingales.

Example 2.3 allows to formulate a slightly more precise notion than Definition 1.1.

Definition 2.4: *Let $\mathcal{F}_t$ be a filtration over Ω.*
A stochastic process (B_t) is said a $\mathcal{F}_t$-brownian motion if it is a brownian motion (following Definition 1.1) such that for all $t \geq 0$, the process $(B_{t+s} - B_t)_{s\geq 0}$ is independent from $\mathcal{F}_t$. In particular, (B_t) is a $\mathcal{F}_t$-martingale.

Observe that Example 2.3 says that a brownian motion is always a brownian motion of its own filtration.

3- Stochastic integrals

We now introduce the stochastic integral with respect to a brownian motion (B_t), following the same kind of construction as for Riemann integral.
Here $(\Omega, \mathcal{F}, P)$ is a probability space and (B_t) is a brownian motion on it. We denote by $\mathcal{F}_t$ its natural filtration (or more generally a filtration with respect to which (B_t) is a brownian motion).

Definition 3.1: *A function* $e : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ *is said an elementary function iff*

$$e(t,\omega) = \sum_{j=0}^{n} e_j(\omega) 1\!\!1_{[t_j, t_{j+1}[}(t)$$

where e_j *is an* $\mathcal{F}_{t_j}$*-measurable random variable and* $0 = t_0 < t_1 < \ldots < t_n = T$

It is then possible to define the stochastic integral

$$\int_0^T e(t,.) dB_t(.) = \sum_{j=0}^{n} e_j(.)(B_{t_{j+1}} - B_{t_j})$$

Observe that the previous integral obviously defines a linear form over the set of elementary functions. The following result claims that it is an isometry between $L^2(dt \otimes dP)$ and $L^2(dP)$.

Theorem 3.2:
$$E(\{\int_0^T e(t,.) dB_t(.)\}^2) = E(\int_0^T e(t,.)^2 dt)$$

The proof is easy, using independence of increments of Brownian motion.

Therefore, one has that if $(e_n(t,.))$ is a Cauchy sequence in $L^2(dt \otimes dP)$ then the random variables $\int_0^T e(t,.) dB_t(.)$ form a Cauchy sequence in $L^2(dP)$. We now look for functions that can be approximated by elementary functions in the previous sense.

Definition 3.3: *A process* $u(t,\omega)$ *is said to belong to* $H^2([0,T])$ *if*

(i) it is adapted i.e. $\forall t \geq 0, u(t,.)$ *is* $\mathcal{F}_t$*-measurable.*

(ii) $E(\int_0^T u(t,.)^2 dt) < \infty$

The following result serves our purpose.

Proposition 3.4: *Let* $u(t,\omega) \in H^2$. *There exists a sequence* (e_n) *of elementary functions such that* $E(\int_0^T [e_n(t,.) - u(t,.)]^2 dt)$ *tends to 0*

The proof - technical but easy- is not of high interest. See [2] pp.22-23.
We can now define the stochastic integral.

Theorem and Definition 3.5: *Let $u \in H^2$. There exists by Proposition 3.4 a sequence (e_n) of elementary functions such that $E(\int_0^T [e_n(t,.) - u(t,.)]^2 dt)$ tends to 0. The sequence of random variables $\int_0^T e_n(t,.)dB_t$ is a Cauchy sequence in $L^2(dP)$ whose limit is independent of the choice of the approximating sequence (e_n). It is called the stochastic integral of u and is denoted by $\displaystyle\int_0^T u(t,.)dB_t(.)$. Moreover*

$$E([\int_0^T u(t,.)dB_t]^2) = E(\int_0^T u(t,.)^2 dt)$$

Proof: The only thing to prove is the independence from the approximating seqence which is a consequence of the isometry result □

An important fact is that the stochastic integral has the martingale property. More precisely, one has

Proposition 3.6: *Let $u \in H^2$. Then $(\int_0^t u(s,.)dB_s, t \geq 0)$ is an $\mathcal{F}_t$-martingale*

Proof: The result is obvious if $u = e$ is an elementary function. Indeed, it suffices to prove that if e_j is $\mathcal{F}_{t_j}$ measurable then $e_j(B_{t\wedge t_{j+1}} - B_{t \wedge t_j})$ is a martingale which is clear.
Now, let $u \in H^2$ and $s < t$. There exists a sequence (e_n) of elementary functions such that $E(\int_0^t [e_n(r,.) - u(r,.)]^2 dr)$ tends to 0. One has

$$E(\int_0^t e_n(r,.)dB_r/\mathcal{F}_s) = \int_0^s e_n(r,.)dB_r$$

and so, by Jensen inequality

$$E[E([\int_0^t e_n(r,.)dB_r - \int_0^t u(r,.)dB_r]/\mathcal{F}_s)^2] \leq$$

$$\leq E([\int_0^t e_n(r,.)dB_r - \int_0^t u(r,.)dB_r]^2) =$$

$$E(\int_0^t [e_n(r,.) - u(r,.)]^2 dr)$$

which tends to 0.
Therefore the above martingale equality tends in L^2 to

$$E(\int_0^t u(r,.)dB_r/\mathcal{F}_s) = \int_0^s u(r,.)dB_r$$

which is what we want. □

To end this paragraph, let us add that one can prove that the previous martingale has a.s. continuous paths.

4- Itô's formula

The major problem with the previous definition is that up to now we have no real way of computation for stochastic integrals. Itô's formula is the main tool of stochastic calculus. Let us begin by a definition.

Definition 4.1: *A stochastic process $(X_t)_{t\geq 0}$ is said to be a real diffusion process if there exist two processes $b(s,\omega) \in L^1(ds \otimes dP)$ and $\sigma \in H^2$ such that*

$$X_t = x_0 + \int_0^t b(s,.)ds + \int_0^t \sigma(s,.)dB_s$$

where $x_0 \in I\!\!R$.

The following theorem tells us that diffusion processes are stable under composition by a sufficiently smooth function.

Theorem 4.2 (Itô's formula): *Let (X_t) be a diffusion process of the form*

$$X_t = x_0 + \int_0^t b(s,.)ds + \int_0^t \sigma(s,.)dB_s$$

Let $f : I\!\!R^+ \times I\!\!R \to I\!\!R$ a continuous function, C^1 on the first variable and C^2 on the second. Then $(f(t,X_t), t \geq 0)$ is a diffusion process and

$$f(t,X_t) = f(0,x_0) + \int_0^t [\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2(s,.)\frac{\partial^2 f}{\partial x^2} + b(s,.)\frac{\partial f}{\partial x}](s,X_s)ds +$$

$$+ \int_0^t \frac{\partial f}{\partial x}(s,X_s)\sigma(s,.)dB_s$$

For the proof, see [2] pp. 35 and seq.

Example 4.3 : Take $f(x) = x^2$ and $X_t = B_t$ (i.e. $b = 0$ and $\sigma = 1$).
Then

$$B_t^2 = B_0 + 2\int_0^t B_s dB_s + t$$

In particular, as $B_0 = 0$,

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

The correcting term $-\frac{1}{2}t$ expresses the difference of behavior between stochastic integrals and Riemann-Stieltjes integral.

5- Stochastic differential equations

We close this chapter by a brief survey of elementary theory of stochastic differential equations. Let us first introduce what is a differential equation in stochastic sense.

Definition 5.1: *Let $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ two measurable functions and $x_0 \in \mathbb{R}$.*

A process (X_t) is said to be a (strong) solution of the stochastic differential equation

$$\text{(SDE)} \begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \end{cases}$$

if for any $t \geq 0$, one has a.s.

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

We shall see later in this course the much more intricate notion of "weak solution". Of course, it is implied in the definition of a solution that all the terms appearing above are well defined (and in particular the stochastic integral).
The following theorem, which is the exact correspondence to Cauchy-Lipschitz theorem for ordinary differential equations, gives existence and unicity.

Theorem 5.2: *Suppose that there exists $K > 0$ such that*

$$| \sigma(t, x) - \sigma(t, x') | \leq K | x - x' |$$

$$| b(t, x) - b(t, x') | \leq K | x - x' |$$

$$| \sigma(t, x) | + | b(t, x) | \leq K(1 + | x |)$$

Then the stochastic differential equation (SDE) has exactly one (strong) solution.

The proof - using the usual fixed point technique - can be found in [2]. An important example is the linear case where the equation can be solved explicitely.

Example 5.3: Suppose $b(t,x) = b(t).x$ and $\sigma(t,x) = \sigma(t)$ where b and σ are continuous bounded functions. Then Theorem 5.2 applies and the unique solution of (SDE) is given by

$$X_t = \exp(\int_0^t b(s)ds)x_0 + \int_0^t \exp(\int_s^t b(u)du)\sigma(s)dB_s$$

Remark 5.4: It is not very difficult to see that Theorem 5.2. can be extended to the case where σ and β depend also on ω (i.e. are random functions). One has in fact just to assume that σ and b are adapted (see [1] 2.7 Th.1).

REFERENCES

[1] Gikhman-Skorokhod: Stochastic Differential Equations, Springer, 1974
[2] Øksendal: Stochastic Differential Equations, Springer, 1985
[3] Revuz- Yor: Continuous martingales and brownian motion, Springer, 1989

Chapter 2: Introduction to control problems

The control problems are widely found in every domain of application of mathematics. The reason is that their object is to look for an optimal regulation of a system evolution. This kind of problem is present in very different frameworks: it may be the action on the power of an engine motor in order to reduce as much as possible the consumption, the action on the price of a financial product in order to get the best return, the choice of parameters of an industrial test in order to get the most reliable result.
Historically, the first real study of a control problem is due to Lagrange when he first set the basis of calculus of variations.
Let us recall this setting: one looks for a function $u : [0,1] \longrightarrow I\!R$, piecewise continuous that minimizes the functional

$$J(u) = \int_0^1 L(t, x_t^u, u_t)dt$$

where x_t^u is solution of the differential equation

$$\begin{cases} \dfrac{dx_t^u}{dt} = u_t \\ x_0^u = x_0 \end{cases}$$

We find in this problem all the ingredients of a control problem: the evolution -the state- of the system is governed by the "process" u_t -the control- in order to minimize a function of u and x^u, denoted $J(u)$, the cost.
Lagrange's ideas for solving this problem, and in particular his extraordinary idea of multipliers in order to find necessary conditions of optimality are the basis for the main ways of resolution of all control problems.
We begin by briefly presenting what a deterministic or a stochastic control problem is.

1- Deterministic and Stochastic Control Problems

a- Deterministic control problem

Let us take again the previous program.
The starting point is a controlled differential equation

$$(I)\begin{cases} \dfrac{dx_t^u}{dt} = f(t, x_t^u, u_t) \\ x_0^u = x_0 \end{cases}$$

where $f : I\!R^+ \times I\!R^p \times I\!R^d \longrightarrow I\!R^p$, $u : I\!R^+ \longrightarrow U \subset I\!R^d$.
One also considers $\mathcal{U} \subset \mathcal{F}(I\!R^+, U)$ - the admissible control set-, which is the set on which we minimize the cost . The admissibility conditions for a control are extremely varied, depending on what one wants to do, but we will always assume the following basic assumption satisfied

|| **Hypothesis 1.1**: $\forall u \in \mathcal{U}$, *there exists a solution* x^u *to (I)* .

One then considers a functional of u, defined on $\mathcal{U}$ by

$$J(u) = \phi(x_T^u)$$

where $\phi : \mathcal{U} \longrightarrow I\!R$, where T is a fixed "terminal" time in $I\!R^+$.
The control problem can then be set as follows: looking for an $u^* \in \mathcal{U}$ realizing $\inf_{u \in \mathcal{U}} J(u)$.
Let us first observe that this form of the cost in fact includes other types of cost functionals. For example, if one takes

$$J(u) = \int_0^T L(s, x_s^u, u_s)ds$$

setting

$$X_t^u = (x_t^u, y_t^u)$$

where

$$y_t^u = \int_0^t L(s, x_s^u, u_s)ds$$

one turns to the previous model with

$$J(u) = \phi(X_T^u) = y_T^u$$

b- Stochastic control

In the stochastic case, the evolution of the system is random. We shall restrict to the case that directly generalizes the deterministic case, which is when the evolution is governed by a stochastic differential equation.
We take the previous model but this time the state of the system is described by the equation

$$(II)\begin{cases} dx_t^u = f(t, x_t^u, u_t)dt + \sigma(t, x_t^u, u_t)dW_t \\ x_0^u = x_0 \end{cases}$$

where $\sigma : I\!R^+ \times I\!R \times I\!R^d \longrightarrow I\!R \times I\!R$ is a diffusion term for the brownian W.

Of course, we immediately see that an adaptation constraint on the control u is required in order that the stochastic integral be defined. It is one of the major differences between the deterministic and stochastic cases.
The set of admissible controls is therefore supposed to be composed with processes adapted to the natural filtration $\mathcal{F}_t$ of W_t (or more generally to a filtration such that W_t is a $\mathcal{F}_t$ brownian motion). In fact, it will be generally necessary to suppose more conditions to guarantee the existence of solutions to (II).
Another big difference between deterministic and stochastic cases comes from the cost: we can no longer hope to optimize the cost for any ω and therefore we try to minimize an average cost

$$J(u) = E(\phi(x_T^u))$$

2- The two deterministic highways: Pontryagin's principle and Dynamic programming.

We now come back to the problem (I). Inspired by Lagrange's study for calculus of variations, we look for necessary conditions of optimality.
When one considers a function $f : I\!\!R \longrightarrow I\!\!R$ of class C^1, in order to write that x^* is a minimum of f, one "modifies" a little bit x^* and writes for example:

$$f(x^* + \varepsilon) \geq f(x^*), \forall \varepsilon > 0$$

and so

$$\frac{f(x^* + \varepsilon) - f(x^*)}{\varepsilon} \geq 0 \Rightarrow_{\varepsilon \to 0} f_d(x^*) \geq 0$$

We can present this result in a dual formulation as follows: x^* is solution to the maximization problem of $H(y) = -f_d(x^*)f(y), y \geq x^*$.
The basic idea is the same for the control problem. We assume that $\mathcal{U}$ is stable by the following transformation: for $u \in \mathcal{U}$,

$$\widetilde{u}(t) = \begin{cases} v_i \text{ if } \tau_i - h_i < t \leq \tau_i \\ u(t) \text{ otherwise} \end{cases}$$

(that is to say that one modifies $\widetilde{u}$ on the intervals $[\tau_i - h_i, \tau_i]$) is a control of $\mathcal{U}$.
A very technical proof allows to obtain the following result that we present in its multidimensional form (see [3]).

Theorem 2.1 (Pontryagin's Principle): *If u_t^* is an optimal control, there exists a function $P(t)$ such that*

$$\begin{cases} \dfrac{d^tP_t}{dt} = -^t P_t.f_x(t, x_t^*, u_t^*) \\ ^tP_T = \phi_x(x_T^*) \end{cases}$$

and, for all t,

$$H(t, x_t^*, u_t^*) = \max_{u \in U} H(t, x_t^*, u)$$

where $H(t, x, u) =^t P(t).f(t, x, u)$ is said the Hamiltonian of the system.

Example 2.2: A fundamental case is when the equation of the system is linear. We now look at it.
Consider the evolution equation

$$\frac{dx_t^u}{dt} = a_t x_t + b_t u_t$$

and the cost

$$J(u) = \int_0^T (m_s x_s^2 + n_s u_s^2) ds + d x_T^2$$

with $m_s > 0, n_s > 0, d > 0$ and set $X_t^u = (x_t^u, y_t^u)$ where

$$y_t^u = \int_0^t (m_s x_s^2 + n_s u_s^2) ds$$

and

$$\phi(X_T^u) = y_T^u + d(x_T^u)^2$$

From Pontryagin principle, if u^* is optimal, one can find $P(t) = (p_t, q_t)$ such that

$$\frac{dp_t}{dt} = -a_t p_t - 2m_t x_t^* q_t$$

$$\frac{dq_t}{dt} = 0$$

with initial conditions

$$p_T = 2x_T^* d, q_T = 1$$

We therefore immediately have $q_t = 1, \forall t$.
The hamiltonian is here

$$H(t, x, u) = p_t(a_t x + b_t u) + (m_t x^2 + n_t u^2)$$

and its maximum is reached by

$$u_t = -\frac{b_t}{2n_t} p_t$$

and that allows to identify u_t by solving the differential system

$$\begin{pmatrix} x_t' \\ p_t' \end{pmatrix} = \begin{pmatrix} a_t x_t - \frac{b_t^2}{2n_t} p_t \\ -a_t p_t + 2m_t x_t \end{pmatrix}$$

We then get, with $A_t = \begin{pmatrix} a_t & -\frac{b_t^2}{2n_t} \\ 2m_t & -a_t \end{pmatrix}$,

$$\begin{pmatrix} x_t \\ p_t \end{pmatrix} = \exp\{-\int_0^t A_s ds\} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$$

The identification is completed by injecting the initial/terminal condition

$$\begin{pmatrix} x^*_T \\ 2dx^*_T \end{pmatrix} = \exp\{-\int_0^T A_s ds\} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$$

or

$$\begin{pmatrix} x_0 \\ p_0 \end{pmatrix} = x^*_T \exp\{\int_0^T A_s ds\} \begin{pmatrix} 1 \\ 2d \end{pmatrix}$$

from which one deduces the value of x^*_T and then the value of p_0. □

A completely different point of view is Dynamic Programming. An important observation is that an optimal trajectory is optimal each time: that means that starting from another point, one can do no better than still follow the optimal trajectory .
More precisely, let $(s, y) \in I\!R^+ \times I\!R$.
One can consider a new control problem in the following way

$$\begin{cases} \dfrac{dx^u_t}{dt} = f(t, x^u_t, u_t), t \geq s \\ x^u_s = y \end{cases}$$

and

$$J_{s,y}(u) = \phi(x^u_T)$$

where $\mathcal{U}_{s,y}$ is the set of controls over $[s, T]$.
One may then consider the *value function of the problem*

$$V(s, y) = \inf_{u \in \mathcal{U}_{s,y}} J_{s,y}(u)$$

Observe in particular that $V(T, y) = \phi(y)$.
The Dynamic Programming Principle can then be set in the following very general form

Theorem 2.3:

$$V(0, x) = \inf_{u \in \mathcal{U}} [\inf_{\tilde{u} \in \mathcal{U}_{s,x^u_s}} J_{s,x^u_s}(\tilde{u})] = \inf_{u \in \mathcal{U}} V(s, x^u_s)$$

This result is true as soon as the set of controls is stable under concatenation: if u and v are two controls of $\mathcal{U}$,

$$(u \mid_t v)_s = \begin{cases} u_s, s \leq t \\ v_s, s \geq t \end{cases}$$

is in $\mathcal{U}$.
Under the previous form, this relation is useful only on a theoretical level. To extract an equation from it, one supposes that V is $C^{1,1}$ (i.e. continuously differentiable with respect

to the two variables). We may then get the "differentiated" form of Dynamic Programming, the Hamilton-Jacobi equation, which gives, a *sufficient condition* of optimality.

Theorem 2.4 (Verification Theorem): *Assume that there is a function V, $C^{1,1}$, solution of*

$$(HJ)\begin{cases} 0 = \inf_{u\in U}[\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x).f(t,x,u)] \\ \phi(x) = V(T,x) \end{cases}$$

with the infemum assumed in $u_t^*(x)$. Then, if u^* is admissible, u^* is an optimal control and V is the value function of the problem

Proof:
For u a fixed admissible control, one can write

$$V(s,x_s^u) = V(0,x) + \int_0^s [\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}.f](t,x_t^u,u_t)dt$$

Observe then that

$$V(s,x_s^*) = V(0,x), \forall s \in [0,T]$$

and so $V(0,x) = V(T,x_T^*) = \phi(x_T^*) = J(u^*)$
On the other hand, for every $u \in \mathcal{U}$,

$$\forall s, V(s,x_s^u) \geq V(0,x) = J(u^*)$$

and so

$$V(T,x_T^u) = \phi(x_T^u) = J(u) \geq J(u^*) \quad \square$$

Example 2.5: Go back to the example of the linear regulator written in the previous form of Example 2.2. The state is still the couple $X_t^u = (x_t^u, y_t^u), s \leq t \leq T$ where $y_t^u = \int_s^t (m_r x_r^2 + n_r u_r^2)dr$.
We write the Hamilton-Jacobi equation

$$0 = \inf_{u\in U}[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(a_t x + b_t u) + \frac{\partial V}{\partial y}(m_t x^2 + n_t u^2)]$$

and, by a genius intuition (!), we look for a solution in the form

$$V(r,X) = y + k_r x^2$$

where $X = (x,y)$ (I must say that the justifications for this form given in the literature are quite strange and not very convincing !...).

The (HJ) equation then becomes

$$0 = \inf_u [k_t' x^2 + 2a_t k_t x^2 + 2b_t k_t u x + m_t x^2 + n_t u^2]$$

The infemum is reached at

$$u = -\frac{b_t k_t}{n_t} x$$

and the value of this minimum is

$$k_t' x^2 + 2a_t k_t x^2 - \frac{k_t^2 b_t^2}{n_t} x^2 + m_t x^2$$

i.e.

$$x^2 [k_t' + 2a_t k_t - \frac{b_t^2}{n_t} k_t^2 + m_t]$$

A sufficient condition for this infemum to be 0 is therefore that k satisfies the following Ricatti differential equation

$$\begin{cases} k_t' + 2a_t k_t - \frac{b_t^2}{n_t} k_t^2 + m_t = 0, s \leq t \leq T \\ k_T = d \end{cases}$$

(see [3] for solutions to this well known Ordinary Differential Equation)
Observe in particular that the optimal cost is given by $V(s,x) = k_s x^2$. As a by-product, we necessarily have $k_s \geq 0$.

Remark 2.6: It is easy to check that one recovers the same optimal control as before, setting $p_t = 2k_t x_t$.

3- What does remain in the stochastic case?

In this paragraph, we try to understand how the two previous ways for the study of a deterministic problem can be adapted to a stochastic control problem.
Let us look first at the Dynamic programming. It is not hard to see at first glance that all the previous results can be extended by means of stochastic calculus.
Let us keep the notations of paragraph 1. As in the deterministic case, it is possible to associate a new control problem.

$$\begin{cases} dx_t^u = f(t, x_t^u, u_t)dt + \sigma(t, x_t^u, u_t)dW_t, t \geq s \\ x_s^u = x_0 \end{cases}$$

and set $J_{s,y}(u) = E(\int_s^T l(r, x_r^u, u_r)dr + \phi(x_T^u))$.

Let us then consider $\mathcal{U}_{s,y}$, the set of controls over $[s,T]$. We set

$$V(s,y) = \inf_{u\in\mathcal{U}_{s,y}} J_{s,y}(u)$$

The (integral) equation of Dynamic Programming is then

Theorem 3.1:

$$V(0,x) = \inf_{u\in\mathcal{U}} E(\int_0^s l(r,x_r^u,u_r)dr + V(s,x_s^u))$$

As in the deterministic case, this result is true as soon as the set of controls is sufficiently stable. (see [2] for the proof which is quite technical).
In order to derive a differentiated form of Dynamic Programming principle, as in the deterministic case, we assume that V is sufficiently regular, which is to say in $C^{1,2}$ (C^1 in time, C^2 in space).

Theorem 3.2: *One supposes that the equation*

$$(HJB)\quad \begin{aligned} 0 =& [\frac{\partial W}{\partial t} + \inf_u(L^u W + l(s,x,u))](s,x,u) \\ \phi(x) =& W(T,x) \end{aligned}$$

admits a $C^{1,2}$ solution and that the minimum in assumed in $u_s^*(x)$ such that u^* is an admissible control. Then u^* is an optimal control and V is the value function of the problem

Proof:
By Itô's formula, one can write

$$W(t,x_t^u) = W(s,x) + \int_s^t (\frac{\partial W}{\partial r} + L^{u_r}W)(r,x_r^u)dr + \mathrm{mg}$$

where "mg" designs a martingale term. Adding $\int_s^t l(r,x_r^u,u_r)dr$ on both sides and taking expectations, we get

$$E(\int_s^t l(r,x_r^u,u_r)dr + W(t,x_t^u)) = W(s,x) + E(\int_s^t [\frac{\partial W}{\partial r} + L^{u_s}W + l](r,x_r^u,u_r))$$

By the equation (HJB), the last expectation is greater than 0. Therefore, taking $t = T$, one has

$$E(\int_s^T l(r,x_r^u,u_r)dr + \phi(x_T^u)) = J_{s,x}(u) \geq W(s,x)$$

and, taking the infemum over u, by definition of the value function, one has $V(s,x) \geq W(s,x)$.
On the other hand, taking $u = u^*$ and using (HJB), we have

$$E(\int_s^T l(r, x_r^{u^*}, u_r^*)dr + \phi(x_t^{u^*})) = J_{s,x}(u^*) = W(s,x)$$

and so we obtain the converse inequality; $\square$

Example 3.3: We look at the linear case, as was done in the deterministic case.
Let us consider the state equation

$$\begin{cases} dx_t^u = (a_t x_t^u + b_t u_t)dt + \sigma_t dW_t, t \geq s \\ x_s^u = y \end{cases}$$

with a cost

$$J_{s,y}(u) = E_{s,y}\Big(\int_s^T [m_t(x_t^u)^2 + n_t(u_t)^2]dt + d(x_T^u)^2\Big)$$

where $n_t > 0, m_t \geq 0, d > 0$.
The generator L^u takes the form

$$L^u f(t,x) = (a_t x + b_t u)\frac{\partial f}{\partial x} + \sigma_t^2 \frac{\partial^2 f}{\partial x^2}$$

Let us look for a solution to (HJB) which is here

$$\inf_u [\frac{\partial}{\partial t}V + L^u V + (m_t x^2 + n_t u^2)] = 0$$

under the form (!) $V(s,x) = k_s x^2 + q_s$.
Observe that as $V(T,x) = dx^2$, one has $k_T = d$ and $q_T = 0$.
Injecting the expression for V, one obtains

$$\inf_u \Big[k_r' x^2 + q_r' + k_r a_r x^2 + k_r b_r x u + a_r k_r x^2 + b_r k_r x u + \sigma_r^2 k_r + m_r x^2 + n_r u^2\Big] = 0$$

The minimum is assumed for

$$u^*(r,x) = \frac{-b_r k_r}{n_r} y$$

Therefore a sufficient condition for u^* to be optimal is that k and q satisfy

$$k_t' + 2a_t k_t - \frac{b_t^2}{n_t} k_t^2 + m_t = 0$$

and

$$q_t' + \sigma_t^2 k_t = 0$$

with boundary conditions $k_T = d, q_T = 0$.
Observe that the equation for k is the same Ricatti equation as in the deterministic case. Therefore , $q_t = \int_t^T \sigma_r^2 k_r dr$ and $q_t \geq 0$ (it was seen at the end of paragraph 2 that $k_r \geq 0, \forall r$). We see in conclusion that the optimal control is the same as in the deterministic case. The difference between the two cases comes from the optimal cost which is greater in the random case by the nonnegative term q_s, a quite intuitive result.

To conclude this chapter, some words about Pontryagin principle. It is easy to see that there is one major problem for its use in the stochastic case, because the natural adjoint process is not adapted and therefore is not solution to a stochastic differential equation. It is then necessary to project this adjoint but the result (a conditional expectation generally impossible to express) is hard to exploid (see [4]). Observe however that recent developments about backward equations could give important results in this direction (see [6]).

REFERENCES

[1] Davis: Linear filtering and control, Chapman and Hall, 1980
[2] El Karoui: Les aspects probabilistes du contrôle stochastique, Ecole d'été de Saint-Flour 1979, Springer
[3] Fleming-Rishel: Deterministic and Stochastic Optimal control, Applications of Mathematics 1, Springer, 1975
[4] Haussmann: The maximum principle, Pitman Research series, 1986
[5] Øksendal: Stochastic Differential Equations, Springer, 1985
[6] Peng: A general stochastic maximum principle for optimal control problems, SIAM Cont.Opt, 28, 1990

Chapter 3: Relaxed controls

In this brief chapter, we present a generalization of the notion of control. Of course, a control problem - and even a deterministic one- does not necessarily have a solution. We begin by an *ad-hoc* famous example taken from [3] in order to illustrate what we are going to do.

1- An example

Consider $U = \{-1, 1\}$ and consider piecewise continuous functions $u : [0,1] \to U$ (the controls).
The dynamic of the problem is given by the differential equation

$$\begin{cases} \dfrac{dx_t^u}{dt} &= u_t \\ x_0^u &= 0 \end{cases}$$

and the cost associated to the problem is

$$J(u) = \int_0^1 (x_t^u)^2 dt$$

First claim: $\inf\limits_u J(u) = 0$

Indeed, consider an integer $n \in \mathbb{N}^*$ and take $u_n(t) = (-1)^k$ if $\frac{k}{n} \le t < \frac{k+1}{n}$ for $0 \le k \le n-1$.
Then, clearly, for all $t \in [0,1]$, $\mid x_t^{u_n} \mid \le \frac{1}{n}$ and so $J(u) \le \frac{1}{n^2}$.

Second claim: There is not an u such that $J(u) = 0$

This is obvious as it would imply that $x_t^u = 0, \forall t$ and so $u_t = 0$ which is impossible.

If we analyze the previous example, we can understand where the trouble is: it is the fact that the sequence (u_n) lacks a limit in the space of controls, limit which should be the natural candidate to optimality. So we look for a space in which this limit exists.
Identify $u_n(t)$ with the Dirac measure on U: $\delta_{u_n(t)}(du)$. Set $q_n(dt, du) = \delta_{u_n(t)}(du)dt$. q_n is a measure over the space $[0,1] \times U$

Lemma 1.1: *q_n converges weakly to*

$$\widetilde{q}(dt, du) = \frac{1}{2}[\delta_{-1} + \delta_1](du)dt$$

Proof: Take f a continuous function on $[0,1]\times U$ (of course only the continuity over [0,1] is meaningful).
One has

$$\int_{[0,1]\times U} f(t,u)q_n(dt,du) = \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t,(-1)^k)dt$$

Suppose first that n is even: $n = 2m$.
As $t \to f(t,1)$ and $t \to f(t,-1)$ are continuous over [0,1], they are uniformly continuous.
Let $\varepsilon > 0$. There is an $M > 0$ such that $\forall m \geq M$, $\mid f(t,u) - f(s,u) \mid < \varepsilon$ if $\mid t-s \mid < \frac{1}{m}$ where u is either 1 or -1.
Fix $m \geq M$.
Then, for every $j = 0,\ldots,m-1$,

$$\mid \int_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt - \int_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt \mid < \frac{\varepsilon}{2m}$$

One has

$$\sum_{j=0}^{m-1}\int_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt + \sum_{j=0}^{m-1}\int_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt = \int_0^1 f(t,u)dt$$

and

$$\mid \sum_{j=0}^{m-1}\int_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt - \sum_{j=0}^{m-1}\int_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt \mid < \frac{\varepsilon}{2}$$

Therefore,

$$\mid \sum_{j=0}^{m-1}\int_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt - \frac{1}{2}\int_0^1 f(t,u)dt \mid < \frac{\varepsilon}{2}$$

and

$$\mid \sum_{j=0}^{m-1}\int_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt - \frac{1}{2}\int_0^1 f(t,u)dt \mid < \frac{\varepsilon}{2}$$

So,

$$\mid \sum_{k=0}^{2m-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t,(-1)^k)dt - \frac{1}{2}[\int_0^1 f(t,1)dt + \int_0^1 f(t,-1)dt] \mid < \varepsilon$$

The case n odd is treated in the same way. □

Now, we can define a "new" control problem associated to such a measure q, which is called a relaxed control.
Consider the dynamic

$$x_t^q = x_0 + \int_0^t \int_U uq(ds,du)$$

and the associated cost is, as before,

$$J(q) = \int_0^1 (x_t^q)^2 dt$$

Then it is clear that the previous control problem is generalized by the present problem by taking measures q of the form

$$q(dt, du) = \delta_{u_t}(du)dt$$

Moreover, if

$$\widetilde{q}(dt, du) = \frac{1}{2}[\delta_1(du) + \delta_{-1}(du)]dt$$

we have $J(\widetilde{q}) = 0$ and so the new problem has $\widetilde{q}$ as an optimal solution.

The remaining part of the chapter is devoted to the study of such relaxed controls.

2- Relaxed controls

We could want to take as controls all the measures $q(dt, du)$. However, for our purpose which is to prove existence of an optimal control, we have in mind to restrict to a compact space containing "classical" controls (by the kind of identification of section 1). This is why the following definition is set.

Definition 2.1: *Let $U \subset I\!R^d$. A relaxed control with values in U is a measure q over $[0, T] \times U$ such that the projection on [0,T] is the Lebesgue measure.*

If there exists $u : [0, T] \to U$ such that $q(dt, du) = \delta_{u_t}(du)dt$, q is identified with (u_t) and said to be a control process.

We have an interesting decomposition of such a relaxed control.

Proposition 2.2: *Let q be a relaxed control with values in U. Then, for all $t \in [0, T]$, there exists a probability measure q_t over U such that*

$$q(dt, du) = dtq_t(du)$$

The proof is an application of Fubini theorem left in exercise. The previous Proposition 2.2 allows us to better interpret what a relaxed control is. In a control process, at a time t, we assign the value u_t. In a relaxed control, the value is "randomly" chosen over the space U with the probability distribution $q_t(du)$.
Another interest of Proposition 2.2 is that we can introduce a canonical decomposition of relaxed controls.

Definition 2.3: *Let V be the space of relaxed controls over U. Let $\alpha \in V$. There exists by Proposition 2.2 a process (α_s) with values in the set of probability measures on U and such that $\alpha(ds, du) = ds\alpha_s(du)$*

The process (q_t) defined on V, which associates the process (α_t) to α is said the canonical process on V.

The filtration $\mathcal{V}_t = \sigma(q_s, s \leq t)$ is said the canonical filtration.

Remark 2.4: It is not hard to see that $\mathcal{V}_t$ is generated by relaxed controls q such that

$$q_{|[t,T]\times U}(dt, du) = \delta_{u_0}(du)dt$$

where u_0 is an arbitrarily fixed point in U.

3- Topology on the space V

V, as a set of measures, is classically equipped with the weak topology.

Definition 3.1: *A sequence (q_n) in V is said to converge to $q \in V$ if for any continuous functions with compact support f on $[0, T] \times U$,*

$$\int f(t, u)q_n(dt, du) \rightarrow \int f(t, u)q(dt, du)$$

This convergence is by definition only valid on continuous functions. However, as all the measures in V have the same marginals on $[0, T]$ (Lebesgue measure), it is possible to considerably improve it.

Proposition 3.2: *Suppose $q_n \rightarrow q$ in V.*

Then, for every measurable function $f(t, u)$ such that $\forall t \in [0, T]$, $u \rightarrow f(t, u)$ is continuous, one has

$$\int f(t, u)q_n(dt, du) \rightarrow \int f(t, u)q(dt, du)$$

("stable convergence")

For the proof, see [4].
Finally, the following result makes clear that the set of relaxed controls has interesting compactness properties. The proof is left as an exercise in measure theory (see [2]).

Proposition 3.3: *Suppose U is a compact set. Then V is compact*

REFERENCES

[1] Billingsley: Probability and Measure, Wiley, 1986
[2] Billingsley: Convergence of Probability measures, Wiley, 1968
[3] Ekeland: Suites minimisantes et solutions optimales de certains problèmes de contrôle, Cahier IRIA 4,mars 1971
[4] Jacod-Mémin: Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilités. Séminaire de Strasbourg 15. Lectures Notes in Math. 851

Chapter 4: Weak solutions of Stochastic Differential Equations and martingale problems

We begin by explaining why we need a less restrictive notion of solution to a SDE rather than a strong solution.
When one studies a control problem, one needs to estimate the cost which is a function of the processes (u_t) and (x_t); therefore only the distribution of the process (u_t, x_t) is important. Therefore, a formulation which would use only these distributions could be more appropriate to our purpose. And, in fact, the weak solutions are solutions "in distribution".
There is also another reason, less immediate. We start again from the previous chapter, namely from relaxed controls.
Consider a controlled equation of the general form

$$dX_t^u = b(t, u_t, X_t)dt + \sigma(t, u_t, X_t)dB_t$$

Observe that this case includes the deterministic case (where $\sigma = 0$). We have seen that such a problem (with a cost given as before) does not necessarily have a solution. In the deterministic case, we have introduced relaxed controls, which are measure-valued controls, and the state equation becomes

$$dX_t^q = [\int_U b(t, a, X_t)q_t(da)]dt$$

If we want to do so with the stochastic equation, we have got a problem because we do not know how to deal with the stochastic integral. If one tries the "natural" choice and writes an expression such as

$$[\int_U \sigma(t, a, X_t)q_t(da)]dB_t$$

supposing that it satisfies the required measurability conditions, it soon appears that one cannot do anything with it. Indeed assume that $q^n(dt, da)$ converges to $q(dt, da)$. One has the following estimates

$$E([\int_0^T \int_U \sigma(t, a, X_t)q_t(da)dB_t - \int_0^T \int_U \sigma(t, a, X_t)q_t^n(da)dB_t]^2) =$$

$$= E([\int_0^T (\int_U \sigma(t, a, X_t)(q_t(da) - q_t^n(da)))dB_t]^2) =$$

$$= E(\int_0^T [\int_U \sigma(t, a, X_t)(q_t(da) - q_t^n(da))]^2 dt)$$

and we are not able to say anything further. We see that the formulation is not satisfactory. Therefore, it could be interesting to be able to formulate the problem not using any stochastic integral. This will be the role of martingale problems.

1- Weak solution of a Stochastic Differential Equation

Let us consider two functions b and σ as in Chapter 1, 5.

Definition 1.1: *A weak solution of the (formal) Stochastic Differential Equation*

$$\text{(SDE)} \begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{cases}$$

is an element $(\Omega, (\mathcal{F}_t), P, (B_t)_{t\geq 0}, (X_t)_{t\geq 0})$ *where*

(i) $(\Omega, (\mathcal{F}_t), P)$ *is a filtered space*

(ii) $(B_t)_{t\geq 0}$ *is a* $(\mathcal{F}_t)_{t\geq 0}$ *brownian motion on* Ω

(iii) $(X_t)_{t\geq 0}$ *is a strong solution to the equation*

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \end{cases}$$

It immediately appears that in a weak solution the probability space nor the brownian motion are prescribed. The solution is therefore the whole element $(\Omega, (\mathcal{F}_t), P, (B_t)_{t\geq 0}, (X_t)_{t\geq 0})$. Of course, any strong solution to an equation is a weak solution: the space and the brownian motion remain just the same.

The interest of this notion is that the converse is not true: there are equations that have weak solutions but not strong ones (see [1]). This is not really surprising, as the requirement on the coefficients of the equation (namely Lipschitzianity) are quite restrictive to get a strong solution.

The major result we shall need in the weakening of the conditions on the coefficients is the following one.

Theorem 1.2: *Suppose that* b *and* σ *are bounded measurable functions, continuous in* x *for every* $t \geq 0$.

Then, for any $x_0 \in I\!R$, *there is a weak solution to the equation (SDE)*

For the proof, see [4], Th.6.1.7 or [1].

2- Canonical space and Martingale problems

The previous formulation is somewhat unpleasant because we do not know how to look for the space Ω and the brownian motion (B_t) defined on it. In fact, what is actually of interest to us is only (X_t) (or even better, the distribution of (X_t)). So, it could be nice to find a formulation that allows us to explicit just this distribution. This can be done through martingale problems.
Let us introduce the generator associated to the previous bounded continuous functions b and σ. It is the operator

$$Lf(t,x) = \frac{\partial f}{\partial t}(t,x) + b(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}$$

where f is a $C_{\mathrm{b}}^{1,2}$ function.
We can easily check by Itô formula that if $(\Omega, (\mathcal{F}_t), P, (B_t)_{t\geq 0}, (X_t)_{t\geq 0})$ is a weak solution of (SDE), then for any $f \in C_{\mathrm{b}}^{1,2}$,

$$f(t, X_t) - \int_0^t Lf(s, X_s)ds$$

is a $\mathcal{F}_t$ martingale.

Definition 2.1: *The canonical space C is the set of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$.*

(x_t) the canonical process on C is defined by $x_t(f) = f(t)$. The canonical filtration is $\mathcal{C}_t = \sigma(x_s, s \leq t)$

The distribution of (X_t) is therefore a probability measure over C.
And, if $(\Omega, (\mathcal{F}_t), P, (B_t)_{t\geq 0}, (X_t)_{t\geq 0})$ is a weak solution to (SDE), then the distribution R of (X_t) is such that for any $f \in C_{\mathrm{b}}^{1,2}$, the process defined on C by

$$f(t, x_t) - \int_0^t Lf(s, x_s)ds$$

is a $\mathcal{C}_t$ martingale under the probability R. It is only a rewriting of the previous martingale property.
We introduce the following definition.

Definition 2.2: *A probability measure R on C is said a solution to the martingale problem associated with the operator L if $\forall f \in C_{\mathrm{b}}^{1,2}$, under R,*

$$f(t, x_t) - \int_0^t Lf(s, x_s)ds$$

is a $\mathcal{C}_t$ martingale

We have just seen that to any weak solution of (SDE) is naturally associated a solution to the martingale problem. In fact, the converse is also true and there is a one-to-one correspondence between weak solutions and solutions of martingale problem.

Theorem 2.3: *Assume that R is a solution to the martingale problem. Then there exists a weak solution $(\Omega, (\mathcal{F}_t), P, (B_t)_{t\geq 0}, (X_t)_{t\geq 0})$ to (SDE) such that X_t has distribution R*

For the proof, see [1] Th.2.7.1'.

The reason why the martingale problem formulation is very pleasant is that it allows to get limit results easily. Below is given one convincing result of that kind.

Proposition 2.4: *Assume that b and σ are bounded and continuous in x. Then the set of solutions of the martingale problem is closed*

Proof: Let (R_n) be a sequence of solutions of the martingale problem, converging to R. One has to prove that under R, for any $f \in C_b^{1,2}$,
$f(t, x_t) - \int_0^t Lf(s, x_s)ds$ is a martingale.
Let $r < t$.
Take h a continuous bounded $\mathcal{C}_r$ measurable function on C.
Then, for any n,

$$R_n([f(t, x_t) - \int_0^t Lf(s, x_s)ds]h) = R_n([f(r, x_r) - \int_0^r Lf(s, x_s)ds]h)$$

Now, as $[f(u, x_u) - \int_0^u Lf(s, x_s)ds]h$ is a bounded continuous function on C, the previous equality goes to the limit with R_n replaced by R and it is what we want. □

In the book [4], one can find a deep study of the properties of solutions to martingale problems. In particular, we shall use the following result.

Theorem 2.5: *Suppose that b and σ are bounded, continuous in x functions, then the solutions of the martingale problem defines a compact set of probability measures on C*

REFERENCES

[1] Ikeda-Watanabe: Stochastic Differential Equations and Diffusion processes, North-Holland, 1981
[2] Øksendal: Stochastic Differential Equations, Springer, 1985
[3] Revuz-Yor: Continuous martingales and Brownian Motion, Springer, 1989
[4] Stroock-Varadhan: Multidimensional Diffusion Processes, Springer, 1979

Chapter 5: Compactification methods in stochastic control problems

We now collect the results of the previous chapters in order to obtain the existence of an optimal control in a very general setting. Let us immediately emphasize the limits of this approach: the theorem we are going to obtain guarantees existence but does not give any hint in order to construct this optimal control. For this, the methods of the analysts (namely the resolution of (HJB) equation) are the most appropriate. However, on the other hand, it is only under very strong conditions that one can solve (HJB) equation and therefore our presentation allows one to "say something" when the analysts are just mute! As always, the validity of the result depends on what you do expect from it.
We begin by a brief summary of the problem.

1- Continuous diffusion control problem

We consider the controlled diffusion equation which appeared in Chapter 2. The state of the problem is therefore governed by the following stochastic differential equation

$$(1)\begin{cases} dx_t^u &= b(t, x_t^u, u_t)dt + \sigma(t, x_t^u, u_t)dW_t \\ x_0^u &= x_0 \end{cases}$$

where $b : I\!R^+ \times I\!R \times U \to I\!R$ and $\sigma : I\!R^+ \times I\!R \times U \to I\!R$ are two measurable functions, continuous in the variables x and u. The control process u_t is, as before, an adapted process that takes values in U and the aim of the problem is to find a control in order to minimize a cost

$$J(u) = E(\int_0^T l(s, x_s^u, u_s)ds + g(x_T^u))$$

where $l : I\!R^+ \times I\!R \times U \to I\!R$ and $g : I\!R \to I\!R$ are measurable functions.

Remark 1.1: Observe that σ may be degenerated (and in particular, the choice $\sigma = 0$ allows to recover the deterministic case). Moreover, σ may depend on the control term u. This is a major difference with the analytical methods where one tries to solve (HJB) equation; they indeed generally require strong assumptions of non degeneracy of the diffusion coefficient (see [4]).

We have already seen in Chapters 3 and 4 why we need to consider relaxed controls and weak controls. As was evoked in Chapter 4, it is the martingale problem formulation that is the best adapted to our purpose and for this reason we introduce now a canonical version of the problem.

2- Canonical formulation: control rules

We first introduce the canonical spaces we shall use to formulate our problem.

$X = C(\mathbb{R}^+, \mathbb{R})$ is the space of continuous real functions, equipped as in Chapter 4 with the canonical process (x_t) and the canonical filtration $(\mathcal{X}_t)$.

V is the set of measures on $\mathbb{R}^+ \times U$ whose projection on $\mathbb{R}^+$ is the Lebesgue measure. As seen in Chapter 3, for any $\alpha \in V$ it is possible to define a kernel (α_t) of probability measures on U such that $\alpha(dt, du) = \alpha_t(du)dt$ and (q_t) a canonical process on V such that $q_t(\alpha) = \alpha_t$. The canonical filtration is $\mathcal{V}_t = \sigma(q_s, s \leq t)$.
The canonical space of the problem is $X \times V$ equipped with the filtration $\mathcal{F}_t = \mathcal{X}_t \otimes \mathcal{V}_t$.
We now set the canonical formulation.

Definition 2.1: *A control rule with initial condition (r, x) is a probability measure R on the space $X \times V$ such that*

(i) $R(x_s = x, \forall s \leq r) = 1$

(ii)$\forall f \in C_b^{1,2}$,

$$f(t, x_t) - \int_r^t \int_U A^a f(s, x_s) q_s(da) ds$$

is an $(\mathcal{F}_t)$-martingale under R

In Definition 2.1, A^a is as in Chapter 2 the operator associated to b and σ i.e.

$$A^a f(t, x) = \frac{1}{2}\sigma^2(t, x, a)\frac{\partial^2 f}{\partial x^2} + b(t, x, a)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}$$

The set of control rules with initial condition (r, x) is denoted by $\mathcal{R}(r, x)$.

The cost associated to the rule R is now

$$\Gamma_r(R) = R(\int_r^T \int_U l(s, x_s, a) q_s(da) ds + g(x_T))$$

Observe first that the set $\mathcal{R}(r, x)$ is not empty as, using the results of Chapter 4 (Theorem 1.2), there exists a solution to the martingale problem associated with a constant control $u_s = u_0$.

Remark 2.2: We shall not study in this course all the properties of the model, as our aim is just to prove an existence result. It is however good to know that one can obtain already, without any further hypothesis a Dynamic Programming principle in the following way.

Theorem 2.3: *Let us denote*

$$V(r,x) = \inf_{R \in \mathcal{R}(r,x)} \Gamma_r(R)$$

the value functions of the problem. Then, $\forall t \in [0,T]$*, one has*

$$V(r,x) = \inf_{R \in \mathcal{R}(r,x)} R(\int_r^t \int_U l(s,x_s,a) q_s(da) ds + V(t,x_t))$$

The proof of this result is quite technical (see for example, [1] or [2]). Of course, the above formulation is not the most general one. In particular, we have set the Dynamic Programming Principle with a deterministic time t. In fact, it is also valid with t replaced by a stopping time, i.e. a random variable τ such that $(\tau \leq s) \in \mathcal{F}_s, \forall s$.

3- Compactness of $\mathcal{R}(r,x)$. Existence of an optimal rule.

We finally arrive at our existence result of an optimal rule. If we consider the problem as minimizing a function $\Gamma(R)$ over a set $\mathcal{R}(r,x)$, the best thing to do is to prove that the function Γ is continuous and the set $\mathcal{R}(r,x)$ is compact. Naturally, we need some extra hypotheses.

Hypotheses 3.0:
- U is a compact set
- l and g are bounded and continuous in (x,u).

Then we have

Proposition 3.1: *The set* $\mathcal{R}(r,x)$ *is compact for any* (r,x)

This is a direct consequence of the results of Chapter 4 (Theorem 2.5).

We have also the following result of regularity for the cost.

Proposition 3.2: *The function* $R \to \Gamma(R)$ *is continuous*

Proof: By definition of the convergence of a sequence of probability measures (which control rules are), it is sufficient to prove that

$$\int_0^T \int_U l(s,x_s,a) q_s(da) ds + g(x_T)$$

defines a continuous bounded function over $X \times V$. This is an easy exercise to check that it is true, due to the continuity and boundedness of l and g. □

We are now ready to state the existence result whose proof is a direct consequence of Propositions 3.1 and 3.2.

|| **Theorem 3.3**: $\forall(r,x), \exists R \in \mathcal{R}(r,x)$ *that realizes the minimum of* Γ

4- Comparison between the control problems

There remains a problem. It is to know whether one has changed the value function of the problem, by allowing to take relaxed controls and weak solutions for the state instead of strong solutions and control processes as is required in the "natural" initial formulation of the problem. We then want to compare the minimal cost in both situations.
An important fact is that these two infima coincide, but it is a difficult result (see [2], part 4). What I would like to evoke here is that this equality is a consequence of an amazing (explaining why it is known as the "chattering lemma") deterministic result due to [5] which says that, in the case when U is a finite set, a step relaxed control can be well approximated by a non-relaxed control.

Proposition 4.1 (Chattering lemma): *Suppose* $U = \{1, 2, \ldots, K\}$ *and let* q *be a step relaxed control with values in* U.

Denote $q_t^i = q_t(\{i\}), \forall i \in \{1, 2, \ldots, K\}$.

Then, $\forall \varepsilon > 0$, $\exists(U_\varepsilon^1, \ldots, U_\varepsilon^K)$ *borelian sets in* $[0,T]$ *such that*

(i) $\sum_{i=1}^{K} 1\!\!1_{U_\varepsilon^i}(t) = 1$ *a.e.*

(ii) $\exists C > 0, \forall f$ *continuous on* $[0,T]$,

$$\sum_{i=1}^{K} \left| \int_0^T f(t)[q_t^i - 1\!\!1_{U_\varepsilon^i}(t)]dt \right| \leq C\omega_f(\varepsilon)$$

where $\omega_f(\varepsilon) = \sup_{|x-y|<\varepsilon} | f(x) - f(y) |$ *is the continuity modulus of* f

In other words, one approximates $(\sum_{i=1}^{K} q_t^i \delta_i(du))dt$ by $\sum_{i=1}^{K}(1\!\!1_{U_\varepsilon^i}(t)\delta_{\{i\}}(du))dt$

Proof: Choose n such that $\frac{T}{2^n} < \varepsilon$. It is possible with no loss of generality to suppose that n is sufficiently large so that q_t^k is constant over dyadic intervals with length $\frac{T}{2^n}$.
Consider $I_n = \{0, \ldots, j\frac{T}{2^n}, \ldots, T\}$ the dyadic subdivision of $[0,T]$ with step $\frac{T}{2^n}$. Denote by $T_j^n =]j\frac{T}{2^n}, (j+1)\frac{T}{2^n}]$ the $(j+1)$ interval in this subdivision. Now divide each interval T_j^n into K intervals $T_{j,k}^n$ with length $\frac{T}{2^n}q^k_{j\frac{T}{2^n}}$, for $1 \leq k \leq K$, and introduce the sets

$U^k_\varepsilon = \bigcup_j T^n_{j,k}$.

It is easy to check that they satisfy property (i) of Theorem 4.1.
Moreover, if f is a continuous function, one has

$$| \int_0^T f(t)(q^k_t - 1\!\!1_{U^k_n}(t))dt | \leq$$

$$\leq \sum_j \int_{T^n_j} | f(t) - f(j\frac{T}{2^n}) || q^k_t - 1\!\!1_{U^k_\varepsilon}(t) | dt + \sum_j | f(j\frac{T}{2^n}) | \int_{T^n_j} | q^k_t - 1\!\!1_{U^k_n}(t) | dt$$

But the second term in the last formula is 0 as $q^k_t = q^k_{j\frac{T}{2^n}}$ on T^n_j and we are easily done □

REFERENCES

[1] El Karoui: Les méthodes probabilistes du contrôle stochastique, Ecole d'été de Saint-Flour 1979, Springer
[2] El Karoui Huu Nguyen Jeanblanc-Picqué: Compactification methods in the Control of Degenerate Diffusions: Existence of an Optimal Control, Stochastics, 20, pp.169-219, 1987
[3] Fleming: Generalized solutions in optimal stochastic control. Differential games and control theory. Kingston Conference 1976, Lect.Notes in Pure and Applied Math.,30, Dekker 1978
[4] Fleming Rishel: Deterministic and Stochastic control theory, Springer, 1975
[5] Ghouila-Houri: Sur la généralisation de la notion de commande d'un système guidable, RIRO 4, 1967