Dynamic Programming for a set-indexed controlled problem

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1 Introduction

The Dynamic Programming Principle (DPP) is an essential tool in the study of a (deterministic or stochastic) problem. In classical situations, it allows to formulate the optimal cost of the optimal control problem (the so-called value function) as the solution of a Partial Differential Equation, the Hamilton-Jacobi-Bellmann equation. See [3] for a complete survey of methods and results in this classical context. Our aim in this paper is to focus on a stochastic control problem for a process indexed by a partially ordered set and more precisely to state a DPP in this situation. This kind of optimization problems has already been studied in the past but only for particular choices of the indexation set (see [6], [7]). We choose the abstract framework of set-indexed processes as introduced in the 1990's by E. Merzbach, G. Ivanoff and their collaborators. Giving an algebraic structure to the index set allows one to define notions such as continuous processes or martingales (and in particular Brownian motion). The recent book by Ivanoff and Merzbach [5] appears as the state of the art in these subjects.
In the classical (real indexed) situation, the basic property that is used to obtain DPP is the separation of the times between the past and the future of an arbitrary stopping time. In our set-indexed context, we need to generalize such a property by using appropriate stopping sets, and in particular we need to define a suitable notion of past and future of such a set that enables to separate the 'time' in past and future and to add a (strong) extra hypothesis on the model (Assumption 3.3). This being done, one obtains the DPP by a (quite technical) transposition of the classical proof as exposed for example in [4].
Our paper is presented in an abstract framework, in order to be at hand for application to special cases. In particular, our hope is to formulate in such a context results of control over branching processes to extend previous studies such as [1].
The paper is organized as follows: after having recalled important notions for set-indexed processes, we formulate the control problem under interest. Then, we prove that the two properties of stability required for the DPP (stability by conditioning and by concatenation - see [2]) are satisfied for this problem and we deduce the DPP.

2 Results on set-indexed processes

In this section, we present the major results from set-indexed process theory we shall need. We advise the interested reader to report to [5] for a most complete overview on the subject and in particular for examples of possible indexation sets $\mathcal{A}$.
An abstract probability space $(\Omega, \mathcal{F}, I\!P)$ is fixed throughout the section.
Let (T, d) be compact connected metric space. As it is observed in [5], all the results can be extended to σ-compact separable, complete metric spaces. We denote by $\mathcal{A}$ a fixed class of compact connected subsets of T and we assume that $\mathcal{A}$ is an indexing collection, in the following sense

(a) $\emptyset, T \in \mathcal{A}$, $A^\circ \neq A$, if $A \neq \emptyset$ or T.

(b) $\mathcal{A}$ is closed under arbitrary intersections.

(c) if $A, B \in \mathcal{A}$ are such that $A, B \neq \emptyset$, then $A \cap B \neq \emptyset$. If (A_i) is an increasing sequence in $\mathcal{A}$ then $\overline{\cup_i A_i} \in \mathcal{A}$.

(d) $\mathcal{A}$ generates $\mathcal{B}(T)$, the Borel σ-algebra on T.

The class of finite unions of elements from $\mathcal{A}$ is denoted by $\mathcal{A}(u)$, $\mathcal{C}$ is the class of elements of the form $A \setminus B$, $A \in \mathcal{A}, B \in \mathcal{A}(u)$ and $\mathcal{C}(u)$ consists of all finite unions in $\mathcal{C}$. Observe that $\mathcal{C}$ is clearly stable by finite intersections. Moreover, if $A \in \mathcal{A}$, $A^c = T \setminus A \in \mathcal{C}$.
We consider a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ on the space Ω and we suppose that it is countably generated. All the processes considered hereafter are indexed by $\mathcal{A}$. We suppose that all the processes admit an additive extension to $\mathcal{C}(u)$.
We also require the following classical assumption.

Assumption 2.1 Separability from above. *There is an increasing sequence of finite sub-semilattices $\mathcal{A}_n = \{A_1^n, \cdots, A_{k_n}^n\}$ of $\mathcal{A}$, closed under intersections, each containing $\emptyset$ and T, and there is a sequence of set-functions, $g_n : \mathcal{A} \to \mathcal{A}_n$ $(n \in I\!N)$ such that, given any $A, A' \in \mathcal{A}$,*

(a) *$g_n(T) = T$ and $A \subset [g_n(A)]^\circ \;\; \forall A \neq T$,*

(b) *$(g_n(A))$ is decreasing with $\bigcap_n g_n(A) = A$,*

(c) *$A \subseteq A'$ implies $g_n(A) \subseteq g_n(A')$,*

(d) *$A \subset A'$ implies $A \subset g_n(A) \cap A'$,*

(e) *$A \cup A' \in \mathcal{A}$ implies $g_n(A \cup A') = g_n(A) \cup g_n(A')$ and*

(f) *g_n preserves countable intersections (i.e., $g_n(\bigcap_{i=1}^\infty A_i) = \bigcap_{i=1}^\infty g_n(A_i)$ for any (A_i) in $\mathcal{A}$).*

$\mathcal{A}_n(u)$ is the set of finite unions of sets in $\mathcal{A}_n$. We denote $\mathcal{C}_n$ the set of $A \setminus B$ with $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_n(u)$. $\mathcal{C}_n(u)$ is the set of finite unions of elements in $\mathcal{C}_n$.
Though the following assumption is quite restrictive, we shall require it systematically.

Assumption 2.2 SHAPE. *If $A, A_1, \cdots, A_n \in \mathcal{A}$ and $A \subseteq \cup_{I=1}^n A_i$, then there is an index i, $1 \leq i \leq n$ such that $A \subseteq A_i$.*

We denote $\emptyset' = \cap_{A \in \mathcal{A}, A \neq \emptyset} A$. We have that $\emptyset' \in \mathcal{A}$ and $\emptyset' \neq \emptyset$. It will play the role played by 0 for martingales indexed by $I\!R_+$ and we suppose that for any process X, $X_{\emptyset'} = 0$.

Definition 2.3 *A function $x : \mathcal{A} \to I\!R$ is said to be* monotone inner-continuous *on $\mathcal{A}$ if for any increasing sequence (A_n) of sets in $\mathcal{A}$ such that $\overline{\cup_n A_n} = A \in \mathcal{A}$, then $\lim_n x(A_n) = x(A)$. $x : \mathcal{A} \to I\!R$ is said to be* monotone outer-continuous *on $\mathcal{A}$ if for any decreasing sequence (A_n) of sets in $\mathcal{A}$, $x(\cap_n A_n) = \lim_n x(A_n)$.*
A function is said to be monotone continuous *if it is monotone inner- and outer-continuous.*

For any set $C \in \mathcal{C}(u) \setminus \mathcal{A}$, we define its strong history to be $\mathcal{G}_C^* = \bigvee_{B \in \mathcal{A}(u), B \cap C = \emptyset} \mathcal{F}_B$.
An important observation is that the σ-algebras $\mathcal{G}_C^*$ are non-increasing: if $C \subseteq C'$, $\mathcal{G}_{C'}^* \subseteq \mathcal{G}_C^*$.

Definition 2.4 *A set-indexed* strong martingale $X = \{X_A, A \in \mathcal{A}\}$ *is an adapted, integrable and additive process on $\mathcal{C}$ that satisfies $I\!E(X_C | \mathcal{G}_C^*) = 0$, for any $C \in \mathcal{C}$.*

Definition 2.5 *Suppose that Λ is a non-negative increasing function defined on $\mathcal{A}$ with $\Lambda_\emptyset = 0$. A process W is a* Brownian motion *with variance measure Λ if $W_\emptyset = 0$, W can be extended to an additive process on $\mathcal{C}(u)$ and if for any disjoint sets $C_1, \cdots, C_n \in \mathcal{C}$, $W_{C_1}, \cdots, W_{C_n}$ are independent mean-zero normal random variables with respective variances $\Lambda_{C_1}, \cdots, \Lambda_{C_n}$.*

We first need to introduce what a stopping set is and then we can explain the approach. Let ξ a stopping set. We recall here some of its properties. A precise formulation is in [5].

Definition 2.6 *Let $\xi : \Omega \to \mathcal{A}(u)$ be of the form $\xi(\omega) = \cup_{i=1}^k \xi_i(\omega), \xi_i : \Omega \to \mathcal{A}, i = 1, \cdots k, k < \infty$. ξ is called a stopping set if, for any $A \in \mathcal{A}$, $\{\omega : A \subseteq \xi(\omega)\} \in \mathcal{F}_A$, $\{\omega : \emptyset = \xi(\omega)\} \in \mathcal{F}_\emptyset$. A stopping set ξ is called simple if $\xi(\omega) \in \mathcal{A}, \forall \omega$.*

Note that since $T \in \mathcal{A}$, stopping sets are trivially bounded. The intersection and under SHAPE, the union of two stopping sets is still a stopping set.

The following lemma is an extension of Theorem 3.3.8 of [5].

Lemma 2.7 *Let (X_A) be a $I\!P$-a.s. outer-continuous martingale and ξ a stopping set. Suppose that for any ξ, X_ξ is uniformly integrable. Then $(X_{\xi \cap A})$ is a strong martingale.*

Proof: Assume that X is a strong martingale and ξ is stopping set taking on a finite number of configurations. Then, given any $C \in \mathcal{C}$, we can write

$$X_{\xi \cap C} = \sum_{i=1}^k X_{C_i \cap C} \mathbf{1}_{F_i}$$

where $F_i \in \mathcal{G}_{C_i} \subseteq \mathcal{G}^*_{C_i}$ for each i. Therefore, since $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$ is decreasing,

$$\mathbb{E}[X_{\xi\cap C}|\mathcal{G}^*_C] = \sum_{i=1}^k \mathbb{E}[X_{C_i\cap C}\, \mathbf{1}_{F_i}|\mathcal{G}^*_C] = \sum_{i=1}^k \mathbb{E}[\mathbf{1}_{F_i}\, \mathbb{E}[X_{C_i\cap C}|\mathcal{G}^*_{C_i\cap C}]|\mathcal{G}^*_C]$$

which equals 0 by the strong martingale property. Actually, as the strong martingale is uniformly continuous and monotone outer continuous on $\mathcal{A}$, we can extend this result to any stopping set. □

Assume that ξ is an $(\mathcal{F}_A)$-stopping set. We recall that for a sake of regularity, the filtration was extended to the class $\tilde{\mathcal{A}}(u)$, countable intersections of sets in $\mathcal{A}(u)$ by $\mathcal{F}_B = \cap_n \mathcal{F}^0_{g_n(B)}$, where $\mathcal{F}^0_B = \vee_{A\in\mathcal{A}, A\subseteq B}\mathcal{F}_A$, $B \in \tilde{\mathcal{A}}(u)$.
When SHAPE is satisfied, or under other conditions, see [5], p. 31, we can define the σ-algebra

$$\mathcal{F}_\xi = \{F \in \mathcal{F} : F \cap \{\xi \subseteq B\} \in \mathcal{F}_B, \forall B \in \mathcal{A}(u)\}.$$

As said before, in the classical situation, DPP requires to divide the space between what happened before a given time t and what will happen after it. In our partially ordered case, a set that is not included in a fixed one does not necessarily contain it. For that reason, we have to deal with a "large" future.
For a fixed stopping set ξ and ω, we denote by $\mathcal{F}^{\xi(\omega)}_l$ the future of ξ. It is the σ-algebra

$$\mathcal{F}^{\xi(\omega)}_l = \sigma\{\omega_B, B \in \mathcal{A} \text{ and } B \not\subseteq \xi(\omega)\}.$$

Recall the following definitions.

Definition 2.8 *Let $B \in \mathcal{A}(u)$. A sub-collection $\{A_1, \cdots, A_k\}$ of $\mathcal{A}$ is said an extremal representation of B if $B = \cup_{j=1}^k A_j$ and given $1 \le i, j \le k \quad i \ne j \Rightarrow A_i \not\subseteq A_j$. Consider now any set $C \in \mathcal{C}$, of the form $C = A \setminus B$, $A \in \mathcal{A}, B \in \mathcal{A}(u)$.*
(a) $A \setminus \cup_{j=1}^n A_j$ is a minimal representation of C if $\cup_{j=1}^n A_j$ is an extremal representation of B and $A_i \subseteq A$ for each $1 \le i \le n$.
(b) $A \setminus \cup_{j=1}^n A_j$ is a maximal representation of C if $\cup_{j=1}^n A_j$ is an extremal representation and given any $A' \in \mathcal{A}$, $A' \cap C = \emptyset \Rightarrow A' \subseteq \cup_{j=1}^n A_j$.

Remark 2.9 *1) C possesses always a minimal representation. Moreover, we will suppose that all sets in $\mathcal{C}$ admit maximal representations. This assumption is however almost always satisfies as was pointed out in [8].*
2)Suppose that $C = A \setminus \cup_{j=1}^n A_j$ is a maximal representation. Then, it is not difficult to check that one has the following σ-algebras equality $\mathcal{G}^*_C = \mathcal{F}_{\cup_{j=1}^n A_j}$.

Assume that ξ is an $(\mathcal{F}_A)$-stopping set. Consider a regular conditional probability distribution of P given $\mathcal{F}_\xi$, denoted by Q_ω (we omit the index ξ for simplicity). It satisfies
(i) For each $\Delta \in \mathcal{F}$, $Q_\omega(\Delta)$ is $\mathcal{F}_\xi$-measurable as a function of ω
(ii) For every $\Delta \in \mathcal{F}$ and $\Gamma \in \mathcal{F}_\xi$, $P(\Delta \cap \Gamma) = \mathbb{E}^P(\mathbb{1}_\Delta(\cdot)Q_\cdot(\Gamma))$.
We intend now to show the following proposition, parallel to Theorem 1.2.10 from [9], which is fundamental for the DPP.

Proposition 2.10 *Let ξ be a stopping set and consider Q_ω a regular conditional probability distribution of $I\!P$ given $\mathcal{F}_\xi$. Let M be an adapted and $I\!P$-a.s. monotone-outer continuous process, such that M_A is integrable for any $A \in \mathcal{A}$. There is equivalence between the two following statements*
(i) $(M_A, \mathcal{F}_A, I\!P)$ is a strong martingale
(ii) $(M_{A\cap\xi}, \mathcal{F}_A, I\!P)$ is a strong martingale and there exists a $I\!P$-null set N such that $I\!E^{Q_\omega}(M_C|\mathcal{G}_C^*) = 0$ for all $\omega \notin N$ and $C \in \mathcal{C}$ such that $C \cap \xi(\omega) = \emptyset$.

Remark 2.11 *The assertion (ii)in the previous Proposition may be interpreted as: 'M is a strong martingale after $\xi(\omega)$'.*

Before proving the Proposition, a technical lemma is necessary.

Lemma 2.12 *The two following properties are equivalent*
(i) $\forall C \in \mathcal{C}, \forall \omega, C \cap \xi(\omega) = \emptyset, I\!E^{Q_\omega}(M_C/\mathcal{G}_C^*) = 0$.
(ii)$\forall C \in \mathcal{C}, I\!E^{Q_\omega}(M_C - M_{C\cap\xi(\omega)}) = 0$ a.s..

Proof: $(ii) \Rightarrow (i)$: Let us consider ω and $C \in \mathcal{C}$ such that $C \cap \xi(\omega) = \emptyset$. Then, $M_{C\cap\xi(\omega)} = M_\emptyset = 0$, Q_ω-a.s. Therefore, $I\!E^{Q_\omega}(M_C - M_{C\cap\xi(\omega)}) = I\!E^{Q_\omega}(M_\emptyset) = 0$.
$(i) \Rightarrow (ii)$: Let $C \in \mathcal{C}$. One has $M_C - M_{C\cap\xi(\omega)} = M_{C\cap\xi(\omega)^c}$. But $C\cap\xi(\omega)^c = \bigcap_{i=1}^n (C\cap\xi_i(\omega)^c)$. Therefore, $C' = C\cap\xi(\omega)^c \in \mathcal{C}$. Moreover, $C'\cap\xi(\omega) = \emptyset$. Hence, applying (i), one obtains $I\!E^{Q_\omega}(M_{C'}/\mathcal{G}_{C'}^*) = 0$. Now, as $C' \subseteq C$, $\mathcal{G}_C^* \subseteq \mathcal{G}_{C'}^*$, and so $I\!E^{Q_\omega}(M_{C'}/\mathcal{G}_C^*) = 0$. □

Proof of Proposition 2.10:
$(ii) \Rightarrow (i)$: Let $C \in \mathcal{C}$. One has to show that for any $F \in \mathcal{G}_C^*$, $I\!E(M_C 1\!\!1_F) = 0$.

$$I\!E(M_C 1\!\!1_F) = I\!E(I\!E^{Q_\omega}(M_C 1\!\!1_F)) = I\!E(I\!E^{Q_\omega}(I\!E^{Q_\omega}(M_C/\mathcal{G}_C^*)1\!\!1_F)).$$

Applying Lemma 2.12, the previous expression is equal to

$$I\!E(I\!E^{Q_\omega}(I\!E^{Q_\omega}(M_{C\cap\xi(\omega)}/\mathcal{G}_C^*)1\!\!1_F)) =$$

$$I\!E(I\!E^{Q_\omega}(M_{C\cap\xi(\omega)}1\!\!1_F)) = I\!E(M_{C\cap\xi}1\!\!1_F) = 0$$

the last equality resulting from the fact that $(M_{A\cap\xi})$ is a strong martingale.
$(i) \Rightarrow (ii)$:
First of all, $(M_{A\cap\xi})$ is a strong martingale due to Lemma 2.7. Consider now $C \in \mathcal{C}$. It admits a maximal representation (Definition 2.8):$C = A \setminus \cup_{j=1}^n A_j$ where $A, A_1, \dots, A_n$ are elements of $\mathcal{A}$. Let $B \in \mathcal{F}_\xi$ and $F \in \mathcal{G}_C^*$. On one hand, $B \cap \{\xi \subseteq \cup_{j=1}^n A_j\} \in \mathcal{F}_\xi$. It is also, as $B \in \mathcal{F}_\xi$, an element of $\mathcal{F}_{\cup_{j=1}^n A_j} = \mathcal{G}_C^*$ (the last equality resulting from Remark 2.9).
Therefore,

$$I\!E(I\!E^{Q_\omega}(M_C 1\!\!1_F)1\!\!1_{B\cap\{\xi\subset\cup_{j=1}^n A_j\}}) =$$

$$= I\!E(M_C 1\!\!1_F 1\!\!1_{B\cap\{\xi\subset\cup_{j=1}^n A_j\}}) =$$

$$= I\!E(I\!E(M_C/\mathcal{G}_C^*)1\!\!1_F 1\!\!1_{B\cap\{\xi\subset\cup_{j=1}^n A_j\}}) = 0$$

as (M_A) is a strong martingale.
Therefore, $\mathbb{E}^{Q_\omega}(M_C 1\!\!1_F) = 0$ a.s. for ω's such that $\xi(\omega) \subset \cup_{j=1}^n A_j$ i.e. $\xi(\omega) \cap C = \emptyset$.
As $\mathcal{F}_{\cup_{j=1}^n} = \mathcal{G}_C^*$ is countably generated, one deduces the existence of a P-null set N_C such that if $\omega \notin N_C$ and $C \cap \xi(\omega) = \emptyset$, $\mathbb{E}^{Q_\omega}(M_C/\mathcal{G}_C^*) = 0$, Q_ω-a.s.
One may therefore find a P-null set N such that if $\omega \notin N, \forall C \in \cup_{n\geq 0}\mathcal{C}_n$, $C \cap \xi(\omega) = \emptyset$, $\mathbb{E}^{Q_\omega}(M_C/\mathcal{G}_C^*) = 0$. As in [5] p.63, we use a reverse martingale convergence result to show the uniform integrability of the martingale, which in addition to its monotone outer-continuity property allows to extend the previous result to every $C \in \mathcal{C}$. This concludes the proof. □

We now introduce canonical spaces for the problem. Let $C(\mathcal{A}, \mathbb{R})$ be the space of monotone continuous functions from $\mathcal{A}$ into $\mathbb{R}$. We denote by (x_A) a canonical process and we endow the with the canonical filtration (χ_A), that is the filtration generated by x.
Let V be the class of adapted controls indexed by $\mathcal{A}$ and with values in $\mathcal{U}$. It is endowed with the filtration $(\mathcal{V}_A)$ and we denote by (u_A) a canonical process. From now on, we set $\Omega = C(\mathcal{A}, \mathbb{R}) \times V$ and the space $(\Omega, \mathcal{F})$ is the canonical one on which we shall systematically work. A canonical element of Ω is denoted by $\omega = (x, u)$. We endow this space with the canonical filtration $\mathcal{F}_A = \chi_A \times \mathcal{V}_A$. This filtration is countably generated and is supposed to be outer-continuous. We denote by $\mathcal{F}_l^A = \sigma\{\omega_B, B \in \mathcal{A} \text{ and } B \not\subset A\}$ the large future of A.
We have the following result that extends to our case Theorem 6.1.2 in [9].

Proposition 2.13 *Let P be a probability measure on $(\Omega, \mathcal{F})$. Let ξ be a stopping set and consider Q_ω a regular conditional probability distribution of $\mathbb{P}$ given $\mathcal{F}_\xi$. There exists a unique probability, denoted by $P \otimes_\xi Q$ such that $P \otimes_\xi Q$ equals P on $(\Omega, \mathcal{F}_\xi)$ and $\delta_\omega \otimes_{\xi(\omega)} Q_\omega$ is a regular conditional probability distribution of $P \otimes_\xi Q$ given $\mathcal{F}_\xi$. That is,*
1. $P \otimes_\xi Q(\Gamma) = P(\Gamma), \forall \Gamma \in \mathcal{F}_\xi$,
2. $\forall A \in \mathcal{F}, P \otimes_\xi Q(A/\mathcal{F}_\xi) = Q_\omega^\xi(A)$ *a.s.*

Proof. The uniqueness is obvious. For the existence, we first check that $\omega \to \delta_\omega \otimes_{\xi(\omega)} Q_\omega^\xi(\tilde{\Gamma})$ is $\mathcal{F}_\xi$-measurable for all $\tilde{\Gamma}$ in $\mathcal{F}$. Due to Proposition 7.2.2 of [5], it is sufficient to consider sets $\tilde{\Gamma}$ of the form

$$\tilde{\Gamma} = (\omega_{A_1} \in \Gamma_1, \ldots, \omega_{A_n} \in \Gamma_n)$$

where $\Gamma_1, \ldots, \Gamma_n$ are borelian sets in $\mathbb{R}$.
For any $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, set

$$\tilde{\Xi}_{i_1,\ldots,i_k} = \left\{\xi \subset A_{i_j}, j = 1, 2, \ldots, k; \xi \not\subset A_j, j \notin \{i_1, \ldots, i_k\}\right\}.$$

These are disjoint $\mathcal{F}_\xi$ sets. Therefore,

$$\tilde{\Gamma} = \bigcup_{\{i_1,\ldots,i_k\}\subset\{1,2,\ldots,n\}} \tilde{\Gamma} \cap \tilde{\Xi}_{i_1,\ldots,i_k}$$

and so

$$\delta_\omega \otimes_{\xi(\omega)} Q_\omega^\xi(\tilde{\Gamma}) = \sum_{\{i_1,\ldots,i_k\}\subset\{1,2,\ldots,n\}} 1\!\!1_{\tilde{\Xi}_{\{i_1,\ldots,i_k\}}}(\omega) Q_\omega^\xi(\tilde{\Gamma})$$

which is clearly $\mathcal{F}_\xi$ measurable.
Consider $R(A) = E^{I\!P}(\delta_\cdot \otimes_{\xi(\cdot)} Q^\xi_\cdot(A))$ or more precisely, $R(A \cap B) = \int_B \delta_\omega \otimes_{\xi(\omega)} Q^\xi_\omega(A) I\!P(d\omega)$.
It is easy to see that R satisfies the properties required for $P \otimes_\xi Q$. □

3 Control Problem

Let us now introduce the optimization problem we study. We deal with a set-indexed stochastic process X whose motion evolves according to the following equation

$$X^U_A = \int_{\mathcal{A}} 1\!\!1_{[\emptyset, A]}(B) b(B, U_B)\lambda(dB) + W_A$$

where λ is a non-negative measure on $\mathcal{A}$, with $\lambda_{\emptyset'} = 0$ and (W_A) is a set-indexed continuous Brownian motion, with variance measure λ. (U_A) is an adapted process with values in $\mathcal{U}$, separable metric space. Our purpose is to study the following optimization problem, when the underlying processes are indexed by sets

$$\min_{U \in \mathcal{U}} J(U)$$

where the expected cost is given by

$$J(U) = I\!E \int_{\mathcal{A}} \ell(B, X^U_B, U_B) 1\!\!1_{[\emptyset, T]}(B)\lambda(dB) + h(T, X^U_T).$$

We intend to extend the dynamic programming result to the context of set-indexed processes. The main difficulty is obviously connected with the lack of total order.

Definition 3.1 *A control rule with initial condition $(A, x) \in \mathcal{A}(u) \times I\!R$ is a probability measure R on $(\Omega, \mathcal{F})$, such that*

1. $R(x_B = x, \forall B \subseteq A) = 1$
2. $(x_B - x - \int_{\mathcal{A}} b(I, U_I) 1\!\!1_{[A,B]}(I)\lambda(dI))$ *is a Brownian motion "after" A.*

Remark 3.2 *1. "after" A means that the strong martingale property holds for any set C such that $C \cap A = \emptyset$.*
2. The first point justifies our consideration of the large future. The martingale is in fact initiated at $\emptyset'$, and we may take into account any set in the class $\mathcal{A}$. Suppose now that B is neither included in A nor containing it. We can even though check the martingale property since under (CI), see [5] one has, $I\!E(X_B|\mathcal{F}_A) = X_{A\cap B} = x$.

In order to be able to get the desired stability results on control rules, we have to make the following assumption on the model.

Assumption 3.3 *Let $A \subset B$ two elements in $\mathcal{A}$ (in particular $A \neq B$). Consider a sequence (B_n) of elements in $\mathcal{A}$ such that $B_n \downarrow B$. Then, there is $N > 0$ such that for $n \geq N, B_n \subset A$.*

Example 3.4 *The following example is developed by Slonowsky and satisfies the previous assumption. Let T be a rooted tree with a finite number of edges. Embed T in $\mathbb{R}^2$ so that T is rooted at 0 and each edge is a line segment. Define $\mathcal{A}$ to be the collection of all $A_t \subseteq T$ where A_t is the unique simple path in T from the origin to T. Obviously, this example can be extended to the σ-compact case in which T has a countable number of edges.*

We denote by $\mathcal{R}_{A,x}$ the collection of control rules with initial condition (A, x). Actually, if $R \in \mathcal{R}_{A,x}$ then the cost associated to R is given by

$$J_{A,x}(R) = R(\int_{\mathcal{A}} \ell(B, X_B^U, U_B) \mathbb{1}_{[A,T]}(B)\lambda(dB) + h(T, X_T^U)).$$

Let $V(A, x) = \inf_{R \in \mathcal{R}(A,x)} J_{A,x}(R)$ be the value function of the problem. We state here our main result which is the set-indexed version of the PDD.

Theorem 3.5 $\forall A \subset \xi$ *stopping set,*

$$V(A, x) = \inf_{R \in \mathcal{R}_{A,x}} R[\int \mathbb{1}_{[A,\xi]} \ell(B, x_B, q_B)\lambda(dB) + V(\xi, x_\xi)].$$

3.1 Closure under conditioning

Consider an initial set A that is included in ξ.
If $R \in \mathcal{R}_{A,x}$ then $(x_B - x - \int_{\mathcal{A}} b(I, U_I)\mathbb{1}_{[A,B]}(I)\lambda(dI))$ is a Brownian motion after A. What we intend to show is that conditionally to $\mathcal{F}_\xi$ the same process started at ξ is still a Brownian motion after ξ.
Define for any $A \in \mathcal{A}(u)$ and $\overline{\omega} \in \Omega$, $\Theta_{A,\overline{\omega}} : \Omega \mapsto \Omega$ by

$$\Theta_{A,\overline{\omega}}(\omega)_B = \begin{cases} \overline{\omega}_B & \text{if} \quad B \subseteq A \\ (x_B(\omega) - x_A(\omega) + x_A(\overline{\omega}), u_B(\omega)) & \text{if} \quad B \not\subset A \end{cases}$$

Lemma 3.6 *If P is a probability on $(\Omega, \mathcal{F}_l^A)$, $A \in \mathcal{A}$, and $\overline{\omega} \in \Omega$, then there exists a unique probability measure, denoted by $\delta_{\overline{\omega}} \otimes_A P$, on $(\Omega, \mathcal{F})$ such that*

$$\delta_{\overline{\omega}} \otimes_A P(\omega : \omega_B = \overline{\omega}_B, B \subseteq A) = 1 \tag{1}$$

$$\delta_{\overline{\omega}} \otimes_A P(\Gamma) = P(\Theta_{A,\overline{\omega}}(\Gamma)), \forall \Gamma \in \mathcal{F}_l^A \tag{2}$$

Lemma 3.7 *Assume that 3.3 holds. For any fixed $A \in \mathcal{A}$ and $\overline{\omega} \in \Omega$ the function Θ is monotone inner and monotone outer continuous on $\mathcal{A}$.*

Proof. Let B be any set in $\mathcal{A}$ and let (B_n) be an increasing sequence in $\mathcal{A}$ such that $B_n \subseteq B$ and $B_n \to B$. If $B \subseteq A$ then $B_n \subseteq A$ and $\overline{x}_{B_n} \to \overline{x}_B$. If $B \not\subset A$ then since the interval $[\emptyset, A]$ is closed for the Hausdorff metric, there exists an N such that for any $n \geq n_0$ $B_n \not\subset A$. Therefore $x_{B_n}(\omega) - x_A(\omega) + x_A(\overline{\omega})$ converges to $x_B(\omega) - x_A(\omega) + x_A(\overline{\omega})$ and the function is monotone inner continuous. Actually, let (B_n) be a decreasing sequence in $\mathcal{A}$ such that $B \subseteq B_n$ and $B_n \to B$. If $B \subseteq A$ and $B_n \subseteq A$ for some n then we use the monotone outer continuity of $\overline{X}$ but if $B_n \not\subset A$ then $\Theta_{B_n} = (x_{B_n}(\omega) - x_A(\omega) + x_A(\overline{\omega}), u_{B_n}(\omega))$ while $\Theta_B = (\overline{x}_B, \overline{u}_B)$. Here we use the assumption 3.3 in order to conclude th outer-continuity of the function. □

Proof of Lemma 3.6. We set $[\emptyset, A] = \{B \in \mathcal{A}, B \subseteq A\}$, $[A,T] = \{B \in \mathcal{A}, B \subseteq T, B \not\subset A\}$, for any $A \in \mathcal{A}$. Let $C([\emptyset, A], I\!R)$ (resp. $C([A,T], I\!R)$) be the subset of $C(\mathcal{A}, I\!R)$ that consists of functions defined on $[\emptyset, A]$ (resp. $[A,T]$) and $V_{[\emptyset,A]}$ (resp. $V_{[A,T]}$) is the subset of V that consists of controls defined on $[\emptyset, A]$ (resp. $[A,T]$).
Let $X_0 \equiv C([\emptyset, A], I\!R) \times V_{[\emptyset,A]}$ and $X_1 \equiv C([A,T], I\!R) \times V_{[A,T]}$ and $\tilde{X} = X_0 \times X$. Define $\Phi_0 : \Omega \mapsto X_0$ and $\Phi : \Omega \mapsto X$ by $\Phi_0(\omega)_B = (x_B, u_B), B \subseteq A$, $\Phi(\omega)_B = (x_B, u_B), B \not\subset A$.Now define the map $\Psi : \tilde{X} \mapsto \Omega$ by $\Psi(\omega_1, \omega_2) = \Theta_{A,\tau_0(\omega_1)}(\tau(\omega_2))$ where $\tau_0 : X_0 \mapsto \Omega, \tau : X \mapsto \Omega$ are defined for $\tilde{\omega} \in X_0$ and $\omega \in X$ by

$$\tau_0(\tilde{\omega})_B = \begin{cases} (\tilde{x}_B, \tilde{u}_B) & \text{if} \quad B \subseteq A \\ (\tilde{x}_A, \tilde{u}_A) & \text{if} \quad B \not\subset A \end{cases}$$

$$\tau(\omega)_B = \begin{cases} (x_A, u_A) & \text{if} \quad B \subseteq A \\ (x_B, u_B) & \text{if} \quad B \not\subset A. \end{cases}$$

Let $\tilde{P}$ be a probability on $\tilde{X}$ defined by $\tilde{P} = \delta_{\overline{\omega}} \circ \Phi_0^{-1} \times P \circ \Phi^{-1}$. Set $\overline{P} = \tilde{P} \circ \Psi^{-1}$. $\overline{P}$ satisfy the required conditions (1) and (2). Indeed,

$$\begin{aligned} \overline{P}(\omega : \omega_B = \overline{\omega}_B, B \subseteq A) &= \tilde{P}((\omega_1, \omega_2) : \Theta_{A,\tau_0(\omega_1)}(\tau(\omega_2))_B = \overline{\omega}_B, B \subseteq A) \\ &= \tilde{P}((\omega_1, \omega_2) : \tau_0(\omega_1)_B = \overline{\omega}_B, B \subseteq A) \\ &= \delta_{\overline{\omega}}(\omega_1 : \tau_0 \circ \Phi_0(\omega_1)_B = \overline{\omega}_B, B \subseteq A) = 1 \end{aligned}$$

since $\tau_0 \circ \Phi_0(\omega_1)_B = \overline{\omega}_B, B \subseteq A$.
Moreover, for $\Gamma \in \mathcal{F}_l^A$,

$$\overline{P}(\Gamma) = \tilde{P}((\omega_1, \omega_2) : \Psi(\omega_1, \omega_2) \in \Gamma) = \tilde{P}((\omega_1, \omega_2) : \Theta_{A,\tau_0(\omega_1)}(\tau(\omega_2)) \in \Gamma)$$

$$= \tilde{P}((\omega_1, \omega_2) : \tau(\omega_2) \in \Theta^{-1}_{A,\tau_0(\omega_1)}(\Gamma)) = \int_{X_0} P \circ \Phi^{-1}(\omega_2 : \tau(\omega_2)) \in \Theta^{-1}_{A,\tau_0(\omega_1)}(\Gamma)) \delta_{\overline{\omega}} \circ \Phi_0^{-1}(d\omega_1)$$

$$= \int_{X_0} P(\omega : \tau \circ \Phi(\omega) \in \Theta^{-1}_{A,\tau_0(\omega_1)}(\Gamma)) \delta_{\Phi_0(\overline{\omega})}(d\omega_1) = P(\omega : \tau \circ \Phi(\omega) \in \Theta^{-1}_{A,\tau_0(\Phi_0(\overline{\omega}))}(\Gamma))$$

$$= P(\Theta^{-1}_{A,\overline{\omega}}(\Gamma))$$

since $\tau \circ \Phi(\omega)_B = \omega_B$, for $B \not\subset A$. We set $\delta_{\overline{\omega}} \otimes_A P = \overline{P}$. □

Remark 3.8 *Let $P \in \mathcal{R}_{A,x}$, there exists a P-null set N_0 such that if $\overline{\omega} \notin N_0$, $\Gamma \in \mathcal{F}_l^A$, then $P(\Gamma) = P(\Theta^{-1}_{A,\overline{\omega}}(\Gamma))$.*

Assume that ξ is an $(\mathcal{F}_A)$-stopping set. A ξ-transition probability is a family $\{Q_\omega, \omega \in \Omega\}$ of probability measures on $(\Omega, \mathcal{F})$ such that $\omega \mapsto Q_\omega(A)$ is $\mathcal{F}_\xi$-measurable, $\forall A \in \mathcal{F}$. In particular, a regular conditional probability with respect to $\mathcal{F}_\xi$ is a ξ-transition probability. Take $\omega \in \Omega$. By Lemma 3.6, there exists a unique probability measure on Ω, $\delta_\omega \otimes_\xi Q_\omega$ such that

$$\delta_\omega \otimes_A Q_\omega(\tilde{\omega} : \tilde{\omega}_B = \omega_B, B \subseteq \xi(\omega)) = 1$$
$$\delta_\omega \otimes_\xi Q_\omega(\Gamma) = Q_\omega(\Theta_{\xi,\omega}(\Gamma)), \forall \Gamma \in \mathcal{F}_l^{\xi(\omega)}$$

We write $\delta_\omega \otimes_\xi Q_\omega = Q^\xi_\omega$.
¿From now on, U^0 is a fixed arbitrary element of $\mathcal{U}$. In the particular case where $\omega = (x(\xi(\omega)), U^0)$, we write $\delta_\omega \otimes_\xi Q_\omega = \overset{\circ}{Q}{}^\xi_\omega$.
Let

$$\Gamma_A(u) = \int_{\mathcal{A}} \ell(B, X^U_B, U_B) 1\!\!1_{[A,T]}(B)\lambda(dB) + h(T, X^U_T).$$

The following equalities hold. $Q_\omega(\Gamma_{\xi(\omega)}) = Q^\xi_\omega(\Gamma_{\xi(\omega)}) = \overset{\circ}{Q}{}^\xi_\omega(\Gamma_{\xi(\omega)})$.

Lemma 3.9 *Suppose Shape. Assume $P \in \mathcal{R}_{A,x}$, ξ is an $(\mathcal{F}_A)$-stopping set, $\overline{\omega} \in \Omega$. Set $W_B = (x_B - x - \int_{\mathcal{A}} b(I, q_I) 1\!\!1_{[A,B]}(I)\lambda(dI)$. $(W_B, \mathcal{F}_B, I\!\!P \circ \Theta^{-1}_{\xi(\overline{\omega}),\overline{\omega}})$ is a Brownian motion, "after" $\xi(\overline{\omega})$.*

<u>*Proof.*</u>
Observe that

$$x_B(\Theta_{\xi(\bar{\omega}),\bar{\omega}}) = \begin{cases} x_B(\bar{\omega}), B \subset \xi(\bar{\omega})) \\ x_B(\omega) - x_{\xi(\bar{\omega})}(\omega) + x_{\xi(\bar{\omega})}(\bar{\omega}), B \not\subset \xi(\bar{\omega}) \end{cases}$$

We want to prove that W_B is a strong martingale after $\xi(\bar{\omega})$ under $P \circ \Theta^{-1}_{\xi(\bar{\omega}),\bar{\omega}}$ which is to say that for any $C \in \mathcal{C}$ such that $C \cap \xi(\bar{\omega}) = \emptyset$ and for any $F \in \mathcal{G}^*_C$,

$$0 = E^{P\circ\Theta^{-1}_{\xi(\bar{\omega}),\bar{\omega}}}(1\!\!1_F.W_C) = E(1\!\!1_F(\Theta_{\xi(\bar{\omega}),\bar{\omega}})W_C(\Theta_{\xi(\bar{\omega}),\bar{\omega}}).$$

We may limit ourselves to take $F \in \mathcal{F}_B, B \cap C = \emptyset$ as $\mathcal{G}^*_C = \bigvee_{B \in \mathcal{A}(u), B\cap C = \emptyset}$ and $\bigcup_{B \in \mathcal{A}(u), B\cap C = \emptyset}$ form a π-system.
Write $\xi(\bar{\omega}) = \bigcup_{j=1}^k \xi_j(\bar{\omega})$ and $C = A \setminus \bigcup_{i=1}^n A_i$ (a maximal representation).
Then,

$$C \cap \xi(\bar{\omega}) = \emptyset \iff \forall 1 \le j \le k, \xi_j(\bar{\omega}) \cap C = \emptyset$$
$$\iff \forall 1 \le j \le k, \xi_j(\bar{\omega}) \subset \bigcup_{i=1}^n A_i \text{ by maximality}$$
$$\iff \xi(\bar{\omega}) \subset \bigcup_{i=1}^n A_i.$$

Now, as $C = A \setminus \bigcup_{i=1}^n (A_i \cap A)$.

$$W_C = W_A - W_{\bigcup_{i=1}^n (A_i \cap A)}$$

$$= x_A - x_{\bigcup_{i=1}^n (A_i \cap A)} - \int_{\mathcal{A}} b(I, U_I) 1\!\!1_{[\emptyset, A]}(I) \lambda(dI) + \int_{\mathcal{A}} b(I, U_I) 1\!\!1_{[\emptyset, \bigcup_{i=1}^n (A_i \cap A)]}(I) \lambda(dI).$$

Observe that $1\!\!1_{[\emptyset, A]} - 1\!\!1_{[\emptyset, \bigcup_{i=1}^n (A_i \cap A)]} = 0$ or 1 as it is impossible to have together $I \subset \bigcup_{i=1}^n (A_i \cap A)$ and $I \not\subset A$. The same is true for $1\!\!1_{[\xi(\bar{\omega}), A]} - 1\!\!1_{[\xi(\bar{\omega}), \bigcup_{i=1}^n (A_i \cap A)]}$. Now,

$$1\!\!1_{[\xi(\bar{\omega}), A]} - 1\!\!1_{[\xi(\bar{\omega}), \bigcup_{i=1}^n (A_i \cap A)]} = 1$$

$$\iff I \not\subset \xi(\bar{\omega}), I \subset A, I \not\subset \bigcup_{i=1}^n (A_i \cap A)$$

$$\iff I \not\subset \xi(\bar{\omega}), I \subset A, I \not\subset \bigcup_{i=1}^n A_i$$

$$\iff I \subset A, I \not\subset \bigcup_{i=1}^n A_i \text{as} \xi(\bar{\omega}) \subset \bigcup_{i=1}^n A_i$$

$$\iff I \subset A, I \not\subset (A_j \cap A), \forall 1 \leq j \leq n \text{ (by SHAPE)}$$

$$\iff 1\!\!1_{[\emptyset, A]}(I) - 1\!\!1_{[\emptyset, \bigcup_{j=1}^n (A_j \cap A)]}(I) = 1.$$

Therefore,

$$W_C = x_C - \int_{\mathcal{A}} b(I, U_I) 1\!\!1_{[\xi(\bar{\omega}), A]}(I) \lambda(dI) + \int_{\mathcal{A}} b(I, U_I) 1\!\!1_{[\xi(\bar{\omega}), \bigcup_{j=1}^n (A_j \cap A)]} \lambda(dI).$$

One has $x_C(\Theta_{\xi(\bar{\omega}), \bar{\omega}}) = x_A(\Theta_{\xi(\bar{\omega}), \bar{\omega}}) - x_{\bigcup_{i=1}^n A_i}(\Theta_{\xi(\bar{\omega}), \bar{\omega}}) = x_A(\omega) - x_{\bigcup_{i=1}^n A_i}(\omega)$ as $A \not\subset \xi(\bar{\omega})$ (otherwise $A \subset \bigcup_{i=1}^n A_i$) and $\bigcup_{i=1}^n A_i \not\subset \xi(\bar{\omega})$).

Therefore,

$$\begin{aligned} E^{P \circ \Theta^{-1}_{\xi(\bar{\omega}), \bar{\omega}}}(1\!\!1_F . W_C) &= E(1\!\!1_F(\Theta_{\xi(\bar{\omega}), \bar{\omega}}) W_C(\Theta_{\xi(\bar{\omega}), \bar{\omega}})) \\ &= E(1\!\!1_F(\Theta_{\xi(\bar{\omega}), \bar{\omega}}) W_C(\omega)). \end{aligned} \tag{3}$$

Take $D \subset B$. One has

$$(\Theta_{\xi(\bar{\omega}), \bar{\omega}})_D = \bar{\omega}_D 1\!\!1_{D \subset \xi(\bar{\omega})} + (x_D(\omega) - x_{\xi(\bar{\omega})}(\omega) + x_{\xi(\bar{\omega})}(\bar{\omega}), u_D(\omega)) 1\!\!1_{D \not\subset \xi(\bar{\omega})}$$

and so $(\Theta_{\xi(\bar{\omega}), \bar{\omega}})_D$ is $\mathcal{F}_D$-measurable so $1\!\!1_F(\Theta_{\xi(\bar{\omega}), \bar{\omega}})$ is $\mathcal{F}_B$-measurable. Therefore, the quantity in (3) is 0 and we get the desired martingale property.

Following the same arguments, we get that $E^{P \circ \Theta^{-1}_{\xi(\bar{\omega}), \bar{\omega}}}([W_C]^2 | \mathcal{G}^*_C) = \Lambda_C$ by definition of the process X and since for any $F \in \mathcal{G}^*_C$, $E^{P \circ \Theta^{-1}_{\xi(\bar{\omega}), \bar{\omega}}}(1\!\!1_F [W_C]^2) = I\!\!E(1\!\!1_F(\Theta_{\xi(\bar{\omega}), \bar{\omega}})[W_C(\omega)]^2)$. Therefore, the *-quadratic variation of this strong martingale is equal to Λ, the variance measure of the Brownian motion. Accordingly, we apply the Brownian characterization obtained by Ivanoff and Merzbach and we get that this strong martingale is in fact a Brownian motion. □

Proposition 3.10 *Let $I\!\!P \in \mathcal{R}_{A,x}$ and ξ be an $(\mathcal{F}_A)$-stopping set. Denote $I\!\!P_{\xi,\omega}$ a regular conditional probability of $I\!\!P$ given $\mathcal{F}_\xi$. There exists a $I\!\!P$-null set N in $\mathcal{F}_\xi$ such that for $\omega \notin N$, $(W_B, \mathcal{F}_B, I\!\!P_{\xi,\omega} \circ \Theta^{-1}_{\xi(\omega),\omega})$ is a Brownian motion, "after" $\xi(\omega)$.*

Proof. From Lemma 3.9 and Proposition 2.10. □

We may conclude that, given a stopping set, the space is separated between the past of ξ, and its large future. After ξ, a "new" problem initiated at ξ starts. This is precisely the idea of stability by conditioning.

Proposition 3.11 *(Closure under conditioning) If $I\!\!P \in \mathcal{R}_{A,x}$ and ξ is a stopping set, then there exists a $I\!\!P$-null set $N \in \mathcal{F}_\xi$ such that $\overset{\circ}{P}{}^{\xi}_{\omega} \in \mathcal{R}_{\xi(\omega),x_{\xi(\omega)}}$, where $\overset{\circ}{P}{}^{\xi}_{\omega} = \delta_{\overline{\omega}} \otimes_\xi I\!\!P_{\xi,\omega}$ for $\overline{\omega} = (x(\xi(\omega)), U^0)$.*

Proof. We know that $(W_B, \mathcal{F}_B, I\!\!P)$ is a Brownian motion after A. Then by Proposition 3.10 there exists a $I\!\!P$-null set $N \in \mathcal{F}_\xi$ such that for $\omega \notin N$, (W_B) is a Brownian motion under $I\!\!P_{\xi,\omega} \circ \Theta^{-1}_{\xi(\omega),\omega}$ after $\xi(\omega)$. As after $\xi(\omega)$, $I\!\!P_{\xi,\omega}$ and $\overset{\circ}{P}{}^{\xi}_{\omega}$ coincide, one has that $(W_B, \mathcal{F}_B, \overset{\circ}{P}{}^{\xi}_{\omega})$ is a strong martingale, that is a Brownian motion after $\xi(\omega)$. Moreover,

$$\overset{\circ}{P}{}^{\xi}_{\overline{\omega}}\,(\omega : x_A = x_{\xi(\omega)}, u_A = U^0, A \subseteq \xi(\omega)) = 1.$$

Therefore, $\overset{\circ}{P}{}^{\xi}_{\omega} \in \mathcal{R}_{\xi(\omega),x_{\xi(\omega)}}$. □

3.2 Closure under concatenation

We now prove that if we concatenate a control rule with an appropriate probability measure then we still get a control rule, associated with earlier initial conditions.

Proposition 3.12 *(Closure under concatenation) Let $I\!\!P \in \mathcal{R}_{A,x}$ and ξ is a stopping set. If Q_ω is a transition probability such that $Q_\omega \in \mathcal{R}_{\xi(\omega),x_{\xi(\omega)}}$, then $I\!\!P_\xi \otimes Q \in \mathcal{R}_{A,x}$*

Proof. We have to show that (W_B) is a Brownian motion after A, under $I\!\!P_\xi \otimes Q$. From Remark 3.8 and the fact that $Q_\omega \in \mathcal{R}_{\xi(\omega),x_{\xi(\omega)}}$ we can verify that $(W_B, \mathcal{F}_B)$ is a strong martingale after ξ. Moreover we know that $Q^\xi_\omega = \delta_\omega \otimes_\xi Q_\omega$ is the regular conditional probability distribution of $I\!\!P \otimes_\xi Q$ given $\mathcal{F}_\xi$. The proof follows from Proposition 2.10 and from the martingale characterization of the Brownian motion. □

3.3 Proof of the main theorem

Let ξ be a stopping set, and let $R \in \mathcal{R}_{A,x}$. From Proposition 3.11 we know that $\overset{\circ}{R}{}^{\xi}_{\omega} \in \mathcal{R}_{\xi(\omega),x_{\xi(\omega)}}$. So

$$\overset{\circ}{R}{}^{\xi}_{\omega}\,(\int 1\!\!1_{[\xi,T]}\ell(B, x_B, q_B)\lambda(dB) + h(T, x_T)) \geq V(\xi, x_\xi).$$

Furthermore, we add to both components $\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)$ and we compute the expectation under R. We finally get

$$R(\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)+V(\xi,x_\xi)]\leq$$

$$\leq R(\int 1\!\!1_{[A,T]}\ell(B,x_B,q_B)\lambda(dB)+h(T,x_T))=J(R).$$

We then take the infimum over $\mathcal{R}_{A,x}$,

$$\inf_{R\in\mathcal{R}_{A,x}} R[\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)+V(\xi,x_\xi)]\leq V(A,x).$$

For the converse inequality, we use the following result, connected with the measurable selection theorem.

Lemma 3.13 *For any $\varepsilon>0$, one can find a measurable mapping R^ε defined on $\mathcal{A}\times I\!\!R$ such that for any (A,x), $R^\varepsilon(A,x)\in\mathcal{R}_{A,x}$ and $J(R^\varepsilon(A,x))\leq V(A,x)+\varepsilon$.*

Let $R\in\mathcal{R}_{A,x}$ and ξ a stopping set. We defined the probability $I\!\!P$ on the product space $\Omega\times\Omega$, by

$$P(A\times B)=R(1\!\!1_A R^\varepsilon(\xi,x_\xi)(B)).$$

Define the process $W_B(\Theta_{\xi(\overline{\omega}),\overline{\omega}})$ as above. We denote by $\hat{R}$its law under $I\!\!P$. Then, by Propositions 3.11 and 3.12, $\hat{R}\in\mathcal{R}_{A,x}$, and is such that $\overset{\circ}{\hat{R}^\xi_\omega}=R^\varepsilon(\xi,x_\xi)$, and coincides with R till ξ. Accordingly, we get

$$V(A,x)\leq\hat{R}[\int 1\!\!1_{[A,T]}\ell(B,x_B,q_B)\lambda(dB)+h(T,x_T)]$$

$$=\hat{R}[\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)+\overset{\circ}{\hat{R}^\xi_\omega}[\int 1\!\!1_{[\xi,T]}\ell(B,x_B,q_B)\lambda(dB)+h(T,x_T)]]$$

$$=\hat{R}[\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)+R^\varepsilon(\xi,x_\xi)\{\int 1\!\!1_{[\xi,T]}\ell(B,x_B,q_B)\lambda(dB)+h(T,x_T)\}]$$

$$\leq\hat{R}[\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)+V(\xi,x_\xi)+\varepsilon]$$

$$=R[\int 1\!\!1_{[A,\xi]}\ell(B,x_B,q_B)\lambda(dB)+V(\xi,x_\xi)]+\varepsilon$$

And we are done since this inequality holds for any ε and any R. □

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