Scaling limit of the invasion percolation cluster on a regular tree

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Abstract

We prove existence of the scaling limit of the invasion percolation cluster (IPC) on a regular tree. The limit is a random real tree with a single end. The contour and height functions of the limit are described as certain diffusive stochastic processes.

These convergence allows us to recover and make precise certain asymptotic results for the IPC. In particular, we relate the limit of the rescaled level sets of the IPC to the local time of the scaled height function.

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1 Introduction

Invasion percolation on an infinite connected graph is a random growth model which is closely related to critical percolation, and is a prime example of self-organized criticality. It was introduced in the eighties by Wilkinson and Willemsen [18]. The relation between invasion percolation and critical percolation has been studied by many authors (see for instance [5, 12]). More recently, Angel, Goodman, den Hollander and Slade [2] have given a structural representation of the invasion percolation cluster on a regular tree, and used it to compute the scaling limits of various quantities related to the IPC such as the distribution of the number of invaded vertices at a given level of the tree.

Fixing a degree $\sigma \geq 2$ we consider $\mathcal{T} = \mathcal{T}_\sigma$: the rooted regular tree with index σ, i.e. the rooted tree where every vertex has σ children. Invasion percolation on $\mathcal{T}$ is defined as follows: Edges of $\mathcal{T}$ are assigned weights which are i.i.d. and uniform on $[0, 1]$. The invasion percolation cluster on $\mathcal{T}$, denoted IPC, is grown inductively by starting I_0 consisting of the

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root of $\mathcal{T}$. At each step I_{n+1} consists of I_n together with the edge of minimal weight in the boundary of I_n. The invasion percolation cluster IPC is the limit $\bigcup I_n$.

We consider the infinite tree IPC as a metric space endowed with the shortest path metric, and consider its scaling limit in the sense of weak limits w.r.t. the Gromov-Hausdorff topology. The limit is a random $\mathbb{R}$-tree — a topological space with a unique rectifiable simple path between any two points. A useful way to describe such $\mathbb{R}$-trees is in terms of their contour or height functions (see below). Note that the IPC is infinite, so that we take a fixed object and only change the metric when taking the scaling limit.

Theorem 1.1. *The IPC has a scaling limit w.r.t. local Gromov-Hausdorff topology, which is a random* $\mathbb{R}$*-tree.*

A key point in our study is that the contour function (as well as height function and Lukaciewicz path) of an infinite tree do not generally encode the entire tree. If the various encodings of trees are applied to infinite trees they describe only the part of the tree to the left of the leftmost infinite branch. We present two ways to overcome this difficulty. Both are based on the fact (see [2]) that the IPC has a.s. a unique infinite branch. Following Aldous we define a *sin-tree* to be an infinite one-ended tree (i.e. with a single infinite branch).

The first approach is to use the symmetry of the underlying graph $\mathcal{T}$ and observe that the infinite branch of the IPC (called the *backbone*) is independent of the metric structure of the IPC. Thus for all purposes involving only the metric structure of the IPC we may as well assume (or condition) that the backbone is the rightmost branch of $\mathcal{T}$. We denote by $\mathcal{R}$ the IPC under this condition. The various encodings of $\mathcal{R}$ encode the entire tree.

The second approach is to consider a pair of encodings, one for the part of the tree to the left of the backbone, and a second encoding the part to the right of the backbone. This is done by considering also the encoding of the reflected tree $\overline{\text{IPC}}$. The reflection of a plane tree is defined to be the same tree with the reversed order for the children of each vertex. The uniqueness of the backbone implies that together the two encodings determine the entire IPC.

In order to describe the limits we first need the process $L(t)$ which is the lower envelope of a Poisson process on $(\mathbb{R}^+)^2$. Given a Poisson process $\mathcal{P}$ in the quarter plane, $L(t)$ is defined by

$$L(t) = \inf\{y \ : \ (x, y) \in \mathcal{P} \text{ and } x \le t\}.$$

Our next results describe the scaling limits of the various encodings of the trees in terms of solutions of

$$Y_t = B_t - \int_0^t L\left(-\underline{Y}_s\right) ds. \tag{$\mathcal{E}(L)$}$$

The reason for the notation is that we will also consider solutions of equations $\mathcal{E}(L/2)$ where in the above, L is replaced by $L/2$.

We work primarily in the space $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ of continuous functions from $\mathbb{R}^+$ to itself with the topology of locally uniform convergence. We will consider three well known and closely related encodings of plane trees, namely the Lukaciewicz path, and the contour and height functions (all are defined below. The three are closely related and indeed their scaling limits are almost the same. The reason for the triplication is that the contour function is the simplest and most direct encoding of a plane tree, whereas the Lukaciewicz path turns out to be easier to deal with in practice. The height function is a middle ground.

For the IPC conditioned on the backbone being on the right, we denote its Lukaciewicz path (resp. height and contour functions) by $V_{\mathcal{R}}$ (resp. $H_{\mathcal{R}}$ and $C_{\mathcal{R}}$. It is interesting that the scaling limit depends on σ only by a multiplicative factor. We use the notation $\gamma = \frac{\sigma-1}{\sigma}$, as it will appear in many formulas.

Theorem 1.2. *We have the weak limits in $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$:*

$$\left(k^{-1}V_{\mathcal{R}}(k^2t)\right)_{t\geq 0} \to \left(\gamma^{1/2}(Y_t - \underline{Y}_t)\right)_{t\geq 0}, \tag{1}$$

$$\left(k^{-1}H_{\mathcal{R}}(k^2t)\right)_{t\geq 0} \to \left(\gamma^{-1/2}(2Y_t - 3\underline{Y}_t)\right)_{t\geq 0}, \tag{2}$$

$$\left(k^{-1}C_{\mathcal{R}}(2k^2t)\right)_{t\geq 0} \to \left(\gamma^{-1/2}(2Y_t - 3\underline{Y}_t)\right)_{t\geq 0}, \tag{3}$$

as $k \to \infty$, where $(Y_t)_{t\geq 0}$ solves $\mathcal{E}(L)$ (and is the same solution in all three limits).

[[omer: perhaps swallow a $\sqrt{\gamma}$ into Y so that the square roots vanish]]

To put this theorem into context, recall that the Lukaciewicz path of a critical Galton Watson tree with finite second moment is a simple random walk. From this it follows that the path of an infinite sequence of critical trees scales to Brownian motion. The height and contour functions of the sequence are easily expressed in terms of the Lukaciewicz path and are seen to scale to reflected Brownian motion. (cf Le Gall [13]). Duquesne and Le Gall generalized this approach in [7], and showed that the genealogical structure of a continuous-state branching process is similarly coded by a height process which can be expressed in terms of a Lévy process, and that this is also the limit of various Galton Watson trees with heavy tails.

The case of sin-trees is considered by Duquesne [6], to study the scaling limit of the range of a random walk on a regular tree. His techniques suffice for analysis of the IIC, but the IPC requires additional ideas. The key difficulty being that the Lukaciewicz path is no longer a Markov process. The scaling limit of the IIC turns out to be an illustrative special case of our results, and we will describe its scaling limit as well.

For the unconditioned IPC we define its left part to be the sub-tree consisting of the backbone and all vertices to its left. The right part is defined as the left part of the reflected IPC. We can now define V_G and V_D to be respectively the Lukaciewicz paths for the left and right parts of the IPC, and similarly define H_G, H_D, C_G, C_D.

Theorem 1.3. *We have the weak limits in $\mathcal{R}(\mathbb{R}^+, \mathbb{R})$*

$$\left(k^{-1}V_G(k^2t)\right)_{t\geq 0} \to \left(\gamma^{1/2}(Y_t - \underline{Y}_t)\right)_{t\geq 0}, \qquad \left(k^{-1}V_D(k^2t)\right)_{t\geq 0} \to \left(\gamma^{1/2}(\widetilde{Y}_t - \underline{\widetilde{Y}}_t)\right)_{t\geq 0},$$

$$\left(k^{-1}H_G(k^2t)\right)_{t\geq 0} \to \left(\gamma^{-1/2}(2Y_t - 3\underline{Y}_t)\right)_{t\geq 0}, \quad \left(k^{-1}H_D(k^2t)\right)_{t\geq 0} \to \left(\gamma^{-1/2}(2\widetilde{Y}_t - 3\underline{\widetilde{Y}}_t)\right)_{t\geq 0},$$

$$\left(k^{-1}C_G(2k^2t)\right)_{t\geq 0} \to \left(\gamma^{-1/2}(2Y_t - 3\underline{Y}_t)\right)_{t\geq 0}, \quad \left(k^{-1}C_D(2k^2t)\right)_{t\geq 0} \to \left(\gamma^{-1/2}(2\widetilde{Y}_t - 3\underline{\widetilde{Y}}_t)\right)_{t\geq 0},$$

where $(Y_t)_{t\geq 0}$ and $\left(\widetilde{Y}_t\right)_{t\geq 0}$ are solutions of $\mathcal{E}(L/2)$. Moreover, conditionally given L, Y and $\widetilde{Y}$ are independent.

1.1 Background

1.1.1 Structure of the IPC

We now give a brief overview of the IPC structure theorem from [2], which is the basis for the present work. First of all, IPC consists of a single infinite branch, called the backbone and denoted BB. The backbone is a uniformly random branch in the tree (in the natural sense). From the backbone emerge, at every height and on every edge away from the backbone, subcritical percolation clusters. This relates the IPC to the incipient infinite cluster (IIC), defined and discussed by Kesten [14] (see also [3]). The IIC consists of an infinite backbone from which emerge *critical* percolation clusters, hence it stochastically dominates the IPC.

The subcritical percolation parameter of the percolation clusters attached to the backbone of the IPC increases to the critical parameter $p_c = \sigma^{-1}$ as one moves up along the backbone. This explains why the IPC and IIC resemble each other far above the root. However, the analysis of [2] shows that the convergence of the parameter of the attached clusters is slow enough that r-point functions and other measurable quantities such as level sizes possess different scaling limits.

Now, the IPC and IIC are infinite discrete trees, and the subcritical, respectively critical, percolation clusters emerging from their respective backbones are all finite ordered trees. Thus they both contain a single infinite branch, or alternately are one-ended. Following Aldous [1] we call one-ended trees *sin-trees*.

All that remains is to describe the percolation parameter for each of the trees attached to the backbone. We will only recall part of the description here. These are given in terms of a certain Markov chain W_n with explicitly stated transition probabilities. W_n is non-increasing and satisfies $W_n \xrightarrow[n\to\infty]{} p_c = \sigma^{-1}$. We than define $\widehat{W}_n = W_n \zeta(W_n)$, where $\zeta(p)$ is the probability that the p-percolation cluster along a particular branch from the root of $\mathcal{T}$ is finite. It is the case that $\widehat{W}_n$ is non-decreasing and converges a.s. to p_c. The percolation parameter for the sub-trees attached to BB_n is $\widehat{W}_n$.

We will only be concerned with the scaling limit of $\widehat{W}_n$, which is the lower envelope process $L(t)$ defined above. To be precise, for any $\varepsilon > 0$ we have

$$
\begin{aligned}
\left(k\left(\sigma W_{[kt]} - 1\right)\right)_{t\geq\varepsilon} &\xrightarrow[k\to\infty]{} \left(L(t)\right)_{t\geq\varepsilon} \\
\left(k\left(1 - \sigma \widehat{W}_{[kt]}\right)\right)_{t\geq\varepsilon} &\xrightarrow[k\to\infty]{} \left(L(t)\right)_{t\geq\varepsilon}
\end{aligned} \tag{4}
$$

The process L_t diverges as $t \to 0$, which somewhat complicates the study of the IPC close to the root.

Note that setting $W_n \equiv p_c$ in the above description gives rise to the IIC on the one hand, while in the scaling limit L is replaced by 0. This enables us to use a common framework for both processes.

1.1.2 Finite tree encodings

For completeness we include here the definition of the various tree encodings we are concerned with. We refer to Le Gall [13] for further details in the case of finite trees and to Duquesne and Le Gall [7] in the caes of sin-trees discussed below.

A rooted plane tree θ (also ordered tree) is a tree with a description as follows. Vertices of θ belong to $\bigcup_{n\geq 0} \mathbb{N}^n$. By convention, $\mathbb{N}^0 = \emptyset$ is always a vertex of θ which is called the root. For a vertex $v \in \theta$, we let $k_v = k_v(\theta)$ be the number of children of v and whenever $k_v = k \in \mathbb{N}$, these children are denoted $v1, \dots, vk$. In particular, the ith child of the root is simply i, and if $vi \in \theta$ then $\forall 1 \leq j < i$, $vj \in \theta$ as well. Edges of θ are the edges (v, vi) whenever $vi \in \theta$. Note that the set of edges of θ are determined by the set of vertices and vice-versa, which allows us to blur the distinction between a tree and its set of vertices. The *kth generation* of a tree contains every vertex $v \in \theta \cap \mathbb{N}^k$, so that the 0'th generation consists exactly of the root. Define $\#\theta$ to be the total number of vertices in θ.

Let $(v^i)_{0\leq i<\#\theta}$ be the vertices of θ listed in lexicographic order, so that $v^0 = \emptyset$. The *Lukaciewicz path* of θ is the continuous function $\big(V_t = V_t^\theta, t \in [0, \#\theta]\big)$ defined as follows: For $n \in \{1, \dots, \#\theta\}$

$$V_n = V_n^\theta := \sum_{i=0}^{n-1} \big(k_{v^i} - 1\big),$$

and between integers V is interpolated linearly.[1]

The values V_n are also given by the following *right description of the Lukaciewicz path.* This description is simpler to visualize, though we do not know of a reference for it. For $v \in \theta$, consider the subtree $\theta^v \subset \theta$ formed by all the vertices which are smaller or equal to v in the lexicographic order. Let $n(v, \theta)$ be the number of edges connecting vertices of θ^v with vertices of $\theta \setminus \theta^v$. Then,

$$V(k) = n(v^k, \theta) - 1.$$

The reason we call this the right description is that $n(v, \theta)$ is the number of edges attached on the right side of the path from $\emptyset$ to v. It is straightforward to check that this description is consistent with other definitions.

The height function is the second encoding we wish to consider. We also define it to be a piecewise linear function[2] with $H(k)$ the height of v^k above the root. It is related to the Lukaciewicz path by

$$H(n) = \#\big\{k < n : V_k = \min\{V_k, \dots, V_n\}\big\}. \tag{5}$$

Finally, the contour function of θ is obtained by considering a walker exploring θ at constant unit speed, starting from the root at time 0, and going from left to right. Each edge is traversed twice (once on each side), so that the total time before returning to the root is $2(\#\theta - 1)$. The value $C^\theta(t)$ of the contour function at time $t \in [0, 2(\#\theta - 1)]$ is the distance between the walker and the root at time t.

It is straightforward to check that the Lukaciewicz path, height function and contour function uniquely determine — and hence represent — any finite tree θ. Figure 1 demonstrates these definitions, as they are easier to understand from a picture.

[1] in [13, 7], the Lukaciewicz path is defined as a piecewise constant, discontinuous function, but there the case when the scaling limit of this path is discontinuous is also treated. Note that only the values of $V_n, n \in \{1, \dots, \#\theta\}$ are needed to recover the tree θ. Moreover, in our case, $\sup_{t\geq 0} |V_{t+1} - V_t|$ is bounded by σ, so that the eventual scaling limit will be continuous. The advantage of our convention is that it allows us to consider locally uniform convergence of the rescaled Lukaciewicz paths in a space of continuous functions.

[2] again, in [13], the height function of a non-degenerate tree is discontinuous.

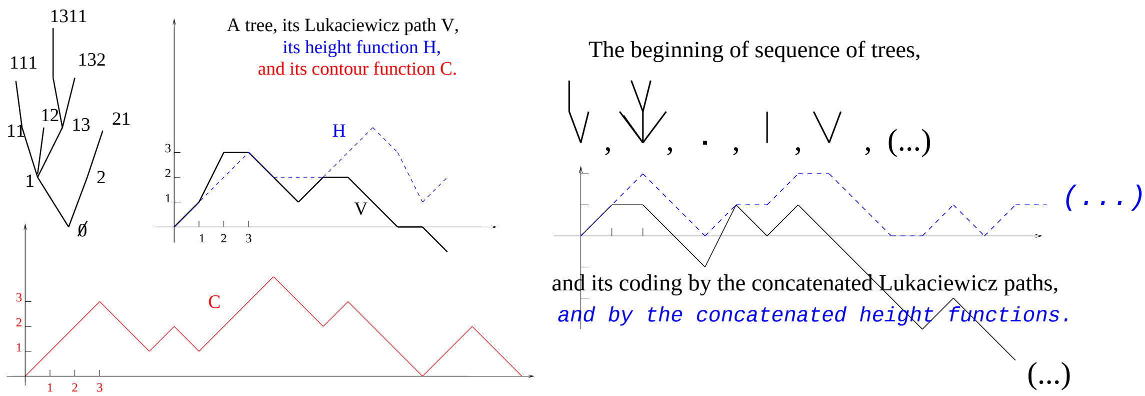

Figure 1: A finite tree and its encodings.

[[**omer: Do we use this:**]] At times it is useful to encode a sequence of finite trees by a single function. This is done by concatenating the Lukaciewicz paths or height function of the trees of the sequence. Note that when coding a sequence of trees, jumping from one tree to the next corresponds to reaching a new infimum in the Lukaciewicz path, while it corresponds to a visit to 0 in the height process.

1.1.3 Encoding sin-trees

While the definitions of Lukaciewicz path, and height and contour functions immediately extend to infinite trees number of vertices, these paths no longer encode a unique infinite tree. For example, all the trees containing the infinite branch $\{\emptyset, 1, 11, 111, \dots\}$ would have the identity function for height function, so that equal paths correspond to distinct infinite trees. In fact, the only part of an infinite tree which one can recover from the the height and contour functions is the sub-tree that lies left of the left-most infinite branch. The Lukaciewicz path encodes additionally the degrees of vertices along the left-most infinite branch.

However, if we restrict the encodings to the class of trees whose only infinite branch is the rightmost branch, then the three encodings still correspond to unique trees. In particular, observe that the IIC_D and $\mathcal{R}$ are fully encoded by their Lukaciewicz paths (as well as by their height, or contour functions). That is the reason we begin our discussion with these conditioned objects.

Not surprisingly, it is possible to encode any sin-tree, such as the IIC and IPC, by using *two* coding paths, one for the part of the tree lying to the left of the backbone, and one for the part lying to its right. More precisely, suppose $\mathcal{T}$ is a sin-tree, and BB denotes its backbone. The left tree is defined as the set of all vertices on or to the left of the backbone:

$$\mathcal{T}_G := \bigcup_{v \in \mathrm{BB}} \mathcal{T}^v = \left\{x \in \mathcal{T} : \exists v \in \mathrm{BB}, x \le v\right\}.$$

We do not define the right-tree of $\mathcal{T}$ as the set of vertices which lies right to the backbone. Rather, because of the way we defined our encodings, it is easier to work with the mirror-image $\overline{\mathcal{T}}$ of $\mathcal{T}$, defined below. We can then define

$$\mathcal{T}_D = (\overline{\mathcal{T}})_G.$$

Since a plane tree is a tree where the children of each vertex are ordered, the mirror image of a tree may be defined as the same tree but with the reverse order on the children at each vertex. [[**omer: Do we need the formal definition? it also does not fit with percolation on a regular tree.**]] The mirror image is defined vertex-wise as follows. $\overline{\emptyset} = \emptyset$. If vi is the i'th child of v then $\overline{vi}$ is $\overline{v}(k_v + 1 - i)$: the $(k_v + 1 - i)$'th child of $\overline{v}$. Note that $\overline{v}$ depends on θ (or on the degrees of the ancestors of v, to be precise). The mirror image of a tree θ is $\overline{\theta} = \{\overline{v} : v \in \theta\}$.

Obviously, only the rightmost branches of $\mathcal{T}_G, \mathcal{T}_D$ are infinite, so the Lukaciewicz paths V_G, V_D, of $\mathcal{T}_G, \mathcal{T}_D$, do encode uniquely each of these two trees (and so do the height functions H_G, H_D and the contour functions C_G, C_G). Therefore, the pair of paths (V_G, V_D) encodes $\mathcal{T}$ (and so do the pairs (H_G, H_D), (C_G, C_D)). Note that H_G, C_G are also respectively the height and contour functions of $\mathcal{T}$ itself, while H_D, C_D are respectively the height and contour functions of $\overline{\mathcal{T}}$.

[[**omer: this seems irrelevant:**]] Note that it is possible to extend the encodings of trees to trees with finitely many ends, though the reflected tree is not as helpful for trees with more than a single end.

1.2 older text

It is easy to see that the Lukaciewicz path of a Galton-Watson tree is simply a random walk (cf [13, Corollary 1.6]). The height function can be explicitly expressed in terms of this random walk as above. Moreover, the path obtained by suitably rescaling concatenated height functions of a sequence of *critical* Galton-Watson trees, converges in distribution to a reflected Brownian path (cf [13, Theorem 1.8]). [[**omer: Do we need this:**]] Duquesne and Le Gall generalized this approach in [7], and showed that the genealogical structure of a continuous-state branching process is similarly coded by a height process which can be expressed in terms of a Lévy process.

Such finite discrete trees can be coded in a number of ways (see for example [13]), the simplest being the so-called *Lukaciewicz path*, and the easiest to visualize being probably the *height* and *contour* functions. We recall in subsection **??** precise definitions of these different codings which we use to describe the IPC.

It is well-known that the Lukaciewicz path of a Galton-Watson tree simply is a part of a certain random walk path (cf Corollary 1.6 of [13]), while its height function can be explicitly expressed in terms of this random walk. Moreover, the path obtained by suitably rescaling concatenated height functions of a sequence of *critical* Galton-Watson trees, converges in distribution to a reflected Brownian path (cf theorem 1.8 of [13]). Duquesne and Le Gall generalized this approach in [7], and showed that the genealogical structure of a continuous-state branching process is similarly coded by a height process which can be expressed in terms of a Lévy process. Duquesne and Le Gall then also establish that these more general continuous trees can be obtained as the scaling limit of certain truncated sequences of Galton-Watson trees with well-chosen branching distributions (see chapter 2 of [7]).

A very natural question is thus to wonder if suitably rescaled versions of IPC, IIC also converge to certain continuous branching structures. One first issue is that we are dealing here with infinite trees. However, in our case, these trees only possess one single infinite branch. Infinite trees with a single backbone are called *sin-trees* (for *single-infinite* backbone, from the terminology of Aldous [1]). Duquesne in [6] studies for instance the scaling limit of the range of a transient random walk on a regular tree, which is as well a sin-tree. As it is shown there, and as we will remind in section 1.1.3, one can code such trees by a pair of infinite paths. In the case of IIC, IPC, we recall that the infinite backbone is uniformly distributed among all infinite paths rising from the root. Hence, we also obtain valuable information by looking at these infinite trees conditioned to have for backbone the rightmost infinite branch of $\mathcal{T}_\sigma$, that is $\{\emptyset, \sigma, \sigma\sigma, \sigma\sigma\sigma, ...\}$; and we denote the conditioned trees $(\mathrm{IIC})_/, (\mathrm{IPC})_/$. Note that, due to the homogeneity of the invasion percolation process, $(\mathrm{IPC})_/$ has the same geometry as the IPC conditioned to have for backbone its rightmost branch. It has therefore the same geometry as the unconditioned IPC, except that the edges at each backbone vertex are permuted to move the backbone to the rightmost child. Similar considerations hold for the $(\mathrm{IIC})_/$.

Using the machinery of [7], it is rather straightforward to find limits for the rescaled paths coding the IIC, and the $(\mathrm{IIC})_/$, (see paragraph 3.6 below). However, the invasion percolation cluster does not exactly fit in the context we just described. Indeed, the random path followed by the Lukaciewicz path of the $\mathrm{IPC}_/$ is not anymore a Markov process.

The main goal of the present work is to overcome this obstacle and show that a suitably rescaled version of the Lukaciewicz path coding the $(\mathrm{IPC})_/$ does converge to a non-degenerate limit (see Theorem 1.1 below). We further show that the pair of Lukaciewicz paths coding the IPC, when suitably rescaled, also converge (cf Theorem 1.3). It is then routine work to show that the rescaled versions of the height and contour functions also converge (cf Corollary 1.2 below) and that the rescaled trees converge (see paragraph 4.5). Similar results hold for the pair of paths coding the IPC, and the IPC itself (see for instance Corollary 1.4 below, and paragraph 4.5). We finally show how this convergences can be used to express asymptotic results for the volume and level estimates of the IPC in terms of the limiting height function (cf Proposition 5.1). In the case of the level sets, this allows us to recover results of [2], moreover we are able to explicit the limiting distribution (see Corollary 5.2).

1.3 Further results

[[omer: I believe everything here is contained in the results as stated above.]]

We now give give some further forms of our main results. Speci state our main results in a , which concern the convergence of rescaled versions of the paths coding the IP and $\mathcal{R}$. The

We will use he following notations. $(V^/(t), t \geq 0)$ will denote the Lukaciewicz path of the $\mathrm{IPC}_/$. Similarly, $V_G(\cdot)$ and $V_R(\cdot)$ will denote the Lukaciewicz paths of IPC_G and IPC_R. The height and contour functions will be denoted by H, C with analoguous variations. We equip the spaces of continuous paths $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+), \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^2)$ with the topology of uniform convergence on compacts. For a real-valued process $(X_t, t \geq 0)$ we let $\underline{X}_t := \inf\{X_s, s \leq t\}$.

The Poisson lower envelope process $(L_t, t > 0)$ is given by $L_t = \inf\{y : (x, y) \in \mathcal{P}, x \leq t\}$, where $\mathcal{P}$ is a Poisson process with unit density in the quarter plane $\mathbb{R}_+^2$. Finally we set

$\gamma := \frac{\sigma-1}{\sigma}$.

Our main results describe the scaling limits of the various encodings of the trees in terms of solutions of

$$Y_t = B_t - \int_0^t L\left(-\underline{Y}_s\right) ds. \tag{$\mathcal{E}(L)$}$$

We will also consider solutions of equations $\mathcal{E}(L/2)$ where in the above, L is replaced by $L/2$.

Note that Theorem 1.1 below implies Brownian scale invariance of $(Y_t, t \geq 0)$. However it is easy to verify this property directly in $\mathcal{E}(L)$, since for any $a > 0$, $(aL(at), t \geq 0)$ and $(L(t), t \geq 0)$ have the same distribution. The same argument holds for solutions of $\mathcal{E}(L/2)$.

Theorem 1.1. *We have the weak convergence in* $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$

$$\left(\frac{1}{k} V^{/}_{k^2 t}, t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\sqrt{\gamma}(Y_t - \underline{Y}_t), t \geq 0\right), \tag{6}$$

where $(Y_t, t \geq 0)$ *solves* $\mathcal{E}(L)$.

Corollary 1.2. *Weakly in* $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$

$$\left(\frac{1}{k} H^{/}_{k^2 t}, \frac{1}{k} C^{/}_{2k^2 t}, t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\frac{1}{\sqrt{\gamma}}(2Y_t - 3\underline{Y}_t), \frac{1}{\sqrt{\gamma}}(2Y_t - 3\underline{Y}_t), t \geq 0\right),$$

where $(Y_t, t \geq 0)$ *solves* $\mathcal{E}(L)$.

For the unconditioned IPC we have the following:

Theorem 1.3. *Weakly in* $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^2)$,

$$\left(\frac{1}{k} V_G(k^2 t), \frac{1}{k} V_D(k^2 t), t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\sqrt{\gamma}(\mathcal{Y}_t - \underline{\mathcal{Y}}_t), \sqrt{\gamma}(\widetilde{\mathcal{Y}}_t - \underline{\widetilde{\mathcal{Y}}}_t), t \geq 0\right), \tag{7}$$

where $(\mathcal{Y}_t, t \geq 0), (\widetilde{\mathcal{Y}}, t \geq 0)$ *are solutions of* $\mathcal{E}(L/2)$. *Moreover, conditionally given* L, $\mathcal{Y}$ *and* $\widetilde{\mathcal{Y}}$ *are independent.*

Corollary 1.4. *Weakly in* $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^2)$,

$$\left(\frac{1}{k} H_G(k^2 t), \frac{1}{k} H_D(k^2 t), t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\frac{2}{\sqrt{\gamma}}(\mathcal{Y}_t - 2\underline{\mathcal{Y}}_t), \frac{2}{\sqrt{\gamma}}(\tilde{\mathcal{Y}}_t - 2\underline{\tilde{\mathcal{Y}}}_t), t \geq 0\right),$$

where $(\mathcal{Y}_t, t \geq 0), (\tilde{\mathcal{Y}}_t, t \geq 0)$ *are as in Theorem 1.3. Similarly,*

$$\left(\frac{1}{k} C_G(2k^2 t), \frac{1}{k} C_D(2k^2 t), t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\frac{2}{\sqrt{\gamma}}(Y_t - 2\underline{Y}_t), \frac{2}{\sqrt{\gamma}}(\tilde{Y}_t - 2\underline{\tilde{Y}}_t), t \geq 0\right),$$

and these two convergences hold jointly.

The convergence of the rescaled trees is discussed in paragraph 4.5.

Finally, our arguments also provide proof of the simpler scaling limit of the IIC. The above results all hold when $V(t)$, etc. are the encodings of the IIC or $\mathrm{IIC}_{/}$, with the only difference that then, L has to be replaced by 0. In this case we obtain $Y_t = \mathcal{Y}_t = B_t$. In particular, the Lukaciewicz path of $\mathrm{IIC}_{/}$ scales to $\sqrt{\gamma}(B_t - \underline{B}_t)$, i.e. a time-changed reflected brownian motion. Moreover, note that the height functions of IIC_G, IIC_D scale to $\frac{2}{\sqrt{\gamma}}(B_t - 2\underline{B}_t)$, i.e. a time-changed Bessel(3) process. See paragraph 2.6 for more details, along with a proof in this simpler case.

1.4 Overview

Let us try to give briefly, and heuristically, some intuition of why Theorem 1.1 holds. For $t > 0$, the tree emerging from $\mathrm{BB}_{[kt]}$ is coded by the $[kt]$th excursion of V above 0. Except from its first step, this excursion has the same transition probabilities as a random walk with drift $\sigma\widehat{W}_{[kt]} - 1$, which, by the convergence (4), is about $-L(t)/k$. Additionally, by [2, Proposition 3.1], W_k is constant for long stretches of time. It is well-known (see for instance [11, Theorem 2.2.1]) that a sequence of random walks with drift c/k, suitably scaled, converges as $k \to \infty$ to a c-drifted Brownian motion. Thus we expect to find segments of drifted Brownian paths in our limit. According to the convergence (4), the drift is expressed in terms of the L-process. This is what the definition of Y expresses.

Thus, the idea when dealing with either the conditioned or the unconditioned IPC is to cut these sin-trees into pieces (which we call segments) corresponding to stretches where W is constant, and to look separately at the convergence of each piece.

In Section 3, we look at the convergence of the rescaled paths coding a sequence of such segments for well-chosen, fixed values of the W-process. In fact, we consider slightly more general settings which allows us to treat the case of the IIC as well as the various flavours of the IPC.

In Section 4, we prove Theorem 1.1. We graft segments together to form the $\mathrm{IPC}_{/}$. To deal with the fact that W is random and exploit the convergence (4), we use a coupling argument (see Subsection 4.3). We then prove that the segments fall into the family dealt with in Section 3.

Finally, because of the divergence of the L-process at the origin, we only perform the above for sub-trees above certain levels, and bound the resulting error separately.

2 Solving $\mathcal{E}(L)$

Claim 2.1. *Solutions to $\mathcal{E}(L), \mathcal{E}(L/2)$ are unique in law.*

A question of independent interest is whether the solutions to $\mathcal{E}(L)$ are a.s. pathwise unique (i.e. strong uniqueness). For our purposes uniqueness in law suffices.

Proof. We prove this claim for equation $\mathcal{E}(L)$. The proof for equation $\mathcal{E}(L/2)$ is identical.

Let Y be a solution of $\mathcal{E}(L)$. Since L is positive, $Y_t \leq B_t$. Since L is non-increasing, $\int_0^t L(-\underline{Y}_s)ds \leq \int_0^t L(-\underline{B}_s)ds$. Also, since a.s. $L_t < t^{-(1+\varepsilon)}$ for all small enough t and $\underline{b}_s > s^{1/2-\varepsilon}$ we find that almost surely $\lim_{t\to 0} \int_0^t L(-\underline{Y}_s)ds = 0$. Thus any solution of $\mathcal{E}(L)$ is continuous.

Let us now consider two solutions Y^1, Y^2 of $\mathcal{E}(B, L)$ and fix $\varepsilon > 0$. Introduce $j^\varepsilon := \inf\{t > 0 : L_t < \varepsilon^{-1}\}$ and

$$t_0^\varepsilon := \inf\{t > 0 : -\underline{B}_t > j^\varepsilon\}, \quad t_1^\varepsilon := \inf\{t > 0 : -\underline{Y^1}_t > j^\varepsilon\}, \quad t_0^\varepsilon := \inf\{t > 0 : -\underline{Y^2}_t > j^\varepsilon\}.$$

From the fact that Y^1, Y^2 are continuous we have $Y^1(t_1^\varepsilon) = Y^2(t_2^\varepsilon) = -j^\varepsilon$. Moreover, by a similar reasoning as above, we have a.s. $t_0^\varepsilon \geq t_1^\varepsilon \vee t_2^\varepsilon$, and therefore,

$$t_1^\varepsilon \vee t_2^\varepsilon \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 0. \tag{8}$$

Introduce a Brownian motion β independent of B and consider the (SDE) :

$$Z_t^\varepsilon = \beta_t - \int_0^t L\,(j^\varepsilon - \underline{Z^\varepsilon}_s)\,ds \qquad \mathcal{E}(\varepsilon, L)$$

By standard arguments $\mathcal{E}(\varepsilon, L)$ is pathwise exact.

We then define

$$Y_t^{1,\varepsilon} = \begin{cases} Y_t^1 & \text{if } t < t_1^\varepsilon \\ Y_{t_1^\varepsilon}^1 + Z_t^\varepsilon & \text{if } t \geq t_1^\varepsilon \end{cases}$$

$$Y_t^{2,\varepsilon} = \begin{cases} Y_t^2 & \text{if } t < t_2^\varepsilon \\ Y_{t_2^\varepsilon}^2 + Z_t^\varepsilon & \text{if } t \geq t_2^\varepsilon \end{cases}$$

Clearly, $Y^{1,\varepsilon}, Y^{2,\varepsilon}$ are a.s. continuous, and moreover, Y^1 and $Y^{1,\varepsilon}$ have the same distribution, and so do Y^2 and $Y^{2,\varepsilon}$. However, $(Y^{i,\varepsilon}(t_1^\varepsilon + t))_{t\geq 0}$ for $i = 1, 2$ have a.s. the same path. From this fact, the continuity of $Y^{1,\varepsilon}, Y^{2,\varepsilon}$ and (8), it follows that for any $F \in \mathcal{C}_b(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathbb{R})$

$$\left| E\left[F(Y^1)\right] - E\left[F(Y^2)\right] \right| = \left| E\left[F(Y^{1,\varepsilon})\right] - E\left[F(Y^{2,\varepsilon})\right] \right|$$

goes to 0 as ε goes to 0, which completes the proof. □

3 Scaling simple sin-trees and their segments

The goal of this section is to establish the convergence of the rescaled paths encoding suitable sequences of well-chosen segments. In order to cover the separate cases at once, we will work in a slightly more general context than might seem necessary. We first look at a sequence of particular sin-trees $\mathbf{T}^k$ for which the vertices adjacent to the backbone generate i.i.d. subcritical (or critical) Galton-Watson trees. The law of such a tree is determined by the branching law on these Galton-Watson trees and the degrees along the backbone. If the degrees along the backbone do not behave too erratically and the percolation parameter scales correctly then the sequence of Lukaciewicz paths $\mathbf{V}^k$ has a scaling limit.

The results for the IIC follow directly. We also deal here with the height and contour functions and with the two sides of the unconditioned IPC.

Finally, we determine the limits of the rescaled paths encoding a sequence of subtrees obtained by truncations at well-chosen vertices on the backbones of $\mathbf{T}^k$. These are important intermediate results in the proofs of Theorems 1.1, 1.3, and their corollaries.

3.1 Notations

Throughout this section we fix for each $k \in \mathbb{Z}_+$ a parameter $w_k \in [0, 1/\sigma]$, and denote by $(\theta_n^k)_{n\in\mathbb{Z}_+}$ a sequence of i.i.d. subcritical Galton-Watson trees with branching law $\mathrm{Bin}(\sigma, w_k)$. We also let Z_k be a sequence of random variables $(Z_{k,n})_{n\geq 0}$ taking values in $\mathbb{Z}_+$. While in our applications the $Z_{k,n}$'s are universally bounded and i.i.d. we need only weaker assumptions on them.

Definition 3.1. The (Z_k, θ^k)*-tree* is the sin-tree, defined as follows. The backbone BB is the rightmost branch. The vertex BB_n has $Z_{k,n}+1$ children, including BB_{n+1}. Letting $v_0, \dots$ be the vertices adjacent to the backbone, in lexicographic order, then v_n is identified with the root of the tree θ^k_n.

Thus the first $Z_{k,0}$ of the θ's are attached to children of BB_0, the next $Z_{k,1}$ to the children of BB_1, and so on. We will use the generic notation T^k to designate the (Z_k, θ^k)-tree, and V^k for its Lukaciewicz path.

Definition 3.2. Let T be a sin-tree whose backbone is its rightmost branch. For $i \in \mathbb{Z}_+$, let BB_i be the vertex at height i on the backbone of T. The *i-truncation* of T is the sub-tree

$$T^i := \{v \in T \ : \ v \leq \mathrm{BB}_i\}$$

We denote by $\mathbf{T}^{k,i}$ the i-truncation of $\mathbf{T}^k$, and by $\mathbf{V}^{k,i}$ its Lukaciewicz path. We further define $\tau^{(i)}$ as the time of the $(i+1)$th return to 0 of $\mathbf{V}^k$. Observe then that $\mathbf{V}^{k,i}$ coincides with $\mathbf{V}^k$ up to the time $\tau^{(i)}$, takes the value -1 at $\tau^{(i)}+1$, and terminates at that time.

It will be useful to study first a special case of such trees, where for any $k \in \mathbb{Z}_+$, Z_k is a sequence of i.i.d. binomial $\mathrm{Bin}(\sigma, w_k)$ random variables. Observe that in this case the subtrees attached to the backbone are i.i.d. Galton-Watson trees (with branching law $\mathrm{Bin}(\sigma, w_k)$). We use calligraphed letters for the various objects in this case. We denote the binomial variables $\mathcal{Z}_{k,n}$, we write $\mathcal{T}^k$ for the corresponding $(\mathcal{Z}_k, \theta^k)$-tree, $\mathcal{T}^{k,i}$ for its i-truncation, and $\mathcal{V}^k, \mathcal{V}^{k,i}$ for the Lukaciewicz paths of these trees.

In the perspective of proving our main results, we note that other special distributions for the variables $Z_{k,n}$ are of interest. When for all $k \in \mathbb{Z}_+$, $Z_{k,n}, n \in \mathbb{Z}_+$ are i.i.d. $\mathrm{Bin}(\sigma-1, w_k)$, then subtrees emerging from the backbone of the (Z_k, θ^k)-tree are independent subcritical percolation clusters with parameter w_k. In particular, for suitably chosen values of w_k, n_k, $\mathbf{T}^{k,n_k}$ has the same law as a certain segment of $\mathrm{IPC}_{/}$.

Moreover, if $w_k := 1/\sigma$ for all k, the corresponding (Z, θ)-tree is simply the IIC.

We will see that the unconditioned IIC, as well as segments of the unconditioned IPC can be treated in a similar way.

The reader will note that in all the above cases the sequence Z_k consists of i.i.d. random variables bounded by σ. Thus the sequences will clearly satisfy the following conditions. In view of possible future extensions we assume only the following weaker versions of boundedness, independence, and consider henceforth only sequences satisfying

$$\mathcal{A}: \begin{cases} \text{for each } k \in \mathbb{Z}_+, \text{ the variables } Z_{k,n}, n \in \mathbb{Z}_+ \text{ are independent.} \\ \text{for some } C, \alpha > 0, \text{ for all } k, n, \ \mathbb{E}[Z_{k,n}^{1+\alpha}] < C, \\ \text{for some } \eta > 0 \text{ for all } k \in \mathbb{Z}_+, n \in \mathbb{Z}_+ \ \ \mathbb{P}[Z_{k,n} \neq 0] \geq \eta. \end{cases}$$

To look at the convergences of rescaled coding paths of truncated trees, we will need some extra assumptions on the sequences Z_k :

$$\mathcal{B}: \begin{cases} \text{for every } k \in \mathbb{Z}_+, \text{ there exists } m_k \text{ s.t. } \frac{1}{n}\sum_{i=0}^n Z_{k,i} \xrightarrow[n\to\infty]{prob.} m_k. \\ m_k \xrightarrow[k\to\infty]{} m \end{cases}$$

In practice we will work with sequences with simple distributions: i.i.d. copies of a bounded random variable. However, the additional generalization has some use. In fact, the independence requirement can be relaxed somewhat, as only weak consequences of independence are used.

3.2 Scaling of segments

Proposition 3.3. *Suppose the sequences Z_k satisfy $\mathcal{A}$, and suppose $k(\sigma w_k - 1) \xrightarrow[k\to\infty]{} -u < 0$. Then, as $k \to \infty$, weakly in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ with the topology of uniform convergence on compact sets,*

$$\left(\frac{1}{k}\mathbf{V}^k_{[k^2 t]}, t \geq 0\right) \xrightarrow[k\to\infty]{} \Big(X_t, t \geq 0\Big), \tag{9}$$

where $X_t = Y_t - \underline{Y}_t$ and Y_t is a drifted Brownian motion: $Y_t = B_{\gamma t} - ut$.

When the sequences Z_k satisfy both assumptions $\mathcal{A}$ and $\mathcal{B}$, we will deduce from Proposition 3.3 that the rescaled Lukaciewicz paths of suitably truncated trees also converge to a non-degenerate limit. Note that it follows from the proposition that this limit has to be a segment of the path of a reflected drifted Brownian motion. The convergence will take place in the space of continuous stopped paths denoted $\mathcal{S}$. An element $f \in \mathcal{S}$ is given by a lifetime $\zeta(f)$ and a continuous $(f(t), t \in [0, \zeta(f)])$. The distance between two continuous stopped paths f_1, f_2 is given by

$$d(f_1, f_2) := |\zeta(f_1) - \zeta(f_2)| + \sup_{t \leq (\zeta(f_1) \wedge \zeta(f_2))} \{|f_1(t) - f_2(t)|\}.$$

This makes $(\mathcal{S}, d)$ a Polish space.

Corollary 3.4. *Asume the conditions of Proposition 3.3 are in force. Assume further that the sequences Z_k satisfy assumption $\mathcal{B}$, and that the sequence of integers $(n_k)_k$ is such that $n_k/k \xrightarrow[k\to\infty]{} x > 0$. Then, weakly in $(\mathcal{S}, d)$*

$$\left(\frac{1}{k}\mathbf{V}^{k,n_k}_{[k^2 t]}, 0 \leq t \leq \frac{\tau^{(n_k)} + 1}{k^2}\right) \xrightarrow[k\to\infty]{} \Big(X_t, 0 \leq t \leq \tau_{mx}\Big), \tag{10}$$

where $\tau^{(i)}$ denotes the ith return to 0 of the path $\mathbf{V}$, X and Y are as in Proposition 3.3 and τ_y is the stopping time $\inf\{t > 0 : Y_t = -y\}$.

It is then straightforward to check that convergence results also hold, under the same assumptions, for the height processes. We let h^k denote the height function coding the tree $\mathbf{T}^k$, h^{k,n_k} the height function of the tree $\mathbf{T}^{k,n_k}$.

Corollary 3.5. *Suppose the assumptions of Corollary 3.4 are in force.*

Weakly in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$,

$$\left(\frac{1}{k}h^k_{[tk^2]}, t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\frac{2}{\gamma}\left(Y_t - \underline{Y}_t\right) - \frac{1}{m}\underline{Y}_t, t \geq 0\right). \tag{11}$$

Weakly in $\mathcal{S}$,

$$\left(\frac{1}{k}h^{k,n_k}_{[tk^2]}, 0 \leq t \leq (\tau^{(n_k)} + 1)/k^2\right) \xrightarrow[k\to\infty]{} \left(\frac{2}{\gamma}\left(Y_t - \underline{Y}_t\right) - \frac{1}{m}\underline{Y}_t, 0 \leq t \leq \tau_{mx}\right). \tag{12}$$

3.3 Proof of Proposition 3.3

We begin with the following Lemma, which relates the Lukaciewicz paths of a sequence of trees, and that of the tree consisting of a backbone to which the trees of the sequence are attached.

Lemma 3.6. *Let $(\theta_n)_{n\geq 0}$ be a sequence of trees, and define the sin-tree T to be the sin-tree with a backbone BB on the right, such that the root of θ_n is identified with BB_n. Let U be the Lukaciewicz path coding the sequence, and let V be the Lukaciewicz path of T. Then*

$$V_n = U_n + 1 - \underline{U}_{n-1},$$

where by convention $\underline{U}_{-1} = 1$.

Proof. The Lemma follows directly from the definition of Lukaciewicz paths. Recall that $\underline{U}$ decreases exactly when the process completes the exploration of a tree in the sequence. The increments of V differ from the increments of U only at vertices of the backbone of T, where the degree in T is one more than the degree in θ_n. □

We first establish the proposition in the special case introduced earlier, where $\mathcal{Z}_k$ is a sequence of i.i.d. $\mathrm{Bin}(\sigma, w_k)$ random variables. Recall that in this case, we write $\mathcal{T}^k$ for the corresponding $(\mathcal{Z}_k, \theta^k)$-tree, and that subtrees attached to the backbone of $\mathcal{T}^k$ form a sequence of independent Galton-Watson trees, whose branching law has expectation σw_k (which tends to 1 as $k \to \infty$), and variance $\sigma w_k(1 - w_k)$ (which tends to γ as $k \to \infty$).

The Lukaciewicz path $\mathcal{U}^k$ of this sequence of Galton-Watson trees is a random walk with drift $\sigma w_k - 1$ and stepwise variance $\sigma w_k(1 - w_k)$. From a well-known result on convergence of random walks (see for instance [11, Theorem II.3.5]) we then deduce that weakly in the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^2)$,

$$\left(\frac{1}{k}\mathcal{U}^k(k^2t), \frac{1}{k}\underline{\mathcal{U}}^k(k^2t), t \geq 0\right) \xrightarrow[k\to\infty]{} \left(Y_t, \underline{Y}_t, t \geq 0\right).$$

It now follows from Lemma 3.6 that

$$\left(\frac{1}{k}\mathcal{V}^k(k^2t), t \geq 0\right) \xrightarrow[k\to\infty]{} \left(X_t, t \geq 0\right).$$

Having established the result for $\mathcal{Z}_{k,n}$, we now extend it to other degree sequences. By the Skorokhod representation theorem, we may assume (by changing the probability space as needed) that the above convergences for $\mathcal{V}^k$ hold almost surely with respect to the topology of uniform convergence on compacts:

$$\left(\frac{1}{k}\mathcal{V}^k_{k^2t}, t \geq 0\right) \xrightarrow[k\to\infty]{a.s.} \left(X_t, t \geq 0\right). \tag{13}$$

By adding all the sequences Z_k to the probability space we get a coupling of the various $\mathcal{T}^k$ and $\mathbf{T}^k$, with the further key assumption that we use the same sequences θ^k in both trees. This allows us to identify each vertex of θ^k_n with one vertex in each of $\mathcal{T}^k, \mathbf{T}^k$, giving also a partial correspondence between $\mathcal{T}^k$ and $\mathbf{T}^k$ (the backbones remaining unmatched).

It will be convenient to consider set of points

$$\mathbf{G}^k := \{(i, \mathbf{V}^k(i)), i \in \mathbb{N}\}, \qquad \mathcal{G}_k := \{(i, \mathcal{V}^k(i)), i \in \mathbb{N}\},$$

which are the integral points in the graphs of $\mathbf{V}^k, \mathcal{V}^k$. In fact, to each vertex $v \in \mathbf{T}^k$ corresponds a point $(\mathbf{x}_v, \mathbf{y}_v)$ in the graph of $\mathbf{V}^k$. From the right description of Lukaciewicz paths introduced in 1.1.2, we see that

$$\begin{aligned}\mathbf{G}^k &= \{(\mathbf{x}_v, \mathbf{y}_v) : v \in \mathbf{T}^k\} = \left\{(\#(\mathbf{T}^k)^v, n(v, \mathbf{T}^k) - 1) : v \in \mathbf{T}^k\right\}, \\ \mathcal{G}^k &= \{(x_v, y_v) : v \in \mathcal{T}^k\} = \left\{(\#(\mathcal{T}^k)^v, n(v, \mathcal{T}^k) - 1) : v \in \mathcal{T}^k\right\},\end{aligned}$$

The next step is to show that these two sets are suitably close to each other. Any $v \in \theta_n^k$ is contained in the natural way in both $\mathbf{T}^k$ and $\mathcal{T}^k$. We first show that $\mathbf{x}_v \approx x_v$ and $\mathbf{y}_v \approx y_v$ for such v, and then show how to deal with the backbones.

Any tree θ_n^k is attached by an edge to some vertex in the backbone of $\mathbf{T}^k$ and $\mathcal{T}^k$. For any vertex $v \in \theta_n^k$ we denote the height of this vertex by $\mathbf{l}_v$ and ℓ_v respectively:

$$\mathbf{l}_v = \sup\{t : \mathrm{BB}_t < v \text{ in } \mathbf{T}^k\} \qquad \ell_v = \sup\{t : \mathrm{BB}_t < v \text{ in } \mathcal{T}^k\}.$$

These values depend implicitly on k. By a slight abuse of notation, we also use $\mathbf{l}_n, \ell_n$ for the same values whenever $v \in \theta_n^k$. Note that this definition does not depend on which $v \in \theta_n^k$ is chosen.

Lemma 3.7. *Assume $v \in \theta_n^k$, then*

$$\begin{aligned}|\mathbf{x}_v - x_v| &= |\mathbf{l}_v - \ell_v|, \\ |\mathbf{y}_v - y_v| &\le \sigma + Z_{k,\mathbf{l}_v}.\end{aligned}$$

Proof. We have

$$x_v = \#(\mathcal{T}^k)^v = \sum_{i<n} \#\theta_i^k + \#(\theta_n^k)^v + \ell_n,$$

and similarly

$$x_v = \#(\mathbf{T}^k)^v = \sum_{i<n} \#\theta_i^k + \#(\theta_n^k)^v + \mathbf{l}_n.$$

The first claim follows.

For the second bound use $\mathbf{y}_v = n(v, \mathbf{T}^k) - 1$. Now, there are $n(v, \theta_n^k)$ edges connecting $(\mathbf{T}^k)^v$ to is complement inside θ_n^k, and at most $Z_{k,\mathbf{l}_n}$ edges connecting $\mathrm{BB}_{\mathbf{l}_n}$ to the complement. Similarly, in $\mathcal{T}^k$ we have the same $n(v, \theta_n^k)$ edges inside θ_n^k and at most $\mathcal{Z}_{k,\ell_n} \le \sigma$ edges connecting BB_{ℓ_n} to the complement. It follows that the difference is at most $\sigma + Z_{k,\mathbf{l}_n}$. $\square$

For a vertex $v \in \mathbf{T}^k$, define $u \in \mathbf{T}^k$ by

$$u = \min\left\{u \in (\mathbf{T}^k \setminus \mathrm{BB}) : u \ge v\right\}.$$

Thus if v is on the backbone then u is the first child of v or of a backbone vertex above v, while if v is off the backbone, $u = v$. Note that $u \in \theta_n^k$ for some n, so we may also consider u as a vertex of $\mathcal{T}^k$. Note that $v \to u$ is a non-decreasing map from $\mathbf{T}^k$ to $\mathcal{T}^k$.

Lemma 3.8. *Let v be a backbone vertex in $\mathbf{T}^k$ and define n by $\theta_n^k < v < \theta_{n+1}^k$. Then*

$$|\mathbf{x}_v - \mathbf{x}_u| \leq 1 + \mathbf{l}_{n+1} - \mathbf{l}_n,$$
$$|\mathbf{y}_v - \mathbf{y}_u| \leq \sigma + Z_{k,\mathbf{l}_{n+1}}.$$

Proof. The only vertices between v and u in the lexicographic order are u and some of the backbone vertices with indices from $\mathbf{l}_n$ to $\mathbf{l}_{n+1}$, yielding the first bound.

Let $w \in \mathrm{BB}$ be u's parent. If v has children apart from the next backbone vertex then $w = v$ and u is v's first child, so $\mathbf{y}_u - \mathbf{y}_v = k_u - 1 \leq \sigma - 1$. If v has no other children then $\mathbf{y}_u - \mathbf{y}_v = (k_u - 1) + (k_w - 1) \leq \sigma + Z_{k,\mathbf{l}_{n+1}}$. ☐

Recall α which was introduced in assumption $\mathcal{A}$, and note that without loss of generality we may assume $\alpha \in (0, 1)$.

Lemma 3.9. *Fix $A > 0$ and let w be the $[Ak^2]$'th vertex of $\mathbf{T}^k$.*

$$\mathbb{P}\left(\sup_{v \in \mathbf{T}^k, v < w} |\mathbf{x}_v - x_u| > 3k^{1+\alpha/2} \right) \xrightarrow[k\to\infty]{} 0.$$

Proof. For a vertex $v \in \theta_n^k$ off the backbone we have $u = v$ and

$$|\mathbf{x}_v - x_u| \leq |\mathbf{l}_v - \ell_v| \leq \mathbf{l}_v + \ell_v \leq \mathbf{l}_w + \ell_w.$$

Since θ is just barely sub-critical, $\mathbb{P}(\#\theta_n^k > k^2) > ck^{-1}$ for some $c > 0$. Consider the first $k^{1+\alpha/2}$ vertices along the backbone in $\mathbf{T}^k$. With overwhelming probability, the number of θ's attached to them is at least $ak^{1+\alpha/2}$ for some $a > 0$. On this event, with overwhelming probability the total size of the θ's attached to the first $k^{1+\alpha/2}$ vertices of the backbone is at least $c/(2k) \cdot ak^{1+\alpha/2} \cdot k^2 = c'k^{2+\alpha/2}$. Thus with high probability $\mathbf{l}_w \leq k^{1+\alpha/2}$. Since the same argument shows $\ell_w \leq k^{1+\alpha/2}$, we find that with high probability $\max |\mathbf{x}_v - x_u| \leq 2k^{1+\alpha/2}$.

It remains to show that if $v < w$ is in the backbone then $|\mathbf{x}_v - \mathbf{x}_u| < k^{1+\alpha/2}$. To this end, note that $\mathbf{l}_{n+1} - \mathbf{l}_n$ is dominated by a geometric random variable with mean η^{-1} (by assumption $\mathcal{A}$). Since only $n < Ak^2$ are relevant to the initial part of the tree, this shows that with high probability $|\mathbf{x}_v - \mathbf{x}_u| < c \log k \ll k^{1+\alpha/2}$. ☐

Lemma 3.10. *Fix $A > 0$ and let w be the $[Ak^2]$'th vertex of $\mathbf{T}^k$. Fix $\beta > 0$ so that $\alpha/2 > \beta(1-\alpha)$. Then*

$$\mathbb{P}\left(\max_{v<w} Z_{k,\mathbf{l}_v} \geq k^{1-\beta} \right) \xrightarrow[k\to\infty]{} 0.$$

Proof. From our assumption, for some c, for every k, n we have $\mathbb{E}Z_{k,n}^{1+\alpha} < c$. By Markov's inequality, for $\mathbb{P}(Z_{k,n} > u) \leq cu^{-1-\alpha}$. Therefore

$$\mathbb{P}\left(\max_{n<k^{1+\alpha/2}} Z_{k,n} \geq u \right) \leq \frac{k^{1+\alpha/2}}{u^{1+\alpha}}.$$

Taking $u = k^{1-\beta}$ gives

$$\mathbb{P}\left(\max_{n<k^{1+\alpha/2}} Z_{k,n} \geq k^{1-\beta} \right) \xrightarrow[k\to\infty]{} 0.$$

As in the previous lemma, with high probability $\mathbf{l}_v \leq \mathbf{l}_w < k^{1+\alpha/2}$, so this implies the claim. ☐

We now finish the proof of Proposition 3.3. Because the path of $\mathbf{V}^k$ is linearly interpolated between consecutive integers, and since for any $A > 0$ the paths of X are a.s. uniformly continuous on $[0, A]$, the proposition will follow if we establish that for any $A, \varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,A]}\left|\frac{1}{k}V^k_{[k^2t]} - X_t\right| > \varepsilon\right) \xrightarrow[k\to\infty]{} 0. \tag{14}$$

Consider first t such that $k^2t \in \mathbb{Z}_+$. Then there is some vertex $v \in \mathbf{T}^k$ so that $\mathbf{x}_v = k^2t$. Let $u \in \mathcal{T}^k$ be as defined above, and suppose $k^2s = x_u$. Then (13) implies that $|k^{-1}y_u - X_s|$ is uniformly small. Lemma 3.9 implies that with high probability $|k^2s - k^2t| = |x_u - \mathbf{x}_v| \leq 3k^{1+\alpha/2}$ for all such v. Thus $|s - t| \leq k^{-1+\alpha/2} \ll 1$. Since paths of X are uniformly continuous we find $|X_s - X_t|$ is uniformly small, and so $|k^{-1}y_u - X_t|$ is uniformly small. Finally, Lemma 3.10 states that with high probability $|y_u - \mathbf{y}_v| \leq k^{1-\beta}$ for some $\beta > 0$, proving our claim.

Next, assume $m < k^2t < m+1$. Then $\mathbf{V}^k(k^2t)$ lies between $\mathbf{V}^k(m)$ and $\mathbf{V}^k(m+1)$. Since both of these are close to the corresponding values of X, and since X is uniformly continuous (and the pertinent points differ by at most k^{-2}) we may interpolate to find that (14) holds for all $t < A$.

At this point, one might be confused by the fact that the convergence in probability stated in (14) seems stronger than the one stated in Proposition 3.3. But this is only an artefact of our use of Skorokhod representation theorem to strengthen the convergence (13), and of our coupling between the trees $\mathbf{T}^k$, $\mathcal{T}^k$.

3.4 Proof of the Corollaries

Proof of Corollary 3.4. In the special case of the tree $\mathcal{T}^k$ we note that the infimum process $\underline{\mathcal{U}}^k$ records the index of the last visited vertex along the backbone. Therefore $\tau^{(n_k)}$ is the time at which $\mathcal{U}^k$ first reaches $-n_k$, and by assumption $n_k \sim xk$. Using the a.s. convergence of $\frac{1}{k}\mathcal{U}^k(k^2t)$ towards Y_t, along with the fact that for any fixed $x > 0, \varepsilon > 0$, one has a.s. $\underline{Y}_{\tau_x-\varepsilon} > -x > \underline{Y}_{\tau_x+\varepsilon}$, we deduce that a.s., $\tau^{(n_k)}/k^2 \to \tau_x$. It then follows that

$$\left(\frac{1}{k}\mathcal{V}^k_{k^2t}, t \leq (\tau^{(n_k)} + 1)/k^2\right) \xrightarrow[k\to\infty]{a.s.} (X_t, t \leq \tau_x).$$

Since, in this case, $m_k = \sigma w_k \to m = 1$, this implies the corollary for this special distribution.

The general case is then a consequence of excursion theory. Indeed $(-\underline{Y}_t, t \geq 0)$ can be chosen to be the local time at its infimum of Y (see for instance paragraph VI.8.55 of [17]), that is a local time at 0 of X, since excursions of Y away from its infimum match those of X away from 0. However, if $N_t^{(\varepsilon)}$ denotes the number of excursions of X away from 0 that are completed before t and reach level ε, then $(\lim_{\varepsilon\to 0}\varepsilon N_t^{(\varepsilon)}, t \geq 0)$ is also "a" local time at 0 of X, which means that it has to be proportional to $(-\underline{Y}_t, t \geq 0)$ (cf for instance, section III.3(c) and theorem VI.2.1 of [4]). In other words, there exists a constant $c > 0$ such that for any $t \geq 0$,

$$\lim_{\varepsilon\to 0}\varepsilon N_t^{(\varepsilon)} = -c\underline{Y}_t.$$

In the special case when $\mathcal{Z}_{k,n} = \mathrm{Bin}(\sigma, w_k)$ we have already proven the corollary. In particular, the number $\mathcal{N}^{k,(\varepsilon)}$ of excursions of $(\frac{1}{k}U^k_{k^2t}, t \leq \tau^{(n_k)})$ which reach level ε is such that, when letting $k \to \infty$, then $\varepsilon \to 0$, $\varepsilon\mathcal{N}^{k,(\varepsilon)} \to cx$.

Let $N^{k,(\varepsilon)}$ be the number of excursions of $\left(\frac{1}{k}V^k_{k^2t}, t \leq \tau^{(n_k)}\right)$ which reach level ε. The third assumption of $\mathcal{A}$ (law of large numbers for the sequences $(Z_{k,n})_{n\in\mathbb{N}}$), along with the assumption $m_k \to m$ ensures that $\varepsilon N^{k,(\varepsilon)} \underset{k\to\infty}{\sim} m\varepsilon\mathcal{N}^{k,(\varepsilon)}$. Therefore, letting first $k \to \infty$, then $\varepsilon \to 0$ we find $\varepsilon N^{k,(\varepsilon)} \to mcx$.

However, Proposition 3.3 implies that the limit of $\left(\frac{1}{k}V^k_{k^2t}, t \leq \tau^{(n_k)}\right)$ has to be a part of the path of $(X_t, t \geq 0)$ stopped at a certain random time τ for which $X_\tau = 0$. From the fact that $\tau^{(n_k)}$ are stopping times, we deduce that τ itself is a stopping time, hence for any $s > 0$, the local time at 0 of X (that is $-\underline{Y}$) increases on the interval $(\tau, \tau + s)$. It follows that for a certain real-valued random variable R, $\tau = \tau_R = \inf\{t \geq 0 : -Y_t = R\}$, and since $\varepsilon N^{k,(\varepsilon)} \to mcx$, we deduce that $R = mx$.

□

Proof of Corollary 3.5. By Proposition 3.3, along with Theorem 2.3.1 and equation (1.7) of [7], the height process of the sequence of trees emerging from the backbone of $\mathbf{T}^k$ converges when rescaled to the process

$$\frac{2}{\gamma}(Y_t - \underline{Y}_t).$$

Moreover, the difference between the height process of $\mathbf{T}^k$ and that of the sequence of trees emerging from the backbone of $\mathbf{T}_k$ is simply $-\underline{U}^k$. As in the proof of Corollary 3.4, one has weakly in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$,

$$\left(-\frac{1}{k}\underline{U}^k_{[k^2t]}, t \geq 0\right) \underset{k\to\infty}{\longrightarrow} \left(-\frac{1}{m}\underline{Y}_t, t \geq 0\right),$$

and (11) follows. The proof of (12) is similar. □

In fact, [7, Corollary 2.5.1] states the joint convergence of Lukaciewicz paths, height, and contour functions. It is thus easy to deduce a strengthening Corollary 3.5 to get the joint convergence.

3.5 Two sided trees

The limit appearing in Proposition 3.3 retains very minimal information about the sequence Z_k. This implies that dependence between two sequences can easily disappear in the scaling limit of the corresponding trees. This remark leads to the following corollary.

For $k \in \mathbb{Z}_+$, let $w_k \in [0, 1/\sigma]$, and denote by $(\theta^k_n)_{n\in\mathbb{Z}_+}$, $(\widetilde{\theta}^k_n)_{n\in\mathbb{Z}_+}$ two *independent* sequences of i.i.d. subcritical Galton-Watson trees with branching law $\mathrm{Bin}(\sigma, w_k)$. We let $Z_k, \widetilde{Z}_k$ be two *(possibly correlated)* sequences of independent random variables $(Z_{k,n})_{n\geq 0}, (\widetilde{Z}_{k,n})_{n\geq 0}$ taking values in $\mathbb{Z}_+$.

Let $\mathbf{T}^k, \widetilde{\mathbf{T}}^k$ designate respectively the (Z_k, θ^k)-tree, $(\widetilde{Z}_k, \widetilde{\theta}^k)$-tree as defined in Section 3.1. Let $\mathbf{V}^k$, resp. $\widetilde{\mathbf{V}}^k$ denote their Lukaciewicz paths. We recall that $\mathbf{T}^{k,n_k}$, $\widetilde{\mathbf{T}}^{k,n_k}$ are respectively the n_k-truncation, of $\mathbf{T}^k$, resp. $\widetilde{\mathbf{T}}^k$, and we denote by $\mathbf{V}^{k,n_k}, \widetilde{\mathbf{V}}^{k,n_k}$ their respective Lukaciewicz paths.

Proposition 3.11. *Suppose $w_k \le 1/\sigma)$ is such that, $\lim_{k\to\infty} k(\sigma w_k - 1) = -u$ and assume that both sequences of variables $Z_k, \widetilde{Z}_k$ satisfy assumption $\mathcal{A}$. Then, as $k \to \infty$, weakly in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^2)$ equipped with the topology of uniform convergence on compact sets,*

$$\left(\frac{1}{k}\mathbf{V}^k_{[k^2t]}, \frac{1}{k}\widetilde{\mathbf{V}}^k_{[k^2t]}, t \ge 0\right) \xrightarrow[k\to\infty]{} \left(X_t, \widetilde{X}_t, t \ge 0\right),$$

where the processes $X, \widetilde{X}$ in the righthand side above are two independent *reflected Brownian motions with drift $-u$ and diffusion coefficient γ.*

Moreover, if $n_k/k \to x > 0$, $m_k \to m$, $\tilde{m}_k \to \tilde{m}$ as $k \to \infty$, we have

$$\left(\frac{1}{k}\mathbf{V}^{k,n_k}_{[k^2t]}, \frac{1}{k}\widetilde{\mathbf{V}}^{k,n_k}_{[k^2t]}, t \ge 0\right) \xrightarrow[k\to\infty]{} \left(X_{t\wedge\tau_{mx}}, \tilde{X}_{t\wedge\tau_{\tilde{m}x}}, t \ge 0\right).$$

The proof is almost identical to that of Proposition 3.3. When the sequences $Z_k, \widetilde{Z}_k$ are independent with $\mathrm{Bin}(\sigma, w_k)$ elements the result follows from Proposition 3.3. For general sequences, the coupling of Section 3.3 shows that the sides have the same joint scaling limit.

3.6 Scaling the IIC

It is known that the IIC is the result of setting $w_k = 1/\sigma$ in the above constructions. Specifically, let us first suppose that Z is a sequence of i.i.d. $\mathrm{Bin}(\sigma - 1, w)$, and $(\theta_n)_n$ is a sequence of i.i.d. $\mathrm{Bin}(\sigma, 1/\sigma)$ Galton-Watson trees. Let $\mathbf{T}$ be a (Z, θ)-tree, then observe that $\mathbf{T}$ has the same distribution as the $\mathrm{IIC}_/$.

The convergence of the rescaled Lukaciewicz path encoding the $\mathrm{IIC}_/$ to a time changed reflected Brownian path is thus a special case of Proposition 3.3. The scaling limits of the height and contour functions follow from Corollary 3.5. We have $m = \gamma$, so both limits are $\frac{2B_{\gamma t} - 3\underline{B}_{\gamma t}}{\gamma}$.

For the IIC, let Y_n be i.i.d. uniform in $\{1, \dots, \sigma\}$. Let $Z_n \sim \mathrm{Bin}(Y_n - 1, 1/\sigma)$ and $\widetilde{Z}_n \sim \mathrm{Bin}(\sigma - Y_n, 1/\sigma)$, independent conditioned on Y_n and independently of all other n. Moreover, suppose that $\theta, \widetilde{\theta}$ are two independent sequences of i.i.d. $\mathrm{Bin}(\sigma, 1/\sigma)$ Galton-Watson trees. Then, $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ are jointly distributed as IIC_G and $\overline{\mathrm{IIC}}_D$.

Since in this case $m = \tilde{m} = \gamma/2$,

From Proposition 3.11, we see that the rescaled Lukaciewicz paths encoding these two trees converge towards a pair of independent time-changed reflected Brownian motions. Corresponding results hold for the right/left height and contour functions of the unconditioned IIC. For example, if H_G, H_D are the left and right height functions of the IIC, then weakly in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^2_+)$,

$$\left(\frac{1}{k}H_G(k^2t), \frac{1}{k}H_D(k^2t), t \ge 0\right) \xrightarrow[k\to\infty]{} \left(\frac{2}{\gamma}(B_{\gamma t} - 2\underline{B}_{\gamma t})\frac{2}{\gamma}(\widetilde{B}_{\gamma t} - 2\underline{\widetilde{B}}_{\gamma t}), t \ge 0\right),$$

where B and $\widetilde{B}$ are two independent Brownian motions, Interestingly, note that the limit $\frac{2}{\gamma}(B_{\gamma t} - 2\underline{B}_{\gamma t})$ is, up to a mutiplicative constant, a 3-dimensional Bessel process.

4 Bottom-up construction

4.1 Right grafting and concatenation

Definition 4.1. Given an ordered discrete tree, its *rightmost-leaf* is the maximal vertex in the lexicographic order; equivalently, it is the last vertex to be visited by the contour process, and is the rightmost leaf of the sub-tree above the rightmost child of the root.

Definition 4.2. The *right-grafting* of a plane tree S on a finite plane tree T, denoted $T \oplus S$ is the plane tree resulting from identifying the root of S with the rightmost leaf of T. More precisely, let v be the rightmost leaf of T. The tree $T \oplus S$ is given by its set of vertices $\{u, vw : u \in T \setminus \{v\}, w \in S\}$.

Note in particular that the vertices of S have been relabeled in $T \oplus S$ through the mapping from S to $T \oplus S$ which maps w to vw.

Definition 4.3. $V = V_1 \oplus V_2$ is the *concatenation* of two functions V_1, V_2 on finite intervals $[0, x_1]$, $[0, x_2]$ such that $V_2(0) = 0$, and is defined by

$$V(t) = \begin{cases} V_1(t) & t \leq x_1, \\ V_1(x_1) + V_2(t - x_1) & t \in [x_1, x_2]. \end{cases}$$

Lemma 4.4. *If $\mathcal{Y} = \bigoplus Y_i$ where each Y_i is a continuous function on an interval $[0, x_i]$ so that $Y_i(0) = 0$, and Y_i attains its minimum at x_i, then*

$$\bigoplus (Y_i - \underline{Y}_i) = \bigoplus Y_i - \underline{\bigoplus Y_i}.$$

Thus it does not matter whether reflection takes place before or after concatenation.

The following is straightforward to check, and may be used as an alternate definition of right-grafting.

Lemma 4.5. *Let $R = T \oplus S$ be finite plane trees, and denote the Lukaciewicz path of R (resp. T, S) by V_R (resp. V_T, V_S). Suppose $\#T = m$, then V_R is the concatenation of V_T and T_S without the values of V_T taken in $[m, m+1]$:*

$$V_R(t) = \begin{cases} V_T(t) & t \leq m, \\ V_S(t - m) & t \geq m. \end{cases}$$

Consider a sin-tree T in which the backbone is the rightmost path (i.e. the path that takes the rightmost child at each step). Given some set $\{x_i\}$ of vertices along the backbone it is possible to cut the tree at these vertices. Let

$$\tilde{T}_i := \{v \in T : v \geq x_i \text{ and } v \leq x_{i+1}\}.$$

In words, $\tilde{T}_i$ contains the segment of the backbone $[x_i, x_{i+1}]$ as well as all the sub-trees connected to any vertex of this segment except to x_{i+1}. Obviously, any vertex of $\tilde{T}_i$ is of the form $x_i v$ for some $v \in \bigcup_{n=0}^{\infty} (\mathbb{N})^n$. Thus $\tilde{T}_i$ can be identified to a finite ordered tree denoted T_i, by relabeling vertices of $\tilde{T}_i$ through the mapping

$$\begin{cases} \tilde{T}_i & \to & T_i \\ x_i v & \to & v. \end{cases}$$

It is clear from the definitions that $T = \bigoplus_{i=0}^{\infty} T_i$. Note that the sequence x_i is arbitrary.

4.2 Notations

In the remainder of the section we will consider both IPC and $(\mathrm{IPC})_/$, which we defined in the introduction as the IPC conditioned to have the rightmost branch of $\mathcal{T}_\sigma$ for backbone. Our goal is to establish the results stated in section 1.3, in particular Theorems 1.1, 1.3. In the following subsection we establish Theorem 1.1 and its corollary 1.2, so we first focus on the $(\mathrm{IPC})_/$. For convenience we use the shorter notation $\mathcal{R}$ to designate this tree, while $\mathcal{V}$ is its Lukaciewicz path.

We construct below a sequence of copies of $\mathcal{R}$ whose scaling limits converge. These will be indexed by k, though the dependence on k will frequently be implicit. Note that the use of Skorokhod representation theorem will in fact allow us to construct the sequence so that the Lukaciewicz paths converge almost surely, rather than just in distribution.

We denote by $\mathcal{R}^k$ the k'th instance of $\mathcal{R}$ in the sequence.

Finally, while $\mathcal{R}$ is close to critical away from the root, the segments close to the root behave differently and need to be dealt with separately. We let $\mathcal{R}^\beta$ (implicitly depending on k) be the subtrees above a certain vertex in the backbone (see below), and let $\mathcal{V}^\beta$ denote its Lukaciewicz path. As $\beta \to \infty$ the trees will get closer to the full trees. Lemma 4.8 below will show that $\mathcal{V}^\beta$ is uniformly close to $\mathcal{V}$ (recalling that both depend implicitly on k.)

4.3 IPC structure and the coupling

In this paragraph we prove Theorem 1.1 and its corollary 1.2.

Recall the $\hat{W}$-process introduced in paragraph 1.1.1, and the convergence (4). The $\hat{W}$-process is constant for long stretches, giving rise to a partition of $\mathcal{R}$ into what we shall call segments. Each segment consists of an interval along the backbone along which $\hat{W}$ is constant, together with all sub-trees attached to such an interval. To be precise, define x_i inductively by $x_0 = 0$ and $x_{i+1} = \inf_{x > x_i}\{\hat{W}_x > \hat{W}_{x_i}\}$. With a slight abuse, we also let x_i designate the vertex along the backbone at height x_i.

Since we have convergence in distribution of the $\hat{W}$'s we may couple the IPC's for different k's so that the convergence is a.s.. More precisely, let J be the set of jump times for $\{L_t\}$. We may assume that a.s., $\{k^{-1}x_i^k\} \xrightarrow[k\to\infty]{} J$ in the sense that there there is a 1-to-1 mapping from the jump times of $\hat{W}$ in $\mathcal{R}^k$ into J that eventually contains every point of J. Furthermore, we may assume that for any $t \notin J$ we have a.s. $k^{-1}(1 - \sigma \hat{W}^k_{[kt]}) \xrightarrow[k\to\infty]{} L_t$.

The backbone is the union of the intervals $[x_i, x_{i+1}]$ for all $i \geq 0$, and the rest of the IPC consists of sub-critical percolation clusters attached to each vertex of the backbone

$y \in [x_i, x_{i+1} - 1]$. We can now write

$$\mathcal{R} = \bigoplus_{i=0}^{\infty} R_i$$

The subset $\tilde{R}_i$ of $\mathcal{R}$ consists in the backbone interval $[x_i, x_{i+1}]$ together with the off-backbone trees attached to each $y \in [x_i, x_{i+1} - 1]$. More precisely, the relabeled R_i has a rightmost branch of length $n_i := x_{i+1} - x_i$. The vertex at height $n \in [0, n_i - 1]$ on this branch has $Z_{k,n}$ children away from the backbone, while the vertex at height n_i has no child. Finally, vertices at distance 1 above the rightmost branch give birth to independent Galton-Watson tree with branching law $\mathrm{Bin}(\sigma, \hat{W}_{x_i})$. Note that $(Z_{k,n})_{n\in[0,n_i-1]}$ are independent and identically distributed, and $Z_{k,n} \sim \mathrm{Bin}(\sigma - 1, \hat{W}_{x_i})$.

In what follows, we say that R_i is a $\hat{W}_{x_i}$-segment of length n_i, and we observe that these segments fall into the family dealt with in section 3.

We may summarize the above in the following lemma:

Lemma 4.6. *Let $\{U_i\}$ be the sequence of distinct values taken by the $\hat{W}$ process and $\{n_i\}$ be the number of times they appear. Then conditioned on $\{U_i, n_i\}$ the trees $\{R_i\}$ are independent and R_i is distributed as a U_i-segment of length n_i.*

A difficulty we must deal with is that in the scaling limit there is no first segment, but rather a doubly infinite sequence of segments. Furthermore, the initial segments are far from critical, and so need to be dealt with separately. This is related to the fact that the Poisson lower envelope process diverges near 0, and has no “first segment”. Because of this we restrict ourselves at first to a slightly truncated invasion percolation cluster. For any $\beta > 0$ we define $x_0^\beta = \min\{x : \sigma\hat{W}_x > 1 - \beta/k\}$. Thus we consider the first vertex along the backbone for which $\sigma\hat{W}_x > 1 - \beta/k$. Let $\tilde{\mathcal{R}}^\beta$ (depending implicitly on k) denote the subtree of $\mathcal{R}^k$ above x_0^β, $\mathcal{R}^\beta$ the relabeled version of $c\tilde{R}^\beta$. If β is large then $\tilde{\mathcal{R}}^\beta$ is almost the complete tree. For any fixed β, as $k \to \infty$ the branches of $\mathcal{R}^\beta$ are all close to critical. As for the entire tree, we define $x_{i+1}^\beta = \inf\{x > x_i^\beta : \hat{W}_x > W_{x_i^\beta}\}$. Note that $x_0^\beta = x_m$ for some m and that $x_i^\beta = x_{m+i}$ for the same m and all i.

If $\beta \notin \{L_t\}$ then β gives rise to a partial indexing of J. Let $j_0^\beta = \inf\{t > 0 : L_t < \beta\}$, and j_{i+1}^β the time of the first jump of L after j_i^β. Under the coupling above we have the limits $k^{-1}x_i^\beta \xrightarrow[k\to\infty]{} j_i^\beta$, and for $y \in [x_i^\beta, x_{i+1}^\beta)$ we have $k(1 - \sigma\hat{W}_y) \xrightarrow[k\to\infty]{} L_{j_i^\beta}$.

Denote by V_i^β (implicitly depending on k) the Lukaciewicz path corresponding to the i'th segment R_i^β in $\mathcal{R}^\beta$. For any β, i, the segment has associated percolation parameters w_k which satisfies $k(1 - \sigma w_k) \xrightarrow[k\to\infty]{} u$ for some value u of L, and their lengths satisfy $k^{-1}n_i^\beta \to x$ for some $x > 0$. By Corollary 3.4, we have the convergence in distribution

$$\left(k^{-1}V_i^\beta(k^2t), 0 \le t \le \tau^{(n_i^\beta)}\right) \xrightarrow[k\to\infty]{} (X_t, 0 \le t \le \tau_{\gamma x})$$

where $X_t = Y_t - \underline{Y}_t$, and Y_t solves

$$dY_t = \sqrt{\gamma}dB_t - udt.$$

As in the previous section, $\tau^{(n_i^\beta)}$ denotes the lifetime of V_i^β (that is its n_i^βth return to 0) and τ_y is the hitting time of $-y$ by Y.

Because this convergence holds for all β, i, we may construct the coupling of the probability spaces so that this convergence too is almost sure, and this is the final constraint in our coupling.

Lemma 4.7. *Fix $\beta > 0$. In the coupling described above we have the scaling limit*

$$k^{-1}\mathcal{V}^\beta(k^2 t) \xrightarrow[k\to\infty]{} X_t,$$

where $X_t = \mathcal{Y}_t^\beta - \underline{\mathcal{Y}}_t^\beta$, and $\mathcal{Y}^\beta$ solves

$$\mathcal{Y}_t^\beta = \sqrt{\gamma}B_t - \int_0^t L\left(j_0^\beta + \frac{1}{\gamma}|\underline{\mathcal{Y}}^\beta{}_s|\right) ds.$$

Proof. Solutions of the equation for $\mathcal{Y}^\beta$ are a concatenation of segments. In each segment the drift is fixed, and each segment terminates when $\underline{\mathcal{Y}}^\beta$ reaches a certain threshold. The corresponding segments of X exactly correspond to the scaling limit of the tree segments R_i^β.

Lemma 4.7 then follows from Lemma 4.4 and Lemma 4.5. □

Lemma 4.8. *Almost surely,*

$$(\mathcal{Y}_t^\beta, t > 0) \xrightarrow[\beta\to\infty]{} \mathcal{Y}_t$$

where $\mathcal{Y}$ solves

$$\mathcal{Y}_t = \sqrt{\gamma}B_t - \int_0^t L\left(\frac{1}{\gamma}|\underline{\mathcal{Y}}_s|\right) ds.$$

Proof. Consider the difference between the solutions for a pair $\beta < \beta'$. We have the relation

$$\mathcal{Y}^{\beta'} = Z \oplus \mathcal{Y}^\beta,$$

where Z is a solution of $Z_t = \sqrt{\gamma}B_t - \int_0^t L\left(j_0^{\beta'} + \frac{1}{\gamma}(|\underline{Z}_s|)\right) ds$, killed when $\underline{Z}$ first reaches $\gamma(j_0^{\beta'} - j_0^\beta)$. In particular Z is a stochastic process with drift in $[-\beta', -\beta]$ (and quadratic variation γ). Thus to show that $\mathcal{Y}^\beta$ is close to $\mathcal{Y}^{\beta'}$, we need to show that Z is small both horizontally and vertically.

The vertical translation of $\mathcal{Y}^\beta$ is $\sqrt{\gamma}k^{-1}(x_0^\beta - x_0^{\beta'})$, which is at most $k^{-1}x_0^\beta$. From [2] we know that this tends to 0 in probability as $\beta \to \infty$. This convergence is a.s. since x_0^β is non-increasing in β.

The values of Z are unlikely to be large, since Z has a non-positive (in fact negative) drift and is killed when $\underline{Z}$ reaches some negative level close to 0.

Finally, there is a horizontal translation of $\mathcal{Y}^\beta$ in the concatenation. This translation is just the time at which $\underline{Z}$ first reaches $\gamma(j_0^{\beta'} - j_0^\beta)$, which is also small, uniformly in $\beta' > \beta$. □

Theorem 1.1 is now a simple consequence of Lemmas 4.7 and 4.8. Indeed, it is straightforward to check that the process $\mathcal{Y} - \underline{\mathcal{Y}}$ coincides with the righthand side of (6), by using scaling properties of Brownian motion and the fact that $(L(at), t \geq 0) \stackrel{\text{(law)}}{=} (a^{-1}L(t), t \geq 0)$. In particular, note that we have found a solution $(\mathcal{Y}_{\gamma^{-1}t}, t \geq 0)$ to equation $\mathcal{E}(L)$. Therefore $\mathcal{E}(L)$ has a unique in law solution, since we already established uniqueness in law in paragraph 2. We shall note that in fact, $\mathcal{Y}$ is the limit of the rescaled Lukaciewicz path coding the sequence of off-backbone trees .

A very similar argument (where in particular one uses Corollary 3.5 instead of Corollary 3.4) leads to the part of Corollary 1.2 concerning convergence of the rescaled height function. From the remark we made in the proof of Corollary 3.5, it is also straightforward to extend this convergence to that of the pair of paths (height and contour functions), as stated in Corollary 1.2,

4.4 The two-sided tree

For convenience we use the shorter notation $\mathcal{T}$ to designate the IPC, and write $\mathcal{V}$ for its Lukaciewicz path.

To deal with $\mathcal{T}$ we introduce the left tree $\mathcal{T}_G$ and the right tree $\mathcal{T}_D$ as introduced in paragraph 1.1.3. They obviously both have the same distribution, but are correlated. As in the previous section we may cut these two trees into segments along which the $\hat{W}$-process is constant. More precisely,

$$\mathcal{T}_G = \bigoplus_{i=0}^{\infty} T_G^i, \qquad \mathcal{T}_D = \bigoplus_{i=0}^{\infty} T_D^i,$$

where the distribution of T_D^i, T_G^i can be precised as follows.

Let $(\mathfrak{T}_n^i)_n, (\tilde{\mathfrak{T}}_n^i)_n$ be two independent sequences of independent Galton-Watson trees with branching law $\text{Bin}(\sigma, \hat{W}_{x_i})$. Let $Y_n, n \in \mathbb{Z}_+$ be independent uniform on $\{1, ..., \sigma\}$, and conditionally on $Y_n, n \in \mathbb{Z}_+$, let Z be a sequence of independent $\text{Bin}(Y_n - 1, \hat{W}_{x_i})$ variables. Moreover, suppose that for any n, $\tilde{Z}_n = \sigma - 1 - Z_n$. Then, the n_i-truncations of the $(Z, \mathfrak{T}^i)$-tree, resp. of the $(\tilde{Z}, \tilde{\mathfrak{T}}^i)$-tree (constructed as in Definition 3.1) have the same distribution as T_G^i, resp. T_D^i.

The rest of the proof of Theorem 1.3 is then almost identical to that of Theorem 1.1. In particular, to deal with the fact that the scaling limit has no first segment, one shall introduce subtrees $\mathcal{T}^\beta$ and consider left and right trees $\mathcal{T}_G^\beta, \mathcal{T}_D^\beta$. We then perform a similar coupling. The convergence for each sequence of segments then follows from the second part of Proposition 3.11. However, note that the value of the expected number of children of a vertex on the backbone is divided by 2 compared to the conditioned case. As a consequence, the limits of the rescaled coding paths of $\mathcal{T}_G^\beta, \mathcal{T}_R^\beta$ will be expressed in terms of solutions to the equation

$$\mathcal{Y}_t^\beta = \sqrt{\gamma} B_t - \int_0^t L\left(j_0^\beta + \frac{2}{\gamma}|\underline{\mathcal{Y}^\beta}_s|\right) ds.$$

Further details are left to the reader.

Finally, Corollary 1.4 is dealt with in a similar fashion, and obviously one could also express the convergence of left and right contour functions as follows :

$$\left(\frac{1}{k}C_G(2k^2t), \frac{1}{k}C_D(2k^2t), t \geq 0\right) \xrightarrow[k\to\infty]{} \left(\frac{2}{\gamma}(Y_{\gamma t} - 2\underline{Y}_{\gamma t}), \frac{2}{\gamma}(\tilde{Y}_{\gamma t} - 2\underline{\tilde{Y}}_{\gamma t}), t \geq 0\right), \qquad (15)$$

where Y and $\tilde{Y}$ are defined as in Theorem 1.3.

4.5 Convergence of trees

In this paragraph we shortly discuss convergence of trees. We refer to chapter 2 of [13] for background on the theory of continuous real trees.

We may define a continuum random sin-tree $\mathcal{T}^{\mathrm{IPC}}$ whose left and right height processes (which can be seen, cf [7] as a continuous analogue of the contour function) are $\frac{2}{\gamma}(Y_{\gamma t/2} - 2\underline{Y}_{\gamma t/2})$ and $\frac{2}{\gamma}(\tilde{Y}_{\gamma t/2} - 2\underline{\tilde{Y}}_{\gamma t/2})$, as follows. For $t \in \mathbb{R}$, let

$$C(t) := \mathbf{1}_{(-\infty,0)}(t)\ \frac{2}{\gamma}(\tilde{Y}_{-\gamma t/2} - 2\underline{\tilde{Y}}_{-\gamma t/2})\ +\ \mathbf{1}_{(0,\infty)}(t)\ \frac{2}{\gamma}(Y_{\gamma t/2} - 2\underline{Y}_{\gamma t/2}).$$

For $s \le t$ let $I(s,t) := \inf\{C(u)\}$, where the infimum is taken over $[s,t]$ if $0 \notin [s,t]$, and over $\mathbb{R} \setminus (s,t)$ if $0 \in [s,t]$. We then introduce the distance $d(s,t) = C(s) + C(t) - 2I(s,t)$, and write $s \sim t$ whenever $d(s,t) = 0$.

The random real tree $\mathcal{T}^{\mathrm{IPC}}$ is defined as the quotient $\mathbb{R}/\sim$. In other words, two real numbers s,t correspond to vertices in $\mathcal{T}^{\mathrm{IPC}}$ whose highest common ancestor is the vertex corresponding to $I(s,t)$.

Recall that for $x > 0$ we defined earlier $\tau_x := \inf\{u > 0 : Y_s = -x\}$. For $x > 0$, we may consider the subtree $\mathcal{T}_x^{\mathrm{IPC}}$ whose height process is defined, when $t \in [0, \frac{2}{\gamma}\tilde{\tau}_{\gamma x/2} + \frac{2}{\gamma}\tau_{\gamma x/2}]$, by

$$\begin{aligned} C_x(t) := &\ \mathbf{1}_{[0,\frac{2}{\gamma}\tau_{\gamma x/2}]}(t)\ \frac{2}{\gamma}(Y_{\gamma t/2} - 2\underline{Y}_{\gamma t/2}) \\ &+\mathbf{1}_{[\frac{2}{\gamma}\tau_{\gamma x/2}, \frac{2}{\gamma}(\tilde{\tau}_{\gamma x/2}+\tau_{\gamma x/2})]}(t)\ \frac{2}{\gamma}(\tilde{Y}_{\tilde{\tau}_{\gamma x/2}+\tau_{\gamma x/2}-\gamma t/2} - 2\underline{\tilde{Y}}_{\tilde{\tau}_{\gamma x/2}+\tau_{\gamma x/2}-\gamma t/2}), \end{aligned}$$

and $C_x(t) = 0$ when $t \notin [0, \frac{2}{\gamma}(\tilde{\tau}_{\gamma x/2} + \tau_{\gamma x/2})]$. One can show that $\mathcal{T}_x^{\mathrm{IPC}}$ is a.s. a compact real tree.

For n_k such that $n_k/k \to x$, consider the n_k-truncation of (IPC). It consists in the vertices at height below n_k on the backbone, along with all descendants of vertices at height strictly below n_k on the backbone. One may consider this tree as a continuous tree, and rescale it so that its edges have length $1/k$. We denote $(\mathrm{IPC})_x^k$ the rescaled tree.

It is then easy to prove that for any $x > 0$, the convergence (15) implies convergence of $(\mathrm{IPC})_x^k$ towards $\mathcal{T}_x^{\mathrm{IPC}}$ in the sense of weak convergence in the space of compact real trees equipped with the Gromov-Hausdorff distance (we refer to chapter 2 of [13] for precise definitions).

A similar construction is obviously also valid in the case of the IIC.

5 Level estimates

The goal of this section is to apply our convergence results to establish asymptotics for level, volume estimates of the invasion percolation cluster. In [2], it was proven that the size of the nth level of the IPC, rescaled by a factor n, converges to a non-degenerate limit. Similarly, the volume up to level n, rescaled by a factor n^2, converges to non-degenerate limit. The Laplace transforms of these limits were expressed as functions of the L-process. However formulas (1.20)–(1.23) of [2] lack insight into the limiting variables. With our convergence theorem for height functions of the $(\mathrm{IPC})_{/}$, we can express the limit in terms of the continuous limiting height function. In the case of the asymptotics of the levels, we also provide an alternative way of expressing the limit.

For $x \in \mathbb{R}_+$ we denote by $C[x]$ the number of vertices of the IPC at height $[x]$. We let $C[0,x] = \sum_{i=0}^{[x]} C[i]$ denote the number of vertices of the IPC up to height $[x]$.

For simplicity, we use the shorter notations $(h_t, t \geq 0)$ and $(H_t, t \geq 0)$, to denote the height processes of the $(\mathrm{IPC})_{/}$, and of the continuous limit of its rescaled version, which appears in the statement of Corollary 1.2. In particular, observe that

$$\frac{1}{n^2} C[0, an] = \int_0^\infty \mathbf{1}_{[0,a]}(h_{sn^2}/n) ds.$$

Also, recall that $\gamma = \frac{\sigma-1}{\sigma}$, and

$$H_t = \frac{2}{\gamma}\left(Y_{\gamma t} - \frac{3}{2}\underline{Y}_{\gamma t}\right) \overset{(\text{law})}{=} \frac{2}{\sqrt{\gamma}}\left(Z_t - \frac{3}{2}\underline{Z}_t\right), \qquad \text{where} \qquad Z_t \overset{(\text{law})}{=} B_t - \int_0^t L\left(-\underline{Z}_s\right) ds.$$

We denote by $l_t^a(H)$ the standard local time at level a, up to time t, of the semimartingale H, that is (since H has quadratic variation $2/\gamma$):

$$l_t^a(H) = \frac{2}{\gamma} \lim_{\eta \to 0} \frac{1}{\eta} \int_0^t \mathbf{1}_{[a,a+\eta]}(H_s) ds.$$

Proposition 5.1. *Let $a > 0$. We have the distributional limits*

$$\frac{1}{n^2} C[0, an] \xrightarrow[n\to\infty]{} \int_0^\infty \mathbf{1}_{[0,a]}(H_s) ds. \tag{16}$$

Furthermore,

$$\frac{1}{n} C[an] \xrightarrow[n\to\infty]{} \frac{\gamma}{4} l_\infty^a(H). \tag{17}$$

The limiting quantity in (17) can be expressed as a sum of independent contributions corresponding to distinct excursions of $Y - \underline{Y}$. These contributions are, conditionally on the L-process, independent exponential random variables. For $c > 0$, let us denote by $\mathbf{e}(c)$ an exponential variable with mean c.

Corollary 5.2. *Let S be a point process such that conditioned on the L-process, S is an inhomogeneous Poisson point process on $[0, a\sqrt{\gamma}]$, with intensity :*

$$\frac{2L\left(s\right) ds}{\exp\left((a\sqrt{\gamma} - s)L(s)\right) - 1}.$$

Then, conditionally on L, and in distribution,

$$\frac{1}{n} C[an] \xrightarrow[n\to\infty]{} \frac{\sqrt{\gamma}}{2} \sum_{s \in S} \mathbf{e}\left\{\frac{L(s)}{1 - \exp\left(-a\sqrt{\gamma} - s)L(s)\right)}\right\}, \tag{18}$$

where the terms in the sum are independent.

From this representation and immediate properties of the L-process, it is straightforward to recover the representation of the asymptotic Laplace transform of level sizes, (1.21) of [2]. Note also that, as we explain below, S is a.s. finite, and so only a finite number of distinct values of L contribute to the sum in (18).

Proof of Proposition 5.1. We start by proving (16). Our objective is the limit in distribution

$$\int_0^\infty \mathbf{1}_{[0,a]}(h_{sn^2}/n)ds \xrightarrow[n\to\infty]{} \int_0^\infty \mathbf{1}_{[0,a]}(H_s)ds.$$

This almost follows from Corollary 1.2. The problem is that $\int \mathbf{1}_{[0,a]}(X_s)ds$ is not a continuous function of the process X, and this is for two reasons. First, because of the indicator function, and second, because the topology is uniform convergence on compacts and not on all of $\mathbb{R}$.

To overcome the second we argue that for any ε there is an A such that

$$\mathbb{P}\left(\int_A^\infty \mathbf{1}_{[0,a]}(h_{sn^2}/n)ds \neq 0\right) < \varepsilon.$$

Indeed, in order for the height function to visit $[0, na]$ after time $n^2 A$ the total size of the $[na]$ sub-critical trees attached to the backbone up to height $[na]$ must be at least $[n^2 A]$. This probability is small even if the trees are replaced by $[na]$ critical trees. Thus it suffices to prove that for every A

$$\int_0^A \mathbf{1}_{[0,a]}(h_{sn^2}/n)ds \xrightarrow[n\to\infty]{dist.} \int_0^A \mathbf{1}_{[0,a]}(H_s)ds. \tag{19}$$

Next we deal with the discontinuity of $\mathbf{1}_{[0,a]}$ by a standard argument. We may bound $f_\varepsilon \leq \mathbf{1}_{[0,a]} \leq g_\varepsilon$ where $f_\varepsilon, g_\varepsilon$ are continuous and coincide with $\mathbf{1}_{[0,a]}$ outside of $[a-\varepsilon, a+\varepsilon]$. Define the operators

$$F_\varepsilon(X) = \int_0^A f_\varepsilon(X_s)ds, \qquad G_\varepsilon(X) = \int_0^A g_\varepsilon(X_s)ds.$$

Then we have a sandwich

$$F_\varepsilon(h_{n^2 s}/n) \leq \int_0^A \mathbf{1}_{[0,a]}(h_{sn^2}/n)ds \leq G_\varepsilon(h_{sn^2}/n),$$

and similarly for H_s. By continuity of the operators

$$F_\varepsilon(h_{sn^2}/n) \xrightarrow[n\to\infty]{dist.} F_\varepsilon(H_s), \qquad F_\varepsilon(h_{sn^2}/n) \xrightarrow[n\to\infty]{dist.} F_\varepsilon(H_s).$$

In the limit we have

$$G_\varepsilon(H_s) - F_\varepsilon(H_s) \xrightarrow[\varepsilon\to 0]{a.s.} 0.$$

and since $G_\varepsilon - F_\varepsilon$ is continuous we also have for any $\delta > 0$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \mathbb{P}\left(G_\varepsilon(h_{sn^2}/n) - F_\varepsilon(h_{sn^2}/n) > \delta\right) = 0.$$

Combining these bounds implies (19).

We now turn to the proof of (17). From (16), we know that for any $\eta > 0$,

$$\frac{1}{\eta n^2} C[an, (a+\eta)n] \xrightarrow[n\to\infty]{(w)} \frac{1}{\eta} \int_0^\infty \mathbf{1}_{[a,a+\eta]}(H_s) ds.$$

Thus, (17) will follow if we can prove that for any $\eta > 0$, in probability,

$$\lim_{n\to\infty} \left| \frac{\eta n C[an] - C[an, (a+\eta)n]}{\eta n^2} \right| = 0. \tag{20}$$

For a given vertex v, let h_v denote the height of v. If v is not on the backbone, we let $\mathrm{perc}(v)$ be the percolation parameter of the off-bacbone percolation cluster to which v belongs. We now single out the vertex on the backbone at height $[an]$, and group together vertices at height $[an]$ which correspond to the same percolation parameter.

More precisely, if $w_1, w_2, w_3, ..., w_{N_n}$ are the distinct values taken by the W-process up to time $[na]$, we let

$$C_n^{(w_i)} := \{v \in \mathrm{IPC} \setminus BB : h_v = [an], \mathrm{perc}(v) = w_i\},$$

so that

$$\mathfrak{C}[an] := \{v \in \mathrm{IPC}, h_v = [an]\} = \bigcup_{i=1}^{N_n} C^{(w_i)} \cup BB_{[an]}, \qquad C[an] = \#\mathfrak{C}[an].$$

Moreover, any vertex between heights $[an]$ and $[(a+\eta)n]$ in the IPC descends from one of the vertices of $\mathfrak{C}[an]$. We let

$$\begin{aligned}
\mathcal{P}_n^{(w_i)} &:= \left\{ v \in (\mathrm{IPC}) \setminus BB : [an] \le h_v \le (a+\eta)n,\ \exists w \in C^{(w_i)} \text{ s.t. } w \le v \right\}, \\
\mathcal{P}_n^{BB_{[an]}} &:= \left\{ v \in (\mathrm{IPC}) : [an] \le h_v \le (a+\eta)n, BB_{[an]} \le v \right\}.
\end{aligned}$$

In particular, $C_n^{(w_i)} \subset \mathcal{P}_n^{(w_i)}$; vertices of the backbone between heights $[an]$ and up to height $[(a+\eta)n])$ are contained in $\mathcal{P}_n^{BB_{[an]}}$ and moreover,

$$\mathfrak{C}[an, (a+\eta)n] := \{v \in \mathrm{IPC}[an] \le h_v \le (a+\eta)n\} = \bigcup_{i=1}^{N_n} \mathcal{P}_n^{(w_i)} \cup \mathcal{P}_n^{BB_{[an]}}.$$

However, the number of distinct values of percolation parameters which one sees at height $[an]$ remains bounded with arbitrarily high probability.

Claim 5.3. *For any $\epsilon > 0$, there is $A > 0$ such that, for any $n \in \mathbb{N}$,*

$$\mathbb{P}\left[\#\{i \in \{1, ..., N_n\} : |C_n^{(w_i)}| \neq 0\} > A\right] \le \epsilon.$$

From Proposition 3.1 of [2], the number of distinct values the $\hat{W}$-process takes between $[na]/2$ and $[na]$ is bounded, uniformly in n, with arbitrarily high probability. Furthermore, it is well-known that with arbitrarily high probability, among $[na]/2$ critical Galton-Watson trees, the number which reaches height $[na]/2$ is bounded, uniformly in n. It follows that the number of clusters rising from the backbone at heights $\{0, ..., [na]/2\}$ and which possess vertices at height $[na]$ is, with arbitrarily high probability, as well bounded for all n. The claim follows.

Claim 5.4. *For any $\eta > 0$, in probability,*

$$\lim_{n\to\infty} \left| \frac{1}{\eta n^2} \mathcal{P}_n^{BB_{[an]}} \right| = 0.$$

Fix η. We observe that $\mathcal{P}_n^{BB_{[an]}}$ is bounded by the total progeny up to height ηn, of ηn critical Galton-Watson trees. If $|B|$ denotes a reflected Brownian motion, and $l_t^0(|B|)$ its local time at 0 up to t, we then deduce from a convergence result for a sequence of such trees (cf formula (7) of [13]) that for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{\eta n^2} \mathcal{P}_n^{BB_{[an]}} > \epsilon\right] \leq \mathbb{P}\left[\frac{1}{\eta} \inf\{t > 0 : l_t^0(|B|) > \eta\} > \epsilon\right],$$

and the claim follows from the fact that $(\inf\{t > 0 : l_t^0(|B|) > u\}, u \geq 0)$ is a half stable subordinator.

Claim 5.5. *For any $t \in (0, a)$, $\eta > 0$, in probability,*

$$\lim_{n\to\infty} \left| \frac{\mathcal{P}_n^{(\hat{W}_{[nt]})}}{\eta n^2} - \frac{\#(C_n^{(\hat{W}_{[nt]})})}{n} \right| = 0.$$

Fix t, η, and define $w_n := \hat{W}_{[nt]}$. We have

$$\begin{aligned}
& \mathbb{P}\left[\left|\frac{\mathcal{P}_n^{(w_n)}}{\eta n^2} - \frac{\#(C_n^{(w_n)})}{n}\right| > \epsilon\right] \\
\leq\ & \mathbb{P}\left[\#(C_n^{(w_n)}) > n\epsilon^{-2}\right] + \mathbb{P}\left[\left|\frac{\mathcal{P}_n^{(w_n)}}{\eta n^2} - \frac{\#(C_n^{(w_n)})}{n}\right| > \epsilon\ ,\ \#(C_n^{(w_n)}) < \epsilon^2 n\right] \\
& + \sum_{k=[\epsilon^2 n]}^{[\epsilon^{-2} n]} \mathbb{P}(\#(C_n^{(w_n)}) = k) \mathbb{P}\left[\left|\frac{\mathcal{P}_n^{(w_n)}}{\eta n^2} - \frac{\#(C_n^{(w_n)})}{n}\right| > \epsilon \;\middle|\; \#(C_n^{(w_n)}) = k\right]
\end{aligned}$$

Using a comparison to critical trees as in the previous argument, the first two terms in the sum above go to 0 as $n \to \infty$. Furthermore, from Corollary 2.5.1 in [7], we know that, conditionally on the processes $\hat{W}$, L, for any $u > 0$, the level sets of $[un]$ subcritical Galton-Watson trees with branching law $\text{Bin}(\sigma, w_n)$ converge to the local time process of a reflected drifted Brownian motion $(|X_s|, s \geq 0)$, with drift $L(t)$, stopped at τ_u. Therefore, for any $u > 0$,

$$\begin{aligned}
& \lim_{n\to\infty} \mathbb{P}\left[\left|\frac{\mathcal{P}_n^{(w_n)}}{\eta n^2} - \frac{\#(C_n^{(w_n)})}{n}\right| > \epsilon \;\middle|\; \#(C_n^{(w_n)}) = [nu]\right] \\
& = \mathbb{P}\left[\left|\frac{1}{\eta}\int_0^{\tau_u} \mathbf{1}_{[0,\eta]}(|X_s|) ds - l_t^0(|X|)\right| > \epsilon\right],
\end{aligned}$$

which for any $\epsilon > 0$, goes to 0 as $\eta \to 0$. Thus by dominated convergence,

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sum_{k=[\epsilon^2 n]}^{[\epsilon^{-2} n]} \mathbb{P}(\#(C^{(w_n)}) = k) \mathbb{P}\left[\left|\frac{\mathcal{P}_n^{(w_n)}}{\eta n^2} - \frac{\#(C^{(w_n)})}{n}\right| > \epsilon \;\middle|\; \#(C^{(w_n)}) = k\right] = 0.$$

Claim 5.5 follows.

From our decompositions of $\mathfrak{C}[an,(a+\eta)n]$, $\mathfrak{C}[an]$, and claims 5.3, 5.4, 5.5, we now deduce (20). This implies (17), and completes the proof of Proposition 5.1. $\square$

Proof of Corollary 5.2. From (17), the corollary will be proven if we manage to express $\frac{\gamma}{4}l_\infty^a(H)$ as the righthand side of (18). Note that, if $l_t^x\left(\frac{\sqrt{\gamma}}{2}H\right)$ denotes the local time up to time t at level x of

$$\frac{\sqrt{\gamma}}{2}H = Z_t - \frac{3}{2}\underline{Z}_t,$$

then

$$\frac{\gamma}{4}l_t^a(H) = \frac{\sqrt{\gamma}}{2}l_t^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right),$$

so that we may as well express $\frac{\sqrt{\gamma}}{2}l_t^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right)$.

To reach this goal, it is convenient to decompose the path of $\frac{\sqrt{\gamma}}{2}H$ according to the excursions above the origin of $Z-\underline{Z}$. Let us introduce a few notations. We let $\mathcal{F}(\mathbb{R}_+,\mathbb{R})$ denote the space of real-valued finite paths, so that excursions of Z and of $Z-\underline{Z}$ are elements of $\mathcal{F}(\mathbb{R}_+,\mathbb{R})$. For a path $e\in\mathcal{F}(\mathbb{R}_+,\mathbb{R})$, we define $\overline{e} := \sup_{s\geq 0} e(s)$, $\underline{e} := \inf_{s\geq 0} e(s)$. For $c\geq 0$, we let $N^{(-c)}$ denote the excursion measure of drifted Brownian motion with drift $-c$ away from the origin, and $n^{(-c)}$ that of reflected drifted Brownian motion with drift $-c$ above the origin.

Claim 5.6. *For any $c>0$, $a>0$, we have*

$$n^{(-c)}(\overline{e}>a) = \frac{2c}{\exp(2ca)-1}, \tag{21}$$

$$N^{(-c)}(\underline{e}<-a) = \frac{c}{1-\exp(-2ca)} \tag{22}$$

This result is well-known, and can be proven by using basic properties of drifted Brownian motion and excursion measures.

We may and will choose $-\underline{Z}$ to be the local time process at 0 of $Z-\underline{Z}$. Using excursion theory (see for instance section VI.8.55 of [17]), we know that for this normalization of local time, conditionally on the L-process, the excursions of $Z-\underline{Z}$ form an inhomogeneous Poisson point process $\mathfrak{P}$ in the space $\mathbb{R}_+\times\mathcal{F}(\mathbb{R}_+,\mathbb{R}_+)$ with intensity $ds\times n^{(-L(s))}$.

For $b\geq 0$, let τ_b denote the hitting time of b by $-\underline{Z}$. Note that for any $s>\tau_b$, $-\underline{Z}_s>b$, from the fact that drifted Brownian motion started at 0 instantaneously visits the negative and the positive half line. Thus, the last visit to $\frac{\sqrt{\gamma}}{2}a$ by $\frac{\sqrt{\gamma}}{2}H$ is $\tau_{a\sqrt{\gamma}}$. Hence, any point of $\mathfrak{P}$ whose first coordinate is larger than $a\sqrt{\gamma}$ corresponds to a part of the path of H which lies strictly above a, and therefore can not contribute to $l_\infty^a(H)$. Moreover, a part of the path of $\frac{\sqrt{\gamma}}{2}H$ which corresponds to an excursion of $Z-\underline{Z}$ starting at a time $t<\tau_{a\sqrt{\gamma}}$ will only reach height $\frac{\sqrt{\gamma}}{2}a$ whenever the supremum of this excursion is greater or equal than $\frac{1}{2}(a\sqrt{\gamma}-\underline{Z}_t)$. Therefore, any excursion of $Z-\underline{Z}$ which gives a nonzero contribution to $l_\infty^a(H)$ corresponds to a point of $\mathfrak{P}$ whose first coordinate is some s, such that $s\leq a\sqrt{\gamma}$, and whose second coordinate is an excursion e such that $\overline{e}\geq\frac{1}{2}(a\sqrt{\gamma}-s)$.

These considerations, along with properties of Poisson point processes, lead to the following claim.

Claim 5.7. *Conditionally on the L-process, the excursions of $Z - \underline{Z}$ which give a nonzero contribution to $\frac{\gamma}{4} l^a_\infty(H) = \frac{\sqrt{\gamma}}{2} l_\infty^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right)$ are points of a Poisson point process $\mathcal{P} \subset \mathfrak{P}$ on $\mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity*

$$\mathbf{1}_{[0,a\sqrt{\gamma}]}(s) n^{(-L(s))}\left(\overline{e} \geq \frac{1}{2}(a\sqrt{\gamma} - s)\right) \; ds \;\times\; n^{(-L(s))}\left(\;\cdot\;,\;\overline{e} \geq \frac{1}{2}(a\sqrt{\gamma} - s)\right)$$

The number of points of $\mathcal{P}$ clearly is almost surely countable, so we may write $\mathcal{P} = (s_i, e_i)_{i\in\mathbb{Z}_+}$. In particular, by (21), $(s_i)_{i\in\mathbb{Z}_+}$ are the points of the Poisson point process on $[0, a\sqrt{\gamma}]$ introduced in Corollary 5.2.

Note that $\{e_i, i \in \mathbb{Z}_+\}$ correspond obviously to distinct excursions of $Z - \underline{Z}$, so that their contributions to $l_\infty^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right)$ are independent.

Claim 5.8. *For any $i \in \mathbb{Z}_+$, the contribution of the excursion e_i to $l_\infty^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right)$ is, conditionally given L, exponentially distributed with parameter*

$$N^{(-L(s))}\left(\underline{e_i} \leq \frac{1}{2}(-a\sqrt{\gamma} + s_i)\right).$$

Fix $i \in \mathbb{Z}_+$, and condition on L. Recall that (s_i, e_i) is one of the points of the Poisson process $\mathcal{P}$, so that e_i is chosen according to the measure $n^{(-L(s_i))}\left(\;\cdot\;, \overline{e} > \frac{1}{2}(a\sqrt{\gamma} - s_i)\right)$. Up to the time at which e_i reaches $\frac{1}{2}(a\sqrt{\gamma} - s_i)$, e_i does not contribute to $l_\infty^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right)$. From the Markov property under $n^{(-L(s_i))}\left(\;\cdot\;, \overline{e} > \frac{1}{2}(a\sqrt{\gamma} - s_i)\right)$, the remaining part of e_i (after it has reached $\frac{1}{2}(a\sqrt{\gamma} - s_i)$ follows the path of a drifted Brownian motion, with drift $-L(s_i)$, started at $\frac{1}{2}(a\sqrt{\gamma} - s_i)$, and stopped when it gets to the origin. Thus, the contribution of e_i to $l_\infty^{\frac{\sqrt{\gamma}}{2}a}\left(\frac{\sqrt{\gamma}}{2}H\right)$ is exactly the local time of this stopped drifted Brownian motion at level $\frac{1}{2}(a\sqrt{\gamma} - s_i)$. By shifting vertically, it is also $l^0_\infty(X)$, the total local time at the origin of X, a drifted Brownian motion, with drift $-L(s_i)$, started at the origin and stopped when reaching $\frac{1}{2}(-a\sqrt{\gamma} + s_i)$. By excursion theory, if $\tilde{\mathfrak{P}}_i$ is a Poisson point process on $\mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+, \mathbb{R})$ with intensity $ds \times N^{(-L(s_i))}$, then $l^0_\infty(X)$ is the coordinate of the first point of $\tilde{\mathfrak{P}}_i$ which falls into the set

$$\mathbb{R}_+ \times \left\{ e \in \mathcal{F}(\mathbb{R}_+, \mathbb{R}) : \underline{e} < \frac{1}{2}(-a\sqrt{\gamma} + s_i)\right\},$$

Claim 5.8 follows.

From Claim 5.7 and the remark which follows it, Claim 5.8 and (22), we deduce Corollary 5.2. □

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