Université Paris Diderot

Monday November 13th, 2017

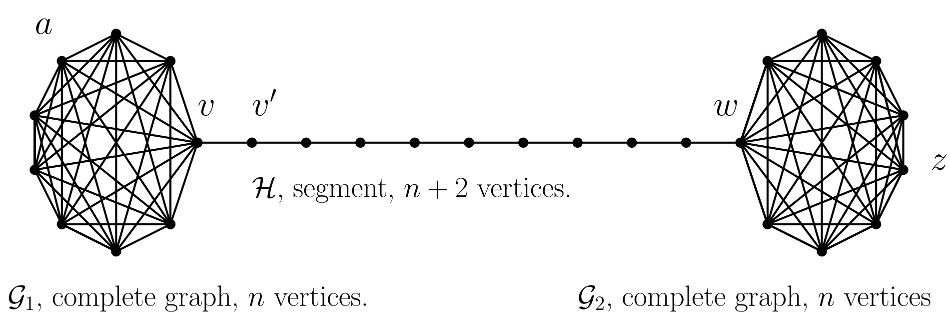
Markov Chains

Exam

Class notes allowed — Books, electronic equipment forbidden

Exercice 1 The barbell graph

Let n be an integer, $n \geq 3$, and the graph $\mathcal{G}$ with $3n$ vertices obtained by linking two copies of the complete graph with n vertices by a segment of length $n+1$.
More precisely let $\mathcal{G}_1, \mathcal{G}_2$ the two copies of the complete graph with n vertices, and $\mathcal{H}$ the segment with $n+2$ vertices linking the two. The vertex of $\mathcal{G}_1$ identified with the first extremity of $\mathcal{H}$ is denoted v, that of $\mathcal{G}_2$ identified with the other extremity is denoted w. Let us further denote a one vertex of $\mathcal{G}_1$ distinct from v and z a vertex of $\mathcal{G}_2$ distinct from w, and finally v' denotes the neighbour of v in $\mathcal{H}$ — see the picture below.



The graph $\mathcal{G}$, for $n = 9$.

We let $(X_t, t \geq 0)$ the continuous-time simple random walk on $\mathcal{G}$ (when at x, it jumps to any neighbour of x, independently at rate 1), and $(Y_k, k \geq 0)$ the associated jump chain. We use the notation $\mathbb{P}^X$ for the law of X, and $\mathbb{P}^Y$ for the law of Y.
We are interested, for both chains, in the hitting time of z starting from a.

1. Is the chain Y irreducible, reversible, aperiodic ? Is it positive recurrent ? Find the set of its invariant distributions.
2. What can be deduced for the continuous-time chain X ? What about its invariant distributions ?
3. Compute $\mathcal{R}(a \leftrightarrow z)$, deduce that

$$\mathbb{E}_a^Y[T_z] = \frac{(n^2+n+4)(n^2+1)}{n},$$

and find an equivalent of this quantity when $n \to \infty$.
Compute $\mathbb{E}_a^X[T_z]$ and find an equivalent of this quantity when $n \to \infty$.

4. In this question we work with the discrete-time chain Y.
   Let $B_1 = \mathcal{G}_1 \setminus \{v\}$, $B_2 = \mathcal{G}_2 \setminus \{w\}$ and recall that v' is the neighbour of v in $\mathcal{H}$. Show that $\mathbb{P}^Y_{v'}(T_{B_2} < T_{B_1}) = \frac{n}{n^2+1}$. Deduce that under $\mathbb{P}^Y_a$, $T_{B_2} \le G$, where $G \sim \text{Geom}\left(\frac{1}{(n-1)(n^2+1)}\right)$. What is, under $\mathbb{P}^Y$, the limit in law of G/n^3 when $n \to \infty$?
   Compare the asymptotic behaviour of $\mathbb{E}[G], \mathbb{E}^Y_a[T_{B_2}], \mathbb{E}^Y_a[T_z]$.
   Deduce the limit in law of T_{B_2}/n^3, and then that of T_z/n^3, as $n \to \infty$.

1. The discrete chain is irreducible because the graph is connex. The graph is finite so the chain is positive recurrent. The graph $\mathcal{G}_1$ has at least three vertices, so starting from a, the chain Y comes back to a in 2 (resp. 3) steps with positive probability, hence the chain Y is aperiodic.
   In fact the chain corresponds to a conductance model (where each edge is equipped with conductance 1) so it is reversible, in particular it possesses a unique invariant distribution (say π) such that $\pi(x) = \frac{d_x}{d_{\mathcal{G}}}, x \in \mathcal{V}$. More precisely
   $$c_{\mathcal{G}} = 2(n-1)^2 + 2n + 2n = 2n^2 + 2,$$
   so if $x \in \mathcal{G}_1 \cup \mathcal{G}_2 \setminus \{v, w\}, \pi(x) = \frac{n-1}{2n^2+2}$, if $x \in \{v, w\}$, $\pi(x) = \frac{n}{2n^2+2}$ and if $x \in \mathcal{H} \setminus \{v, w\}$, $\pi(x) = \frac{2}{2n^2+2}$.
2. Since Y is, the chain X is irreducible, positive recurrent. It has a unique invariant distribution, and it is (say λ) the uniform distribution on $\mathcal{V}$. Indeed $Q(x, y) = Q(y, x)$, so X is reversible with respect to λ.
3. Let us compute $\mathcal{R}(a \leftrightarrow z)$ by looking at the potential associated with a current from a to z. We notice that vertices of $\mathcal{G}_1$ distinct from a, v play symmetric roles, so they must have same potential and we can identify them as a single vertex, say y. Now, vertex a is connected to y by $n-2$ edges of conductance 1, equivalent to a unique conductance $n-2$, and y is also connected to v by $n-2$ edges of conductance 1. The two resistances in series add up to a resistance $2/(n-2)$, but let us not forget the one edge between a and v, so the effective conductance between a and v is $(n-2)/2 + 1 = n/2$. By a similar reasoning the effective resistance between w and z is also $n/2$, and of course the effective resistance between v and w is $n+1$. It remains to sum up resistances in series to obtain
   $$\mathcal{R}(a \leftrightarrow z) = \frac{4}{n} + n + 1 = \frac{n^2+n+4}{n}.$$
   By symmetry $\mathbb{E}^Y_a[T_z] = \mathbb{E}^Y_z[T_a]$, so using the commute time identity for discrete-time chains we find
   $$\mathbb{E}^Y_a[T_z] = \frac{c_{\mathcal{G}}\mathcal{R}(a \leftrightarrow z)}{2} = \frac{(n^2+1)(n^2+n+4)}{n},$$
   as required, and $\mathbb{E}^Y_a[T_z] \sim n^3$ as $n \to \infty$.
   On the other hand, again by symmetry $\mathbb{E}^X_a[T_z] = \mathbb{E}^X_z[T_a]$, and so by the commute time identity for continuous-time chains,
   $$\mathbb{E}^X_a[T_z] = |\mathcal{V}|\frac{\mathcal{R}(a \leftrightarrow z)}{2} = \frac{3n}{2}\frac{n^2+n+4}{n},$$
   so that $\mathbb{E}^X_a[T_z] \sim \frac{3n^2}{2}$ as $n \to \infty$.

4. In this part we only deal with the discrete-time chain, and so we drop the superscript Y from our notation.

   Let us first compute, as suggested, $\mathbb{P}_{v'}(T_{B_2} < T_{B_1})$. Identifying vertices of B_1 as, say, y_1, and those of B_2 as, say, y_2, it is easily seen that the effective conductance between v' and y_1 is $(n-1)/n$, and that between v' and y_2 is $(n-1)/(n^2-n+1)$, thus

   $$\mathbb{P}_{v'}(T_{B_2} < T_{B_1}) = \frac{\frac{n-1}{n}}{\frac{n-1}{n} + \frac{n-1}{n^2-n+1}} = \frac{n}{n^2+1}$$

   Now, for the chain to go from any $a \in B_1$ to B_2 before returning to B_1, it must go at time 1 to v and at time 2 to v' (otherwise it has returned to B_1 in the first or second step), and then it must go from v' to B_2 before it returns to B_1. Thus

   $$\mathbb{P}_a(T_{B_2} < T^+_{B_1}) = \frac{1}{n-1}\frac{1}{n}\frac{n}{n^2+1} = \frac{1}{n-1}\frac{1}{n^2+1},$$

   thus, thanks to the Markov property the number of visits to B_1 before hitting B_2, that is $G := \sum_{k=0}^{T_{B_2}-1} \mathbb{1}_{\{Y_k \in A\}}$, is geometric with parameter $1/((n-1)(n^2+1))$. Of course T_{B_2} is larger than G, which establishes the desired result.

   Now, by e.g. looking at moment generating functions, it is easily seen that G/n^3 converges in distribution as $n \to \infty$ towards an exponential variable with parameter 1.

   Finally we observe that $n^3 \sim \mathbb{E}[G] \le \mathbb{E}_a[T_{B_2}] \le \mathbb{E}_a[T_z] \sim n^3$, so that the expected time spent outside of B_1 before reaching B_2, $\mathbb{E}_a[T_{B_2} - G]$ is such that $n^{-3}\mathbb{E}_a[T_{B_2} - G] \to 0$. Since we are looking at a nonnegative variable this implies $n^{-3}(T_{B_2} - G) \to 0$ in probability, and so $n^{-3}G$ and $n^{-3}T_{B_2}$ share the same limit in distribution, and we conclude that

   $$n^{-3}T_{B_2} \xrightarrow[n\to\infty]{\text{(law)}} \mathbf{e}_1,$$

   with $\mathbf{e}_1$ is an exponential variable with parameter 1.

Exercice 2 Let $(X(t))_{t\ge0}$ be the continuous-time simple random walk on the hypercube $\{-1,1\}^d$, which, when at x waits for an exponential random time of parameter 1, and then jumps to one of the d neighbours of x choosen uniformly at random.
When $x \in \{-1,1\}^d$ and $i \in \{1,...,d\}$, we write $x_i \in \{-1,1\}$ for the ith coordinate of x and $x^i = (x_1, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_d)$ for the neighbour of x along the ith coordinate.
Moreover we let $\mathcal{F}_t = \sigma(X_s, s \ge 0)$ for $t \ge 0$.
Finally we write $\mathbf{1} = (1, ..., 1) \in \{-1,1\}^d$ and $\mathbb{P}_{\mathbf{1}}$ for the law of the chain started at $\mathbf{1}$.

1. Express the generator Q of the chain.
2. For $f : \{-1,1\}^d \to \mathbb{R}$, and $x \in \{-1,1\}^d$, establish that

   $$Qf(x) = \frac{1}{d}\sum_{i=1}^{d}(f(x^i) - f(x)).$$

3. Does the convergence theorem apply, and if so, what does it state ?

4. Is $(X_1(t), t \geq 0)$ a Markov chain ?
Let $g(t) = \mathbb{P}_{\mathbf{1}}(X_1(t) = 1), t \geq 0$, show that

$$\exp(t/d)g(t) = 1 + \frac{1}{d}\int_0^t \exp(u/d)(1 - g(u))du,$$

and deduce that $g(t) = \frac{1}{2}(1 + \exp(-2t/d))$ for any $t \geq 0$.

5. For $J \subset \{1, ..., d\}$, let $f_J(x) = \prod_{j \in J} x_j$. Show that $Qf_J = \lambda_J f_J$, for a λ_J which you shall compute. Find all eigenvalues of Q with multiplicity.

6. Establish that $\left(M_t^J := f_J(X(t))\exp(2t|J|/d)\right)_{t \geq 0}$ is an $(\mathcal{F}_t)$-martingale, where $|J|$ denotes the cardinal of J. Deduce $\mathbb{P}_{\mathbf{1}}\left(\prod_{j \in J} X_j(t) = 1\right)$ for $t \geq 0$. Check in particular that you recover the result of question 3.

7. Explain why we could also have established directly that

$$\mathbb{P}_{\mathbf{1}}\left(\prod_{j \in J} X_j(t) = 1\right) = \sum_{I \subset J,\ |I| \text{even}} \prod_{i \in I} \mathbb{P}_{\mathbf{1}}(X_i(t) = -1) \prod_{j \in J \setminus I} \mathbb{P}_{\mathbf{1}}(X_j(t) = 1),$$

and recover the result of question 5. You may first check that if $j = \lfloor |J|/2 \rfloor$,

$$\left(\frac{1+x}{2} + \frac{1-x}{2}\right)^{|J|} + \left(\frac{1+x}{2} - \frac{1-x}{2}\right)^{|J|} = 2\sum_{k=0}^{j} \binom{|J|}{2k} \left(\frac{1-x}{2}\right)^{2k} \left(\frac{1+x}{2}\right)^{|J|-2k}.$$

8. Let $(Y(n), n \in \mathbb{N})$ be the *discrete-time, lazy* simple random walk on $\{-1, 1\}^d$, i.e. with jump kernel $P = \frac{Q}{2} + I$. Establish that

$$\mathbb{P}_{\mathbf{1}}\left(\prod_{j \in J} Y_j(n) = 1\right) = \frac{1 + \left(1 - \frac{|J|}{d}\right)^n}{2}.$$

1. For any $x \in \{-1, 1\}^d$, , we have

$$Q_{x,x} = -1, \quad Q_{x,x^i} = \frac{1}{d}, i \in \{1, ..., d\}.$$

2. It follows directly that for any $x \in \{-1, 1\}^d$,

$$Qf(x) = \frac{1}{d}\sum_{i=1}^{d}(f(x^i) - f(x))$$

3. The chain is irreducible, the state space is finite, theorem convergence applies. Since the invariant distribution is the uniform one (which gives mass $2{-}d$ to each element of $\{-1, 1\}^d$), the theorem states that whatever the initial distribution μ, whatever $x \in \{-1, 1\}^d$,

$$\mathbb{P}_\mu(X_t = x) \to \frac{1}{2^d}.$$

4. Since $Q(x, x^1) = 1/d$ whatever x, the process $(X_1(t), t \geq 0)$ is itself a continuous-time Markov chain on $\{-1, 1\}$, which changes sign at rate $1/d$. We write $\mathbb{P}_1$ for the law of this chain when started at 1. By Markov at the first jump of X_1 (when it happens before t), we have

$$\begin{aligned} g(t) &= \exp(-t/d) + \int_0^t \frac{1}{d} \exp(-s/d) \mathbb{P}_1(X_1(t) = 1 \mid X_1(s) = -1) ds \\ &= \exp(-t/d) + \int_0^t \frac{1}{d} \exp(-s/d) \left(1 - \mathbb{P}_1(X_1(t-s) = 1)\right) ds \end{aligned}$$

where we used Markov property at time s and the fact that 1 and -1 play symmetric roles so that $\mathbb{P}_{-1}(X_1(t-s) = 1) = \mathbb{P}_1(X_1(t-s) = -1) = 1 - \mathbb{P}_1(X_1(t-s) = 1)$. Now, timing by $\exp(t/d)$ and changing variables $u = t - s$ we get

$$g(t) \exp(t/d) = 1 + \frac{1}{d} \int_0^t \exp(u/d)(1 - g(u)) du,$$

as desired. Differenciating, then timing by $\exp(-t/d)$, it comes that

$$g'(t) = \frac{1}{d} - \frac{2}{d} g(t), \quad g(0) = 1,$$

and it is then easy to check that the unique solution to this ODE is

$$g(t) = \frac{1}{2} \left(1 + \exp(-2t/d)\right), \quad t \geq 0.$$

5. By question 2, we have $Qf_J(x) = \frac{1}{d} \sum_{i=1}^d (f_J(x^i) - f_J(x))$. Now observe that whatever $x \in \{-1, 1\}^d$, if $i \in J$, $f_J(x^i) = -f_J(x)$, while if $i \notin J$, $f_J(x^i) = f_J(x)$. It follows that for any $x \in \{-1, 1\}^d$,

$$Qf_J(x) = \frac{1}{d} \sum_{i \in J} (-2 f_J(x)) = -\frac{2|J|}{d} f_J(x),$$

that is, f_J is an eigenfunction of Q associated with the eigenvalue $\lambda_{|J|} = -\frac{2|J|}{d}$. For $k \in \{0, ..., d\}$, there are $\binom{d}{k}$ manners of choosing $J \subset \{1, ..., d\}$ such that $|J| = k$, and it is easily seen that the corresponding eigenfunctions form a linearly independent family, so eigenvalue λ_k has multiplicity at least $\binom{d}{k}$. Now $\sum_{k=0}^d \binom{d}{k} = 2^d$, hence we have determined all eigenvalues and corresponding multiplicities.

6. Observe first that f_J is bounded so integrability condition is obvious. Recall further that f_J is a eigenfunction of Q associated with eigenvalue $-\frac{2|J|}{d}$, so it is also an eigenfunction of $P(t) = \exp(tQ)$ associated with eigenvalue $\exp(-2t|J|/d)$, and it follows that

$$\begin{aligned} \mathbb{E}[\exp(2(t+s)|J|/d) f_J(X(t+s)) \mid \mathcal{F}_s] &= \exp(2(t+s)|J|/d) P(t) f_J(X(s)) \\ &= \exp(2(t+s)|J|/d) \exp(-2t|J|/d) f_J(X(s)) = M_s^J \end{aligned}$$

One could also have directly quoted exercise IV.12.

Now for any $t \geq 0$,

$$\begin{aligned} 1 &= \mathbb{E}_{\mathbf{1}}[M_t^J] \\ &= \mathbb{P}_{\mathbf{1}}(f_J(X(t)) = 1)\exp(2t|J|/d) - (1 - \mathbb{P}_{\mathbf{1}}(f_J(X(t)) = 1))\exp(2t|J|/d), \end{aligned}$$

and thus

$$\mathbb{P}_{\mathbf{1}}\left(\prod_{j\in J} X_j(t) = 1\right) = \mathbb{P}_{\mathbf{1}}(f_J(X(t)) = 1) = \frac{1 + \exp(2t|J|/d)}{2}.$$

For $J = \{1\}$ we recover the result of question 3.

7. Jump times of X are those of a Poisson process with rate 1. For any $i \in \{1, ..., d\}$, each jump is a jump of X_i with probability $1/d$, independently of other jumps. By properties of Poisson processes, jumps of $\{X_i, i \in \{1, ..., d\}\}$ are those of d independent Poisson processes with the same rate $1/d$. It follows that $\{X_i, i \in \{1, ..., d\}\}$ are d independent copies of X_1. Moreover,

$$\begin{aligned} \mathbb{P}_{\mathbf{1}}\left(\prod_{j\in J} X_j(t) = 1\right) &= \mathbb{P}_{\mathbf{1}}\left(\text{ an even number of } \{X_j(t), j \in J\} \text{ equal } -1\right) \\ &= \sum_{I\subset J, |I| \text{even}} \mathbb{P}_{\mathbf{1}}\left(X_i(t) = -1\ \forall i \in I,\ X_j(t) = 1\ \forall j \in J \setminus I\right) \\ &= \sum_{I\subset J, |I| \text{even}} \mathbb{P}_1(X_1(t) = 1)^{|J|-|I|}(1 - \mathbb{P}_1(X_1(t) = 1))^{|I|} \end{aligned}$$

By question 3 or 6, $\mathbb{P}(X_1(t) = 1) = \frac{1}{2}(1 + \exp(-2t/d))$, and there are $\binom{|J|}{2k}$ subsets of J with cardinality $2k$, hence

$$\begin{aligned} \mathbb{P}_{\mathbf{1}}\left(\prod_{j\in J} X_j(t) = 1\right) &= \sum_{k=0}^{\lfloor |J|/2} \binom{|J|}{2k}\left(\frac{1 + \exp(-2t/d)}{2}\right)^{|J|-2k}\left(\frac{1 - \exp(-2t/d)}{2}\right)^{2k} \\ &= \frac{1 + \exp(-2t|J|/d)}{2} \end{aligned}$$

where the last line comes from the general formula suggested in the statement of the exercise, which itself follows directly from binomial expansion.

8. Since the jump kernel of Y is $P = \frac{Q}{2} + I$, P has the eigenfunction f_J associated with eigenvalue $1 - \frac{|J|}{d}$. Thus if $|J| < d$, $\left(R_n^J := f_J(Y(n))(1 - \frac{|J|}{d})^{-n}, n \geq 0\right)$ is an $\mathcal{F}^Y$-martingale, and it follows easily that

$$\mathbb{P}(f_J(Y(n)) = 1) = \frac{1 + (1 - \frac{|J|}{d})^n}{2}.$$

When $J = \{1, ..., d\}$, we simply have $Pf_J = 0$, in particular for any $n \in \mathbb{N}^*$ (no matter what the starting point) $\mathbb{E}[f_J(Y(n)) \mid \mathcal{F}_{n-1}] = Pf_J(Y(n-1)) = 0$, so $\mathbb{E}[f_J(Y(n))] = 0$, and this implies, for this particular choice of J, $\mathbb{P}_{\mathbf{1}}(f_J(Y(n)) = 1) = \frac{1}{2}, n \in \mathbb{N}^*$, agreeing with the general formula.

Exercice 3 Let $(X_t), t \geq 0$ a Markov process on $\mathbb{N}$ with X_t representing the size of a population at time $t \geq 0$. Each individual, independently of others, dies at rate 1, and at its death is immediately replaced by an independent random number of individuals, more precisely by $0, 2$ or 3 individuals with respective probabilities p_0, p_2, p_3, with $p_0 + p_2 + p_3 = 1$. We assume in addition that $0 < p_0 < 1$.

1. Check that the generator of $(X_t)_{t\geq 0}$ is such that for any $n \in \mathbb{N}$,
$$Q_{n,n} = n, \quad Q_{n,n-1} = np_0,$$
$$Q_{n,n+1} = np_2, \quad Q_{n,n+2} = np_3$$
Draw the diagramm of X.
2. Find the communication classes of the chain.
3. In the rest of the exercise we aim at computing $h(t) := \mathbb{P}_1(X_t = 0)$. Establish first that $\mathbb{P}_i(X_t = 0) = h(t)^i, i \in \mathbb{N}$. Deduce, using Markov property at the first jump of the chain, that
$$h(t) = \int_0^t e^{-s}(p_0 + p_2 h(t-s)^2 + p_3 h(t-s)^3)ds.$$
4. Establish that
$$h'(t) = (1 - h(t))\left(p_0 - (p_2 + p_3)h(t) - p_3 h(t)^2\right), \quad h(0) = 0.$$
5. Let α, β the roots of $p_3 X^2 + (p_2 + p_3)X - p_0$, with $\alpha < 0$ and $\beta > 0$. We assume in this question that $\beta \neq 1$. Show that if we set $a = \frac{1}{(1-\alpha)(1-\beta)}, b = \frac{1}{(\beta-\alpha)(1-\alpha)}, c = \frac{1}{(\beta-\alpha)(1-\beta)}$, one finds that for $t \geq 0$,
$$(1 - h(t))^{-a}(h(t) - \alpha)^b(\beta - h(t))^{-c} = (-\alpha)^b \beta^{-c} \exp(p_3 t).$$
6. In this last question we assume that $p_0 = 1 - p$ and $p_3 = 0$. By a reasoning similar as in the above, show that
$$h'(t) = 1 - p - h(t) + ph(t)^2, \quad h(0) = 0.$$
Compute $h(t)$ (one shall distinguish the cases $p \neq 1/2, p = 1/2$, and prove for example that if $p \neq 1/2$, $h(t) = \frac{1-\exp((1-2p)t)}{\frac{p}{1-p} - \exp((1-2p)t)}$). Discuss the asymptotic behaviour of $h(t)$ when $t \to \infty$. What does it mean for the chain X ?

1. This is straightforward (as in IV.3 of the class notes).
2. First observe that $Q_{0,0} = 0$ so 0 always is an absorbing state. Now fix $n \in \mathbb{N}^*$. Since $p_0 > 0$, we have $n \to n-1 \to ... \to 1$. Also since $p_0 < 1$ either p_2 or p_3 is positive. If $p_2 > 0$ then $1 \to 2 \to 2... \to n$, and then $1 \leftrightarrow n$. Otherwise $p_3 > 0$ and then $1 \to 3 \to ... \to 2\lfloor n/2 \rfloor + 1$. But even if n is even, $n+1 \to n$ so in the end 1 and n are always in the same class.

   We conclude that the chain has two classes : $\{0\}$ which corresponds to an absorbing state and $\mathbb{N}^*$ which is transient.

3. Since the descendances of different individuals in the population at a given time are independent, the process started with i individuals corresponds to the sum of i independent copies of the process started with a single individual. Thus under $\mathbb{P}_i$, $X_t = \sum_{k=1}^{i} X_t^k$, where $X^1, ..., X^k$ are i.i.d with the same law as X under $\mathbb{P}_1$. It follows in particular that

$$\mathbb{P}_i(X_t = 0) = (\mathbb{P}_1(X_t = 0))^i = h(t)^i, \ \forall t \geq 0.$$

Let J_1 denotes the first jump time of the chain, so J_1 is a stopping time. Moreover, $\mathbb{P}_1(X_t = 0, J_1 > t) = 0$, finally J_1 is exponential with parameter one, and at time J_1 the chain jumps to 0 with probability p_0, to 2 with probability p_2 and to 3 with probability p_3. By using the Markov property at J_1, then the beginning of the question, we find

$$\begin{aligned} h(t) &= \int_0^t \exp(-s) \left[p_0 + p_2\mathbb{P}_2[X_{t-s} = 0] + p_3\mathbb{P}_3[X_{t-s} = 0]\right] ds \\ &= \int_0^t \exp(-s) \left[p_0 + p_2 h(t-s)^2 + p_3 h(t-s)^3\right] ds, \end{aligned}$$

as required.

4. Changing variables $u = t - s$, multiplying by $\exp(t)$ and differentiating yields

$$h(0) = 0, \quad h'(t)\exp(t) + h(t)\exp(t) = \exp(t)\left[p_0 + p_2 h(t)^2 + p_3 h(t)^3\right],$$

so that

$$h(0) = 0, \quad h'(t) = p_0 - h(t) + p_2 h(t)^2 + p_3 h(t)^3 = (1 - h(t))\left(p_0 - (p_2 + p_3)h(t) - p_3 h(t)^2\right).$$

5. We have thus

$$h(0) = 0, h'(t) = p_3(1 - h(t))(h(t) - \alpha)(\beta - h(t))$$

It is then straightforward to check that

$$\frac{1}{(1-x)(x-\alpha)(\beta-x)} = \frac{a}{1-x} + \frac{b}{x-\alpha} + \frac{c}{\beta-x},$$

so we have

$$h(0) = 0, \quad \frac{ah'(t)}{1-h(t)} + \frac{bh'(t)}{h(t)-\alpha} + \frac{ch'(t)}{\beta-h(t)} = p_3$$

hence

$$(1-h(t))^{-a}(h(t)-\alpha)^b(\beta-h(t))^{-c} = (-\alpha)^b\beta^{-c}\exp(p_3 t).$$

6. By the same reasoning as above

$$h(0) = 0, \quad h'(t) = (1-p) - h(t) + ph(t)^2 = (1-h(t))((1-p) - ph(t)).$$

Assume first $p \neq 1/2$, then

$$\frac{1-2p}{(1-x)(1-p-px)} = \frac{1}{1-x} - \frac{p}{1-p-px},$$

thus

$$h(0) = 0, \quad \frac{h'(t)}{1 - h(t)} - \frac{ph'(t)}{1 - p - ph(t)} = 1 - 2p.$$

It follows that

$$\frac{1 - p - ph(t)}{1 - h(t)} = (1 - p)\exp((1 - 2p)t),$$

and finally

$$h(t) = \frac{1 - \exp((1 - 2p)t)}{\frac{p}{1-p} - \exp((1 - 2p)t)}.$$

When $p > 1/2$, $p/(1 - p) > 1$, and the exponentials in the above converge to 0 as $t \to \infty$. Thus $h(t) \to \frac{1-p}{p}$, and the convergence is exponentially fast. In particular, there is a positive probability that the process survives forever, and this probability equals $1 - \lim_{t\to\infty} h(t) = 1 - \frac{1-p}{p} = \frac{2p-1}{p}$.

When $p < 1/2$, $\exp((1 - 2p)t)$ diverges, so $h(t)$ is better expressed by multiplying numerator and denominator by $-\exp((2p - 1)t)$ to get

$$h(t) = \frac{1 - \exp((2p - 1)t}{1 - \frac{p}{1-p}\exp((2p - 1)t)}$$

which is $1 - \frac{1-2p}{1-p}(\exp((2p - 1)t)) + o(\exp((2p - 1)t))$, yielding that for $p < 1/2$, the probability that the population has gone extinct by time t converges to 1 at exponential speed.

Finally, when $p = 1/2$, we have

$$h(0) = 0, \quad h'(t) = \frac{1}{2}(1 - h(t))^2,$$

so that $\frac{1}{1-h(t)} = 1 + \frac{t}{2}$ and $h(t) = \frac{1}{1+2/t}$. Again here the probability that the population has gone extinct by time t converges to 1, only this time the convergence speed is only polynomial, since $h(t) = 1 - \frac{2}{t} + o(1/t)$.