Université Paris Diderot

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Markov chains

Exercises

1 Linear algebra for Markov chains

Exercise 1 Assume P is a $N \times N$ stochastic matrix. Depending on context, we may use p_{ij}, P_{ij} or $P(i,j)$ to designate the entry of P at the ith row and jth column. Recall P is stochastic whenever

— all entries of P are nonnegative : for any $1 \leq i, j \leq N$, $p_{ij} \geq 0$,

— for any $1 \leq i \leq N$, $\sum_{j=1}^{N} p_{ij} = 1$.

In other words, every row of P can be thought of as a probability distribution on $E = \{1, ..., N\}$, and P as a transition kernel on E.
Similarly, we may and will confuse a $1 \times N$ matrix $(\mu(1) \ldots \mu(N)))$, with nonnegative entries such that $\sum_{i=1}^{N} \mu(i) = 1$ with the probability measure μ on E. Finally, we may and will confuse a $N \times 1$ $\mathbb{C}$-valued matrix $\begin{pmatrix} f(1) \\ \vdots \\ f(N) \end{pmatrix}$ with the function $f : E \to \mathbb{C}$.

1. Show that if P is stochastic, so is P^n for any $n \in \mathbb{N}$.
2. Show that if μ is a probability measure on E (described by a $1 \times N$ matrix as above), so is μP^n for any $n \in \mathbb{N}$.
3. Let X be a (discrete-time) Markov chain with kernel P, μ a probability measure on $\{1, ..., N\}$ and $f : \{1, ..., N\} \to \mathbb{C}$, Express as synthetically as possible , for $n \in \mathbb{N}$, $i, j, x_0, \ldots x_n$ elements of $\{1, ..., N\}$, μ and f as above
   (a) $\mathbb{P}_\mu(X_0 = x_0, ..., X_n = x_n)$,
   (b) $\mathbb{P}_i(X_n = j)$,
   (c) $\mathbb{P}_\mu(X_n = j)$,
   (d) $\mathbb{E}_i[f(X_n)]$,
   (e) $\mathbb{E}_\mu[f(X_n)]$.

1. We use an induction. The claim is obvious when $n = 0$ since $P^0 = Id$. Now assume the claim for some $n \in \mathbb{N}$. Obviously P^{n+1} possesses nonnegative entries. Moreover, for any $i \in \{1, ..., N\}$,

$$\sum_{j=1}^{N} P^{n+1}(i,j) = \sum_{j=1}^{N} \sum_{\ell=1}^{N} P^n(i,\ell) P(\ell,j) = \sum_{\ell=1}^{N} P^n(i,\ell) \sum_{j=1}^{N} P(\ell,j),$$

now for any ℓ, $\sum_{j=1}^{N} P(\ell,j) = 1$ since P is stochastic, and $\sum_{\ell=1}^{N} P^n(i,\ell) = 1$ by induction hyspothesis, so we are done.

2. The reasoning is similar as that of the previous question, using $\sum_{j=1}^N \mu P^{n+1}(j) = \sum_{\ell=1}^N \mu P^n(\ell)P(\ell,j)$.

3. (a) $\mu(x_0)\prod_{i=0}^{n-1} P(x_i,x_{i+1})$.
   (b) $P^n(i,j)$
   (c) $\mu P^n(j)$
   (d) $(P^n f)_i$
   (e) $\mu P^n f$.

Exercise 2 Let P be a $N \times N$ stochastic matrix (if you prefer, a transition kernel on $E = \{1, ..., N\}$). Let $\lambda_1, ..., \lambda_N$ the (possibly complex, possibly multiple) eigenvalues of P.

1. What are the eigenvalues of P^T ?
2. Show that $1 \in \{\lambda_1, ..., \lambda_N\}$
3. Show that $r(P) := \max_{1\le i\le N} |\lambda_i| = 1$. What is $r(P^T)$?
4. Let P be stochastic and such that $P(N,1) = 1$, $P(i,i+1) = 1$ for any $i \in \{1, ..., N-1\}$. Compute the successive powers of P, then show that $\{\lambda_1, ..., \lambda_N\} = \{\exp(2i\pi k/N, 0 \le k \le N-1\}$. Note that in this example all eigenvalues have modulus one.

1. P and P^T have the same eigenvalues.
2. Since P is stochastic, we have $P\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$, thus $\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$ is an eigenvector of P, associated with eigenvalue 1.
3. Let $\lambda \in \mathbb{C}$ an eigenvalue of P, and $x \in \mathbb{C}^N \setminus \{0\}$ an associated eigenvector. For i such that $|x_i| = ||x||_\infty$ we have
$$|\lambda|||x||_\infty = |\lambda x_i| = |(Px)_i| = \sum_{j=1}^n p_{ij}x_j \le ||x||_\infty,$$
where we used that P is stochastic to get the last inequality. We conclude that $|\lambda| \le 1$. Since 1 is an eigenvalue (cf previous question), it follows that $r(P) = 1$, so is $r(P^T)$ since P and P^T have the same eigenvalues.
4. Define $\omega = \exp(2i\pi/N)$, and let $x^{(k)} \in \mathbb{C}^N$, $k \in \{0, ..., N-1\}$ be such that $x_i^{(k)} = \omega^{ik}$. We easily check that
$$Px^{(k)} = \omega^k x^{(k)},$$
hence the spectrum of P is indeed $\{1, \omega, \omega^2, ..., \omega^{N-1}\}$.

Exercise 3 **[Perron's theorem]** Let A be a $N \times N$ matrix with real positive entries. For x, y in $\mathbb{R}^N$ write $x \ge y$ iff $x_i \ge y_i$ for any $1 \le i \le N$. We let $\lambda_1, ..., \lambda_N$ the (possibly complex, possibly multiple) eigenvalues of A, and $r(A) := \max_{1\le i\le N} |\lambda_i|$.

1. Assume that all entries of P are nonzero, introduce $X = \{x \in \mathbb{R}^n : x \ge 0, ||x||_1 = 1\}$ and for $x \in X$, $\theta(x) = \max\{t \in \mathbb{R}_+ : tx \le Ax\}$.

(a) Show that $r_0 = \sup_{x\in X} \theta(x) \in \mathbb{R}_+^*$.

(b) Assume $x \in X$ is such that $\theta(x) = r_0$. Show that $Ax = r_0 x$. *Hint* : You may assume by contradiction that $Ax \neq r_0 x$, then show that for small enough ε we must have $A(Ax - (r_0 + \varepsilon)x) > 0$. Deduce that if $y = Ax/||Ax||_1 \in X$, then $\theta(y) \geq r_0 + \varepsilon$, a contradiction with (a).

(c) Using compacity of X, show that there exists $x_0 \in X$ such that $Ax_0 = r_0 x_0$. Deduce that $r_0 = r(A)$. By looking at Ax_0, check that $x_0 > 0$.

(d) Show that if $Av = \lambda v$ with $\lambda = r(A)$, it must be that $v = \alpha x_0$ for some $\alpha \in \mathbb{C}$.

em Conclusion : $r(A)$ is an eigenvalue of A, The corresponding eigenspace has dimension 1 and is generated by $x_0 > 0$. Any other eigenvalue $\lambda \neq |||A|||$ of A is such that $|\lambda| < r(A)$.

2. Assume now that for some $k \in \mathbb{N}^*$, all entries of A^k are nonzero. Check that the same conclusion as in the previous question still holds for A.

Remarks : Note that the assumption of 2. is that A is strongly irreducible.
The slightly more difficult Frobenius theorem states that if A is irreducible (i.e. for any $i, j \in \{1, ..., N\}$, there exists $k \in \mathbb{N}^*$ such that $A^k_{ij} > 0$), then $r(A)$ is an eigenvalue of A, and moreover the corresponding eigenspace has dimension 1 and is generated by some $x_0 > 0$. Note that this part of the conclusion holds under the weaker assumption of irreducibility. However, when A is only assumed irreducible, there may exist several eigenvalues whose modulus equals $r(A)$ (see last question of exercise 2). In fact, this situation where $r(A)$ is not a dominant eigenvalue only occurs when A is irreducible but *periodic*, that is when $\gcd\{n \in \mathbb{N}^* : A^n(i,i) > 0\} \geq 2$.

1. (a) The first element of the canonical basis e_1 is in X, and $(Ae_1)_1 = a_{11} > 0$ so that $\theta(e_1) = a_{11} > 0$. Thus $r_0 > 0$. Moreover, if $x \in X$, $||Ax||_\infty \leq \max_{1\leq i,j\leq N} a_{ij}$, and therefore, $r_0 \leq \max_{1\leq i,j\leq N} a_{ij} < \infty$. We conclude that $r_0 \in \mathbb{R}_+^*$.

(b) Assume $x \in X$ is such that $\theta(x) = r_0$, by definition of θ it must be that for some $i \in \{1, ..., N\}$, $x_i > 0$ and $(Ax)_i = r_0 x_i$. As suggested, assume by contradiction that $Ax \neq r_0 x$, that is there exists $j \neq i$ such that $(Ax)_j < r_0 x_j$. Under this assumtion $Ax - r_0 x \geq 0$ is nonzero, and since all entries of A are positive, it follows that $A(Ax - r_0 x) > 0$, hence (A is continuous) for some ε small enough, $A(Ax - (r_0 + \varepsilon)x) > 0$. Since A is linear, it follows that $Ay \geq (r_0 + \varepsilon)y$, that is $\theta(y) \geq r_0 + \varepsilon$, a contradiction with the definition of r_0.

(c) By definition of r_0 there exists a sequence $(x_n)_{n\in\mathbb{N}^*} \in X^{\mathbb{N}}$ such that the sequence $(\theta(x_n))_{n\in\mathbb{N}}$ is non-decreasing and has limit r_0. By compacity of X there must be a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ which converges in X towards some $x_0 \in X$, and by continuity of θ it must be that $\theta(x_0) = r_0$, as required. By the previous question, we must have $Ax_0 = r_0 x_0$. It follows that r_0 is an eigenvalue of A so $r_0 \geq r(A)$. Now, if λ is an eigenvalue of A, and x an associated eigenvector with $||x||_1 = 1$, setting $|x| = (|x_1|, \ldots, |x_N|)$, observe that for any $i \in \{1, ..., N\}$

$$|(Ax)_i| = |\lambda||x_i| \leq (A|x|)_i \leq r_0|x_i|,$$

thus $|\lambda| \leq r_0$ and $r(A) \leq r_0$.

Finally, since A only has positive entries, we have $r_0(x_0)_i = (Ax_0)_i > 0$ for any $i \in \{1, ..., N\}$, so that $x_0 > 0$.

(d) Assume λ is an eigenvalue such that $|\lambda| = r(A) = r_0$ and v is an associated eigenvector. Letting $w = v/||v||_1$, note that $|w| \in X$ so $A|w| \leq r_0|w|$ thus by linearity of A, $A|v| \leq r_0|v|$. As in the previous question

$$r_0|v_i| = |(Av)_i| = \left|\sum_{j=1}^{N} a_{ij}v_j\right| \leq \sum_{j=1}^{N} a_{ij}|v_j| \leq r_0|v_j|,$$

so we must have the equality

$$\left|\sum_{j=1}^{N} a_{ij}v_j\right| = \sum_{j=1}^{N} a_{ij}|v_j|,$$

which, since all a_{ij} are positive, can only happen if all v_j have the same argument. It follows that $v = \exp(i\gamma)|v|$, and that $|v|$ is an eigenvector of eigenvalue $|\lambda| = r_0$. Now since $(A|v|)_i = r_0|v_i|$ and A only has positive entries it must be that $|v_i| > 0$ for any $i \in \{1, ..., N\}$. In fact, all nonnegative real-valued eigenvectors associated with eigenvalue r_0 must have all positive entries.

Now if, by contradiction, we assume that x_0 and $|v|$ are non-colinear, it must be that for some $t > 0$, $x_0 - t|v| \geq 0$, is nonzero, but has at least one null coordinate (simply take $t = \inf\{s > 0 : \exists i \in \{1, ..., N\}(x_0)_i - sv_i = 0\}$). We have found an eigenvector associated with eigenvalue r_0 with a null coordinate, a contradiction.

2. Observe that A^k satisfies the assumptions of the previous question. Thus $r(A^k) = r(A)^k$ is an eigenvalue of A^k, the corresponding eigenspace has dimension 1 and is generated by some $x > 0$, and all other eigenvalues of A^k have modulus less than $r(A)^k$. Now $Ax > 0$, and if, by contradiction, x isn't an eigenvector for A, it must be that for some $i \in \{1, ..., N\}$, $0 < (Ax)_i < r(A)x_i$. But then we would deduce that $(A^2x)_i < r(A)^2x_i$, etc... until $(A^kx)_i < r(A)^kx_i$, a contradiction. It follows that x is also an eigenvector of A, associated with eigenvalue $r(A)$. If there is y such that $Ay = r_0y$, then $A^ky = r(A)^ky$, so it must be that x and y are colinear. Thus the eigenspace for A associated with eigenvalue $r(A)$ is $\mathrm{Vect}\{x\}$. Finally if y is an eigenvector of A associated with eigenvalue $\lambda \neq r(A)$, it is also an eigenvector of A^k associated with eigenvalue λ^k, thus $|\lambda^k| = |\lambda|^k < r(A)^k$ and therefore $|\lambda| < r(A)$, as required.

Exercise 4 [A few properties of matrix exponentials] For A an $N \times N$ matrix (with, say, complex entries) we define $\exp(A) := \sum_{k\geq 0} \frac{A^k}{k!}$. Note this is always well-posed since the series is normally convergent.

1. Show that $\exp(A^T) = \exp(A)^T$, $\exp(A^*) = \exp(A)^*$.
2. Assume P is invertible. Express $\exp(P^{-1}AP)$ in terms of $P, P^{-1}, \exp(A)$. *Remark* This gives a practical method to explicitly compute $\exp(A)$ when A is diagonalizable. Explain why.
3. Show that if A and B commute, then $\exp(A)\exp(B) = \exp(B)\exp(A) = \exp(A + B)$.
4. Show that for any $s > 0$ sufficiently large, we have $\int_0^\infty \exp(-st)\exp(tA)dt = (sI - A)^{-1}$.

5. Show that the unique solution to the differential system $Y'(t) = AY(t), Y(0) = Y_0$ is given by $Y_0 \exp(At)$.

1. Since for any $k \in \mathbb{N}$, $(A^k)^T = (A^T)^k$, $(A^k)^* = (A^*)^k$, this follows from the definition.
2. Since $(P^{-1}AP)^k = P^{-1}A^kP$ for any $k \in \mathbb{N}$, we have

$$\exp(P^{-1}AP) = P^{-1}\exp(A)P.$$

   Of course, when A is diagonalizable, that is $A = P^{-1}DP$ for some invertible P, we find

$$\exp(A) = \exp(PDP^{-1}) = P\exp(D)P^{-1}.$$

   Now it is easy to check that $\exp(D)$ simply is the diagonal matrix whose diagonal entries are the exponentials of those of D.
3. If A and B commute, so do A^k and B^k for any k, thus $\exp(A)$ and $\exp(B)$. Now $(A+B)^k = \sum_{i=0}^k \binom{k}{i} A^iB^{k-i}$, so

$$\begin{aligned} \exp(A+B) &= \sum_{k\geq 0}\sum_{i=0}^{k} \frac{\binom{k}{i}A^iB^{k-i}}{k!} \\ &= \sum_{i\geq 0}\sum_{k\geq i} \frac{A^iB^{k-i}}{i!(k-i)!} \\ &= \sum_{i\geq 0}\frac{A^i}{i!}\sum_{k'\geq 0}\frac{B^{k'}}{k'!} = \exp(A)\exp(B). \end{aligned}$$

4. First observe that for any $k \in \mathbb{N}$, by an easy induction, we find

$$\int_0^\infty t^k \exp(-st)dt = \frac{k!}{s^{k+1}}.$$

   Thus

$$\int_0^\infty t^k \exp(-st)\frac{A^k}{k!}dt = \frac{A^k}{s^{k+1}}.$$

   It follows that for $s > |||A|||$ so that the infinite sum below is guaranteed to converge, we find

$$\int_0^\infty \exp(-st)\exp(tA)dt = \sum_{k\geq 0}\frac{A^k}{s^{k+1}},$$

   and thus

$$(sI - A)\sum_{k\geq 0}\frac{A^k}{s^{k+1}} = \sum_{k\geq 0}\frac{A^k}{s^k} - \sum_{k\geq 0}\frac{A^{k+1}}{s^{k+1}} = I,$$

   as wished.
5. Since $\max_{1\leq i,j\leq N}|a_{ij}| = C < \infty$, a direct application of Cauchy-Lipschitz theorem ensures existence and unicity of the solution. It is then straightforward to check that $t \to Y_0\exp(At)$ is indeed a solution.

2 Properties of exponential and Poisson distributions

Exercise 5 Let $X_1, X_2, ...$ be independent exponential variables with respective parameters $\lambda_1, \lambda_2, ...$ Show that

$$\sum_{n\geq 1} \frac{1}{\lambda_n} < \infty \Rightarrow \mathbb{P}\left(\sum_{n\geq 1} X_n < \infty\right) = 1$$

$$\sum_{n\geq 1} \frac{1}{\lambda_n} = \infty \Rightarrow \mathbb{P}\left(\sum_{n\geq 1} X_n = \infty\right) = 1$$

One has $\mathbb{E}[\sum_{n\geq 1} X_n] = \sum_{n\geq 1} \frac{1}{\lambda_n}$, and it is obvious that

$$\sum_{n\geq 1} \frac{1}{\lambda_n} < \infty \Rightarrow \mathbb{P}\left(\sum_{n\geq 1} X_n < \infty\right) = 1.$$

On the other hand assume that $\sum_{n\geq 1} \frac{1}{\lambda_n} = +\infty$. Then

$$\mathbb{E}\left[\exp\left(-\sum_{n\geq 1} X_n\right)\right] = \prod_{n\geq 1} \frac{1}{1/\lambda_n + 1} = 0,$$

so that $\mathbb{P}\left(\sum_{n\geq 1} X_n = \infty\right) = 1$.

Exercise 6 Let $X_1, X_2, ...$ be independent exponential variables with respective parameters $\lambda_1, \lambda_2, ...$

1. What is the law of $\min(X_1, X_2)$? What about that of $\min(X_1, ..., X_k)$, for $k \geq 2$?
2. Show that if $\Lambda := \sum_k \lambda_k < \infty$, then $X = \inf_{k\in\mathbb{N}^*} X_k$ defines a positive random variable. Find its law.
   Further fhow that the infimum is reached at a random index K, independent of X. What is the law of K ?
3. What happens when $\Lambda = \sum_k \lambda_k = +\infty$?

1. By independence

   $$\mathbb{P}(\min_{1\leq i\leq k} X_i > t) = \mathbb{P}(X_i > t, i = 1, ..., k) = \prod_{i=1}^{k} \mathbb{P}(X_i > t) = \prod_{i=1}^{k} e^{-\lambda_i t} = \exp\left(-\sum_{i=1}^{k} \lambda_i t\right),$$

   so that $\min_{1\leq i\leq k} X_i \sim \exp\left(\sum_{i=1}^{k} \lambda_i\right)$.
2. By the same proof as in the previous question $X \sim \exp(\Lambda)$. Denote

   $$K = \begin{cases} i \text{ if } X = X_i \\ 0 \text{ if infimum is not reached or reached by multiple indices} \end{cases},$$

For $i \geq 1$

$$\begin{aligned}
\mathbb{P}(X > t, K = i) = \mathbb{P}(X_j > X_i \forall j \neq i) &= \int_t^{+\infty} dx \lambda_i \exp(-\lambda_i x) \mathbb{P}(X_j > x, \forall j \neq i) \\
&= \int_t^{+\infty} dx \lambda_i \exp(-\lambda_i x) \prod_{j \neq i} \mathbb{P}(X_j > x) \\
&= \int_t^{+\infty} dx \lambda_i \exp(-\lambda_i x) \exp\left(-\sum_{j \neq i} \lambda_j x\right) \\
&= \int_t^{+\infty} dx \lambda_i \exp(-\Lambda x) = \frac{\lambda_i}{\Lambda} \exp(-\Lambda t).
\end{aligned}$$

We deduce that K is independent of X, and that the law of K is given by $\mathbb{P}(K = i) = \lambda_i / \Lambda, i \geq 1$ (in particular $\sum_{i \geq 1} \mathbb{P}(K = i) = 1$ so that $\mathbb{P}(K = 0) = 0$, i.e. the infimum is a.s. reached by a single X_i).

3. When $\sum_{i \geq 1} \lambda_i = +\infty$, we have for any $t > 0$,

$$\mathbb{P}(X > t) = \prod_{i \geq 1} \mathbb{P}(X_i > t) = 0,$$

and in this case $X = 0$ a.s.

Exercise 7 Let $X_1, X_2, ...$ be i.i.d exponentials with parameter $\lambda > 0$, $S_0 = 0$ and for $n \geq 1$, $S_n = \sum_{k=1}^n X_k$.

1. For $n \geq 1$, compute the density of S_n.
2. Fix $t \geq 0$ and denote
$$N_t := \#\{n \in \mathbb{N}^* : S_n \leq t\}.$$
What is the distribution of N_t ?
3. Fix $k \in \mathbb{N}^*$ and $t_1 \leq t_2 \leq \ldots t_k$. Find the joint distribution of $\left(N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_k} - N_{t_{k-1}}\right)$

1. Since $S_1 = X_1$, S_1 has the $\exp(\lambda)$ density, that is $\lambda \exp(-\lambda t) \mathbb{1}_{\{t \geq 0\}}$. Now Since $\{S_n > t\} = \{S_{n-1} > t\} \cup \{S_{n-1} \leq t\} \cap \{X_n > t - S_{n-1}\}$, we have for any $n \geq 2$
$$\mathbb{P}(S_n > t) = \mathbb{P}(S_{n-1} > t) + \mathbb{P}(S_{n-1} \leq t, X_n > t - S_{n-1}).$$
For example
$$\mathbb{P}(S_2 > t) = \exp(-\lambda t) + \int_0^t \lambda \exp(-\lambda u) \exp(-\lambda(t - u)) du = (1 + \lambda t) \exp(-\lambda t),$$
so that S_2 has density $\lambda^2 t \exp(-\lambda t) \mathbb{1}_{\{t \geq 0\}}$ (that is, the $\Gamma(2, \lambda)$ density). Let us prove by induction that S_n has the $\Gamma(n, \lambda)$ density. It is indeed the case for $n = 1, 2$ as we just showed. Assume the induction assumption holds for S_n. Of course S_{n+1} is also supported on $\mathbb{R}_+$. We find that for any $t \geq 0$,
$$\begin{aligned}
\mathbb{P}(S_{n+1} > t) &= \int_t^\infty \frac{\lambda^n u^{n-1}}{(n-1)!} \exp(-\lambda u) du + \int_0^t \frac{\lambda^n u^{n-1}}{(n-1)!} \exp(-\lambda u) \exp(-\lambda(t - u)) du \\
&= \int_t^\infty \frac{\lambda^n u^{n-1}}{(n-1)!} \exp(-\lambda u) du + \frac{\lambda^n t^n}{n!} \exp(-\lambda t)
\end{aligned}$$

so that S_{n+1} has density

$$\left(\frac{\lambda^n t^{n-1}}{(n-1)!}\exp(-\lambda t) - \frac{\lambda^n t^{n-1}}{n!}\exp(-\lambda t) + \frac{\lambda^{n+1}t^n}{n!}\exp(-\lambda t)\right)\mathbb{1}_{\{t\geq 0\}} = \frac{\lambda^{n+1}t^n}{n!}\exp(-\lambda t)\mathbb{1}_{\{t\geq 0\}},$$

as required.

2. We have, by definition of N_t.

$$\begin{aligned}\mathbb{P}(N_t = n) &= \mathbb{P}(S_n \leq t < S_{n+1}) = \mathbb{P}(S_n \leq t, X_{n+1} > t - S_n)\\ &= \int_0^t \frac{\lambda^n u^{n-1}}{(n-1)!}\exp(-\lambda t)du = \frac{(\lambda u)^n}{n!}\exp(-\lambda u),\end{aligned}$$

so that $N_t \sim \text{Poisson}(\lambda t)$.

3. We are going to prove by induction on k that these are independent Poisson with respective parameters $\lambda(t_i - t_{i-1}), 1 \leq i \leq k$, where we have set $t_0 = 0$. This is true for $k = 1$ by the previous question. Now assume it is true for some $k \in \mathbb{N}^*$, we have

$$\begin{aligned}&\mathbb{P}(N_{t_1} = n_1, ..., N_{t_k} - N_{t_{k-1}} = n_k, , N_{t_{k+1}} - N_{t_k} = n_{k+1})\\ = \;&\mathbb{P}(N_{t_1} = n_1, ..., N_{t_k} - N_{t_{k-1}} = n_k, , N_{t_k} - N_{t_{k-1}} = n_k)\\ &\times\mathbb{P}(N_{t_{k+1}} - N_{t_k} = n_{k+1} \mid N_{t_1} = n_1, ..., N_{t_k} - N_{t_{k-1}} = n_k, , N_{t_k} - N_{t_{k-1}} = n_k)\end{aligned}$$

Now if we set for $\ell \geq 1$, $N_\ell = \sum_{i=1}^{\ell} n_i$, we can rewrite the event $\{N_{t_1} = n_1, ..., N_{t_k} - N_{t_{k-1}} = n_k, , N_{t_k} - N_{t_{k-1}} = n_k\}$ as $\{S_{N_1} < t_1, S_{N_1+1} > t_1, S_{N_2} \leq t_2, S_{N_2+1} > t_2, ..., S_{N_k} \leq t_k, S_{N_k+1} > t_k\}$, and therefore, by independence of the $X_i, i \geq 1$, we are interested in

$$\begin{aligned}&\mathbb{P}\Big(S_{N_{k+1}} \leq t_{k+1}, S_{N_{k+1}+1} > t_{k+1} \mid\\ &\qquad S_{N_1} < t_1, S_{N_1+1} > t_1, S_{N_2} \leq t_2, S_{N_2+1} > t_2, ..., S_{N_k} \leq t_k, S_{N_k+1} > t_k\Big)\\ = \;&\mathbb{P}\left(S_{N_{k+1}+1} > t_{k+1} \mid S_{N_k} \leq t_k, S_{N_k+1} > t_k\right)\\ = \;&\mathbb{P}\Big(X_{N_k+1} - (t_k - S_{N_k}) + \sum_{i=N_k+2}^{N_{k+1}} X_i \leq t_{k+1} - t_k,\\ &\qquad X_{N_k+1} - (t_k - S_{N_k}) + \sum_{i=N_k+2}^{N_{k+1}+1} X_i > t_{k+1} - t_k \mid X_{N_k+1} > t_k - S_{N_k}\Big)\end{aligned}$$

Using again the independence of the X_i (which implies the independence of X_{N_k+1} and S_{N_k}), and the lack of memory property of X_{N_k+1}, we may rewrite this as

$$\mathbb{P}\left(\tilde{X}_{N_k+1} + \sum_{i=N_k+2}^{N_{k+1}} X_i \leq t_{k+1} - t_k, \tilde{X}_{N_k+1} + \sum_{i=N_k+2}^{N_{k+1}+1} X_i > t_{k+1} - t_k\right),$$

with $\tilde{X}_{N_k+1} \sim \exp(\lambda)$ independent of $X_i, i \geq N_k + 2$. By the previous question, this is $\frac{(\lambda(t_{k+1}-t_k))^{N_{k+1}}}{(N_k+1)!}\exp(-\lambda(t_{k+1} - t_k))$, as we wished to complete our proof.

Exercise 7

1. Consider $(X_i)_{i\geq 1}$ a sequence of i.i.d random variables taking values in $\{1, ..., d\}$, with $p_k := \mathbb{P}(X_1 = k), k = 1, ..., d$. Let also $N \sim$ Poisson(λ) be independent of $(X_i)_{i\geq 1}$. What is the distribution of $(N_1, ..., N_d)$, where, for $k = 1, ..., d$,
$$N_k := \sum_{i=1}^{N} \mathbb{1}_{\{X_i = k\}}?$$
2. What if the $(X_i)_{i\geq 1}$ are still i.i.d but now take values in $\mathbb{N}^*$ with $p_k = \mathbb{P}(X_1 = k), k \in \mathbb{N}^*$?
3. Let I a finite or countable set of indices, $(N_i, i \in I)$ be independent Poisson variables with respective parameters $\lambda_i, i \in I$, such that $\Lambda = \sum_{i \in I} \lambda_i = \Lambda$. What is the distribution of $N = \sum_{i\in I} N_i$?

Sachant $\{N = n\}$, l'expérience correspond exactement au schéma multinômial de paramètres $n, p_1, ..., p_d$. On a donc, , quitte à poser $n = k_1 + ... + k_d$,

$$\begin{aligned}
\mathbb{P}(N_1 = k_1, ..., N_d = k_d) &= \mathbb{P}(N_1 = k_1, ..., N_d = k_d, N = n) \\
&= \frac{\lambda^n \exp(-\lambda)}{n!} \binom{n}{k_1 \dots k_d} p_1^{k_1} ... p_d^{k_d} \\
&= \prod_{i=1}^{d} \frac{(\lambda p_i)^{k_i} \exp(-\lambda p_i)}{i!},
\end{aligned}$$

de sorte que $(N_1, ..., N_d)$ est un d-uplet de variables de Poisson indépendantes de paramètres respectifs $\lambda p_1, ..., \lambda p_d$.

Remarque 1 : Il n'est pas difficile de généraliser ce résultat pour X_1 ayant une loi sur $\mathbb{N}^*$ plutôt qu'une partie finie de $\mathbb{N}^*$ comme plus haut. On aura alors que les $(N_i, i \geq 1)$ forment une suite de variables indépendantes avec $N_k \sim$ Poisson(λp_k). En effet, pour tout $d \in \mathbb{N}^*$, $(k_1, ..., k_d) \in (\mathbb{N}^*)^d$, $K := k_1 + \cdots + k_d$, on a

$$\begin{aligned}
&\mathbb{P}(N_1 = k_1, ..., N_k = k_d) \\
= &\sum_{n\geq K} \mathbb{P}(N_1 = k_1, ..., N_k = k_d, N = n) \\
= &\frac{\lambda^n \exp(-\lambda)}{n!} \binom{n}{k_1 \dots k_d \; n-K} p_1^{k_1} ... p_d^{k_d} \left(1 - \sum_{i=1}^{d} p_i\right)^{n-K} \\
= &\sum_{n\geq K} \prod_{i=1}^{d} \frac{(\lambda p_i)^{k_i} \exp(-\lambda p_i)}{i!} \exp\left(-\lambda\left(1 - \sum_{i=1}^{d} p_i\right)\right) \frac{\left(\lambda\left(1 - \sum_{i=1}^{d} p_i\right)\right)^{n-K}}{(n-K)!} \\
= &\prod_{i=1}^{d} \frac{(\lambda p_i)^{k_i} \exp(-\lambda p_i)}{i!},
\end{aligned}$$

,et, ce résultat étant valable pour $d \geq 1$ quelconque on obtient la conclusion souhaitée. Bien évidemment, on peut alors facilement généraliser à X_1 ayant une loi sur un ensemble dénombrable quelconque.

Remarque 2 : A l'inverse, si I est une ensemble d'indices fini ou dénombrable, et $(N_i, i \in I)$ sont des variables de Poisson indépendantes de paramètres respectifs $\lambda_1, \lambda_2, ...$ avec $\sum_{i\in I} \lambda_i =: \Lambda < \infty$, alors $N = \sum_{i\in I} N_i \sim \text{Poisson}(\Lambda)$. La preuve la plus directe utilise les fonctions génératrices des moments (et le fait que la fonction génératrice caractérise la loi d'une v.a. à valeurs dans $\mathbb{N}$) : pour tout $t \in [0,1]$,

$$\mathbb{E}[t^N] = \prod_{i\in I} \mathbb{E}[t^{N_i}] = \prod_{i\in I} \exp(\lambda_i(t-1)) = \exp(\Lambda(t-1).$$

On peut également faire une preuve élémentaire en décomposant $\mathbb{P}(N = n)$ selon toutes les valeurs possibles des $(N_i, i \in I)$. Lorsque $I = \{1, ..., d\}$, on retrouve, bien entendu, des étape du calcul effectué plus haut.

Exercise 8 Let $X_1, X_2, ...$ be i.i.d exponentials with parameter $\lambda > 0$, and N independent of $(X_i, i \geq 1)$, having a geometric(β) distribution. Find the distribution of $X := \sum_{k=1}^{N} X_k$.
On a pour $t \geq 0$

$$\begin{aligned}
\mathbb{E}[\exp(-tX)] &= \mathbb{E}\left[\prod_{i=1}^{N} \exp(-tX_i)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N} \exp(-tX_i) \mid N\right]\right] \\
&= \mathbb{E}\left[\left(\frac{\lambda}{\lambda+t}\right)^N\right] \\
&= \frac{\frac{\beta\lambda}{\lambda+t}}{1-(1-\beta)\frac{\lambda}{\lambda+t}} \\
&= \frac{\beta\lambda}{t+\beta\lambda},
\end{aligned}$$

de sorte que $X \sim \exp(\beta\lambda)$.
Si vous n'êtes pas familiers avec l'espérance conditionnelle, la deuxième ligne du calcul ci-dessus peut se comprendre de manière plus élémentaire en décomposant suivant les valeurs possibles de N puis en appliquant Fubini, et l'indépendance des variables $(N, X_i, i \geq 1)$

$$\begin{aligned}
\mathbb{E}\left[\prod_{i\leq N} \exp(-tX_i)\right] &= \mathbb{E}\left[\sum_{n\geq 1} \prod_{i=1}^{n} \exp(-tX_i) \mathbb{1}_{\{N=n\}}\right] \\
&= \sum_{n\geq 1} \mathbb{E}\left[\prod_{i=1}^{n} \exp(-tX_i) \mathbb{1}_{\{N=n\}}\right] \\
&= \sum_{n\geq 1} \prod_{i=1}^{n} \mathbb{E}[\exp(-tX_i)] \mathbb{P}(N=n) \\
&= \sum_{n\geq 1} \left(\frac{\lambda}{\lambda+t}\right)^n \mathbb{P}(N=n)
\end{aligned}$$

Remarque – option A : On peut également déduire ce résultat des propriétés des processus de Poisson : soit $(N_t, t \geq 0)$ un processus de Poisson d'intensité λ, i.e. un processus croissant, à valeurs dans $\mathbb{N}$, issu de 0 et qui saute de 1 en chacun de ses sauts et dont les temps de saut sont espacés par des variables i.i.d, exponentielles de paramètre λ. Une autre caractérisation de ce processus est que N est à accroissements indépendants et stationnaires avec $N_t \sim \text{Poisson}(\lambda)$ (voir l'exercice ci-dessus).
Considérons alors le processus $\tilde{N}$, également croissant, à valeurs dans $\mathbb{N}$, et qui, comme N, saute de 1 en chacun de ses sauts. Les temps de saut de $\tilde{N}$ sont un sous-ensemble de ceux de N et ils sont déterminés de la faqn suivante : on introduit $(\xi_i, i \geq 1)$ des variables i.i.d Bernoulli de paramètre β. Au i-ème temps de saut de N, $\tilde{N}$ saute également ssi $\xi_i = 1$. Autrement dit, on colorie de manière i.i.d, en vert (resp. rouge) les temps de saut de N suivant que $\xi_i = 1$ (resp. 0). Les temps de saut du processus $\tilde{N}$ sont les sauts "verts" de N. En utilisant en particulier un des exercices précédents (boîte de peinture), on peut montrer que $\tilde{N}$ reste à accroissements indépendants, et que $\tilde{N}_t \sim \text{Poisson}(\beta\lambda)$. Autrement dit, $\tilde{N}$ reste un processus de Poisson, d'intensité $\lambda\beta$. Les temps de saut de $\tilde{N}$ sont donc espacés par des variables i.i.d, exponentielles de paramètre $\beta\lambda$. Et le premier saut est justement donné par la variable X de l'exercice.

1. Par une récurrence on montre que la densité f_n de S_n est
$$f_n(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda \exp(-\lambda t) \mathbb{1}_{\{t \geq 0\}}.$$

2. On a donc par indépendance de X_{n+1} et S_n, et la question précédente
$$\begin{aligned} \mathbb{P}(N_t = n) = \mathbb{P}(X_{n+1} > t - S_n, S_n \leq t) &= \int_0^t \lambda \exp(-\lambda u) \frac{(\lambda u)^{n-1}}{(n-1)!} \exp(-\lambda(t-u)) du \\ &= \frac{(\lambda t)^n}{n!} \exp(-\lambda t), \end{aligned}$$
de sorte que $N_t \sim \text{Poisson}(\lambda t)$.

3 Classical examples of discrete-time Markov chains, classification

Exercise 10
Let P be a transition kernel on state space E (finite or countable).

1. We say state x leads to y and write $x \to y$ iff there exists $n \in \mathbb{N}$ such that $P^n(x, y) > 0$. Is $\to$ an equivalence relation ?
2. We say states x and y communicate and write $x \leftrightarrow y$ iff $x \to y$ and $y \to x$. Show that $\leftrightarrow$ is an equivalence relation. We call communication classes the corresponding equivalence classes.
3. Assume there exists $k \in \mathbb{N}^*$ such that $P^k(x, y) > 0$ for any $x, y \in E$. Describe the partition of E into communication classes.

4. Consider the transition kernel P given in the last question of exercise 2. Describe the partition of $E = \{1, ..., N\}$ into communication classes. What about those associated with the transition kernel P^N ?
5. Consider the chain on $E = \mathbb{Z}$ such that $P(n, n+1) = 1$ for any $n \in \mathbb{Z}$. Describe the partition of E into communication classes.

1. The relation $\to$ is reflexive ($P^0 = Id$), transitive (if $\exists n, n'$ such that $P^n(x, y) > 0, P^{n'}(y, z) > 0$, then $P^{n+n'}(x, z) \geq P^n(x, y)P^{n'}(y, z) > 0$. But it isn't symmetric : e.g. for $P = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$ we have $2 \to 1$ but $1 \nrightarrow 2$.
2. $\leftrightarrow$ is clearly reflexive and symmetric because $\to$ is, and it is symmetric by design. It is indeed an equivalence relation.
3. If for some $k \in \mathbb{N}^*$, all entries of P^k are positive, then obviously for any $\{x, y\} \in E^2$, $x \leftrightarrow y$, that is, there is only one communication class. We'll say in that case that the chain is *irreducible*. In fact, when this stronger assumption ($\exists k \in \mathbb{N}^*$ such that for any $(x, y) \in E^2, P^k(x, y) > 0$) holds, we'll say that the chain is *strongly irreducible*.
4. We have $1 \to 2 \to \cdots \to N \to 1$, so the chain is irreducible. Now $P^N = Id$, so for the chain with transition kernel P^N, there are N communication classes (every state is by itself in its communication class).
   More generally, one can easily show that any communication class of P^k is always included in one communication class of P, but as the above example shows, the inclusion may be strict.
5. For any $n \in \mathbb{Z}, k \in \mathbb{N}$, $n \to n + k$, however, for any $k \in \mathbb{N}^*$, $n + k \nrightarrow n$ since for any $j \in \mathbb{N}$, $P^j(n + k, n) = 0$. Thus for any $(n, m) \in \mathbb{Z}^2, n \neq m$, $n \nleftrightarrow m$, and therefore the communication class of any given $n \in \mathbb{Z}$ is reduced to $\{n\}$.

Exercise 11 For $p, q \in [0, 1]$, let X be the two-state $(1, 2)$ chain, with transition matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$.

1. For which values of p, q is the chain irreducible ? aperiodic ?
2. For each p, q, find the set $\mathcal{D}$ of all invariant distributions of X.
3. Compute $P^n, n \in \mathbb{N}$.
4. When X is irreducible, compute

$$d_1(n) := \frac{1}{2} \left(|\mathbb{P}_1(X_n = 1) - \pi(1)| + |\mathbb{P}_1(X_n = 2) - \pi(2)| \right).$$

   and

$$d_2(n) := \frac{1}{2} \left(|\mathbb{P}_2(X_n = 1) - \pi(1)| + |\mathbb{P}_2(X_n = 2) - \pi(2)| \right).$$

5. Draw the shape of the graphs of $n \to d_i(n), i = 1, 2$ for $p = q = 0.5$, for $p = 0.4, q = 0.1$, for $p = 0.9, q = 0.95$ and finally for $p = q = 1$.

1. irreducible iff $p > 0, q > 0$, aperiodic if $(p, q) \neq (1, 1)$.
2. If $p = q = 0$, $\mathcal{D} = \{\alpha\delta_1 + (1 - \alpha)\delta_2, \alpha \in [0, 1]\}$.
   Otherwise $\mathcal{D} = \{\frac{q}{p+q}\delta_1 + \frac{p}{p+q}\delta_2\}$

3. If $p = q = 0$, $P^n = I_2$ for any $n \in \mathbb{N}$.
   Otherwise $P = A^{-1}DA$ with

$$A = \begin{pmatrix} 1 & -p \\ 1 & q \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix} A^{-1} = \frac{1}{p+q} \begin{pmatrix} q & p \\ -1 & 1 \end{pmatrix},$$

   so one obtains

$$P^t = AD^nA^{-1} = \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{pmatrix} + (1-p-q)^n \begin{pmatrix} \frac{p}{p+q} & \frac{-p}{p+q} \\ \frac{-q}{p+q} & \frac{q}{p+q} \end{pmatrix}.$$

4. For $n \in \mathbb{N}$,

$$d_1(n) = \frac{1}{2} \left(|P^n(1,1) - \pi(1)| + |P^n(1,2) - \pi(2)| \right) = \frac{p}{p+q} |1-p-q|^n.$$

   Using a symmetry argument

$$d_2(t) = \frac{q}{p+q} |1-p-q|^n.$$

Exercise 12 Let X be the chain on $\{0, 1, ..., n\}$ with transition matrix P such that

$$P(0,k) = \frac{1}{2^{k+1}}, k \in \{0, ..., n-1\}, \quad P(0,n) = \frac{1}{2^n}$$
$$P(k, k-1) = 1, 1 \le k \le n-1, \quad P(n,n) = P(n, n-1) = 1/2.$$

1. Compute the unique invariant distribution π of the chain.
2. Show that, for any $x_0 \in \{0, 1, ..., n-1\}$, $P^{(x_0+1)}(x_0, \cdot) = \pi$.
3. For any $x_0 \in \{0, 1, ..., n\}$, establish that $P^{(n)}(x_0, \cdot) = \pi$.
4. For $t \in \mathbb{N}$ compute

$$d(t) := \frac{1}{2} \sum_{x=0}^{n} \left| P^{(t)}(n,x) - \pi(x) \right|,$$

   and draw the shape of $t \to d(t)$.

1. The chain is clearly irreducible ($n \to n-1 \to .... \to 0 \to n$) and aperiodic ($0 \to 0$). It therefore possesses a unique invariant distribution π such that $\pi(k) - \pi(k+1) = \frac{\pi(0)}{2^{k+1}}, k = 0, ..., n-1$, $\pi(n) = \frac{\pi(0)}{2^n}$. It follows that

$$\pi(k) = \frac{\pi(0)}{2^k}, k = 0, ..., n-1, \quad \pi(n) = \frac{\pi(0)}{2^n},$$

   hence $\pi(0) = 1/2$, and

$$\pi(k) = \frac{1}{2^{k+1}}, k = 0, ..., n-1, \quad \pi(n) = \frac{1}{2^n}.$$

2. By the above $P(0, \cdot) = \pi$, which is the assertion for $x_0 = 0$.
   For $x_0 \in \{1, ..., n-1\}$, the x_0 first steps are deterministic and one unit to the left, so $P^{x_0}(x_0, 0) = 1$, and $P^{x_0+1}(x_0, \cdot) = P(0, \cdot) = \pi$, as required.

3. According to the above if $x_0 = 0, ..., n-1$,

$$P^n(x_0, \cdot) = P^{n-x_0-1}\pi = \pi.$$

Under $\mathbb{P}_n$, the variable $G = \inf\{n \geq 1 : X_n = n-1\}$ is Geom(1/2). It follows, under $\mathbb{P}_n$, that

$$X_n = (G-1)\mathbb{1}_{\{G \leq n\}} + n\mathbb{1}_{\{G>n\}}.$$

Thus

$$P^t(n,k) = \mathbb{P}_n(X_n = k) = \mathbb{P}_n(G = k+1) = \frac{1}{2^{k+1}}, k = 0, ..., n-1,$$
$$P^t(n,n) = \mathbb{P}_n(X_n = n) = \mathbb{P}_n(G > n) = \frac{1}{2^n},$$

and finally $P^n(n, \cdot) = \pi$.

4. If $t \geq n$, $P^t(n, \cdot) = \pi$ so that $d(t) = 0$.
Assume now $t \leq n-1$. Under $\mathbb{P}_n$, using the previous question

$$X_t = (n-t+G-1)\mathbb{1}_{\{G \leq t\}} + n\mathbb{1}_{\{G>t\}},$$

thus

$$P^t(n,k) = \mathbb{P}_n(G = k-n+t+1) = \frac{1}{2^{k-n+t+1}}, k = n-t, ..., n-1, \quad P^t(n,n) = \mathbb{P}_n(G > t) = \frac{1}{2^t}.$$

We obtain, for $t \leq n-1$,

$$d(t) = \sum_{x=0}^{n-t-1} \pi(x) = 1 - \frac{1}{2^{n-t}}.$$

Exercise 13

1. Fix $p \in [0,1]$, and introduce i.i.d variables $(X_i, i \geq 1)$ such that $\mathbb{P}(X_1 = 1)$, $\mathbb{P}(X_1 = -1) = 1-p$. Let $S_n = S_0 + \sum_{k=1}^n X_i$, and $\mathbb{P}_k$ the law of $(S_n)_{n \geq 0}$ under which $S_0 = k$.
Let $N \in \mathbb{N}$ be fixed and denote $\tau^N = \inf\{n \geq 0 : S_n \in \{0, N\}\}$. For $k \in \{0, ..., N\}$, compute $\mathbb{P}_k(S_{\tau^N} = N)$ (one should distinguish between the cases $p = 1/2$, $p \neq 1/2$).
2. How does the previous question imply that S is recurrent iff $p = 1/2$? When $p = 1/2$, is S positive recurrent or null recurrent ?
3. Fix $p \in [0,1]$, $q \in [0, 1-p)$, and assume now that $\mathbb{P}(X_1 = 1) = p$, $\mathbb{P}(X_1 = 0) = q$, $\mathbb{P}(X_1 = -1) = 1-p-q$. Answer the previous question for this lazy version of the walk.
4. Let $T_1 = \inf\{n \in \mathbb{N} : S_T = 1\}$. For $p > 1/2$, $s > 0$, compute $\mathbb{E}_0[\exp(-sT_1)]$ (it may be useful to introduce an exponential martingale).

1. In both cases it is easy that τ^N is bounded by NG with G some geometric variable. It follows that $\mathbb{E}[\tau^N] < \infty$.
   If $p = 1/2$, then $(S_{n\wedge\tau^N}, n \geq 0)$ is a martingale, Doob's optional stopping theorem then implies, for $k \in \{0, ..., n\}$, that
   $$\mathbb{P}_k(S_{\tau^N} = N) = \frac{k}{N}.$$
   For $p \neq 1/2$, then $\left(Y_n := \left(\frac{1-p}{p}\right)^{S_{n\wedge\tau^N}}, n \geq 0\right)$ is a martingale. Now Doob's theorem allows to conclude et le théorème de Doob that for any $k \in \{1, ..., n\}$,
   $$\mathbb{P}_k(S_{\tau^N} = N) = \frac{1 - \left(\frac{1-p}{p}\right)^k}{1 - \left(\frac{1-p}{p}\right)^N}.$$
2. When $p > 1/2$, we have $(1-p)/p < 1$ hence
   $$\mathbb{P}_1(T_0 = +\infty) = \mathbb{P}_1(\bigcap_N S_{\tau^N} = N) = \lim_{N\to\infty} \mathbb{P}_1(S_{\tau^N} = N) = \frac{2p-1}{p} > 0,$$
   so the walk is transient. By a symmetry argument it is also transient when $p < 1/2$. However if $p = 1/2$,
   $$\mathbb{P}_1(T_0 = +\infty) = \lim_{N\to\infty} \mathbb{P}_1(S_{\tau^N} = N) = 0,$$
   hence the walk is then recurrent.
   Since steps are all of unit size, a walk started at 2 must necessarily go through 1 before reaching the origin. Moreover by translation invariance $\mathbb{E}_2[T_1] = \mathbb{E}_1[T_0]$ so $\mathbb{E}_2[T_0] = 2\mathbb{E}_1[T_0]$, and finally Markov property at time 1 for the walk started at 1 implies
   $$\mathbb{E}_1[T_0] = 1 + \frac{1}{2}2\mathbb{E}_1[T_0].$$
   It follows that $\mathbb{E}_1[T_0] = +\infty$, and that simple symmetric random walk on $\mathbb{Z}$ is null recurrent.
3. If one simply forgets the time steps when the lazy walk does not move, one recovers the preceding model with $p' = \frac{p}{1-q}, 1 - p' = \frac{1-p-q}{1-q}$.
4. Let $g(\lambda) = \ln(p\exp(\lambda) + (1-p)\exp(-\lambda))$. One can easily check that
   $$\left(M_n^\lambda = \exp(\lambda S_n - ng(\lambda)), n \geq 0\right)$$
   is a martingale, and that $M_{n\wedge T_0}^\lambda$ is uniformly integrable provided $\lambda > 0$. By Doob's theorem,
   $$\mathbb{E}_0[\exp(-T_1 g(\lambda))] = \exp(-\lambda).$$
   It only remains to compute $\lambda(s)$ such that $g(\lambda(s)) = s$ in order to conclude.

Exercise 14 A commercial promotion game consists in collecting n coupons to win a prize. We are interested in the time τ necessary for a given customer to collect all n coupons, assuming he receives each day exactly one coupon, choosen independently, and uniformly amongst all.

1. Show that $\tau = \tau_1 + ... + \tau_n$ with $\tau_i \sim \text{Geom}(\frac{n-i+1}{n})$.
2. Letting $c > 0$, show that

$$\mathbb{P}(\text{ not draw coupon 1 in } \lfloor n\log(n) + cn \rfloor \text{ days }) = \left(1 - \frac{1}{n}\right)^{\lfloor n\log(n)+cn \rfloor}$$

and deduce that

$$\mathbb{P}(\tau > \lfloor n\log(n) + cn \rfloor) \le \exp(-c)$$

3. Compute $\mathbb{E}[\tau], \text{Var}[\tau]$. Find equivalents for these quantities when $n \to \infty$. Deduce a bound on

$$\mathbb{P}(|\tau - \mathbb{E}[\tau]| > A\sqrt{\text{Var}[\tau]}).$$

Conclude.

1. For convenience let us label the coupons $\{1, ..., n\}$. Let c_k the label of the coupon which is received day $k \in \mathbb{N}^*$. Let $\mathcal{C}_k = \{i : \exists l \le k \; c_l = i\}$ the set of coupons collected until day $k \in \mathbb{N}^*$. Finally let

$$\sigma_0 = 0, \sigma_i = \inf\{k : |\mathcal{C}_k| = i\}, i = 1, ..., n$$

so that $\tau_i := \sigma_i - \sigma_{i-1}$ is the number of days between the collection times of $i-1$ and i distinct coupons distincts. Since draws are independent and uniform, variables $\tau_i, i = 1, ..., n$ are indeed independent and geometric with respective parameters $\frac{n-i+1}{n}, i = 1, ..., n$.

2. First equality follows directly from the fact that draws are independent and uniform. Then

$$\begin{aligned}
\mathbb{P}(\tau > \lfloor n\log(n) + cn \rfloor) &= \mathbb{P}\left(\bigcup_{i=1}^{n} \text{not draw } i \text{ in } \lfloor n\log(n) + cn \rfloor \text{ days}\right) \\
&\le n\left(1 - \frac{1}{n}\right)^{\lfloor n\log(n)+cn \rfloor} \\
&= n\exp\left(\lfloor n\log(n) + cn \rfloor \log\left((1 - \frac{1}{n})\right)\right) \le \exp(-c).
\end{aligned}$$

3. One has

$$\mathbb{E}[\tau] = \sum_{i=1}^{n} \mathbb{E}[\tau_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} \sim_{n\to\infty} n\log(n),$$

$$\begin{aligned}
\text{Var}[\tau] &= \sum_{i=1}^{n} \text{Var}[\tau_i] = \sum_{i=1}^{n} \left(1 - \frac{n-i+1}{n}\right)\left(\frac{n}{n-i+1}\right)^2 \\
&= n\sum_{i=1}^{n} \frac{i-1}{(n-i+1)^2} \sim_{n\to\infty} \frac{n^2\pi^2}{6}.
\end{aligned}$$

By Chebychev, one finds

$$\mathbb{P}(|\tau - \mathbb{E}[\tau]| > A\sqrt{\mathrm{Var}[\tau]}) \leq \frac{1}{A^2}.$$

Chernoff would allow for an even more precise inequality.
Anyhow the first order of τ remains close to its expectation, which itself remains close to $n\log(n)$; more precisely with probability close to 1 when $A \to \infty$, fluctuations of τ around $n\log(n)$ do not exceed An.

Exercise 15 Let $(X_n)_{n\geq 0}$ the Markov chain on $E = \mathbb{N}$ with transition kernel P such that

$$P(0,0) = r_0,\ P(0,1) = p_0,\ \text{and } \forall i \geq 1,\ P(i,i-1) = q_i,\ P(i,i) = r_i,\ P(i,i+1) = p_i,$$

with $p_0, r_0 > 0$, $p_0 + r_0 = 1$ and for any $i \geq 1$, $p_i > 0, q_i > 0, p_i + r_i + q_i = 1$. Such a chain is usually refered to as a *birth-and-death* chain.

1. Show that X is irreducible, aperiodic.
2. Show that X is reversible and that it has a unique stationary distribution iff $\sum_{i\geq 1} \frac{p_0\cdots p_{i-1}}{q_1\cdots q_i} < \infty$. In that case, express this stationary distribution as a function of $\{p_i, i \geq 0\}, \{q_i, i \geq 1\}$.
3. Consider the case when $p_i = p > 0$, $q_i = q > 0$ for any $i \geq 1$. Compute $\mathbb{E}_i[T_i^+]$, for any $i \in E$.

1. Irreducibility follows immediatly from the assumption $p_i > 0, i \in \mathbb{N}, q_i > 0, i \in \mathbb{N}^*$. Aperiodicity follows from $r_0 > 0$.
2. For reversibility one must check the detailed balance equations, which imply

   $$\frac{\pi_{i+1}}{\pi_i} = \frac{p_i}{q_{i+1}},\ \forall i \in \mathbb{N}.$$

   For $i \geq 1$, we can always set

   $$\pi_i = \pi_0 \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}},$$

   so the chain is reversible.
   For such π to be a distribution it is necessary and sufficient that

   $$S = \sum_{i\geq 1} \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}} < \infty.$$

   In this case $\pi_0 = \frac{1}{1+S}$,

   $$\pi_i = \frac{1}{1+S} \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}}, i \geq 1,$$

   and π is the unique (by irreducibility) invariant distribution. By theorem, this also corresponds to the case when X is positive recurrent.

3. When $p_i = p, q_i = q$ we have
$$S = \sum_{i \geq 1} \left(\frac{p}{q}\right)^i$$
which is finite if $p < q$ (unsurprising : we are looking at an asymmetric simple random walk reflected at the origin, and the condition $p < q$ corresponds indeed to the positive recurrent case).
When $p < q$ we have $1 + S = \frac{1}{1-p/q} = \frac{q}{q-p}$, and
$$\pi_i = \frac{q}{q-p}\left(\frac{p}{q}\right)^i, i \geq 0,$$
hence $\mathbb{E}_i[T_i^+] = \frac{1}{\pi_i} = \frac{q-p}{q}\left(\frac{q}{p}\right)^i$.
When $p = q$ the chain is null recurrent, and when $p > q$ it is transient, in both cases $\mathbb{E}_i[T_i^+] = \infty$.

Exercise 16

1. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a locally finite connected graph (locally finite means $\mathcal{V}$ is at most countable and each node has finitely many neighbours). Write $x \sim y$ iff $(x, y) \in \mathcal{E}$. To each edge $e \in \mathcal{E}$ a conductance $c_e > 0$ is assigned.
Let X be the chain on $\mathcal{V}$ with transition kernel P satisfying
$$P(x, y) = \frac{c(x, y)}{\sum_{z \sim x} c(x, z)}, \forall y \sim x.$$
Show that X is irreducible and reversible, and find its unique invariant distribution.
2. Let X be a reversible irreducible chain on the countable space E. Assume for any $x \in E$ there are only finitely many $y \in E$ such that $P(x, y) > 0$. Show one can find $\mathcal{G}$ locally finite and connected, and conductances $(c_e, e \in \mathcal{E})$ such that P can be expressed as in the previous question.

1. Irreducibility of X follows from connectivity of $\mathcal{G}$. Setting $c(x) := \sum_{y \sim x} c(x, y), x \in \mathcal{V}$, and $c_{\mathcal{G}} = \sum_{x \in \mathcal{V}} c(x)$, then X is reversible with $\pi(x) = \frac{c(x)}{c_{\mathcal{G}}}, x \in \mathcal{V}$
2. The graph is the one usually associated with the chain through its diagramm, it is connected because X is irreducible. Choose (no matter how) $x_0 \in \mathcal{V}, y_0 \sim x_0$, and fix $c(x_0, y_0) = 1$ (in fact the model does is invariant when multiplying conductances by a positive constant, so this choice is arbitrary).
For P to be expressed as in the previous question, we must have, for any $y \sim x, y \neq y_0$, $c(x_0, y) = P(x_0, y)/P(x_0, y_0)$. This fixes $c(x_0) = \sum_{y \sim x_0} P(x_0, y)/P(x_0, y_0)$ (finite thanks to our assumptions), and reversibility of X allows to uniquely determine $c(y)$ for any $y \sim x_0, y \neq x_0$, i.e. for any $y : d_{\mathcal{G}}(x_0, y) = 1$. Finally for any $y : d_{\mathcal{G}}(x_0, y) = 1$, and $z \sim y$, $d_{\mathcal{G}}(x_0, z) = 2$, one sets $c(y, z) = P(y, z)c(y)$.
By the same reasoning, if $\{c(z), c(z, z') : z \sim z', d_{\mathcal{G}}(x_0, z) = k, d_{\mathcal{G}}(x_0, z') = k+1\}$ are determined, reversibility allows to determine $\{c(z') : d_{\mathcal{G}}(x_0, z') = k+1\}$, then the knowledge of $P(z, z')$ allows to fix $c(z', z'')$ for any $z' \sim z''$ such that $d_{\mathcal{G}}(x_0, z') = k+1, d_{\mathcal{G}}(x_0, z'') = k+2$.

Exercise 17 Assume $(G, \cdot)$ is a group with at most countably many elements, μ a distribution on G, and X the chain on G such that $P(g, h \cdot g) = \mu(h)$. We refer to X as the random walk on G with jump kernel μ.

1. Explain why SRW on $\mathbb{Z}^d$ is an example of such chain.
2. Explain why symmetric simple random walk on $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^d$ is another.
3. Consider the following shuffling of a deck of $n \geq 2$ cards : at each time step, pick two uniform independent positions in the deck, independently of the past, and exchange the cards at these respective positions. Show that this constitutes a third example of a random walk on a group.
4. Let $\mathcal{H} = \{h_1 \cdot h_2 \cdot \dots \cdot h_n, \mu(h_i) > 0, i = 1, ..., n, n \in \mathbb{N}\}$. What can be said of X according to wether $\mathcal{H} \subsetneq G$ or $\mathcal{H} = G$? Find examples of chains corresponding to each of these two cases.
5. Show that any uniform measure on G is stationary.
6. Assume X is irreducible. Find the set of invariant distributions (one may distinguish between the cases when G is finite or infinite).
7. Assume X is irreducible. Show X is reversible iff μ satisfies

$$\mu(h^{-1}) = \mu(h) \quad \forall h \in G.$$

8. Give an example of a shuffling of a deck of n cards corresponding to a chain that is irreducible, but non reversible.

1. Random walk on $G = (\mathbb{Z}, +)$ is obtained with $\mu(1) = 1 - \mu(-1) = p$.
2. Symmetric SRW on $G = (\mathbb{Z}/n\mathbb{Z})^d$ is obtained with $\mu(e_i) = \mu(-e_i) = \frac{1}{2d}, i = 1, ..., d$.
3. Here $G = \mathfrak{S}_n$ and $\mu_{(ij)} = \frac{1}{n(n-1)}, i \neq j \in \{1, ..., n\}^2$, with (ij) the transposition of i and j.
4. A walk on G with jump kernel μ started at its neutral element id_G can only reach elements of $\mathcal{H}$, so if $\mathcal{H} \subsetneq G$ the walk can not be irreducible.
   On the other hand assume $\mathcal{H} = G$, and fix $g \in G$. Since $g \in \mathcal{H}$ there exists $h_1, ..., h_n$ s.t. $g = h_1 ... h_n$ so $id_G \to g$. Also since $g^{-1} \in \mathcal{H}$ and $id_G = g^{-1}g$ we also find $g \to id_G$. In the end every state communicates with id_G and the walk is irreducible.
   Walk on $\mathbb{Z}$ (cf first question) is irreducible when $p \in (0, 1)$, it is not when $p \in \{0, 1\}$.
5. Assume $\pi(g) = c$ for any $g \in G$, so π is a uniform measure on G. Since $g = hg' \Leftrightarrow h = g(g')^{-1}$, we have

$$\pi P(g) = \sum_{g' \in G} \pi(g') P(g', g) = c \sum_{g' \in G} P(g', gg'^{-1}g') = c \sum_{g' \in G} \mu(gg'^{-1}) = c,$$

   since μ is a distribution on G.
6. When X is irreducible there is a unique invariant measure which attributes a given mass c to id_G. Thus, if G is finite there exists a unique invariant distribution hich gives mass $1/|G|$ to each element of G.
   If G is infinite, there exists no invariant distribution. In particular this implies that an irreducible walk on an infinite group can not be positive recurrent.

7. When X is irreducible, there is a unique invariant measure which attributes a given mass c to id_G, by the above it has to be the corresponding uniform measure. Detailed balance now reads
$$cP(g,g') = cP(g',g) \forall g,g' \in G,$$
hence for any $h \in G$,
$$\mu(h) = P(g,hg) = P(hg,h^{-1}hg) = \mu(h^{-1}).$$
Conversely if $\mu(h) = \mu(h^{-1})$ detailed balance equations are satisfied with respect to a uniform measure.
8. Let $n \geq 3$, $G = \mathfrak{S}_n$ and assume that a jump of the chain consists in placing the top card at one of the n positions choosen uniformly at random (i.e. $\mu((1k(k-1)...2)) = \frac{1}{n}, k = 1,...,n)$. Such walk is clearly irreducible, however for $k \geq 3, \mu((1k(k-1)...2)^{-1}) = \mu(12...k)) = 0$, so it is not reversible.

Exercise 18 An admissible q-coloring of the graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is an application $\chi : \mathcal{V} \to \{1,...,q\}$ such that for any $(x,y) \in \mathcal{E}$ one has $\chi(x) \neq \chi(y)$.

1. If $\mathcal{G}$ is a tree show it admits a q-coloring, for any $q \geq 2$. Does the converse hold ?
2. Let $\mathcal{T}$ be a finite tree. Consider $\mathbb{G}_q(\mathcal{T})$ the graph whose nodes are the admissible q-colorings of $\mathcal{T}$ and edges are between pairs of admissible colorings which only differ at one node. How many nodes does $\mathbb{G}_2(\mathcal{T})$ possess ? Is it connected ? Is $\mathbb{G}_3(\mathcal{T})$ connected ?

1. Obviously, having an admissible q-coloring implies having admissible q'-colorings for any $q' \geq q$. For a tree, there are exactly two admissible 2-colorings of a tree : the first is obtained by coloring each node at even (resp. odd) distance from the root with color 1 (resp. 2), the second is obtained by doing the exact opposite.

   Converse does not hold : an even-length cycle is not a tree, but it also admits two 2-colorings.
2. By the above $\mathbb{G}_2$ has 2 noeuds. It is not connected as soon as $\mathcal{T}$ counts at least two nodes.

   By induction on the depth of $\mathcal{T}$, we are going to show that $\mathbb{G}_3$ is connected.

   More precisely if $\mathcal{T}$ has depth n and if c_1, c_2 are two admissible 3-colorings we should establish show that
   - when c_1 and c_2 coincide at the root of $\mathcal{T}$, one can go from c_1 to c_2 in $\mathbb{G}_3(\mathcal{T})$ without changing the color of the root.
   - when c_1 and c_2 differ at the root of $\mathcal{T}$, one can go from c_1 to c_2 in $\mathbb{G}_3(\mathcal{T})$ without ever using the third color for the root.

   If $n = 0, \mathcal{T}$ only has its root and the assertion is obvious.

   Let us now assume the above assertion holds for a tree of depth at most n, and fix $\mathcal{T}$ of depth $n+1$ and to admissible colorings of $\mathcal{T}$, say $c_1, c_2 : \mathcal{T} \to \{1,2,3\}$.

   Denote by $\mathcal{T}_1,...,\mathcal{T}_d$ the subtrees of $\mathcal{T}$ below its root $\emptyset$.

   If $c_1(\emptyset) = c_2(\emptyset)$, apply induction assumption to $\mathcal{T}_i, i = 1,...,d$ to go from c_1 to c_2 inside each subtree without using the color of the root of $\mathcal{T}$ for nodes at depth 1 : it is obviously possible for a subtree whose roots colors match in c_1 and c_2, and it is still possible when the do not, since by assumption one can match colors in $\mathcal{T}_i$ without using the third color which has to be that of the root of $\mathcal{T}$.

If $c_1(\emptyset) \neq c_2(\emptyset)$ (to fix ideas and w.l.o.g., let us say that $c_1(\emptyset) = 1, c_2(\emptyset) = 2$). By the above, from c_1, we can reach the 2-coloring using only colors 1 (at even depths) and 3 (at odd depths) without ever changing the color of the root

Now change the root color to 2.

By induction, we can now reach c_2 from this coloring without having to change the root's color.

So we have gone from c_1 to c_2 without using the third color, and we are done.

4 Complements on discrete-time chains

Exercise 19 Let E be countable, $(Z_n, n \geq 1)$ i.i.d taking values in Λ and $\phi : E \times \Lambda \to E$. Let X be such that

$$X_{n+1} = \phi(X_n, Z_{n+1}),\ n \geq 0,$$

and denote by $\mathbb{P}_{x_0}$ the law of X when $X_0 = x_0$.

1. Show that X is Markov, and find its transition kernel P.
2. Here $Z_n = (j_n, B_n)$ with $j_n \sim \text{Unif}\{1, ..., N\}$ independent of $B_n \sim \text{Ber}(1/2)$. How should one choose ϕ so as to recover the lazy SRW on the hypercube ?
3. What difference is there between filtrations $(\mathcal{F}_n), (\mathcal{G}_n)$ where $\mathcal{F}_n = \sigma(X_0, ..., X_n), n \geq 0$ and $\mathcal{G}_n = \sigma(X_0, Z_1, ..., Z_n), n \geq 0$?
4. For the example of question 2, show that
$$T = \inf\{n \geq 0 : \{j_1, ..., j_n\} = \{1, ..., N\}\}$$
is a $(\mathcal{G}_n)$-stopping time. Is it a $(\mathcal{F}_n)$-stopping time ?
5. What is the asymptotic behaviour of T as $n \to \infty$ (one may use a previous exercise) ?
6. In general if X is Markov and $f : E \to F$, is the process $(Y_n := f(X_n), n \geq 0)$ an F-valued Markov chain ?

1. Fix $x, y \in E$, and let $A_{x,y} = \{z \in E : \phi(x, z) = y\}$. Then $P(x, y) = \int_{A_{x,y}} d\mathbb{P}_{Z_1}(z)$.
2. The hypercube is $E = \{0, 1\}^n$. To recover lazy SRW on E it suffices to choose
$$\phi : \begin{cases} E \times \{1, ..., n\} \times \{0, 1\} \to E \\ (x, j, b) \to (x_1, ..., x_{j-1}, b, x_{j+1}, ..., x_n) \end{cases},$$
so that ϕ replaces the jth coordinate of x with b.
3. Since $X_n = \phi(...\phi(\phi(X_0, Z_1), Z_2)..., Z_n)$ is a function of $X_0, Z_1, ..., Z_n$, filtration $(\mathcal{G}_n)$ is finer than $(\mathcal{F}_n)$.
4. T clearly is a $(\mathcal{G}_n)$-stopping time. However it is not a $(\mathcal{F}_n)$-stopping time. Indeed for $x \in E$, the event (in $\mathcal{F}_n$), $\{X_0 = X_1 = ... = X_n = x\}$ intersects both $\{\tau = n\}$ and $\{\tau > n\}$.

5. The law of T is exactly that of the collection time of n coupons. We had seen that $T = n\log(n) + o(n\log(n))$ with probability tending to 1 as $n \to \infty$.
6. In general this is not a Markov chain. For instance take X SRW on $\frac{\mathbb{Z}}{3\mathbb{Z}}$ absorbed at 0 and $f \equiv \text{mod}2$. Then $(Y_n = f(X_n), n \geq 0)$ is not a 2-state chain : $\mathbb{P}_1(Y_4 = 1 \mid Y_1 = 0, Y_2 = 1, Y_3 = 0) > 0$ but $\mathbb{P}_1(Y_4 = 1 \mid Y_1 = Y_2 = Y_3 = 0) = 0$.

Exercise 20 Assume $(X_n)_{n\geq 0}$ is Markov (λ, P), $(\mathcal{F}_n)_{n\geq 0}$ is the natural filtration of X, and T is an $(\mathcal{F}_n)_{n\geq 0}$-stopping time.

1. Define the trace field $\mathcal{F}_T := \{A \in \mathcal{F} : \forall n \in \mathbb{N}, A \cap \{T = n\} \in \mathcal{F}_n\}$. Show that $\mathcal{F}_T = \sigma(X_0, ..., X_T)$.
2. Establish that if $B \in \mathcal{F}_T$, $m \in \mathbb{N}$, $x \in E$ we have
$$\begin{aligned}\mathbb{P}_\lambda(X_T = j_0, X_{T+1} = j_1, ..., X_{T+n=j_n} \cap B \cap \{T = m\} \cap \{X_T = x\})\\ = \mathbb{1}_{\{j_0 = i\}}\mathbb{P}_i(X_1 = j_1, ..., , X_n = j_n)\mathbb{P}_\lambda(B \cap \{T = m\} \cap \{X_T = x\})\end{aligned}$$
3. Deduce that
$$\begin{aligned}\mathbb{P}_\lambda(X_T = j_0, X_{T+1} = j_1, ..., X_{T+n=j_n} \cap B \mid T < \infty, X_T = x)\\ = \mathbb{P}_\lambda(B \mid T < \infty, X_T = x)\mathbb{1}_{\{j_0 = i\}}\prod_{k=0}^{n-1} P(j_k, j_{k+1}).\end{aligned}$$
4. What is the conditional law of $(X_{T+n}, n \geq 0)$ given $\{T < \infty, X_T = x\}$?

1. For A to be in $\sigma(X_0, ..., X_T)$ it is necessary and sufficient that for any $n \in \mathbb{N}$, $A \cap \{T = n\} \in \sigma(X_0, ..., X_n) = \mathcal{F}_n$. In other words $\mathcal{F}_T$ contains exactly the events which can be decided before T.
2. This is a direct application of Markov property since $B \cap \{T = m\} \in \mathcal{F}_m$.
3. Summing over $m \in \mathbb{N}$ both sides of the equation obtained in the previous question (there are no problems with the infinite sums since all terms are positive) we find
$$\begin{aligned}\mathbb{P}_\lambda(X_T = j_0, X_{T+1} = j_1, ..., X_{T+n=j_n} \cap B \cap \{T < \infty\} \cap \{X_T = x\})\\ = \mathbb{1}_{\{j_0 = i\}}\mathbb{P}_i(X_1 = j_1, ..., , X_n = j_n)\mathbb{P}_\lambda(B \cap \{T < \infty\} \cap \{X_T = x\})\end{aligned}$$
Dividing by $\mathbb{P}(\{T < \infty\} \cap \{X_T = i\})$ we obtain the desired equality.
4. The previous question implies that conditionally given $\{T < \infty, X_T = x\}$, $(X_{T+n}, n \geq 0)$ is Markov (δ_x, P)

Exercise 21 Let X be an E-valued Markov chain, with kernel P. For $x \in E$, let $\mathcal{T}(x) := \{n \in \mathbb{N}^* : P^n(x, x) > 0\}$, and $d(x) = \text{pgcd}(\mathcal{T}(x))$

1. Show that, if x and y both belong to the same communication class, then $d(x) = d(y)$. One may start by noticing that if n, m are such that $P^{(n)}(x, y) > 0, P^{(m)}(y, x) > 0$ then $d(x)$ and $d(y)$ divide $n + m$.
2. Show that if E is finite and X is irreducible and aperiodic, one can find $r > 0$ such that $P^{(r)}(x, y) > 0$ for any $x, y \in E$.

1. Consider, as suggested n, m such that $P^{(n)}(x,y) > 0, P^{(m)}(y,x) > 0$. Tnen $P^{(n+m)}(x,x) \geq P^{(n)}(x,y)P^{(m)}(y,x) > 0$, hence $d(x)$ divides $s+t$. Similarly $P^{(n+m)}(y,y) \geq P^{(n)}(y,x)P^{(m)}(x,y) > 0$ and $d(y)$ also divides $n+m$.
   Now if $P^{(r)}(x,x) > 0$ we also have $P^{(n+m+t)}(y,y) \geq P^{(m)}(y,x)P^{(r)}(x,x)P^{(n)(x,y)} > 0$, so that when $r \in \mathcal{T}(x)$, $n+m+r \in \mathcal{T}(y)$, hence $d(y)$ divides $n+m+r$. But $d(y)$ divides $n+m$ so $d(y)$ also divides r. Since the reasoning holds for any $r \in \mathcal{T}(x)$, we conclude that $d(y)$ divides $d(x)$.
   By a symmetric argument $d(x)$ divides $d(y)$ et we finally conclude that $d(x) = d(y)$.
2. Let $x \in E$. Since $\mathrm{pgcd}(\mathcal{T}(x)) = 1$, we are going to show the existence of $n_x \in \mathbb{N}$ such that $\mathcal{T}(x) \supset \{n_x, n_x+1, n_x+2, ...\}$.
   The sequence $(\mathrm{pgcd}(\mathcal{T}(x) \cap \{1, ..., m\}))_m$ is integer valued and non increasing, it must be constant above some rank, so it must be 1 above some rank. Hence there exists m_0 such that $\mathrm{pgcd}(\mathcal{T}(x) \cap \{1, ..., m_0\}) = 1$. Write $\mathcal{T}(x) \cap \{1, ..., m_0\} = \{k_1, ..., k_r\}$. By Bezout one can find $a_1, ..., a_r$ such that $\sum_{i=1}^r a_i k_i = r$. Even if it means taking $K \geq k_1 \max |a_i|$, a $K' \geq 0$ and $k < k_1$ any $n \geq n_x K \sum_{i=1}^r k_i$ can be written as

   $$n = K \sum_{i=1}^r k_i + K' k_1 + k(\sum_{i=1}^r a_i k_i),$$

   so that

   $$n = (K + K' + k a_1) k_1 + \sum_{i=2}^r (K + k a_i) k_i,$$

   where the integers $K + K' + k a_1, K + k a_2, ..., K + k a_r$ are all nonnegative. In the end $P^n(x,x) > 0$ for any $n \geq n_x$, as required.
   Now under the assumption that E is finite, $N := \max_x n_x$ also is, and $P^n(x,x) > 0$ for any $n \geq N$, and any $x \in E$. Write $n_{x,y} := \min\{k : P^k(x,y) > 0\}$, and $N' = \max_{x,y \in E} n_{x,y}$, to conclude finally that if $n \geq N + N'$, whatever $x, y \in E$, $n - n_{x,y} \geq N$ then

   $$\forall x, y \in E \quad P^n(x,y) \geq P^{n-n_{x,y}}(x,x) P^{n_{x,y}}(x,y) > 0.$$

Exercise 22 Let X be an irreducible E-valued chain with kernel P, and period $d \geq 2$.

1. Show that E can be partitioned into d classes $C_0, ..., C_{d-1}$ which satisfy the following : for any distribution λ on E s.t. $\lambda(C_0) = 1$, and for any $r \in \{0, ..., r-1\}$, the chain $(Y_n^{(r)} := X_{dn+r})_{n \geq 0}$ is C_r-valued. Is the chain $Y^{(r)}$ irreducible ? aperiodic ?
2. Let λ be a distribution on E s.t. $\lambda(C_0) = 1$. Establish that

   $$\mathbb{P}_\lambda(X_{dn+r} = j) \underset{n \to \infty}{\longrightarrow} \frac{d}{\mathbb{E}_j[T_j^+]}.$$

1. Fix an $x_0 \in E$ and write

   $$C_i = \{x \in E : \exists n \geq 0 P^{nd+i}(x_0, x) > 0\}, \quad i = 0, 1, ..., d-1.$$

   Since the chain is irreducible the union of these non empty sets cover E. If $i, j \in \{0, ..., d-1\}$ and $x \in C_i \cap C_j$ one may find n, n' such that

$P^{nd+i}(x_0,x)>0, P^{n'd+j}(x_0,x)>0$. By irreducibility there exists r s.t. $P^r(x,x_0)>0$, so both integers $nd+i+r, n'd+j+r$ are in $\mathcal{T}(x)$, hence they, their difference and $i-j$ must be divisible by d. Since $|i-j|<d-1$, we conclude that $i=j$.
Hence the $C_i, i=0,...,n-1$ are disjoint, and we have checked they form a partition of E.
Let x_0 as in the previous question and $x\in C_0$. Consider $\lambda=\delta_{x_0}$. Suppose y is reached by $Y^{(r)}$, i.e. $P^{dn+r}(x,y)>0$ for some $n\geq 0$. Since $x\in C_0$, there exists m such that $P^{md}(x_0,x)>0$, but then $P^{(m+n)d+r}(x_0,y)>0$ so that $y\in C_r$. Hence in this case $Y^{(r)}$ is indeed C_r-valued.
A λ such that $\lambda(C_0)=1$ can be decomposed as $\sum_{x\in C_0}\alpha(x)\delta_x$. By the above if X starts from λ the chain $Y^{(r)}$ is still C_r-valued.
Note that, by the same reasoning, if $\lambda(C_i)=1$, for X started from λ, the chain $Y^{(r)}$ is $C_{i+r \mathrm{mod} d}$-valued.
Now fix $x\in C_0$, we have $d(x)=\mathrm{pgcd}(\mathcal{T}(x))=d$ (cf first question of previous exercise). By the reasoning of the second question of that exercise, there exists n_x sufficiently large so that for any $n\geq n_x$, $P^{nd}(x,x)>0$ (observe that this part of the reasoning remains valid even if E is only assumed countable).
Now fix $y_1,y_2\in C_r$, and $x\in C_0$. Since the chain X is irreducible, and using the above we may find $k_1=n_1d+r, k_2=n_2d+r$ such that
$P^{(k_1)}(y_1,x)>0, P^{(k_2)}(x,y_2)>0$. Then for any $n\geq n_x$, denoting $n'=n+n_1+n_2$ we have $P^{n'd}(y_1,y_2)=P^{nd+k_1+k_2}(y_1,y_2)>P^{k_1}(y_1,x)P^{nd}(x,x)P^{k_2}(x,y_2)>0$. Hence for any $n'\geq n_1+n_2+n_x$, $P^{n'd}(y_1,y_2)>0$, and one concludes that $Y^{(r)}$ is irreducible and aperiodic.

2. Denote $\mathbb{Q}_y$ the law of $Y^{(r)}$ started at $y\in C_r$. Since one step of $Y^{(r)}$ corresponds to d steps of X, we find that $d\mathbb{E}_{\mathbb{Q}_y}[T_y^+]=\mathbb{E}_{\mathbb{P}_y}[T_y^+]$. In particular positive recurrence (resp. null recurrence, resp. transience) of X is equivalent to that of $Y^{(r)}, r\in\{0,...,d-1\}$.
1st case : Both chains are recurrent positive, then the stationary distribution of $Y^{(r)}$, denoted $\pi^{(r)}$, must satisfy
$$\pi^{(r)}(y)=\frac{1}{\mathbb{E}_{\mathbb{Q}_y}[T_y^+]}=\frac{d}{\mathbb{E}_{\mathbb{P}_y}[T_y^+]}.$$
Apply convergence theorem to $Y^{(r)}$ to deduce that whatever $\mu^{(r)}$ distribution on C_r, we have
$$\mathbb{Q}_{\lambda^{(r)}}(Y^{(r)}(n)=y)\underset{n\to\infty}{\longrightarrow}\pi^{(r)}(y)=\frac{d}{\mathbb{E}_{\mathbb{P}_y}[T_y^+]},$$
which is the desired conclusion.
2nd case : If both chains are transient $\mathbb{P}_\lambda(X_{nd+r}=y)$ goes to 0 whatever λ,y and the desired conclusion easily follows.
3rd case If both chains are null recurrent let us focus on $Y=Y^{(r)}$: an irreducible, aperiodic, null recurrent chain. Fix $A>0$.
Since$\mathbb{E}_{\mathbb{Q}_y}[T_y^+]=\sum_{n\in\mathbb{N}}\mathbb{Q}_y(T_y^+>n)=+\infty$, one can find N such that
$$\sum_{n=0}^{N-1}\mathbb{Q}_y(T_y^+>n)\geq A.$$

Now let $n \geq N$, decompose according to the different possible values of the last passage time at y before n, and use Markov at time k :

$$\begin{aligned} 1 &\geq \sum_{k=n-N+1}^{n} \mathbb{Q}_y\left(Y_k = y, y \notin \{Y_{k+1}, ...Y_n\}\right) \\ &\geq \sum_{k=n-N+1}^{n} \mathbb{Q}_y(Y_k = y)\mathbb{Q}_y\left(T_y^+ > n-k\right) = \sum_{k=0}^{N-1} \mathbb{Q}_y(Y_{n-k} = y)\mathbb{Q}_y\left(T_y^+ > k\right). \end{aligned}$$

One concludes that for any $n \geq N$, there exists $k \in \{0, ..., N-1\}$ such that $\mathbb{Q}_y(Y_{n-k} = y) \leq 1/A$.
It remains, as in the proof of the convergence theorem to use that for μ, ν distributions on E, when $n \to \infty$ $d_{TV}(\mu P^n, \nu P^n) \to 0$ and deduce that for any fixed k, $\mu = \lambda, \nu = \lambda P^k$

$$d_{TV}(\lambda P^{n-k}, \lambda P^n) \to 0,$$

as $n \to \infty$. In particular

$$\mathbb{Q}_y(Y_{n-k} = y) - \mathbb{Q}_y(Y_n = y) \to 0,$$

so that $\mathbb{Q}_y(Y_n = y) \leq 2/A$ for sufficiently large n. Since A is arbitrary large, we conclude, as required, that $\mathbb{Q}_y(Y_n = y) \to 0$.

Exercise 23 Let X be an E-valued Markov chain with initial distribution μ, $(\mathcal{F}_n)_n$ the natural filtration of X and T an $(\mathcal{F}_n)$-stopping time. We assume T to be $\mathbb{P}_\mu$-a.s. finite. We further assume that the law of X_T under $\mathbb{P}_\mu$ is μ.

1. What can be said of the measure

$$\nu(x) := \mathbb{E}_\mu\left[\sum_{k=0}^{T-1} \mathbb{1}_{\{X_k = x\}}\right], \quad x \in E?$$

2. Let θ_k the shift operator by k time steps, that is, if $(x_n)_{n\geq 0} \in E^{\mathbb{N}}$,

$$\theta_k((x_n)_{n\geq 0}) = (x_{n+k}, n \geq 0).$$

   We then define $T = T_1$ and $T_{k+1} = T_k + T \circ \theta_{T_k}$.
3. How can the trajectory of X be decomposed into identically distributed pieces ?
4. In general, are these pieces independent ?
5. Consider $x, y \in E$, and assume throughout the remainder of the exercise that the chain is irreductible, positive recurrent positive, and denote by π its unique invariant probability.
   Let $T = \inf\{n \geq T_y : X_n = x\}$. Show that $\nu = (\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x])\pi$, and then that $\nu(x) = \frac{1}{\mathbb{P}_x(T_y < T_x^+)}$. Deduce that

$$\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x] = \frac{\mathbb{E}_x[T_x^+]}{\mathbb{P}_x(T_y < T_x^+)}.$$

6. For $x, y, z \in E$, by introducing $\sigma = \inf\{n \geq T_y : X_n = z\}$ and $\tau := \inf\{n \geq \sigma : X_n = x\}$, use a similar method to establish that

$$\mathbb{E}_x\left[\#\{\text{visites en } z \text{ avant } T_y\}\right] = \frac{\mathbb{E}_x[T_y] + \mathbb{E}_y[T_z] - \mathbb{E}_x[T_z]}{\mathbb{E}_z[T_z^+]},$$

and

$$\mathbb{E}_x[T_y] + \mathbb{E}_y[T_z] + \mathbb{E}_z[T_x] = \mathbb{E}_x[T_x^+]\left(\frac{1}{\mathbb{P}_x(T_y < T_x^+)} + \frac{\mathbb{P}_y(T_x < T_z)}{\mathbb{P}_x(T_z < T_x^+)}\right).$$

1. We are going to show that the measure ν is invariant : by Markov property at time k,

$$\begin{aligned}
\nu P(x) &= \sum_{y \in E} \mathbb{E}_\mu\left[\sum_{k=0}^{T-1} \mathbb{1}_{\{X_k = y\}}\right] P(y, x) \\
&= \sum_{y \in E} \mathbb{E}_\mu\left[\sum_{k=0}^{\infty} \mathbb{1}_{\{X_k = y, X_{k+1} = x\}} \mathbb{1}_{\{T > k\}}\right] \\
&= \mathbb{E}_\mu\left[\sum_{k=0}^{\infty} \sum_{y \in E} \mathbb{1}_{\{X_k = y, X_{k+1} = x, T > k\}}\right] \\
&= \mathbb{E}_\mu\left[\sum_{k'=1}^{T} \mathbb{1}_{\{X'_k = x\}}\right] = \nu(x).
\end{aligned}$$

   Interversions of $\mathbb{E}_\mu$ and $\sum$ are justified by the fact that integrated quantities are all nonnegative ; and the last equality above comes from the fact that the law of X_T is μ, so that $\mu(x) = \mathbb{E}_\mu(\mathbb{1}_{\{X_0 = x\}}) = \mathbb{E}_\mu(\mathbb{1}_{\{X_T = x\}})$.

2. Thanks to the strong Markov property, $(X_i, i = T_k, ..., T_{k+1})_k$ are identically distributed.

   Remark : Beware that in invoking Markov, we make use of the property that T, thus $T_k, k \geq 1$ are $\mathbb{P}_\mu$-a.s. finite. In the case when $\mathbb{P}_\mu(T = \infty) > 0$, there is only a geometric number of pieces, and there are no longer identically distributed (all except the last are conditioned on having finite length, while the last is conditioned on having infinite length).

3. However, these pieces are a priori not independent (as soon as the support of μ counts at least two states, the law of $(X_i, i = T_k, ..., T_{k+1})$ obviously depends on the last value of the previous piece of the trajectory).

   Note however that when $\mu = \delta_x$, then for any k, $X_{T_k} = x$ and strong Markov property ensures in this particular case that the pieces are i.i.d. This is very useful in establishing the ergodic theorem for Markov chains.

4. Le temps d'arrêt T vérifie les hypothèses de l'énoncé. La mesure ν associée est donc invariante de poids total $\mathbb{E}[T] = \mathbb{E}_x[T_y] + \mathbb{E}_y[T_x]$, et par unicité de la distribution stationnaire, on déduit que $\nu = (\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x])\,\pi$. Par ailleurs, au point x, $\pi(x) = \frac{1}{\mathbb{E}_x[T_x^+]}$, tandis que

$$\nu(x) = \mathbb{E}_x\left[\#\{\text{ visites en } x \text{ avant } T_y\}\right].$$

Or, par la propriété de Markov forte aux temps successifs de retour en x, on a que $\#\{$ visites en x avant $T_y\}$ est une variable géométrique de paramètre $\mathbb{P}_x(T_y < T_x^+)$, d'espérance $\frac{1}{\mathbb{P}_x(T_y<T_x^+)}$, et on obtient donc

$$\frac{1}{\mathbb{P}_x(T_y < T_x^+)} = \nu(x) = (\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x])\,\pi(x) = \frac{\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x]}{\mathbb{E}_x[T_x^+]},$$

l'égalité souhaitée.

5. Le temps d'arrêt τ vérifie également les hypothèses de l'énoncé et donc la mesure ν associée est invariante de poids total

$$\mathbb{E}[\tau] = \mathbb{E}_x[T_y] + \mathbb{E}_y[T_z] + \mathbb{E}_z[T_y],$$

on déduit que $\nu = (\mathbb{E}_x[T_y] + \mathbb{E}_y[T_z] + \mathbb{E}_z[T_x])\,\pi$. Au point z, $\pi(z) = \frac{1}{\mathbb{E}_z[T_z^+]}$, tandis que

$$\begin{aligned}\nu(z) &= \mathbb{E}_x\left[\#\{\text{ visites en } z \text{ avant } T_y\}\right] + \mathbb{E}_z\left[\#\{\text{ visites en } z \text{ avant } T_x\}\right]\\ &= \mathbb{E}_x\left[\#\{\text{ visites en } z \text{ avant } T_y\}\right] + \frac{\mathbb{E}_x[T_z] + \mathbb{E}_z[T_x]}{\mathbb{E}_z[T_z^+]}\end{aligned}$$

où on a utilisé la question précédente à la dernière ligne. On déduit donc

$$\frac{\mathbb{E}_x[T_y] + \mathbb{E}_y[T_z] + \mathbb{E}_z[T_x]}{\mathbb{E}_z[T_z^+]} = \nu(z) = \mathbb{E}_x\left[\#\{\text{ visites en } z \text{ avant } T_y\}\right] + \frac{\mathbb{E}_x[T_z] + \mathbb{E}_z[T_x]}{\mathbb{E}_z[T_z^+]},$$

ce qui conduit à la première égalité souhaitée.
Par ailleurs, au point x, on a d'une part $\nu(x) = \frac{\mathbb{E}_x[T_y]+\mathbb{E}_y[T_z]+\mathbb{E}_z[T_x]}{\mathbb{E}_x[T_x^+]}$, et d'autre part

$$\begin{aligned}\nu(x) &= \mathbb{E}_x\left[\#\{\text{ visites en } x \text{ avant } T_y\}\right] + \mathbb{E}_y\left[\#\{\text{ visites en } x \text{ avant } T_z\}\right]\\ &= \frac{1}{\mathbb{P}_x(T_y < T_x^+)} + \mathbb{E}_y\left[\#\{\text{ visites en } x \text{ avant } T_z\}\right]\end{aligned}$$

Reste à voir (par Markov fort en T_x, que sous $\mathbb{P}_y$, le nombre de visites en x avant T_z est 0 sur l'événement $\{T_z < T_x\}$, et prend la valeur d'une géométrique de paramètre $\mathbb{P}_x(T_z < T_x^+)$ sur l'événement $\{T_x < T_z\}$. On déduit que

$$\mathbb{E}_y\left[\#\{\text{ visites en } x \text{ avant } T_z\}\right] = \frac{\mathbb{P}_y(T_x < T_z)}{\mathbb{P}_x(T_z < T_x^+)},$$

ce qui conduit à la deuxième égalité souhaitée.

Exercise 24 Assume π is the invariant distribution for an E-valued irreducible chain X. Show that $\pi(x) > 0$ for any $x \in E$.
Since E is at most countable there exists at least one $y \in E$ such that $\pi(y) > 0$.
Now fix $x \in E$. Since the chain is irredcible, we may find a k such that $P^k(y,x) > 0$. But then $\pi(x) = \pi P^k(x) \geq \pi(y)P^k(y,x) > 0$.

Exercise 25 Let X be an E-valued Markov chain whose transition kernel P is assumed to be symmetric.

1. Assume that E is finite. Show that uniform distribution on E is invariant.
2. Assume again that E is finite. Upon what condition can it be said that the uniform distribution is the unique invariant distribution ? What happens if this condition is not fullfilled ?
3. Can you find an example with E infinite (countably), P irreducible and symmetric, in which the chain X does not possess any invariant distribution.
4. Can you find an example with E infinite (countably), P irreducible and symmetric, in which the chain X possesses a unique invariant distribution ?

1. Let π the uniform distribution on E. As P is symmetric,
$$\sum_{x\in E}\pi(x)P(x,y)=\frac{1}{n}\sum_{x\in E}P(y,x)=\frac{1}{n}=\pi(y).$$
2. If X is irreducible, its stationary distribution is unique and it must therefore be π. Also since P is symmetric the chain can only have closed finite hence recurrent classes.
   Thus if the chain is not irreducible, then there must be at least two positive recurrent classes, so there must be infinitely many invariant distributions, that are the convex linear combinations of uniform distributions on each class. In that case, π is only one example of such combination.
3. SRW on $\mathbb{Z}^d$.
4. We prove by contradiction that there can not be such an example.
   By irreducibility, if there is an invariant distribution π for the chain, any invariant measure has to be a multiple of π (note that the next exercise provides another proof of this fact). Now the uniform (infinite) measure μ attributing mass one to each element in E must be invariant, by the same reasoning as in question 1, but since E is infinite, μ can not be a multiple of π and we have reached a contradiction.

Exercise 26 Assume X is an irreducible E-valued Markov chain admitting an invariant distribution π. For μ a nonnegative measure on E, and $f:\mathbb{R}_+\to\mathbb{R}$ strictly convex and bounded, we define
$$\mathrm{Ent}(\mu\mid\pi)=\sum_{x\in E}f\left(\frac{\mu(x)}{\pi(x)}\right)\pi(x).$$

1. Show that $\mathrm{Ent}(\mu P\mid\pi)\le\mathrm{Ent}(\mu\mid\pi)$.
2. When is the above inequality an equality ? Deduce that any invariant measure of X is a multiple of π.

1. Since π is invariant, for any x, $\nu_x(y)=\frac{\pi(y)P(y,x)}{\pi(x)}$ defines a distribution. Thus by the

convexity of f and Jensen's inequality

$$\begin{aligned}
\mathrm{Ent}(\mu P \mid \pi) &= \sum_{x\in E} f\left(\frac{\sum_{y\in E}\mu(y)P(y,x)}{\pi(x)}\right)\pi(x) \\
&= \sum_{x\in E} f\left(\sum_{y\in E}\nu_x(y)\frac{\mu(y)}{\pi(y)}\right)\pi(x) \\
&\leq \sum_{x\in E}\sum_{y\in E}\nu_x(y) f\left(\frac{\mu(y)}{\pi(y)}\right)\pi(x) \\
&= \sum_{y\in E}\pi(y)\left(\sum_{x\in E}P(y,x)\right) f\left(\frac{\mu(y)}{\pi(y)}\right) = \mathrm{Ent}(\mu \mid \pi)
\end{aligned}$$

2. Since f is strictly convex, and $\mathrm{Ent}(\mu \mid \pi)$ is finite because f is bounded and π is a distribution, equality in the inequality above can only occur if $y \to \frac{\mu(y)}{\pi(y)}$ is constant, that is if $\mu = C\pi$. If μ is invariant one must have $\mu = \mu P$, by the above reasoning one must have $\mu = C\pi$.

Exercise 27 Let X be an E-valued Markov chain with kernel P. We assume $\sim$ to be an equivalence relation on E. For $x \in E$, let $\tilde{x}$ denote its equivalence classe in $E/\sim$.

1. Assume that for any $\tilde{a}, \tilde{b} \in E/\sim$, the application $\left\{\begin{array}{l} \tilde{a} \to \mathbb{R}_+ \\ x \to \sum_{y\in\tilde{b}} P(x,y) \end{array}\right.$ remains constant. Establish that under this assumption $\tilde{X}$ is an $\tilde{E}$-valued Markov chain, whose transition kernel $\tilde{P}$ shall be precised. Such chain is usually refered to as the *projected chain.*
2. Let $n \in \mathbb{N}, n \geq 2$ and $E = \mathbb{Z}/(2n\mathbb{Z})$, let X the SRW (not necessarily symmetric) on E, and finally denote by $\sim$ the equivalence relation

$$x \sim y \Leftrightarrow x + y = 0 \ \ [2n].$$

Upon what condition can the projected chain be defined ?

3. Let X be SRW on the hypercube $E = \{0,1\}^d$, with $d \geq 1$, and $\sim$ be the equivalence relation

$$x \sim y \Leftrightarrow \sum_{i=1}^d x_i = \sum_{i=1}^d y_i.$$

Describe the corresponding projected chain.

1. Let $n \in \mathbb{N}$. We have

$$\mathbb{P}(X_0 \in \tilde{a_0}, ..., X_n \in \tilde{a}, X_{n+1} \in \tilde{b}) = \sum_{x_0\in\tilde{a_0},...,x_n\in\tilde{a_n},y\in\tilde{b}} \mathbb{P}(X_0 = x_0, ..., X_n = a_n)P(x,y).$$

From the assumption of the statement, $f(\tilde{a_n}, \tilde{b}) := \sum_{y\in\tilde{b}} P(x,y)$ does not depend on the choice of $a_n \in \tilde{a_n}$ so that

$$\mathbb{P}(X_0 \in \tilde{a_0}, ..., X_n \in \tilde{a}, X_{n+1} \in \tilde{b}) = f(\tilde{a}, \tilde{b})\mathbb{P}(X_0 \in \tilde{a_0}, ..., X_n \in \tilde{a_n}).$$

We may therefore define $\tilde{P}(\tilde{a},\tilde{b}) = f(\tilde{a},\tilde{b})$, and it is therefore straightforward that $\tilde{X}$ is indeed Markov, takes values in $\tilde{E}$ and has kernel $\tilde{P}$.

2. We have $\tilde{E} = \{0, ..., n\}$ (class $i, 1 \leq i \leq n$ corresponding to points i and $2n - i$ in E). For $\sum_{y\in\tilde{b}} P(x, y)$ not to depend on x it must be that the initial walk is symmetric. In that case the projected chain is SRW on $\{0, ..., n\}$, reflected at the boundaries 0 and n

3. Here $\tilde{E} = \{0, ..., d\}$ and one recovers Ehrenfest's model, more precisely

$$\tilde{P}(i, i+1) = \frac{d-i}{d}, i = 0, ..., d-1, \quad \tilde{P}(i, i-1) = \frac{i}{d}, i = 1, ..., d.$$

Exercise 28 For $j = 1, ..., d$ it is assumed that $X^{(j)}$ is a Markov chain taking values in the countable E_j, with transition kernel P_j.
Assume further that ν is a distribution on $\{1, ..., d\}$, and define the *product chain associated with* ν $X = (X^{(1)}, ..., X^{(d)})$ with kernel P s.t.

$$P(x, y) = \sum_{j=1}^{d} \nu(j) P_j(x_j, y_j) \prod_{i\neq j} \mathbb{1}_{\{x_i = y_i\}}.$$

1. Find a NSC for irreductibility of X. We will assume that this condition is satisfied in the remainder of the exercise.
2. Find a NSC for aperiodicity of X.
3. Show that the existence of a unique invariant distribution π for X is equivalent to the existence of a unique invariant distribution π_j for each coordinate chain $X_j, j = 1, ..., d$. Then, express such π in terms of $\pi_j, j = 1, ..., d$ and ν.
4. What is X when ν is uniform on $\{1, 2\}$ and $X_j, j = 1, 2$ are both SRW on $\mathbb{Z}/n\mathbb{Z}$?
5. Can one choose $X_1, ..., X_d$ and ν such that X is SRW on the hypercube $\{0, 1\}^d$? What about lazy SRW on the hypercube ?

1. X is irreducible iff for each $i \in \{1, ..., d\}$, either $\nu(i) > 0$ and X_i is irreducible, or E_i is a singleton.
   Indeed, assume this condition holds. Fix x and y in the product space. Let k_j be an integer such that $P_j^{k_j}(x^{(j)}, y^{(j)}) > 0$. Then for $k = k_1 + ... + k_d$ we have $P^{(k)}(x, y) > 0$. On the other hand if for some i, the coordinate chain is not irreducible then clearly X can not be : $a \nrightarrow b$ in E_i then $x \nrightarrow y$ in E as soon as $x^{(i)} = a, y^{(i)} = b$. Moreover if for some i, $\nu(i) = 0$, the ith coordinate of the chain remains unchanged, and unless E_i is a singleton, this forbids irreducibility of X.
2. Fix $x \in E$.
   Let $k_i = \mathrm{pgcd}\{n \geq 1 P_i^n(x^{(i)}, x^{(i)}) > 0\}, i = 1..., d$ the respective periods of coordinate chains. For X to be aperiodic it is necessary and sufficient that $\mathrm{pgcd}(k_1, ..., k_d) = 1$. Indeed one can reach the starting point in k_i steps, for any $i \in \{1, ..., d\}$, even if it means only moving along one coordinate. Thus the period of X divides $\mathrm{pgcd}(k_1, ..., k_d)$.

On the other hand if for $k>0$, $P^k(x,x)>0$, and if the trajectory involves coordinates $i_1,...,i_\ell$, then we must have

$$k=\sum_{j=1}^{\ell} n_{i_j}k_{i_j},$$

so $\mathrm{pgcd}(k_1,...,k_d)$ must divide the period of X.

3. This product chain is simply SRW on the 2-dimensional discrete torus.
4. In the case ν uniform, $E_i=\{0,1\}$, and $P_i(0,1)=P_i(1,0)=1$ the product chain is indeed SRW on the hypercube. For the lazy version simply take $P_i(0,1)=P_i(1,0)=1/4$.

Exercise 29 Consider an E-valued irreducible Markov chain. For $x,y\in E$ set

$$G(x,y):=\mathbb{E}_x\left[\sum_{t=0}^{\infty}\mathbb{1}_{\{X_t=y\}}\right]\in\overline{\mathbb{R}_+}.$$

The function G is called *Green function* of the chain X.

1. Show the following are equivalent
   - X is recurrent.
   - $\exists x\in E: G(x,x)=\infty$.
   - $\forall x,y\in E, G(x,y)=\infty$.
2. Compute $G(0,0)$ when X is SRW on $\mathbb{N}$: $\mathbb{P}(X_{n+1}=1\mid X_n=0)=1$ and for $k\geq 1$ $\mathbb{P}(X_{n+1}=k+1\mid X_n=k)=p\in[0,1])$ as a function of p.
3. Assume $\mathcal{T}$ is an infinite d-regular tree (where each node is of degree d, except the root $\emptyset$ of degree d), and consider the λ-biaised walk on this tree (with $\lambda\in[0,1]$). More precisely, if at the root, such walk chooses uniformly one of its d descendants, and from any other node the walk goes to its parent with probability $\lambda/(\lambda+d)$, otherwise it chooses one of the d offsprings uniformly. Compute $G(\emptyset,\emptyset)$ as a function of d,λ.

1. Fix x and y and assume X is recurrent, i.e for some $x_0\in E$, T_{x_0} is a.s. finite under $\mathbb{P}_{x_0}$. Since the chain is irreducible $P^k(x_0,x)>0$ for some $k>0$, and by the strong Markov property at the successive returns at x_0, the chain a.s. visits x infinitely often. Thus $\mathbb{P}_x(T_{x_0}=\infty)=0$ otherwise the chain would not be recurrent. Therefore the chain started at x must visit x_0 (hence infinitely often), and by the same reasoning as above it must visit y infinitely often. In the end it must be that $G(x,y)=\infty$.

   On the other hand if $G(x,y)=\infty$ for any x,y, even if it means taking $y=x$ we get the second assertion.

   Finally if X is transient then $\mathbb{P}_x(T_x^+=\infty)=p>0$, but then by the strong Markov property at return times at x, the total number of visits at x is geometric with parameter p, and then $G(x,x)=1/p<\infty$.
2. In this case

$$\mathbb{P}(X_{2n}=0)=\binom{2n}{n}\frac{1}{2^{2n}}\sim_{n\to\infty}\frac{1}{\sqrt{\pi n}},$$

   so $G(0,0)=\infty$ and the chain is recurrent.

3. When X is reflected SRW, for $p \le 1/2$ it is recurrent by comparing to the case $p = 1/2$ of the previous question, so $G(0,0) = \infty$. Assume now $p > 1/2$. By a preceding exercise, $\mathbb{P}_0(T_0^+ = \infty) = \frac{2p-1}{p}$. Using strong Markov, the total number visits at the origin is geometric with parameter $\frac{2p-1}{p}$, hence if $p > 1/2$,
$$G(0,0) = \frac{p}{2p-1}.$$
4. Letting $|v|$ denote the height (distance from the root) of a vertex v, it is easily seen that $(|X_n|, n \ge 0)$ remains Markov, in fact it is rreflected SRW with $p = d/(d+\lambda)$. Thus
$$G(\emptyset, \emptyset) = \begin{cases} \frac{d}{d-\lambda} \text{ if } d > \lambda \\ +\infty \text{ otherwise.} \end{cases}.$$

Exercise 30

1. Soit X chaîne de Markov à valeurs dans E, de noyau P, $(\mathcal{F}_n)_n$ la filtration naturelle de X, et $f : E \to \mathbb{R}$ bornée Montrer que le processus
$$\left(M_n^f := \sum_{k=1}^n (f(X_i) - Pf(X_{i-1})) \,,\ n \ge 0 \right)$$
est une $(\mathcal{F}_n)$-martingale.
2. Etablir la réciproque : si pour toute f bornée, $(M_n^f, n \ge 0)$ est une $(\mathcal{F}_n)$-martingale, alors X est Markov de noyau P.

1. Les propriétés de mesurabilité et d'intégrabilité de M_n^f sont évidentes. Notons d'autre part que $M_n^f - Pf(X_n)$ est $\mathcal{F}_n$-mesurable. De plus par Markov au temps n la loi de X_{n+1} sachant $\mathcal{F}_n$ est exactement celle de X_{n+1} sachant X_n, i.e. $P(X_n, \cdot)$. On a donc
$$\begin{aligned} \mathbb{E}[M_{n+1}^f \mid \mathcal{F}_n] &= M_n^f - Pf(X_n) + \mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] \\ &= M_n^f - Pf(X_n) + \sum_{x \in E} f(x) P(X_n, x) = M_n^f, \end{aligned}$$
et on conclut que $(M_n^f)_n$ est bien une $(\mathcal{F}_n)$-martingale.
2. Supposons que pour toute f bornée, $(M_n^f, n \ge 0)$ est une martingale. Fixons $x \in E$, pour $f = \mathbb{1}_x$, on obtient que
$$\mathbb{E}[\mathbb{1}_{X_{n+1}=x} \mid \mathcal{F}_n] = P\mathbb{1}_x(X_n) = P(X_n, x).$$
Autrement dit, pour tout $(x, x_0, ..., x_n) \in E^{n+2}$,
$$\mathbb{P}(X_{n+1} = x \mid X_0 = x_0, ..., X_n = x_n) = \mathbb{P}(X_{n+1} = x \mid X_n = x_n) = P(x_n, x),$$
et (X_n) est donc bien une chaîne de Markov homogène de noyau P.

Exercise 31 Soit X une chaîne de Markov sur E fini de matrice de transition P. On dit que $h : E \to \mathbb{R}$ est harmonique au point $x \in E$ ssi $\sum_{y \in E} P(x,y) h(y) = h(x)$.
Dans tout l'exercice on suppose que la chaîne X est irréductible.

1. Montrer qu'une fonction h harmonique sur E est constante.
2. Montrer qu'une fonction h harmonique sur $A \subsetneq E$ vérifie

$$\max_{x\in E} h(x) = \max_{x\in E\setminus A} h(x).$$

3. Soit $B \subset\neq E$, et $h_B : B \to \mathbb{R}$. Montrer que $h(x) = \mathbb{E}_x[h(X_{\tau_B})], x \in E$ est l'unique extension de h_B à E qui est harmonique sur $A = E \setminus B$

1. L'espace étant fini, h atteint son maximum M, disons en $x \in E$. Puisqu'elle est harmonique en x, $M = h(x) = \sum_{y_1} P(x, y_1)h(y_1)$ et donc $h(y_1) = M$ pour tout y_1 tel que $P(x, y_1) > 0$. En répétant ce raisonnement, on voit par récurrence que pour tout $n \in \mathbb{N}$, et pour tout y_n tel que $P^{(n)}(x, y_n) > 0$ on a $h(y_n) = M$. On conclut que h est constante égale à M puisque la chaîne est irréductible.
2. La fonction h atteint son maximum. Si c'est sur $E \setminus A$ il n'y a rien à démontrer. Sinon elle l'atteint sur A et on peut répeter la preuve de la question précédente pour voir que les points de $\partial A = \{y \in E \setminus A :\ \exists x \in A\ P(x, y) > 0\} \subset E \setminus A$ réalisent également ce maximum.
3. Pour vérifier qu'une telle fonction est bien définie on va d'abord montrer que $\tau_B < \infty$ p.s. Comme la chaîne est irréductible, on peut trouver k tel que $P^{(k)}(x_1, x_2) > 0$ quelque soient $x_1, x_2 \in E$.
   Fixons alors $y \in B$, comme E est fini $p = \min\{P^{(k)}(x, y), x \in E\} > 0$, la propriété de Markov aux temps $k, 2k, 3k, ...$ entraîne alors que $\tau_B \leq kG$ où $G \sim \text{Geom}(p)$.
   Si $x \in B$ il est évident que $\tau_B = 0$ sous $\mathbb{P}_x$ et donc $h(x) = h_B(x)$.
   Vérifions maintenant que h_B est harmonique sur $A = E \setminus B$. Soit $x \in A$. Par Markov au temps 1,
   $$h(x) = \mathbb{E}_x[h_B(X_{\tau_B})] = \sum P(x, y)\mathbb{E}_y[h_B(X_{\tau_B})],$$
   comme souhaité.
   Enfin, il reste à vérifier l'unicité de h. Supposons qu'il existe deux fonctions h_1, h_2 harmoniques sur A et coïncidant avec h_B sur B. La différence $h_1 - h_2$ reste bien entendu harmonique sur A, et elle vaut 0 sur B. D'après la question précédente elle atteint son maximum sur B et donc elle est négative ou nulle sur A. Par le même raisonnement $h_2 - h_1$ reste également négative ou nulle sur A et on conclut finalement que $h_1 = h_2$.

5 Continuous-time chains, finite state space

Exercise 32 Consider a continuous-time taking values in $\{1, 2, 3\}$, with generator

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

1. Draw the diagramm of the chain. Check that if $A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ one has $A^{-1}QA = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Deduce that $P(t) = e^{tQ}$ for $t \geq 0$.
2. For $t \geq 0$, compute $\mathbb{P}_1(X_t = 1), \mathbb{P}_1(X_t = 2), \mathbb{P}_1(X_t = 3)$. What happens as $t \to \infty$?

1. With $A^{-1} = \frac{1}{6}\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 2 & 2 & 2 \end{pmatrix}$ checking the matrix product is straightforward, it follows that

$$\begin{aligned} P(t) &= A \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} \\ &= \frac{1}{6}\begin{pmatrix} 2+4e^{-3t} & 2-2e^{-3t} & 2-2e^{-3t} \\ 2-2e^{-3t} & 2+4e^{-3t} & 2-2e^{-3t} \\ 2-2e^{-3t} & 2-2e^{-3t} & 2+4e^{-3t} \end{pmatrix} \end{aligned}$$

2. $\mathbb{P}_1(X_t = 1), \mathbb{P}_1(X_t = 2), \mathbb{P}_1(X_t = 3)$ are given, respectively by first, second and third entries of the first row of $P(t)$. As $t \to \infty$, these three quantities converge towards $1/3$, i.e. towards the weights given to states $1, 2, 3$ by the stationary distribution of the chain (it is indeed an invariant distribution by immediate considerations of symmetry). Hence we have checked the convergence theorem in this particular case for the chain started at 1. However, the computation of the previous question gives more information : since this convergence holds for all the rows of $P(t)$, there is convergence towards the stationary distribution whatever the initial distribution ; moreover, the exact computation of $P(t)$ also yields that the speed of this convergence towards the invariant distribution is exponentially fast.

Exercise 33 Let X be the continuous-time chain on $\{1, 2, 3\}$, with generator

$$\begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

1. Check that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ are eigenvectors of Q.
2. For $t \geq 0$, compute $\mathbb{P}_1(X_t = 1), \mathbb{P}_1(X_t = 2), \mathbb{P}_1(X_t = 3)$, what happens as $t \to \infty$?

1. This is straightforward, corresponding eigenvalues are $0, -5, -4$.
2. As in the previous exercise, one can diagonalize Q and compute

$$P(t) = \frac{1}{5}\begin{pmatrix} 3+2e^{-4t} & 1-e^{-4t} & 1-e^{-4t} \\ 3-8e^{-4t}+5e^{-5t} & 1+4e^{-4t} & 1+4e^{-4t}-5e^{-5t} \\ 3+2e^{-4t}-5e^{-5t} & 1-e^{-4t} & 1-e^{-4t}+5e^{-5t} \end{pmatrix}.$$

The values we look for then correspond to the entries in the first row of $P(t)$. As $t \to \infty$, regardless of the initial distribution, the law of X_t converges to the stationary distribution of the chain $\lambda = (3/5\ 1/5\ 1/5)$.

Exercise 34 Let X the continuous-time chain on $\{1, 2, 3, 4\}$, with generator

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Is the chain irreducible? Find the set of its invariant distributions. Compute $\lim_{t\to\infty} \mathbb{P}_1(X_t = x), x \in E$.

The corresponding transition matrix is

$$\Pi = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which is clearly irreducible as $1 \to 2 \to 3 \to 4 \to 1$. An invariant measure of X must satisfy $\lambda Q = 0$, i.e. $Q^T \lambda^T = 0$, so we can start by looking at $\ker(Q^T)$. We find (by using pivot algorithm)

$$\ker(Q^T) = \ker \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{Vect} \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1 \end{pmatrix}.$$

It remains to check that $\lambda_i \geq 0, i = 1, 2, 3, 4, \quad \sum_{1 \leq i \leq 4} \lambda_i = 1$,and we finally conclude that the unique invariant distribution is

$$\lambda = \left(\frac{1}{6}\ \frac{1}{6}\ \frac{1}{3}\ \frac{1}{3} \right).$$

Our chain is irreducible and positive recurrent, convergence theorem applies to ensure that for $i = 1, 2, 3, 4$,

$$\lim_{t\to\infty} \mathbb{P}_i(X_t = j) = \lambda_j, \quad j = 1, 2, 3, 4.$$

Exercise 35 Let $X_1, X_2, \ldots$ i.i.d exponential variables with parameter λ and N independent of $\{X_i, i \geq 1\}$, following a geometric law with parameter β. What is the distribution of $X := \sum_{k=1}^{N} X_k$?

For $t \geq 0$

$$\begin{aligned}
\mathbb{E}[\exp(-tX)] &= \mathbb{E}\left[\prod_{i \leq N} \exp(-tX_i)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{i \leq N} \exp(-tX_i) \mid N\right]\right] \\
&= \mathbb{E}\left[\left(\frac{\lambda}{\lambda + t}\right)^N\right] \\
&= \frac{\frac{\beta\lambda}{\lambda+t}}{1 - (1-\beta)\frac{\lambda}{\lambda+t}} \\
&= \frac{\beta\lambda}{t + \beta\lambda},
\end{aligned}$$

so that $X \sim \exp(\beta\lambda)$.

6 Poisson process

We will use the Theorem seen in class which gives three equivalent definitions/characterizations of a Poisson process.

Exercise 36

1. Let $(X_t, t \geq 0), (Y_t, t \geq 0)$ be two independent Poisson processes with respective parameters λ, μ. What is the law of $(Z_t := X_t + Y_t, t \geq 0)$?
2. How can we generalize the previous question to a (at most countable) family of independent Poisson processes ?
3. Assume $(J_n)_{n \geq 1}$ are the jump times of a Poisson process with parameter λ, and introduce $(\xi_n)_{n \geq 1}$ i.i.d, Bernoulli(p), independent of (J_n). For $n \geq 1$, set

$$u_n = \inf\{k \in \mathbb{N} : \sum_{i=1}^{k} \xi_i = n\}, \quad K_n = J_{u_n},$$
$$v_n = \inf\{k \in \mathbb{N} : \sum_{i=1}^{k} (1 - \xi_i) = n\}, \quad L_n = J_{v_n}.$$

   Show that $(J_{u_n}, J_{v_n}, n \geq 1)$ are the jump times of two independent Poisson processes whose respective parameters shall be computed.
4. How can one generalize the previous question to form, from one Poisson process, a (at most countable) family of independent Poisson processes.

1. We are going (e.g.) to use the second characterization of a Poisson process. First we see that increments of Z must be independent, since those of X and Y are. Moreover, uniformly in t,

$$\mathbb{P}(Z_{t+h} - Z_t = 0) = \mathbb{P}(X_{t+h} - X_t = 0, Y_{t+h} - Y_t = 0) = (1 - \lambda h + o(h))(1 - \mu h + o(h)) = 1 - (\lambda + \mu)h + o(h),$$

$$\mathbb{P}(Z_{t+h}-Z_t=1)=\mathbb{P}(X_{t+h}-X_t=0,Y_{t+h}-Y_t=1)+\mathbb{P}(X_{t+h}-X_t=1,Y_{t+h}-Y_t=0)=(\lambda+\mu)h+o(h),$$

so that $\mathbb{P}(Z_{t+h}-Z_t\geq 2)=o(h)$. Hence Z satisfies second characterization of a Poisson process with parameter $\lambda+\mu$.

Note that we could as well have used the third characterization (see next question).

In fact, a proof using the first characterization, the memoryless property of exponential distributions, and the fact that the minimum of independent exponential variables remains exponential could also be performed.

2. By a similar proof, we obtain a similar result for an at most countable family of independent Poisson processes, as long as their respective parameters sum up to a finite quantity (see also the corresponding Proposition stated in class). In the case when the family is finite, an immediate induction allows to conclude. For the proof in the general countable case, use the third characterization. Increments of X must be independent and stationary since those of each of the processes $(X_i, i\geq 1)$. Moreover one can easily check (use moment generating functions) that a sum of independent Poisson variables with respective parameters $\lambda_i, i\geq 1$ remains Poisson with parameter $\Lambda=\sum_i \lambda_i$. It follows that X satisfies the third characterization of a Poisson process with paramater Λ.
3. See the following question.
4. General case (paintbox process), given a Poisson process X with parameter λ and a distribution $(p_i, i\in I)$ amounts to setting

$$X_i(t)=\sum_{n=1}^{X_t}\xi_{n,i},$$

where $(\xi_{n,i}, n\in\mathbb{N})$ are i.i.d Ber(p_i) conditioned on $\sum_{i\in I}\xi_{n,i}=1$, and for any $n\in\mathbb{N}$, $(\xi_{n,i}, i\in I)$ is independent of $\{\xi_{n',i}, n'\neq n, i\in I\}$.

Increments of X are independent and stationary ; those of X_i for $i\in I$ must be as well. For the joint law of $(X_i(t), i\in I)$ one could e.g. wprk with moment generating functions : for $u_i\in[0,1], i\in I$, one has

$$\begin{aligned}
\mathbb{E}\left[\prod_{i\in I}u_i^{X_i(t)}\right] &= \mathbb{E}\left[\prod_{i\in I}u_i^{\sum_{n=1}^{X_t}\xi_{n,i}}\right]\\
&= \sum_{p\in\mathbb{N}}\frac{(\lambda t)^p}{p!}\exp(-\lambda t)\mathbb{E}\left[\prod_{i\in I,1\leq n\leq p}u_i^{\xi_{n,i}}\right]\\
&= \sum_{p\geq 1}\frac{(\lambda t)^p}{p!}\exp(-\lambda t)\left(\sum_{i\in I}p_iu_i\right)^p\\
&= \exp\left(\lambda t\sum_{i\in I}p_iu_i-\lambda t\right)=\prod_{i\in I}\exp\left(\lambda p_it(u_i-1)\right)
\end{aligned}$$

so that $(X_i(t), i\in I)$ are independent Poisson with respective parameters $\lambda p_i t, i\in I$. One concludes $(X_i, i\in I)$ are independent and by the third characterization, Poisson processes with repective parameters $\lambda p_i, i\in I$.

Exercise 37 Let $(X_t, t \geq 0)$ be a Poisson process with parameter λ. Show that knowing $\{X_t = n\}$, the first n jump times of X_t are distributed as the order statistics of n i.i.d Unif$[0,t]$ variables.
The $S_i, i \geq 1$ are i.i.d $\exp(\lambda)$, so joint density of $(S_1, ..., S_{n+1})$ is $\lambda^{n+1}\exp(-\lambda(s_1 + ... + s_{n+1}))\mathbb{1}_{\{s_1 \geq 0, ..., s_{n+1} \geq 0\}}$,. Hence, by changing variables the joint density of $(J_1, ..., J_{n+1}) = (S_1, S_1 + S_2, ..., S_1 + ... + S_{n+1})$ is

$$\lambda^{n+1}\exp(-\lambda t_{n+1})\mathbb{1}_{\{0 \leq t_1 \leq ... \leq t_{n+1}\}}.$$

It follows that for $A \in \mathcal{B}([0,t]^n)$, we have

$$\begin{aligned}
\mathbb{P}((J_1, ..., J_n) \in A, X_t = n) &= \mathbb{P}((J_1, ..., J_n) \in A, J_{n+1} > t) \\
&= \int_A dt_1 ... dt_n \mathbb{1}_{\{0 \leq t_1 \leq ... \leq t_n\}} \int_t^\infty dt_{n+1} \lambda^{n+1} \exp(-\lambda t_{n+1}) \\
&= \int_A dt_1 ... dt_n \mathbb{1}_{\{0 \leq t_1 \leq ... \leq t_n\}} \lambda^n \exp(-\lambda t).
\end{aligned}$$

Moreover $\mathbb{P}(X_t = n) = \exp(-\lambda t)\frac{(\lambda t)^n}{n!}$, and we deduce that

$$\mathbb{P}((J_1, ..., J_n) \in A \mid X_t = n) = \int_A dt_1 ... dt_n \mathbb{1}_{\{0 \leq t_1 \leq ... \leq t_n\}} \frac{n!}{t^n},$$

as required.

Exercise 38 Assume the passage times of bus 27 at the stop "Place d'Italie" are the jump times of a Poisson process X with parameter α, those of bus 21 at the same stop are the jump times of a Poisson process Y with parameter β, independent of X. We use the hour as a the unit time.

1. One starts waiting for a bus at 7am. What is the probability to wait for the next bus (from either line) more than 30 minutes ?
2. What is the probability that a total of exactly (resp. at least) 50 buses arrive at the stop between 7am and 9am ?
3. What is the probability that at least k buses of line 21 arrive at the stop before the first bus of line 27 ?
4. What is the probability that exactly k buses of line 21 arrive during first hour knowing that a total of n bus arrive during that period of time ?
5. Introduce $A := \{100$ buses arrive at the stop between 7am and 9am$\}$. Conditionally given A, what is the probability :
   (a) that 30 of these buses have arrived between 7am and 8am ?
   (b) that one waits for more than 30 minutes to see the first bus arrive after 7am ?

   Conditionally given A, what is the expectation of the waiting time after 7am of the first arrival ?
6. Assume in this question that due to a strike, each bus is independently canceled with probability 1/2. There is a big line-up, so one decides to let the first one pass and to take the second. What is the distribution of the waiting time ? What is the distribution of the waiting time of the first (resp. the nth) bus of line 27 ?

W.l.o.g we fix the origin of times at 7am in what follows. Memoryless property of exponential distribution ensures that passage times of buses of lines 27, 21 are still the jump times of (say X_1, X_2) independent Poisson processes of respective parameters α, β. By exercise 7, passage times of buses are the jump times of a Poisson process (say X) with parameter $\alpha+\beta$. Note that, again by exercise 7, if from X, we perform the paintbox procedure with $p_1 = \frac{\alpha}{\alpha+\beta}, p_2 = 1-p_1$, then the pair of resulting processes, say $(\tilde{X}_1, \tilde{X}_2)$, has same distribution as (X_1, X_2).

1. The probability we look for is that of a Poisson variable with parameter $(\alpha+\beta)/2$ to take value 0, that is $\exp(-(\alpha+\beta)/2)$.
2. We are looking at events $\{X(2) = 50\}, \{X(2) \geq 50\}$, having respective probabilities
$$\exp(-2(\alpha+\beta))\frac{(2(\alpha+\beta))^{50}}{50!}, \quad \sum_{k\geq 50} \exp(-2(\alpha+\beta))\frac{(2(\alpha+\beta))^k}{k!}.$$
3. Consider the paintbox version $(\tilde{X}_1, \tilde{X}_2)$ (which does not affect the joint distribution, as we explained) i.e. consider all arrival times, and "colour" them independently as a "bus 27" with probability $\alpha/(\alpha+\beta)$, otherwise as a "bus 21". The event $\{k \text{ bus of line 21 arrive before the first bus of line 27}\}$ corresponds exactly to the event $\{\text{the first } k \text{ arrival times are "coloured" as "bus 21"}\}$, and the probability of the latter is clearly $(\beta/(\alpha+\beta))^k$.
4. Again working with the paintbox version, we search here for the conditional probability $\{k \text{ points in } [0,1] \text{ are "coloured" as "bus 21"}\}$ knowing $[0,1]$ has a total n points. Since colourings are independent, this is the probability that a Binomial$(n, \beta/(\alpha+\beta))$ takes value k, i.e.
$$\binom{n}{k}\left(\frac{\beta}{\alpha+\beta}\right)^k\left(\frac{\alpha}{\alpha+\beta}\right)^{n-k}$$
5. As above
$$\mathbb{P}(A) = \exp(-2(\alpha+\beta))\frac{(2(\alpha+\beta))^{100}}{100!}.$$
By exercise ..., conditionally given A, the distribution of passage times of these 100 bus is that of the order statistics of 100 i.i.d Unif$[0,2]$. The probability that one of these takes its value in $[a,b] \subset [0,2]$ is $(b-a)/2$, thus the probability that k passage times fall between a and b is the probability that a Binomial$(100, (b-a)/2)$ takes value k. Hence
   (a) $\binom{100}{30}\frac{1}{2^{100}}$
   (b) $\frac{3^{100}}{4^{100}}$

   Conditionally given A, the passage time (say T) of the first of the 100 buses is the minimum of 100 i.i.d Unif$[0,2]$. Thus
$$\mathbb{P}(T > t) = \left(\frac{(2-t)^+}{2}\right)^{100},$$
$$\mathbb{E}[T] = \int_0^2 \left(\frac{2-t}{2}\right)^{100} dt = \frac{2}{101}.$$

6. Again we can use the paintbox representation : colour points of X (parameter $\alpha+\beta$) as "striker", "bus 27, non striker", and "bus 21, non striker", with respective probabilities $1/2$, $\alpha/2(\alpha+\beta)$, $\beta/2(\alpha+\beta)$. Buses now arrive according to the jump times of a Poisson process with parameter $(\alpha+\beta)/2$, while buses of line 27 arrive according to the jump times of a Poisson process with parameter $\alpha/2$.

   Passage time of second bus is therefore the sum of two independent exponential variables with parameter $(\alpha+\beta)/2$, that is, a Gamma$(2,(\alpha+\beta)/2)$, its expectation is $4/(\alpha+\beta)$.

   The passage time of the first (resp. the nth) bus of line 27 has an exponential distribution with parameter $\alpha/2$ (resp. Gamma$(n,\alpha/2)$), its expectation is $2/\alpha$ (resp. $2n/\alpha$).

7 Explosion ?

For a continuous-time chain X, with jump times $(J_n, n\geq 1)$ (by convention we let $J_0=0$), recall that its *explosion time* is defined as $\zeta := \sup_{n\geq 1} J_n$ taking values in $\mathbb{R}_+^* \cup \{+\infty\}$. The chain is said non-explosive if $\zeta=+\infty$ a.s. whatever the initial distribution.

Exercise 39 Fix $\theta>0$ and for $x\in E$ let $z_x := \mathbb{E}_i[\exp(-\theta\zeta)]$.

1. Show that $(z_x, x\in E)$ satisfies
   — $|z_x|\leq 1$ for any $x\in E$, i.e. $||z||_\infty \leq 1$.
   — $Qz=\theta z$.
2. Show that if $\tilde{z}$ satisfies the two above properties then $\tilde{z}_x \leq z_x$ for any $x\in E$
3. Deduce that a chain is non-explosive iff $[Q\tilde{z}=\theta\tilde{z}, ||\tilde{z}||_\infty\leq 1$ implies $\tilde{z}_x=0$ for any $x\in E]$.

1. The first property is obvious. For the second, use the fact that $J_1\sim\exp(q_x)$ and the strong Markov property at time J_1 for the chain started at x to get

$$\begin{aligned} z_x &= \sum_{y\in E, y\neq x} \Pi(x,y)\mathbb{E}_x\left[\exp(-\theta\zeta) \mid X_{J_1}=y\right] \\ &= \sum_{y\in E, y\neq x} \Pi(y,x)\frac{q_x}{q_x+\theta} z_y = \sum_{y\in E, y\neq x} \frac{q_{x,y}}{q_x+\theta} z_y \end{aligned}$$

   so that, indeed $(Qz)_x = \theta z_x$.

2. Assume $\tilde{z}$ is such that $||\tilde{z}||_\infty\leq 1$ and $Q\tilde{z}=\theta\tilde{z}$. An immediate induction shows that for any n, $\tilde{z}_x \leq \mathbb{E}[\exp(-\theta J_n)]$. Indeed if this holds, then

$$\begin{aligned} \tilde{z}_x = \sum_{y\in E, y\neq x} \frac{q_{x,y}}{q_x+\theta}\tilde{z}_y & \\ &\leq \sum_{y\in E, y\neq x} \frac{q_{x,y}}{q_x+\theta}\mathbb{E}[\exp(-\theta J_n)] \\ &= \mathbb{E}[\exp(-\theta J_{n+1})] \end{aligned}$$

   Now the righthand side above converges to $\mathbb{E}[\exp(-\theta\zeta)]$ by dominated convergence, which yields the desired result.

3. Assume $[Qz = \theta z, ||z||_\infty \le 1$ implies $z_x = 0$ for any $x \in E]$. By the first question z such that $z_x = \exp(-\theta\zeta)$ satisfies the two conditions so it must be that $\mathbb{E}_x[\exp(-\theta\zeta)] = 0$ for any $x \in E$, i.e. $\zeta = +\infty$ $\mathbb{P}_x$-a.s.
   Conversely, assume the chain is non-explosive so $z_x = \mathbb{E}_x[\exp(-\theta\zeta)] = 0$ for any $x \in E$. Assume $\tilde{z}$ is such that $Q\tilde{z} = \theta\tilde{z}$ and $||\tilde{z}||_\infty \le 1$, by the previous two questions we must have $\tilde{z}_x \le z_x = 0$, so $\tilde{z} \le 0$. By applying the same argument to $-\tilde{z}$, we must also have $\tilde{z} \ge 0$ hence $\tilde{z} = 0$.

Exercise 40 Show that an irreducible chain is non-explosive provided one of the three following conditions holds :

- E finite.
- $q_{\max} := \sup_{x\in E} q_x < \infty$
- the chain is recurrent.

Obviously the first condition implies the second.
Assume the second holds, then the sequence of holding times $(J_n - J_{n-1}, n \ge 1)$ can be coupled with a sequence $(e_n, n \ge 1)$ of i.i.d exponential variables with parameter $q_{\max}$, so that $J_n - J_{n-1} \ge e_n$. Now, a.s. $+\infty = \sum_{n\ge1} e_n \le \zeta$, so the chain is non-explosive. Note that we could also have directly applied the result of exercise 4.
Assume the third condition holds, and that the chain is started at x. By assumption the chain must visit x infinitely often, so the total holding time at x is infinite, hence ζ (the sum of all holding times) as well.

Exercise 41 Assume in this exercise that X is a birth chain, i.e. it takes values on $\mathbb{N}$ and has $\Pi_{i,i+1} = 1$ for any $i \in \mathbb{N}$. Show that the chain is explosive iff $\sum_{i\ge0} \frac{1}{q_i} < \infty$.
This is a simple application of exercise 6.

8 General case, further results

Exercise 42 Consider a continuous-time chain with generator Q, $A \subset E$ and T_A the hitting time of A. Show that

$$\mathbb{E}_x[T_A] = \begin{cases} 0 \text{ if } x \in A \\ \frac{1}{q_x} + \sum_{y\neq x} \pi_{xy}\mathbb{E}_y[T_A] \text{ if } x \notin A \end{cases}$$

Notice that for $x \notin A$ the above equation can be rewritten as $\sum_{y\in E} q_{xy}\mathbb{E}_y[T_A] + 1 = 0$.
Application Compute $\mathbb{E}_1[T_3]$ for the chains of exercises 1,2.
If $x \in A$, $T_A = 0$ so that $\mathbb{E}_x(T_A) = 0$.
Otherwise the time spent at x before the first jump time J_1 of the chain is exponential with parameter q_x. For $y \neq x$, $\mathbb{P}_x(X_{J_1} = y) = \frac{q_{xy}}{q_x} = \pi_{xy}$. By strong Markov at time J_1, the law of T_A under $\mathbb{P}_x$ is that of $J_1 + \tilde{T}_A$, where with probability π_{xy}, $\tilde{T}_A$ has the law of T_A under $\mathbb{P}_y$. Thus, for $x \notin A$.

$$\mathbb{E}_x[T_A] = \frac{1}{q_x} + \sum_{y\neq x} \pi_{xy}\mathbb{E}_y[T_A].$$

Even if it means multiplying by $q_x = -q_{xx}$, we deduce that for $x \notin A$,

$$-q_{xx}\mathbb{E}_x[T_A] = \sum_{y\neq x} q_{xy}\mathbb{E}_y[T_A] + 1,$$

yielding the desired equation.
Application : In both cases, we get the system

$$q_[11]\mathbb{E}_1[T_3] + q_{12}\mathbb{E}_2[T_3] + 1 = 0, \quad q_{21}\mathbb{E}_1[T_3] + q_{22}\mathbb{E}_2[T_3] + 1 = 0.$$

Its resolution in the setting of exercise 1 gives $\mathbb{E}_1[T_3] = \mathbb{E}_1[T_2] = 1$. As for the setting of exercise 2, we find $\mathbb{E}_1[T_3] = 5/4, \mathbb{E}_2[T_3] = 3/2$.

Exercise 43 Jobs arrive at a server according to the jump times of a Poisson process X with parameter λ, service times are assumed independent of X and i.i.d exponential with parameter μ. Let $(Y_t, t \geq 0)$ the number of jobs in the queue at time t, $(J_n, n \geq 1)$ the jump times of Y and $S_1 = J_1, S_n = J_n - J_{n-1}, n \geq 1$.

1. Find the distribution of f S_{n+1} knowing $Y_{J_n} = 0$?
2. Find the distribution of S_{n+1} knowing $Y_{J_n} > 0$?
3. Find the distribution of $Y_{J_{n+1}}$ knowing Y_{J_n}.
4. Find the generator of Y.
5. Find a necessary and sufficient condition for the chain to be recurrent.
6. Show that for $C \geq 0$, $\pi(k) = C\left(\frac{\lambda}{\mu}\right)^k$, $k \in \mathbb{N}$ is an invariant measure of the chain. Does the chain possess a stationary distribution ?

1. Conditionally given $Y_{J_n} = 0$, there are no jobs in the queue at time J_n. The jump time J_{n+1} then corresponds necessarily to the arrival of the next job, i.e. to the next jump time of X. Thus, knowing $Y_{J_n} = 0$, $S_{n+1} \sim \exp(\lambda)$. In other words if Q is the generator of Y we have found that $q_0 = q_{01} = \lambda$.
2. Conditionally given $Y_{J_n} > 0$, there is at least one job in the queue waiting for the current job being serviced. The nezt jump of Y corresponds either to the end of the service time or to the arrival of the next job in the queue. Service times being independent of X, it follows that knowing $Y_{J_n} > 0$, $S_{n+1} = \min(e, e')$ where $e \sim \exp(\lambda)$, independent of $e' \sim \exp(\mu)$. Thus, knowing $Y_{J_n} > 0$, $S_{n+1} \sim \exp(\lambda + \mu)$. In other words, for $k > 0$, $q_{k,k+1} = \lambda, q_{k,k-1} = \mu, q_k = \lambda + \mu$.
3. By the above

$$\mathbb{P}(Y_{J_{n+1}} = 1 \mid Y_{J_n} = 0) = 1, \quad \mathbb{P}(Y_{J_{n+1}} = Y_{J_n} + 1 \mid Y_{J_n} > 0) = 1 - \mathbb{P}(Y_{J_{n+1}} = Y_{J_n} - 1 \mid Y_{J_n} > 0) = \frac{\lambda}{\lambda + \mu}.$$

4. See above.
5. The corresponding jumo chain, say Z, has transition matrix Π s.t.

$$\pi_{01} = 1, \quad \forall k \in \mathbb{N}^*, \quad \pi_{k,k+1} = 1 - \pi_{k,k-1} = \frac{\lambda}{\lambda + \mu}.$$

We deduce that Z is SRW on $\mathbb{N}$ reflected at 0, with $p := \frac{\lambda}{\lambda+\mu}$, hence it is recurrent iff $p \leq 1/2$ i.e. $\lambda \leq \mu$.
What we just showed is not very surprising : when the average service time $1/\mu$ is greater than the average waiting time $1/\lambda$ between two successive jobs arrivals, the queue length will tend to infinity corresponding to the transient case. In the opposite case there will be arbitrarily large times when the server is inactive, corresponding to the recurrent case.

6. Recall that $q_{00} = -\lambda, q_{10} = \mu, q_{\ell 0} = 0 \forall \ell \geq 2$, so

$$(\pi Q)_0 = \sum_{\ell=0}^{\infty} C \left(\frac{\lambda}{\mu}\right)^{\ell} q_{\ell 0} = -C q_0 + C \frac{\lambda}{\mu} q_{10} = 0.$$

For $k \in \mathbb{N}^*$ (recall that $q_{kk} = -\lambda - \mu, q_{k-1,k} = \lambda, q_{k+1,k} = \mu$ and $q_{\ell,k} = 0 \forall \ell : |\ell - k| \geq 2$)

$$\begin{aligned}(\pi Q)_k &= \sum_{\ell=0}^{\infty} C \left(\frac{\lambda}{\mu}\right)^{\ell} q_{\ell k} \\ &= C \left(\frac{\lambda}{\mu}\right)^{\ell} \left(\lambda \frac{\mu}{\lambda} - \lambda - \mu + \mu \frac{\lambda}{\mu}\right) = 0.\end{aligned}$$

In the end $\pi Q = 0$, so π is indeed invariant for Q. Observe it is easy to deduce an invariant measure for Z, namely

$$\tilde{\pi}(k) = q_k \pi(k), k \in \mathbb{N}$$

If $\lambda > \mu$ chains Y, Z are transient and cannot possess an invariant distribution.
If $\lambda \leq \mu$ the invariant measure is unique up to a constant multiple. When $\lambda = \mu$ the total mass of π is infinite and there cannot be an invariant distribution (null recurrent case). When $\lambda < \mu$ (positive recurrent case), one can set $C = \frac{\mu - \lambda}{\mu}$ to express the unique invariant distribution

$$\pi_0(k) = \frac{\mu - \lambda}{\mu} \left(\frac{\lambda}{\mu}\right)^k, k \in \mathbb{N}.$$

Remarks : Convergence theorem for continuous-time chains (see the exercise below) ensures then the law of Y_t converges to π as $t \to \infty$.
In fact (see exercise ...) for a recurrent chain,

$$\lambda_x(y) := \mathbb{E}_x \left[\int_0^{T_k^+} ds \mathbb{1}_{\{X_s = y\}}\right], y \in E$$

defines a stationary measure (the unique one which attributes mass $1/q_x$ to state x, and it has total mass $\mathbb{E}_x[T_x^+]$). In the positive recurrent case, the unique invariant distribution is therfore given by

$$\pi_0(x) = \frac{\lambda_x(x)}{\lambda_x(E)} = \frac{1}{q_x \mathbb{E}_x[T_x^+]}.$$

Finally, this last result can be generalized into a continuous-time version of ergodic theorem. In the poisitive recurrent case, it ensures that as $t \to \infty$, $\frac{1}{t} \int_0^t f(X_s) ds \longrightarrow \sum_{x \in E} f(x) \lambda(x)$, where $f : E \to \mathbb{R}$ is s.t. $\sum_{x \in E} f(x) \lambda(x) < \infty$ and λ is the unique stationary distribution of the chain.

Exercise 44 Show that if $\lambda_x q_{xy} = \lambda_y q_{yx}$ for any $x, y \in E$, then $\lambda Q = 0$.
For any $x \in E$,

$$(\lambda Q)_x = \sum_{y \in E} \lambda(y) q_{yx} = \sum_{y \in E} \lambda(x) q_{xy} = \lambda_x \sum_{y \in E} q_{xy} = 0,$$

since Q is a generator.

Exercise 45 Assume X with generator Q is an irreducible recurrent continuous-time chain. Denote by $T_x^+ := \inf\{t \geq J_1 : X_t = x\}$ the return time at x.

1. Show that
$$\mu(y) := \mathbb{E}_x \left[\int_0^{T_x^+} \mathbb{1}_{\{X_s = y\}} \right], y \in E$$
defines a stationary measure.
2. Explain why it is unique up to a constant multiple.
3. Show that for $s > 0$,
$$\mu_y = \mathbb{E}_x \left[\int_s^{T_x^+ + s} \mathbb{1}_{\{X_u = y\}} du \right]$$
and deduce that $\mu = \mu P(s)$.

1. For $y \in E$, denoting τ_x^+ the return time at x for the *jump chain*, and ν_x the stationary measure of the *jump chain* which attributes mass 1 to x (such measure exists and is unique since the jump chain clearly also is irreducible and recurrent, and it satisfies therefore $\nu_x \Pi = \nu_x$),
$$\begin{aligned}
\mu(y) &= \mathbb{E}_x \left[\sum_{n=0}^{\tau_x^+ - 1} S_{n+1} \mathbb{1}_{\{Y_n = y\}} \right] \\
&= \frac{1}{q_y} \mathbb{E}_x \left[\sum_{n=0}^{\tau_x^+ - 1} \mathbb{1}_{\{Y_n = y\}} \right] \\
&= \frac{1}{q_y} \nu_x(y) \\
&= \frac{1}{q_y} \sum_{z \in E} \nu_x(z) \pi(zy) \\
&= \frac{1}{q_y} \sum_{z \in E} \nu_x(z) \frac{q_{zy}}{q_z} \\
&= \frac{1}{q_y} \sum_{z \in E, z \neq y} \mu(z) q_{zy}.
\end{aligned}$$
Now since $q_y = -q_{yy}$ we deduce that
$$\sum_{z \in E} \mu(z) q_{zy} = 0,$$
so $\mu Q = 0$, and μ is invariant for chain X.

2. If ν is stationary for Π, then $\mu(y) = \frac{\nu(y)}{q_y}, y \in E$ is stationary for X. Unicity, up to a constant multiple, for invariant measure of the jump chain hence implies unicity up to a constant multiple for invariant measure of X.
3. Let $y \in E$. By strong Markov property at T_x^+ we find

$$\int_0^s \mathbb{1}_{\{X_u=y\}} du \stackrel{\text{(loi)}}{=} \int_{T_x^+}^{T_x^+ + s} \mathbb{1}_{\{X_u=y\}} du,$$

yielding the desired equality. Setting $v = u - s$ and applying Markov at v,

$$\begin{aligned}
\mu_y &= \mathbb{E}_x\left[\int_s^{T_x^+ + s} \mathbb{1}_{\{X_u=y\}} du\right] \\
&= \mathbb{E}_x\left[\int_0^{T_x^+} \mathbb{1}_{\{X_{v+s}=y\}} dv\right] \\
&= \mathbb{E}_x\left[\int_0^{T_x^+} \mathbb{1}_{\{X_{v+s}=y\}} dv\right] \\
&= \mathbb{E}_x\left[\int_0^{T_x^+} \sum_{z \in E} \mathbb{1}_{\{X_v=z\}} dv\right] p_{zy}(s) \\
&= \sum_{z \in E} \mu(z) p_{zy}(s) = (\mu P(s))_y,
\end{aligned}$$

as required.

Exercise 46 Let X be a continuous-time chain with kernel Q. Let $h > 0$ and $(Z_n := X_{nh}, n \geq 0)$.

1. Show that Z is a discrete time chain, what is its transition kernel ?
2. Show that X is irreducible and recurrent iff Z also is.
3. Assume in this question X to be irreducible, positive recurrent, with invariant distribution λ. What is the invariant distribution of Z ? Establish convergence theorem for Z.
4. Explain why the above implies the convergence theorem for X.

1. Markov property at time nh for X implies Markov property for Z at time n, it follows easily that Z indeed is a Markov chain. Its kernel, say R, is such that $r_{xy} = \mathbb{P}_x(X_h = y) = (P(h))_{xy}$, so that $R = P(h)$.
2. Successive states visited by Z are subsets of the successive states visited by X, so if Z is irreducible and recurrent, X must be as well.

   Conversely if X is irreducible it exactly means that its jump chain Y also is, i.e. for any $x, y, \in E$ there exists $n \in \mathbb{N}$ such that $\Pi_{xy}^{(n)} > 0$. But then for any $x, y \in E$, $(P(t))_{xy} \geq \mathbb{P}_x(J_n \leq t < J_{n+1}, Y_n = y) > 0,\ \forall x, y \in E,\ \forall t > 0$, so that $(P(h))_{xy} > 0$. We conclude that irreducibility of X implies that of Z.

Since $\int_0^\infty p_{xx}^(t)dt = \frac{1}{q_x}\sum_{n\geq 0}\pi_{xx}^{(n)}$ and recurrence of X is equivalent to recurrence of Y, we deduce that recurrence of X is equivalent to having $\int_0^\infty p_{xx}^(t)dt = +\infty$. By Markov at time $t \in [nh, (n+1)h]$, $p_{xx}((n+1)h) \geq p_{xx}(t)e^{-q_x h}$, so

$$\sum_{k\geq 1} p_{xx}(kh) \geq \frac{e^{-q_x h}}{h}\int_0^\infty p_{xx}(t),$$

and we conclude that recurrence of X implies that of Z.

3. Recall that if $x \in E$, T_x^+ denotes return time at x for the chain X. The previous exercise implies that if X is positive recurrent, $\lambda(y) = \frac{\mathbb{E}_x\left[\int_0^{T_x^+} \mathbb{1}_{\{X_s=y\}}ds\right]}{\mathbb{E}_x[T_x^+]}$, $y \in E$ is the unique stationary distribution of X, and for any $s > 0$, $\lambda P(s) = \lambda$. In particular $\lambda P(h) = \lambda$, so λ is also one (hence the unique since Z is irreducible and recurrent) stationary distribution of Z. The fact that Z possesse a stationary distribution now implies it is positive recurrent. Finally, we have seen that $(P(h))_{xy} > 0$ for any $x, y \in E$ so Z clearly is aperiodic, and satisfies the convergence theorem : for any $x, y \in E$,
$$\mathbb{P}_x(Z_n = y) = (P(nh))_{xy} \underset{n\to\infty}{\longrightarrow} \lambda(y).$$
4. We first argue that for any $x, y \in E$, $(P(t))_{xy}$ is uniformly continuous, indeed $1 - \mathbb{P}_x(X_s = x) \leq \mathbb{P}_x(J_1 \leq s) = e^{-q_x s}$, so for $t \geq 0$,
$$|(P(t+s))_{xy} - (P(t))_{xy}| \leq 4(1 - \mathbb{P}_x(X_s = x)) \leq 4e^{-q_x s}.$$
Now, setting $n_t = \lfloor t/h \rfloor$, the preceding inequality imples $|(P(t))_{xy} - (P(n_t h))_{xy}| \leq 4q_x h$. Fix $\varepsilon > 0$, and h sufficiently small so that $4q_x h \leq \varepsilon/2$. As $t \to \infty$, $n_t \to \infty$ so for t large enough, convergence theorem for Y implies that $|(P(n_t h))_{xy} - \lambda(y)| \leq \varepsilon/2$. In the end
$$|(P(t))_{xy} - \lambda(y)| \leq \varepsilon.$$
We have established convergence theorem for the chain X started at a given $x \in E$. It is then straightforward to generalize to an arbitrary initial condition.

Exercise 47 Fix $\lambda \in (0,1)$, $\{q_i, i \in \mathbb{N}\} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, and consider the continuous-time Markov chain on $\mathbb{N}$ with generator such that $q_{0,1} = q_0$, $q_{i,i+1} = \lambda q_i, i \geq 1$, $q_{i,i-1} = (1-\lambda)q_i, i \geq 1$.

1. Show that X is irreducible, and transient iff $\lambda > 1/2$.
2. Show that the chain is reversible with respect to an invariant measure ν which will be specified.
3. Assume for some $a > 0$ that $q_i = a^i$. Upon what condition on λ, a does the chain possess an invariant distribution ν (i.e. a distribution ν such that $\nu Q = 0$) ?
4. Is it true that a continuous-time chain possessing an invariant distribution must be positive recurrent ?

1. The corresponding jump chain is ASRW reflected at 0 with $p = \lambda$. Thus X is clearly irreducible, and transient iff $p = \lambda > 1/2$.

2. Detailed balance reads

$$\nu(0)q_0 = \nu_1(1-\lambda)q_1, \qquad \nu(i)\lambda q_i = \nu(i+1)(1-\lambda)q_{i+1}, i \geq 1.$$

Setting

$$\nu(i) = \nu(0)\frac{\lambda^{i-1}}{(1-\lambda)^i}\frac{q_0}{q_i}$$

allows to check detailed balance, and the chain is reversible with respect to such ν.

3. If $q_i = a^i$, setting $\frac{\lambda}{1-\lambda} = b$, we find that

$$\nu(i) = K\left(\frac{b}{a}\right)^i,$$

so that the chain possesses an invariant distribution iff $b < a$ i.e. iff $\frac{\lambda}{1-\lambda} < a$.

4. No : when $a > 1$ it is therefore possible to find a $\lambda > 1/2$ such that $\frac{\lambda}{1-\lambda} < a$, and then the chain is transient AND possesses an invariant distribution. Note however that when it is this case, the chain is necessarily explosive.

9 Simple random walks and their continuous-time versions

Exercise 48 Consider a connected unoriented finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and the continuous-time chain which, when at x, jumps at rate 1 to any of the neighbours of x, independently.

1. Write the generator of the chain.
2. Show it is reversible and express the unique invariant distribution of the chain. Is the chain positive recurrent ?
3. Compute the expected return time at the starting point in the following cases, for *both* the discrete and the continuous-time versions of the chain :
   — $\mathcal{G}$ is the d-dimensional hypercube.
   — $\mathcal{G}$ is the complete graph with n nodes.
   — $\mathcal{G}$ is the n-star (consider both cases when starting point is the center of the star and when it is not).
   — $\mathcal{G}$ is a d-regular tree of height n (consider different cases depending on the starting point).
   — X is the rook's walk on an $n \times n$ chessboard (each jump of the chain consists in one allowed move choosen uniformly at random).

1. We have $Q(x,x) = -d_x$, and $Q(x,y) = \mathbb{1}_{\{x,y\}\in\mathcal{E}}$ for $x \neq y$, with d_x the degree of x, i.e. the number of neighbours of x in the graph.
2. The chain is clearly reversible with respect to a uniform measure on vertices of the graph. Since we assumed $\mathcal{G}$ finite, there exists an invariant distibution : the uniform one. The chain is irreducible because the graph is connected, and since $\mathcal{G}$ is finite, it is positive recurrent.

3. Let λ, resp. π denote the invariant distributions associated with the continuous-time, and the discrete-time chain, so that (recall $d_{\mathcal{G}} = \sum_{x\in\mathcal{V}} d_x = 2|\mathcal{E}|$).

$$\lambda(x) = \frac{1}{d_x|\mathcal{V}|}, \quad \pi(x) = \frac{d_x}{2|\mathcal{E}|}, \quad \forall x \in \mathcal{V}.$$

The expected return time of the continuous-time chain to its starting point, say x, in general, satisfies $\mathbb{E}_x[T_x^+] = \frac{1}{q_x\lambda(x)} = \frac{|\mathcal{V}|}{d_x}$, and for the discrete-time version we have $\mathbb{E}_x T_x^+ = \frac{1}{\pi(x)} = \frac{d_{\mathcal{G}}}{d_x} = \frac{2|\mathcal{E}|}{d_x}$. Thus we only need to compute $d_x, |\mathcal{V}|, |\mathcal{E}|$ to evaluate these quantities.

— Here $|\mathcal{V}| = 2^d, d_x = d\ \forall x \in E, 2|\mathcal{E}| = d2^d$ so $\mathbb{E}_x[T_x^+]$ equals $2^d/d$ in continuous-time and 2^d in discrete-time.

— Set $|\mathcal{V}| = n$, so $2|\mathcal{E}| = n(n-1)$, $d_x = n-1$ for any x, so $\mathbb{E}_x[T_x^+]$ equals 1 in continuous-time and n in discrete-time.

— Here $|\mathcal{V}| = n+1$, $2|\mathcal{E}| = 2n$, $d_x = n$ for x the center and $d_x = 1$ otherwise. When x is the center of the center, $\mathbb{E}_x[T_x^+] = \frac{n+1}{n}$ in continuous-time and 2 in discrete-time. When x is not the center, $\mathbb{E}_x[T_x^+] = n+1$ in continuous-time, $2n$ in discrete time.

— Here $|\mathcal{V}| = \frac{d^{n+1}-1}{d-1} =: N$, $2|\mathcal{E}| = 2|\mathcal{V}| - 2$, $d_x = d$ if x is the root, $d+1$ if x is at height $1, ..., n-1$ and 1 if x is a leaf. For the continuous-time version, $\mathbb{E}_x[T_x^+] = \frac{N}{d}, \frac{N}{d+1}, N$ resp, when starting point is the root, an intermediate vertex, a leaf, resp. For the discrete-time version, $\mathbb{E}_x[T_x^+] = \frac{2N-2}{d}, \frac{2N-2}{d+1}, 2N-2$ resp.

— Here $|\mathcal{V}| = n^2$, $d_x = 2(n-1)$ for any $x \in \mathcal{V}$, $2|\mathcal{E}| = 2n^2(n-1)$, so $\mathbb{E}_x[T_x^+] = \frac{n^2}{2(n-1)}$ for the continuous-time version, and n^2 for the discrete-time version.

Exercise 49 Consider $(X_t, t \geq 0)$ the continuous-time version of the symmetric simple random walk on $\mathbb{Z}^d$ which jumps at rate λ. We also introduce its rescaled version $(X_t^N)_{t\geq 0}$ with $N \in \mathbb{N}^*$ and $X_t^N := X_{[Nt]}/\sqrt{N}, t \geq 0$.

1. Express the generator Q of X, and the generator Q_N of X^N.
2. For $f : \mathbb{R}^d \to \mathbb{R}$, write the expression of $Qf(x), x \in \mathbb{Z}^d$, and then that of $Q_N f(x), x \in \mathbb{Z}^d/\sqrt{N}$.
3. When $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, what can be said of the limit of $Q_N f(x)$ as $N \to \infty$?

1. For any $x \in \mathbb{Z}^d$,

$$Q(x, x+e_i) = \frac{\lambda}{2d}, Q(x, x-e_i) = \frac{\lambda}{2d}, i = 1, ..., d, \quad Q(x,x) = -\lambda.$$

Similarly for $x \in \mathbb{Z}^d/\sqrt{N}$,

$$Q_N(x, x+e_i/\sqrt{N}) = \frac{N\lambda}{2d}, Q(x, x-e_i/\sqrt{N}) = \frac{N\lambda}{2d}, Q(x,x) = -N\lambda.$$

2. For $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{Z}^d$,

$$Qf(x) = \frac{\lambda}{2d}\left(\sum_{i=1}^{d}[f(x+e_i) + f(x-e_i) - 2f(x)]\right),$$

and for $x \in \mathbb{Z}^d/\sqrt{N}$,

$$Q_N f(x) = \frac{N\lambda}{2d}\left(\sum_{i=1}^{d}\left[f(x+e_i/\sqrt{N}) + f(x-e_i/\sqrt{N}) - 2f(x)\right]\right).$$

3. When $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, as $N \to \infty$ $\sum_{i=1}^{d}\left[f(x+e_i/\sqrt{N}) + f(x-e_i/\sqrt{N}) - 2f(x)\right] \sim \sum_{i=1}^{d}\frac{1}{N}\frac{\partial^2 f}{\partial x_i^2}(x)$, so

$$Q_N f(x) \to \frac{\lambda}{2}\Delta f(x).$$

Exercise 50 Consider $(X_t^N, t \geq 0)$ a continuous-time version of the asymmetric simple random walk which jumps at rate λ and to the right with probability $\frac{1}{2}(1+\frac{\alpha}{N})$ for some $\alpha \in \mathbb{R}$ fixed and $N \in \mathbb{N}^*$ large enough that $\frac{|\alpha|}{N} < 1$. We again introduce a rescaled version $(Y_t^N)_{t \geq 0}$ with $N \in \mathbb{N}^*$ as before and $Y_t^N := \frac{X^N_{[N^2 t]}}{N}, t \geq 0$.

1. Express the generator Q_N of Y^N.
2. For $f : \mathbb{R}^d \to \mathbb{R}$, write the expression of $Q_N f(x), x \in \mathbb{Z}/N$.
3. When $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$, what can be said of the limit of $Q_N f(x)$ as $N \to \infty$?

1. for $x \in \mathbb{Z}/N$,

$$Q_N(x,x) = -N^2\lambda, Q(x, x+1/Ne_i) = \frac{N^2\lambda}{2}(1+\frac{\alpha}{N}), Q(x, x-1/N) = \frac{N^2\lambda}{2}(1-\frac{\alpha}{N}).$$

2. For $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{Z}^d/N$,

$$Q_N f(x) = \frac{N^2\lambda}{2}\left[(1+\frac{\alpha}{N})f(x+1/N) + (1-\frac{\alpha}{N})f(x-1/N) - 2f(x)\right].$$

3. When $N \to \infty$ and $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$,

$$\left[(1+\frac{\alpha}{N})f(x+1/N) + (1-\frac{\alpha}{N})f(x-1/N) - 2f(x)\right] \sim \frac{2\alpha f'(x)}{N^2} + \frac{f''(x)}{N^2},$$

so

$$Q_N f(x) \to \alpha\lambda f'(x) + \frac{\lambda}{2}f''(x).$$

10 Queues

Exercise 51 In this exercise, we consider the $M/M/1/n-1$ queue, which is described almost exactly as the $M/M/1$ queue of exercise III.14 : customers arrive at rate $\lambda > 0$, service times are independent exponentials with parameter $\mu > 0$ except now the queue has a maximal size $n-1$ (in other words, customers arriving when the queue is of size $n-1$ are turned away and do not add up to the queue).

1. Express the generator of the chain, or/and draw its diagramm.

2. Show the chain is reversible, positive recurrent and compute its unique invariant distribution.
3. Consider a customer arriving when the queue is at equilibrium. What is the probability that he is turned away ? Conditionally given he is not turned away, what is his expected waiting time ? Find an equivalent of this conditional waiting time as $n \to \infty$.

1. We have

$$Q(x, x+1) = \lambda, x \in \{0, ..., n-2\}, \qquad Q(x, x-1) = \mu, x \in \{1, ..., n-1\},$$
$$Q(x, x) = \begin{cases} -\lambda & \text{if } x = 0 \\ -\lambda - \mu & \text{if } x \in \{1, ..., n-2\} \\ -\mu & \text{if } x = n-1 \end{cases}.$$

2. The chain is clearly irreducible on a finite state space so it is positive recurrent. It is reversible w.r.t the measure π as soon as π satisfies,

$$\pi(x)\lambda = \pi(x+1)\mu, x \in \{0, ..., n-2\}, \text{ i.e. } \pi(x) = C\left(\frac{\lambda}{\mu}\right)^x,$$

for some constant C. The unique invariant distribution is π as above with $C = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^n}$ if $\lambda \neq \mu$, and $C = 1/n$ if $\lambda = \mu$. From now on we use notation π to designate this unique invariant distribution.
3. The probability that a customer finding the queue at equilibrium is turned away is $\pi(n-1)$. Conditionally given he is not, he finds a queue of $k \in \{0, ..., n-2\}$ customers with probability $\pi(k)/(1-\pi(n-1))$, and then his waiting time is that of $k+1$ service times, of expectation $(k+1)/\mu$. Thus, conditionally he is not turned away, his expected waiting time is

$$\mathbb{E}[T] = \frac{1}{1-\pi(n-1)} \sum_{k=0}^{n-2} (k+1) C\left(\frac{\lambda}{\mu}\right)^k = \frac{1}{1-C\left(\frac{\lambda}{\mu}\right)^{n-1}} \sum_{k=1}^{n-1} k\left(\frac{\lambda}{\mu}\right)^{k-1}.$$

If $\lambda = \mu$ we find

$$\mathbb{E}[T] = \frac{n}{n-1}\frac{n(n-1)}{2} = \frac{n^2}{2}.$$

On the other hand if $x \neq 1$, $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$ so $\sum_{k=1}^{n-1} kx^{k-1} = \frac{(n-1)x^n - nx^{n-1}+1}{(1-x)^2}$, and therefore when $\lambda \neq \mu$.

$$\mathbb{E}[T] = \frac{1-\left(\frac{\lambda}{\mu}\right)^n}{1-\left(\frac{\lambda}{\mu}\right)^{n-1}} \frac{(n-1)\left(\frac{\lambda}{\mu}\right)^n - n\left(\frac{\lambda}{\mu}\right)^{n-1} + 1}{\left(1-\frac{\lambda}{\mu}\right)^2}$$

Now when $\lambda > \mu$, a simple computation provides

$$\mathbb{E}[T] \sim \frac{\mu}{\lambda-\mu} n \left(\frac{\lambda}{\mu}\right)^n.$$

Finally when $\lambda < \mu$,

$$\mathbb{E}[T] \sim \frac{\mu^2}{(\mu - \lambda)^2}$$

Exercise 52 In this exercise we consider the $M/M/s$ queue, where there are now $s \in \mathbb{N}^* \cup \{+\infty\}$ servers. Customers arrive at rate λ, if they find an available server at arrival they go directly to it, otherwise they wait until one gets free. Service times are independent exponentials of parameter μ, and X_t denotes the total number of customers in the system at time t, including both the customers currently being served and those waiting.

1. Express the generator of the chain, or/and draw its diagramm.
2. Show that the chain is non-explosive (one may prefer to distinguish between the cases $s < \infty$ and $s = +\infty$).
3. Show that the chain is reversible, with respect to an invariant measure π which satisfies
$$\frac{\pi(k)}{\pi(k-1)} = \begin{cases} \frac{\lambda}{k\mu} \text{ if } 1 \leq k \leq s \\ \frac{\lambda}{s\mu} \text{ if } k > s \end{cases}.$$
4. Find a necessary and sufficient condition on s, λ, μ for the chain to be positive recurrent.
5. In the case $s = \infty$, find the expected waiting time of a customer who enters the queue at equilibrium.

1. We have are dealing with a birth-and-death chain and $Q(x, x+1) = \lambda, \ \forall x \geq 0$, $Q(x, x-1) = x\mu, x = 1, ..., s$, $Q(x, x-1) = s\mu$ for $x \geq s$, and of course $Q(x, x) = -\lambda - x\mu, x = 0, ..., s, Q(x, x) = -\lambda - s\mu, x \geq s$.
2. If $s < \infty$ we have $q_{\max} < \infty$ so by theorem, the chain is non explosive.
If $s = \infty$ let $\theta > 0$ and z a bounded nonnegative solution to $Qz = \theta z$. Assume for now $z_0 > 0$. Then we have $\lambda(z_1 - z_0) = \theta z_0$ so $z_1 - z_0 > 0$, and since for any $x \geq 1$,
$$\lambda(z_{x+1} - z_x) = \theta z_x + x\mu(z_x - z_{x-1}),$$
$z_{x+1} - z_x > 0$, and
$$(z_{x+1} - z_x) > \frac{x\mu}{\lambda}(z_x - z_{x-1})$$
so $z_{x+1} - z_x$ increases at least as $c2^x$ as soon as $x\mu \geq 2$, contradicting our assumption that z is bounded. Hence $z_0 = 0$, and then $z_x = 0$ for all x. Hence the chain is non explosive.
3. One easily checks that for such π, and any $x \in \mathbb{N}$, $\pi(x)Q(x, x+1) = \pi(x+1)Q(x, x-1)$, so detailed balance is satisfied for such π.
Alternatevely, we may have observed that the chain is recurrent as soon as $s\mu \geq \lambda$ (indeed the chain restricted to $\{s, s+1, ...\}$ is a continuous-time simple random walk with rate $\lambda + \mu s$ and probability to jump to the right $\frac{\lambda}{\lambda + \mu s}$, it is recurrent as soon as $p \leq 1/2$). Also there is an obvious coupling for chains X, X' with parameters $s \leq s'$ such that $X'_0 \leq X_0$ implies $X'_t \leq X_t \forall t \geq 0$, so that the chain with $s = \infty$ has to be recurrent, hence non explosive.

4. For finite s observe that for $k \geq s$ $\pi(k) = C\frac{\lambda^k s^s}{s!(\mu s)^k}$, and for $s = \infty$, $\pi(k) = C\frac{\lambda^k}{\mu^k k!}$. Thus there exists an invariant distribution for the chain as soon as $\lambda < \mu s$. But for $\lambda > \mu s$ we have seen that the chain is recurrent hence it must be positive recurrent. On the other hand when $\lambda \leq \mu s$ non trivial invariant measures all have infinite mass, so the chain can not be positive recurrent.
5. When $s = \infty$, the new customer always finds an available server, so his expected waiting time is his expected servive time μ.

Exercise 53 In this exercise we consider the $M/M/s/s$ queue, with $s \in \mathbb{N}^*$, but here customers are turned away when all s servers are busy. Express the generator of the chain, show it is reversible, positive recurrent, and compute the unique invariant distribution of this chain.
Here $Q(x, x+1) = \lambda, x = 0, ..., s-1$, $Q(x, x-1) = x\mu, x = 1, ..., s$ and $Q(x, x) = -\lambda - \mu x, x = 0, ..., s-1, Q(s, s) = -\mu s$. The chain is clearly reversible with respect to $\pi(k) = C\frac{\lambda^k}{k!\mu^k}, k = 0, ...s$. It is irreducible on a finite state space so it clearly is positive recurrent. The unique invariant distribution is the one such measure with $C = \left(\sum_{k=0}^{s} \frac{\lambda^k}{\mu^k k!}\right)^{-1}$.

Exercise 54 In this exercise we briefly introduce the $M/G/1$ queue. Customers still arrive at the jump times of a Poisson process with rate λ, service times are still i.i.d but now they are distributed as a random variable T with mean μ.

1. In general, is the process a continuous-time Markov chain ?
2. Consider X_n the number of customers in the queue when the nth customer leaves the system. Show that
$$X_{n+1} = X_n + Y_{n+1} - \mathbb{1}_{\{X_n > 0\}},$$
where Y_{n+1} is the number of extra arrivals between the times the nth and the $(n+1)$th customers leave the system. Is (X_n) a discrete-time chain ?
3. Compute the expectation of Y_{n+1} given $\{X_n > 0\}$ and deduce a necessary and sufficient condition for positive recurrence of the chain.

1. If service times do not have the memoryless property the process can not satisfy Markov property.
Indeed starting from 1 customer, $J_1 = \min(T, e)$ with e exponential with parameter λ independent of T. Markov property would entail for any $s, t > 0$ that
$$\frac{\mathbb{P}(T > t + s)\mathbb{P}(e > t + s)}{\mathbb{P}(T > t)\mathbb{P}(e > t)} = \mathbb{P}_1(J_1 > t + s \mid J_1 > t) = \mathbb{P}_1(J_1 > s) = \mathbb{P}(T > s)\mathbb{P}(e > s),$$
which implies memoryless property for T.
2. Let T_n the time at which the nth customer leaves the system, and let T denote the service time of the $(n+1)$th customer. T_n clearly is a stopping time. Conditionally given $\{X_n > 0\}$, Y_{n+1} is the number of arrivals (jumps of a Poisson process with rate λ) which fall in $[T_n, T_n + T]$ and thus is independent of $\mathcal{F}_{T_n}$. Given $\{X_n = 0\}$, one must first wait for the next client to arrive (say at $T_n + e$), and then for his service

time, so Y_{n+1} equals one plus the number of arrivals in $[T_n + e, T_n + t + e]$ (that is the previous distribution shifted by one).
In other words

$$X_{n+1} = X_n + \mathbb{1}_{\{X_n=0\}} + Z_{n+1} - \mathbb{1}_{\{X_n>0\}},$$

with $Z_k, k \geq 1$ i.i.d and having the law of the number of points of a Poisson process with rate λ that fall in $[0, T]$, and clearly (X_n) is a discrete-time Markov chain.

3. We have $\mathbb{E}[Z_k] = \mathbb{E}[\mathbb{E}[Z_k \mid T]] = \mathbb{E}[\lambda T] = \lambda\mu$ so $\mathbb{E}[Y_{n+1} \mid X_n > 0] = \mathbb{E}[Z_k] = \lambda\mu$.
If X is started at 0, its first step brings it to $1 + Z_1$. Then, as long as X remains positive, it performs a random walk on the integers where the increments are the $Z_k - 1, k = 2, 3, ...,$ of expectation $\lambda\mu - 1$. By standard arguments, the chain started at 0 will then return a.s. to zero in finite time iff $\lambda\mu \leq 1$. Also the expectation of the return time is finite (i.e. the walk is positive recurrent) iff $\lambda\mu < 1$.

11 Branching processes

Exercise 55 In this exercise we consider a continuous-time Galton-Watson process with rate λ and a reproduction law charging only 0 and 2. More precisely at rate λ, each present individual dies and independently of the past, is replaced by 0, resp. 2 individuals with probabilities $1 - p$, resp p.

1. Write the generator of the chain.
2. With $g_u(x) := u^x$, establish that

$$Qg_u(y) = \lambda g'_u(y)\phi(u),$$

where $\phi(u) = pu^2 - u + 1 - p$.

3. Let $f(t, u) := \mathbb{E}_1(u^{Z_t})$, so $f(0, u) = u$. Check that $f(t, u) = P(t)g_u(1)$, and use the forward equation to establish that

$$\frac{\partial f}{\partial t}(t, u) = \phi(u)\frac{\partial f}{\partial u}(t, u).$$

4. Explain why $P(t)g_u(x) = (P(t)g_u(1))^x$. Use the backward equation to establish that

$$\frac{\partial f}{\partial t}(t, u) = \lambda\phi(f(t, u)).$$

1. We have $Q(0, 0) = 0$ and for $x \in \mathbb{N}^*$, $Q(x, x) = -\lambda x$,
$Q(x, x - 1) = \lambda x(1 - p), Q(x, x + 1) = \lambda xp$.
2. We have, for $y \in \mathbb{N}^*$,

$$Qg_u(y) = \lambda y((1 - p)u^{y-1} + pu^{y+1} - u^y) = \lambda y u^{y-1}\phi(u),$$

as required. For $y = 0$, $Qg_u(y) = 0 = g'_u(0)\phi(u)$, and we are done.

3. We have indeed
$$P(t)g_u(1) = \sum_{y\in\mathbb{N}} P_{1y}(t)u^y = \sum_{y\in\mathbb{N}} \mathbb{P}_1(X_t = y)u^y = \mathbb{E}_1[u^{Z_t}] = f(t,u).$$
Now
$$\frac{\partial f}{\partial t}(t,u) = P'(t)g_u(1) = P(t)Qg_u(1) = \phi(u)P(t)g_u'(1) = \phi(u)\mathbb{E}_1[Z_t u^{Z_t-1}]$$
which yields the desired result.
4. Since every individual's descendance is independent of the rest of the population, the process started from $x \in \mathbb{N}^*$ is the sum of x independent copies of the process started at 1, hence $P(t)g_u(x) = \mathbb{E}_x[u^{Z_t}] = \left(\mathbb{E}_1[u^{Z_t}]\right)^x = (P(t)g_u(1))^x$. Now using backward equation
$$\begin{aligned}\frac{\partial f}{\partial t}(t,u) &= P'(t)g_u(1) = QP(t)g_u(1)\\ &= \lambda((1-p)P(t)g_u(0) + pP(t)g_u(2) - P(t)g_u(1)) = \lambda\phi(P(t)g_u(1)),\end{aligned}$$
as required.

Exercise 56 In this exercise we consider a general continuous-time Galton-Watson process with rate λ and reproduction law μ. Prove that the results of the previous exercise still hold for an adequate function ϕ which you will explicit.
By the exact same reasoning all results hold with $\phi(u) = \mathbb{E}[u^\xi]$, where ξ has law μ.

Exercise 57 Consider, for $N \in \mathbb{N}^*$ the continuous-time Galton-Watson process $(Z_t^N)_{t\geq 0}$ with rate 1, and reproduction law μ_N such that $\mu_N(0) = (1+p_N)/2, \mu(2) = (1-p_N)/2$, with $p_N \in (0,1)$. Introduce $X_t^N := \frac{Z^N_{v_N t}}{m_N}, t \geq 0$, where for any $N \in \mathbb{N}$, v_N, m_N are positive reals.
1. Write the generator of the chain X^N.
2. Assume that as $N \to \infty$, $m_N \to \infty$, $p_N v_N \to \alpha$, $v_N m_N^{-1} \to \beta$, and that $f : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable. What can be said of the limit of $Q_N f$ as $N \to \infty$?

1. When the process X^N is at $x \in \mathbb{N}^*/m_N$, this means there are $m_N x$ individuals in the population, also observe that the time change causes rates for X^N to be accelerated by v_N compared to those of Z^N. We thus can write the generator Q_N for the chain X^N, for $x \in \mathbb{N}^*/m_N$ (note of course that $Q(0,0) = 0$),
$$Q_N(x, x-1/m_N) = v_N m_N x(1+p_N)/2,\ Q_N(x, x+1/m_N) = v_N m_N x(1-p_N)/2,\ Q(x,x) = v_N m_N x.$$
2. Now if $f : \mathbb{R}_+ \to \mathbb{R}$, $x \in \mathbb{N}^*/m_N$,
$$Q_N f(x) = \frac{v_N m_N x}{2}\left((1+p_N)f(x-1/m_N) + (1-p_N)f(x+1/m_N) - 2f(x)\right)$$
As $N \to \infty$, e find that
$$Q_N f(x) \sim \frac{v_N m_N x}{2}\left(-2p_N f'(x)/m_N + f''(x)/m_N^2\right),$$
and with our assumptions
$$Q_N f(x) \to -\alpha x f'(x) + \frac{\beta}{2} f''(x).$$

12 Harmonic functions

Exercise 58 Let X be an E-valued Markov chain with transition kernel P. The function $h : E \to \mathbb{R}$ is said to be harmonic at $x \in E$ iff $\sum_{y \in E} P(x,y)h(y) = h(x)$.
We assume in this exercise that X is irreducible.

1. Assume in this question that E is finite. Show that if h is harmonic on the whole of E it is constant. Does the property still hold when E is infinite ?
2. If h is harmonic on a finite subset A of E show that
$$\sup_{x \in E} h(x) = \sup_{x \in E \setminus A} h(x).$$
3. Let $B \subsetneq E$, and assume $A = E \setminus B$ is finite. The function $h_B : B \to \mathbb{R}$ being given, show that $h(x) = \mathbb{E}_x[h_B(X_{\tau_B})], x \in E$ defines the unique extension of h_B to E that is harmonic on A.

1. Is E is finite, h reaches its maximum M at some $x \in E$. By harmonicity at x, $M = h(x) = \sum_{y_1} P(x,y_1)h(y_1)$ so that $h(y_1) = M$ for any y_1 such that $P(x,y_1) > 0$. By induction, for any $n \in \mathbb{N}$, and any y_n such that $P^n(x,y_n) > 0$, we must have $h(y_n) = M$. Since the chain is irreducible it follows that h equals M on the whole of E.
In the case of SSRW on $E = \mathbb{Z}$ (that is $P(x,x+1) = 1 - P(x,x-1) = 1/2$ for any $x \in \mathbb{Z}$), any affine function $f : x \to a + b$ is harmonic on the whole of $\mathbb{Z}$:
$$Pf(x) = \frac{1}{2}(a(x+1)+b) + \frac{1}{2}(a(x-1)+b) = ax + b = f(x), \quad \forall x \in \mathbb{Z}$$
in particular functions that are harmonic on the whole of $\mathbb{Z}$ are not necessarily constant.
2. Function h reaches its maximum M_A on A which is assumed finite. One can repeat the proof of the previous question to see that there must necessarily be a point of $\partial A = \{y \in E \setminus A : \exists x \in A \ P(x,y) > 0\} \subset E \setminus A$ where h is at least M_A. The desired claim follows, it is refered to as *the maximum principle*.
3. Because the chain is irreducible and A is finite it must be that $\tau_B < \infty$ a.s. (otherwise the chain would have to visit a point of $x \in A$ infinitely often, and by irreducibility there must exist $n \in \mathbb{N}$, $y \in B$ with $P^n(x,y) > 0$, yielding a contradiction).
We now check h is harmonic on A : for $x \in A$, using Markov property at time 1,
$$h(x) = \mathbb{E}_x[h_B(X_{\tau_B})] = \sum P(x,y)\mathbb{E}_y[h_B(X_{\tau_B})],$$
as required.
Finally let us check unicity. Assume h_1, h_2 are harmonic on A and coincida with h_B on B. It is immediate that the difference $h_1 - h_2$ remains harmonic on A, and is 0 on B. By the above question it reaches its maximum on B so it must be nonpositive on A. By the same reasoning $h_2 - h_1$ must also be nonpositive on A and we conclude that $h_1 = h_2$.

13 Martingales

Exercise 59 Let X an E-valued Markov chain with kernel P, $(\mathcal{F}_n)_n$ its natural filtration.

1. Assume $f : E \to \mathbb{R}$ is such that for any $n \in \mathbb{N}$, $\mathbb{E}[|f(X_n)|] < \infty$. Show that

$$\left(M_n^f := f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P - I) f(X_k),\ n \geq 0 \right)$$

   is a $(\mathcal{F}_n)$-martingale.

2. What can be said of $(f(X_n))$ when f is harmonic on E, and for any $n \in \mathbb{N}$, $\mathbb{E}[|f(X_n)|] < \infty$?
3. Assume $g : E \to \mathbb{R}$ to be such that for any $n \in \mathbb{N}$, $\mathbb{E}[|g(X_n)|] < \infty$ and further assume that for some $\lambda \in \mathbb{R}^*$, $Pg = \lambda g$. Show that $(g(X_n)/\lambda^n)$ is a $(\mathcal{F}_n)$-martingale.
4. Assume $h : \mathbb{N} \times E \to \mathbb{R}$ to be such that or any $n \in \mathbb{N}$, $\mathbb{E}[|h(n, X_n)|] < \infty$, and further assume that

$$\forall x \in E,\ \forall n \in \mathbb{N}\ Ph(n+1, x) = \sum_{y \in E} P(x, y) h(n+1, y) = h(n, y).$$

   Show that $(h(n, X_n))_n$ is a $(\mathcal{F}_n)$-martingale.

1. Measurability and integrability properties of M_n^f are straightforward from the assumptions. Note in addition that $M_n^f - Pf(X_n)$ is $\mathcal{F}_n$-mesurable. By Markov property at time n the law of X_{n+1} conditionally given $\mathcal{F}_n$ is $P(X_n, \cdot)$. Thus

$$\begin{aligned} \mathbb{E}[M_{n+1}^f \mid \mathcal{F}_n] &= M_n^f - Pf(X_n) + \mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] \\ &= M_n^f - Pf(X_n) + \sum_{x \in E} f(x) P(X_n, x) = M_n^f, \end{aligned}$$

   and we conclude that $(M_n^f)_n$ is indeed a $(\mathcal{F}_n)$-martingale.

2. In this case $M_n^f = f(X_n) - f(X_0)$, so $(f(X_n))$ also is a $(\mathcal{F}_n)$-martingale.
3. Again measurability and integrability follow easily from the assumptions. Moreover

$$\mathbb{E}[g(X_{n+1}) \mid \mathcal{F}_n] = Pg(X_n) = \lambda g(X_n)$$

   so $(g(X_n)/\lambda^n)$ indeed is a $(\mathcal{F}_n)$-martingale.

4. Again measurability and integrability follow easily from the assumptions, moreover

$$\mathbb{E}\left[h(n+1, X_{n+1}) \mid \mathcal{F}_n\right] = Ph(n+1, X_n) = h(n, X_n),$$

   as required.

Exercise 60
Let (X_n) be SRW on $\mathbb{Z}$ with $p = P(x, x+1) = 1 - P(x, x-1) = 1 - q$, $(\mathcal{F}_n)_n$ its natural filtration

1. Show that $(X_n - (p - q)n)$ is a $(\mathcal{F}_n)$-martingale.

2. Show that $\left((\frac{q}{p})^{X_n}\right)$ is a $(\mathcal{F}_n)$-martingale.
3. Explain how to recover gambler's ruin probabilities from the above.
4. Show that $((X_n - (p-q)n)^2 - 4pqn)$ is a $(\mathcal{F}_n)$-martingale.
5. For $\lambda \in \mathbb{R}$, let $\Phi(\lambda) = \log(p\exp(\lambda) + q\exp(-\lambda))$. Show that $\exp(\lambda X_n - n\Phi(\lambda))$ is a $(\mathcal{F}_n)$-martingale.

1. Simply note that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n + p - q$ and apply the third point of the preceding exercise.
2. We have $\mathbb{E}[(q/p)^{X_{n+1}}] = (q/p)^{X_n}\left(p \times (q/p) + q \times (q/p)^{-1}\right) = (p/q)^{X_n}$ so the desired result follows.
3. Let $T = T_0 \wedge T_N$, and consider the chain started at $k \in \{0, ..., N\}$. Even if it means bounding T by nG with $G \sim \mathrm{Geom}(p^n)$ one finds that $T < \infty$ $\mathbb{P}_k$-a.s.

   In the case $p = q = 1/2$, $X_{n\wedge T}$ remains bounded, Doob's optional stopping can be applied and
$$N\mathbb{P}_k(X_T = N) = \mathbb{E}_k[X_T] = k,$$
   it follows that
$$\mathbb{P}_k(X_T = 0) = 1 - \mathbb{P}_k(X_T = N) = \frac{N-k}{N}.$$
   In the case $p \neq 1/2$, $(q/p)^{X_{n\wedge T}}$ remaining bounded, and Doob's optional stopping can again be applied to show
$$\mathbb{P}_k(X_T = 0) + (1 - \mathbb{P}_k(X_T = 0))(q/p)^N = \mathbb{E}_k[(p/q)^{X_T}] = \left(\frac{q}{p}\right)^k,$$
   and it follows that
$$\mathbb{P}_k(X_T = 0) = \frac{(q/p)^k - (q/p)^N}{1 - (q/p)^N}.$$
4. We find
$$\begin{aligned}\mathbb{E}[(X_{n+1} - (p-q)(n+1))^2 \mid \mathcal{F}_n] &= p(X_n - (p-q)n + 2q)^2 + q(X_n - (p-q)n - 2p)^2 \\ &= (X_n - (p-q)n)^2 + 4pq\end{aligned}$$
   and we conclude to the desired result.
5. The result follows from the fact that
$$\mathbb{E}[\exp(\lambda X_{n+1}) \mid \mathcal{F}_n] = \exp(\lambda X_n)\Phi(\lambda),$$
   and from the second point of the preceding exercise.

Exercise 61 Let X be an E-valued Markov chain with transition kernel P. Assume further that $A \subsetneq E$ is finite, $B = E \setminus A$, and $h_{\partial A} : \partial A \to \mathbb{R}$ bounded. Letting h be the unique function on $A \cup \partial A$ which is harmonic on A and coincides with $h_{\partial A}$ on ∂A. Show that $M_n = \mathbb{E}_x[h(X_{n\wedge T_{\partial A}})]$ is a $(\mathcal{F}_n)$-martingale.

By the maximum principle h reaches its maximum on ∂A and is therefore bounded on $A \cup \partial A$. Since $T = T_{\partial A}$ is a $(\mathcal{F}_n)$-stopping time, $\mathbb{1}_{\{T>n\}}, \mathbb{1}_{\{T \le n\}}$ are $\mathcal{F}_n$-mesurable, so

$$\begin{aligned}
\mathbb{E}[h(X_{n+1\wedge T}) \mid \mathcal{F}_n] &= \mathbb{E}[h(X_{n+1})\mathbb{1}_{\{T>n\}} \mid \mathcal{F}_n] + \mathbb{E}[h(X_T)\mathbb{1}_{\{T\le n\}} \mid \mathcal{F}_n] \\
&= Ph(X_n)\mathbb{1}_{\{T>n\}} + h(X_T)\mathbb{1}_{\{T\le n\}} \\
&= h(X_{T\wedge n}),
\end{aligned}$$

where at the last line we used that $\{X_n \in A\} \subset \{T > n\}$ and the fact that h is harmonic on A.

Exercise 62 Let (Z_n) be a Galton-Watson process un processus with reproduction law μ, i.e.

$$Z_0 = 1, \quad Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n,i}, n \ge 1,$$

where $(\xi_{n,i}, n \ge 0, i \ge 1)$ are i.i.d with law μ. For $n \ge 0$ et $\mathcal{F}_n = \sigma(\xi_{k,i}, k \le n, i \ge 1)$.

1. Let $m = \sum_{k\ge 0} k\mu(k)$, and assume it is finite. Show that $(m^{-n}Z_n)$ is a $(\mathcal{F}_n)$-martingale.
2. When $m > 1$, and $\mu(0) > 0$, establish the existence of $\zeta \in (0,1)$ such that $G_\mu(\zeta) = \zeta$, where $G_\mu(x) = \sum_{n\ge 0} \mu(k)k^n$. Show that (ζ^{Z_n}) is a $(\mathcal{F}_n)$-martingale.

1. We have

$$\begin{aligned}
\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\sum_{k=1}^{Z_n} \xi_{n+1,k} \mid \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\sum_{k\ge 1} \xi_{n+1,k}\mathbb{1}_{\{Z_n \ge k\}}\right] \\
&= \sum_{k\ge 1} \mathbb{1}_{\{Z_n\ge k\}} m = mZ_n,
\end{aligned}$$

and the result follows.

2. The existence of ζ follows from the fact that G_μ is convex, increasing, and satisfies $G_\mu(0) = \mu(0), G_\mu(1) = 1, G'_\mu(1) = m > 1$. We then see that

$$\mathbb{E}[\zeta^{Z_{n+1}} \mid \mathcal{F}_n] = \prod_{i=1}^{Z_n} \mathbb{E}[\zeta^{\xi_{n+1,i}}] = G_\mu(\zeta)^{Z_n} = \zeta^{Z_n}$$

as required.

Exercise 63 For $N \ge 2$, consider X a Wright-Fisher chain on $\{0, ..., N\}$, i.e. for any $n \ge 0$, conditionally given X_n, $X_{n+1} \sim \text{Bin}(N, X_n/N)$.

1. Find kernel P of X. Is the chain irreducible ?
2. Show $(X_n)_n$ is a $(\mathcal{F}_n)$-martingale (with $(\mathcal{F}_n)$ being the natural filtration of X).
3. Let $\tau = T_0 \wedge T_N$. Compute $\mathbb{P}_k(X_\tau = N)$ as a function of $k \in \{0, ..., N\}$ and N.

4. Show $\left(\frac{N}{N-1}\right)^N X_n(N-X_n)$ is a $(\mathcal{F}_n)$-martingale. Deduce that

$$\left(\frac{N}{N-1}\right)^n k(N-k)\frac{4}{N^2} \leq \mathbb{P}_k(\tau > n) \leq \left(\frac{N}{N-1}\right)^n k(N-k)\frac{1}{N-1}.$$

1. For fixed $i, j \in \{0, ..., N\}$, $P(i,j)$ is the probability that a binomial with parameters $N, i/N$ takes value j, hence

$$P(i,j) = \binom{N}{j}\left(\frac{i}{N}\right)^j \left(\frac{N-i}{N}\right)^{N-j}.$$

2. If $X \sim \mathrm{Bin}(n,p)$ one has $\mathbb{E}[X] = np$, thus

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = N\frac{X_n}{N} = X_n,$$

and (X_n) is a (bounded) martingale.

3. The stopping time τ can be e.g. bounded by a geometric variable with parameter N^{-N}, it is therefore a.s. finite and by Doob's optional stopping we deduce

$$\mathbb{P}_k(X_\tau = N) = \frac{k}{N}.$$

4. If $X \sim \mathrm{Bin}(n,p)$ we have $\mathbb{E}[X^2] = np(1-p) + n^2p^2$, thus $\mathbb{E}[X(n-X)] = n^2p - npq - n^2p^2 = (n^2-n)pq$. It follows that

$$\mathbb{E}[X_{n+1}(N-(X_{n+1})) \mid \mathcal{F}_n] = (N^2-N)\frac{X_n}{N}\frac{N-X_n}{N} = \left(1-\frac{1}{N}\right)X_n(N-X_n),$$

which yields that $\left(M_n = \left(\frac{N}{N-1}\right)^N X_n(N-X_n)\right)$ is a martingale. Evidently $(M_{n\wedge\tau})$ also is a martingale. Noticing that $M_\tau = 0$ we find

$$k(N-k) = \mathbb{E}_k[M_{\tau\wedge n}] = \mathbb{E}[\mathbb{1}_{\{\tau>n}M_n] = \left(\frac{N}{N-1}\right)^N \mathbb{E}[\mathbb{1}_{\{\tau>n}X_n(N-X_n)].$$

Moreover if $X_n \in \{1, ..., N-1\}$, $N-1 \leq X_n(N-X_n) \leq \frac{N^2}{4}$, so the desired inequalities follow.

14 Discrete potential theory

Exercise 64 Consider X an E-valued irreducible chain and $D \subset E$. Let $T = \inf\{n \geq 0 : X_n \in D^c\}$, and set, for $x, y \in E$,

$$G_D(x,y) := \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=y, T>n\}}\right]$$

Note that G_D generalies the Green function of the last exercise of the previous sheet (in that exercise, we were in the case $D = E$). For $c : D \to \mathbb{R}_+$ introduce

$$u_D(x) = \sum_{y\in D} c(y)G_D(x,y).$$

1. Show that $u_D(x) = \mathbb{E}_x\left[\sum_{n=0}^{T-1} c(X_n)\right]$.
2. Establish that u_D is solution to

$$(\star) \quad u(x) = \begin{cases} Pu(x) + c(x) & x \in D \\ 0 & x \in D^c \end{cases}$$

3. Show that if u is solution to $(\star)$, then $\left(M_n := u(X_{n\wedge T}) + \sum_{k=0}^{n-1} c(X_k)\mathbb{1}_{\{T>k\}}\right)$ is a $(\mathcal{F}_n)$-martingale (with $(\mathcal{F}_n)$ the natural filtration of X) provided $\mathbb{E}[|M_n|] < \infty\ \forall n$.
4. Assume in this question that c is bounded on D and that for any $x \in D$ $T < \infty$ $\mathbb{P}_x$-a.s. Show then that u_D is the unique bounded solution of $(\star)$.

1. Since c is nonnegative one can use Fubini-Tonelli to see that

$$\begin{aligned} \sum_{y\in D} c(y)G_D(x,y) &= \sum_{y\in D} c(y)\mathbb{E}_x\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{X_n=y,T>n\}}\right] \\ &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} c(X_n)\mathbb{1}_{\{T>n\}}\right], \end{aligned}$$

as required.
2. Obvisouly $u_D(x) = 0$ if $x \in D^c$.
By this fact, the expression of the first question and the Markov property at time 1, we find for $x \in D$ that

$$u_D(x) = c(x) + \sum_{y\in D} P(x,y)\mathbb{E}_y\left[\sum_{n=0}^{T-1} c(X_n)\right] = c(x) + Pu_D(x).$$

3. Since $u(X_T)\mathbb{1}_{\{T\le n\}}$ is $\mathcal{F}_n$-mesurable, $\{T > n\} \in \mathcal{F}_n$, and on this event $X_n \in D$ we find

$$\begin{aligned} \mathbb{E}[u(X_{(n+1)\wedge T}) \mid \mathcal{F}_n] &= \mathbb{E}\left[u(X_{n+1})\mathbb{1}_{\{T>n+1\}} + u(X_T)\mathbb{1}_{\{T\le n\}} \mid \mathcal{F}_n\right] \\ &= \mathbb{1}_{\{T>n\}}Pu(X_n) + u(X_T)\mathbb{1}_{\{T\le n\}} \\ &= u(X_{T\wedge n}) - c(X_n)\mathbb{1}_{\{T>n\}}. \end{aligned}$$

Finally since $\sum_{k=0}^{n} c(X_k)\mathbb{1}_{\{T>k\}}$ is $\mathcal{F}_n$-mesurable, we obtain that

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = u(X_{T\wedge n}) - c(X_n)\mathbb{1}_{\{T>n\}} + \sum_{k=0}^{n} c(X_k)\mathbb{1}_{\{T>k\}} = M_n.$$

4. Let u be a bounded solution of $(\star)$. As c is also bounded, M_n is integrable for any n and by the previous question (M_n) is a martingale (started at 0). In particular

$$u(x) = \mathbb{E}_x[M_n] \Rightarrow \mathbb{E}_x[u(X_{T\wedge n})] = u(x) - \mathbb{E}_x\left[\sum_{k=0}^{n-1} c(X_k)\mathbb{1}_{\{T>k\}}\right].$$

As $n \to \infty$, using that $T < \infty$ $\mathbb{P}_x$-a.s. along with the boundedness of u, and dominated convergence, the above lefthand side converges to $\mathbb{E}_x[u(X_T)] = 0$ (using

$X_T \in D^c$). Now, by monotone convergence (c being nonnegative) the righthand side converges to $u(x) - \mathbb{E}_x\left[\sum_{k=0}^{T-1} c(X_k)\right] = u(x) - u_D(x)$. This reasoning holds for any $x \in D$. Of course both u u_D cancel on D^c, in the end $u \equiv u_D$.
Remark : When $c \equiv 0$, we proved that if for any $x \in D$, $T < \infty$ $\mathbb{P}_x$-a.s., a function that is harmonic on D and vanishes on D^c also vanishes on D. This slightly extends the result of exercise 1 above for null boundary conditions, which required D to be finite.

Exercise 65 We make the same assumptions as in the previous exercise and recall that

$$\partial D := \{y \in E : \exists x \in D \quad P(x,y) > 0\}.$$

Introduce

$$G_{\to \partial D}(x,y) := \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = y, T = n\}}\right], \quad x \in D, y \in \partial D,$$

and for $\phi : \partial D \to \mathbb{R}_+$,

$$u_{\partial D}(x) = \sum_{y \in \partial D} \phi(y) G_{\to \partial D}(x,y).$$

1. Show that $u_{\partial D}(x) = \mathbb{E}_x\left[\phi(X_T)\mathbb{1}_{\{T<\infty\}}\right], x \in D$. On what set is $u_{\partial D}$ harmonic ?
2. Assume here that $\phi : \partial D \to \mathbb{R}_+$, $c : D \to \mathbb{R}_+$ are bounded. Assume further that for any $x \in D$ $T < \infty$ $\mathbb{P}_x$-a.s. Establish that $u_D + u_{\partial D}$ is the unique bounded solution to

$$u(x) = \begin{cases} Pu(x) + c(x) & x \in D \\ \phi(x) & x \in \partial D \end{cases}.$$

1. Using Fubini-Tonelli, for any $x \in D$,

$$\begin{aligned} u_{\partial D}(x) &= \sum_{y \in \partial D} \phi(y) G_{\to \partial D}(x,y) \\ &= \sum_{y \in \partial D} \phi(y) \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = y, T = n\}}\right] \\ &= \mathbb{E}_x\left[\sum_{y \in \partial D} \phi(y) \mathbb{1}_{\{X_T = y, T < \infty\}}\right] \\ &= \mathbb{E}_x\left[\phi(X_T)\mathbb{1}_{\{T<\infty\}}\right]. \end{aligned}$$

2. If $x \in \partial D$, $u_{\partial D}(x) = \phi(x)$. If $x \in D$, $P(x,y) = 0$ for $y \notin D \cup \partial D$ thus by Markov property at time 1,

$$u_{\partial D}(x) = \sum_{y \in D \cup \partial D} P(x,y) \mathbb{E}_y\left[\phi(X_T)\mathbb{1}_{\{T<\infty\}}\right] = P u_{\partial D}(x),$$

hence $u_{\partial D}$ coincides with ϕ on ∂D and is harmonic on D.

3. Thanks to our assumption on ϕ, $u_{\partial D}$ is bounded, hence so is $u - u_{\partial D}$. But then $u - u_{\partial D}$ satisfies equation $(\star)$ of the previous exercise, and thus $u - u_{\partial D} = u_D$ as required.

   Remark : In the case $c \equiv 0$, we have shown that provided ϕ is bounded and for any $x \in D$, $T < \infty$ $\mathbb{P}_x$-a.s., the unique function that is harmonic on D and coincides with ϕ on ∂D is $u_{\partial D}$. This extends the result of exercise 1 which required D to be finite. Note further that maximum principle is self-evident with this approach since obviously

$$\max_{x \in D \cup \partial D} \mathbb{E}_x \left[\phi(X_T)\mathbb{1}_{\{T<\infty\}}\right] \leq \max_{x \in \partial D} \phi(x).$$

Exercise 66 Let X an E-valued chain with kernel P, irreducible and recurrent. What can be said of the set of nonnegative functions that are harmonic on the hole of E ?

From a previous exercise if $Ph = h$ then $(h(X_n))$ is a nonnegative martingale, so it must $\mathbb{P}_x$-a.s. converge to some H. Since the chain is irreducible and recurrent, it visits any $y \in E$ infinitely often, so the sequence $(h(X_n))$ takes value $h(y)$ infinitely often. It follows (by considering a subsequence) that we must have $H = h(y)$ a.s., and since the reasoning is valid for any y, it follows that h must be constant.

15 Time reversal

Let X be an irreducible E-valued Markov chain with kernel P, and π a non trivial invariant measure of X. Define, for any $x, y \in E$,

$$\widehat{P}(x,y) := \frac{\pi(y)P(y,x)}{\pi(x)}.$$

Exercise 67

1. Show that $\widehat{P}$ is well-defined and that it is the transition kernel of an E-valued chain $\widehat{X}$, also irreducible. write $\widehat{\mathbb{P}}_\mu$ for the law of $\widehat{X}$ when started at μ.
2. Show that π is also invariant for $\widehat{X}$.
3. Show that for any $t \geq 0, x_0, ..., x_t \in E$, we have

$$\pi(x_0)\mathbb{P}_{x_0}\left(X_0 = x_0, ..., X_t = x_t\right) = \pi(x_t)\widehat{\mathbb{P}}_{x_t}\left(\widehat{X}_0 = x_t, ..., \widehat{X}_t = x_0\right).$$

   Deduce that if π is an invariant *distribution* of X, for any $n \geq 0$, the law of $(X_0, ..., X_n)$ under $\mathbb{P}_\pi$ matches that of $(X_n, ..., X_0)$ under $\widehat{\mathbb{P}}_\pi$.
4. What happens when X is reversible ?

1. X being irreducible implies that $\pi(x) > 0$ for any $x \in E$ (cf an eercise of the first sheet), which ensures that $\widehat{P}(x,y)$ is well defined for any $x, y \in E$.
   To check that it is a transition kernel, fix $x \in E$ and compute

$$\begin{aligned}\sum_{y \in E} \widehat{P}(x,y) &= \sum_{y \in E} \frac{\pi(y)P(y,x)}{\pi(x)} \\ &= \frac{\pi P(x)}{\pi(x)} = 1,\end{aligned}$$

using that π is invariant.
Since $\pi(x) > 0$ for any $x \in E$, $P^r(x,y) > 0$ implies $\widehat{P}^r(y,x) > 0$, thus irreducibility of X implies that of $\widehat{X}$.

2. We have

$$\begin{aligned}\pi\widehat{P}(y) &= \sum_{x\in E}\pi(x)\widehat{P}(x,y)\\ &= \sum_{x\in E}\pi(y)P(y,x) = \pi(y),\end{aligned}$$

so π also is invariant for $\widehat{X}$.

3. Using definition of $\widehat{P}$,

$$\begin{aligned}\pi(x_t)\widehat{\mathbb{P}}_{x_t}\left(\widehat{X}_0 = x_t, ..., \widehat{X}_t = x_0\right) &= \pi(x_t)\widehat{P}(x_t, x_{t-1})...\widehat{P}(x_1, x_0)\\ &= P(x_{t-1}, x_t)...P(x_0, x_1)\pi(x_0)\\ &= \pi(x_0)\mathbb{P}_{x_0}\left(X_0 = x_0, ..., X_t = x_t\right)\end{aligned}$$

Since π is a distribution, we deduce that for any $n \geq 0$,

$$\widehat{\mathbb{P}}_\pi\left(\widehat{X}_0 = x_n, ..., \widehat{X}_n = x_0\right) = \mathbb{P}_\pi\left(X_0 = x_0, ..., X_n = x_n\right),$$

and the desired equality of laws.

4. When X is reversible, we find $\widehat{P} = P$, and then whatever μ distibution on E, the law of X under P_μ matches that of $\widehat{X}$ under $\widehat{\mathbb{P}}_\mu$. In particular, a reversible chain and its time-reversal, when at stationarity, are indistinguishable.

Exercise 68 Let X be an E-valued Markov chain, π a non trivial invariant measure of X, $\widehat{X}$ the corresponding time-reversal. Consider further μ a nonnegative measure on E.

1. Establish that
$$\frac{\mu P}{\pi} = \widehat{P}\left(\frac{\mu}{\pi}\right).$$

2. Let $g : E \to \mathbb{R}_+^*$. Show that the following assertions are equivalent

   i The function g is $\widehat{P}$-harmonic on E.

   ii If μ is the measure on E such that for any $x \in E$, $\mu(x) = g(x)\pi(x)$, then it is invariant P.

3. What happens when X is irreducible and recurrent ?

1. For each $x \in E$,

$$\begin{aligned}\widehat{P}\left(\frac{\mu}{\pi}\right)(x) &= \sum_{y\in E}\frac{\mu(y)}{\pi(y)}\widehat{P}(x,y)\\ &= \sum_{y\in E}P(y,x)\frac{\mu(y)}{\pi(x)} = \frac{\mu P}{\pi}(x).\end{aligned}$$

2. From the above, if $g = \frac{\mu}{\pi}$ then

$$\widehat{P}g = \widehat{P}\left(\frac{\mu}{\pi}\right) = \frac{\mu P}{\pi},$$

so that

$$g = \widehat{P}g \Leftrightarrow \mu P = \mu,$$

as required.

3. When X (thus $\widehat{X}$) is irreducible and recurrent, the only nonnegative harmonic functions are the constants, and we recover the fact that any invariant measure must be a multiple of π.

16 Reversible chains, electric networks analogy

In this paragraph we let X be irreducible, with kernel P, reversible with respect to the nonnegative measure π.
Recall such chain is equivalent to a conductance model on the graph $\mathcal{G} = (\mathcal{V} = E, \mathcal{E} = \{(x, y) : P(x, y) > 0\})$, with conductance function $c : \mathcal{E} \to \mathbb{R}_+$ satisfying that for some constant $K > 0$

$$c(x, y) = K\pi(x)P(x, y), \ \forall (x, y) \in \mathcal{E}.$$

We further remind that we write $c(x) = \sum_{y \sim x} c(x, y)$, and $c_{\mathcal{G}} = \sum_{x \in E} c(x)$.
When looking at the dependence in K, we will add the superscript (K) to these notations.

Exercise 69 (Influence of the constant K)
Recall

$$c^{(K)}(x, y) = K\pi(x)P(x, y), x, y \in E,$$

and fix $a \in E, Z \subset E, a \notin Z$. We assume that the time to reach $\{a\} \cup Z$ from any point in $E \setminus (\{a\} \cup Z)$ is a.s. finite.

1. Let $\alpha > \beta$ be real. How does the function $V = V^{(K)}_{\alpha,\beta}$ satisfying

$$V(a) = \alpha, V(z) = \beta \ \forall z \in Z, V(\cdot) P\text{-harmonic on } E \setminus (\{a\} \cup Z)$$

depend on the choice K ? How does it depend on the choice of α, β ?

2. Express the current $I^{(K)}_{\alpha,\beta}$ associated with $V^{(K)}_{\alpha,\beta}$ as a function of $I^{(1)}_{\alpha,\beta}$, then as a function of K, α, β and $I^{(1)}_{1,0}$.
3. How does the unit current from a to Z depend on the choice of K ?
4. How does the effective resistance between a and Z, denoted $\mathcal{R}^{(K)}(a \leftrightarrow Z)$, depend on the choice of K ?

1. Kernel P being fixed, it does not depend on K, neither does $V^{(K)}_{\alpha,\beta}$. We write it $V_{\alpha,\beta}$ in what follows.
Note however that $V = \beta + (\alpha - \beta)V_{1,0}$ takes value α at a, β on Z, and remains harmonic on $E \setminus (\{a\} \cup Z)$ (as $V_{1,0}$ is). Solution to the Dirichlet problem is unique from our assumption and exercise 9, hence $V_{\alpha,\beta} = \beta + (\alpha - \beta)V_{1,0}$.

2. By Ohm's law, for $x, y \in E$,

$$\begin{aligned} I_{\alpha,\beta}^{(K)}(x,y) &= c^{(K)}(x,y)(V_{\alpha,\beta}(x) - V_{\alpha,\beta}(y)) \\ &= K(\alpha-\beta)\pi(x)P(x,y)(V_{1,0}(x) - V_{1,0}(y)) \end{aligned}$$

so that

$$I_{\alpha,\beta}^{(K)} = KI_{\alpha,\beta}^{(1)},$$

$$I_{\alpha,\beta}^{(K)} = K(\alpha-\beta)I_{1,0}^{(1)}.$$

3. The force $||I_{\alpha,\beta}^{(K)}||$ of the current $I_{\alpha,\beta}^{(K)}$ is by definition

$$\begin{aligned} \operatorname{div}_a(I_{\alpha,\beta}^{(K)}) &= \sum_{y\sim a} I_{\alpha,\beta}^{(K)}(a,y) \\ &= K(\alpha-\beta)||I_{1,0}^{(1)}|| \end{aligned}$$

Thus we must choose a potential difference $\alpha - \beta = \frac{1}{K||I_{1,0}^{(1)}||}$ between a and Z to create a unit current (i.e. of force 1) between a and Z. For this choice of α, β, the corresponding current if

$$I_{\alpha,\beta}^{(K)} = K(\alpha-\beta)I_{1,0}^{(1)} = \frac{I_{1,0}^{(1)}}{||I_{1,0}^{(1)}||},$$

so the unit current between a and Z does not depend on the choice of K.

4. Effective resistance between a and Z is such that for any $\alpha > \beta$,

$$\mathcal{R}^{(K)}(a \leftrightarrow Z) = \frac{\alpha-\beta}{||I_{\alpha,\beta}^{(K)}||} = \frac{1}{||I_{1,0}^{(K)}||} = \frac{1}{K}\mathcal{R}^{(1)}(a \leftrightarrow Z).$$

Exercise 70
Let $a \in E, Z \subset E$ with $a \notin Z$. Assume $\tau_Z = \inf\{t \geq 0 : X_t \in Z\}$ is a.s. finite regardless of the starting point, and set for $x \in E$,

$$G_Z(a,x) := \mathbb{E}_a\left[\sum_{t=0}^{\tau_Z - 1} \mathbb{1}_{\{X_t = x\}}\right].$$

Introduce $V(x) = \frac{G_Z(a,x)}{c(x)}, x \in E$ (beware of the fact that even if it does not appear in our notation, the function V should a priori depend on a and on the choice of the constant $K > 0$ such that $c(x,y) = K\pi(x)P(x,y), x, y \in E$).

1. What value(s) takes the function V on Z?
2. Establish that $V(a) = \mathcal{R}^{(K)}(a \leftrightarrow Z)$.
3. Show V to be harmonic on $E \setminus \{a\} \cup Z$.
4. Show that

$$\mathbb{E}_a[\tau_Z] = \sum_{x\in E} c(x)V(x)$$

5. For $a, y \in E$ establish that
$$\mathbb{E}_a[\tau_y] + \mathbb{E}_y[\tau_a] = c_{\mathcal{G}} \mathcal{R}(a \leftrightarrow y).$$

6. For $x, y \in E$ write
$$S_{xy} = \sum_{t=0}^{\tau_Z - 1} \mathbf{1}_{\{X_t = x, X_{t+1} = y\}}.$$
Show then that $\mathbb{E}_a[S_{xy}] = G_Z(a, x)P(x, y)$, and deduce that if I is the unit current from a to Z,
$$I(x, y) = \mathbb{E}_a[S_{xy} - S_{yx}].$$

1. By definition of τ_Z and G_Z, $G_Z(a, z) = 0$ for any $z \in Z$, hence V cancels on Z.
2. We have $V(a) = \frac{G_Z(a,a)}{c(a)}$. Using notation of the previous exercise $V_{1,0}(x) = \mathbb{P}_x(\tau_a < \tau_Z)$ (because $x \to \mathbb{P}_x(\tau_a < \tau_Z)$ satisfies the same Dirichlet problem as $V_{1,0}$, and the solution to that problem is unique by exercise 9). Letting $\tau_a^+ = \inf\{t > 0 : X_t = a\}$, we obtain in particular that
$$\begin{aligned}
\mathbb{P}_a(\tau_a^+ < \tau_Z) &= \sum_{x \in E} P(a, x) V_{1,0}(x) \\
&= \frac{1}{\pi(a)} \sum_{x \in E} \pi(a) P(a, x) V_{1,0}(x) \\
&= 1 + \frac{1}{\pi(a)} \sum_{x \in E} \pi(a) P(a, x)(V_{1,0}(x) - V_{1,0}(a)) \\
&= 1 - \frac{1}{\pi(a)} ||I_{1,0}^{(1)}||
\end{aligned}$$
By definition of $\mathcal{R}^{(1)}(a \leftrightarrow Z)$ and the previous exercise we deduce that
$$\mathbb{P}_a(\tau_a^+ < \tau_Z) = 1 - \frac{1}{\pi(a)\mathcal{R}^{(1)}(a \leftrightarrow Z)} = 1 - \frac{1}{c(a)\mathcal{R}^{(K)}(a \leftrightarrow Z)}.$$
The above expression does not depend on K because $c(a) = K\pi(a)$ which is why we shall forget the dependence in K in the notation and write $c(a)\mathcal{R}(a \leftrightarrow Z)$.
It remains to be seen, using Markov property at the successive visit times at a before reaching Z, that $G_Z(a, a)$ is the expectation of a geometric random variable with parameter $\mathbb{P}_a(\tau_Z < \tau_a^+)$ so that
$$V(a) = \frac{G_Z(a, a)}{c(a)} = \mathcal{R}^{(K)}(a \leftrightarrow Z),$$
as required.

3. If $x \in E \setminus (\{a\} \cup Z)$, using that $c(\cdot) = K\pi(\cdot)$, reversibility of the chain and the Markov property at time t :

$$\begin{aligned}
c(x) \sum_{y \in E} P(x,y) V(a,y) &= \sum_{y \in E} \frac{1}{c(y)} \mathbb{E}_a \left[\sum_{t=0}^{\tau_Z - 1} K\pi(x) P(x,y) \mathbb{1}_{\{X_t = y\}} \right] \\
&= \sum_{y \in E} \mathbb{E}_a \left[\sum_{t=0}^{\tau_Z - 1} \frac{K}{c(y)} \pi(y) P(y,x) \mathbb{1}_{\{X_t = y\}} \right] \\
&= \sum_{y \in E} \mathbb{E}_a \left[\sum_{t=0}^{\tau_Z - 1} \mathbb{1}_{\{X_t = y, X_{t+1} = x\}} \right] \\
&= \mathbb{E}_a \left[\sum_{t=0}^{\tau_Z - 1} \mathbb{1}_{\{X_{t+1} = x\}} \right] = G_Z(a,x),
\end{aligned}$$

where in the last equality we used $x \neq a, x \notin Z$ to see that under $\mathbb{P}_a$, $\sum_{t=0}^{\tau_Z-1} \mathbb{1}_{\{X_{t+1}=x\}} = \sum_{t=0}^{\tau_Z-1} \mathbb{1}_{\{X_t=x\}}$.

4. Simply
$$\mathbb{E}_a[\tau_Z] = \sum_{x \in E} G_Z(a,x) = \sum_{x \in E} c(x) V(x).$$

5. If $a = y$ the desired equality is straightforward. Otherwise, even if it means setting $Z = \{y\}$, the above points imply
$$\mathbb{E}_a[\tau_y] = \sum_{x \in E} c(x) V^{a,y}(x),$$
where $V^{a,y}$ is the unique function that is harmonic on $E \setminus \{a,y\}$ and such that
$$V^{a,y}(a) = \mathcal{R}^{(K)}(a \leftrightarrow y), V^{a,y}(y) = 0.$$
Similarly
$$\mathbb{E}_y[\tau_a] = \sum_{x \in E} c(x) V^{y,a}(x),$$
where $V^{y,a}$ is the unique function that is harmonic on $E \setminus \{a,y\}$ and such that
$$V^{y,a}(y) = \mathcal{R}^{(K)}(a \leftrightarrow y), V^{y,a}(a) = 0.$$
The function $W = \mathcal{R}^{(K)}(a \leftrightarrow y) - V^{a,y}$ satisfies the same Dirichlet problem, so that $V^{y,a} = \mathcal{R}^{(K)}(a \leftrightarrow y) - V^{a,y}$. We finally obtain
$$\begin{aligned}
\mathbb{E}_a[\tau_y] + \mathbb{E}_y[\tau_a] &= \sum_{x \in E} c(x)(V^{a,y}(x) + V^{y,a}(x)) \\
&= \sum_{x \in E} c(x) \mathcal{R}(a \leftrightarrow y) = \mathcal{R}(a \leftrightarrow y) c_{\mathcal{G}},
\end{aligned}$$
as required (again $c(\cdot)\mathcal{R}(\cdot \leftrightarrow \cdot)$ does not depend on K, which is why we droped the superscripts).

6. Observe that the current I_V from a to Z associated with V satisfies

$$\mathcal{R}(a \leftrightarrow Z) = \frac{V(a) - V(z)}{||I_V||} = \frac{\mathcal{R}(a \leftrightarrow Z)}{||I_V||},$$

hence it it the unit current from a to Z (by the preceding exercise it does not depend on K).
By Markov property at t,

$$\begin{aligned} \mathbb{E}_a[S_{xy}] &= \mathbb{E}_a\left[\sum_{t=0}^{\tau_Z - 1} \mathbf{1}_{\{X_t = x, X_{t+1} = y\}}\right] \\ &= \mathbb{E}_a\left[\sum_{t=0}^{\tau_Z - 1} \mathbf{1}_{\{X_t = x\}} P(x,y)\right] = G_Z(a,x)P(x,y). \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}_a[S_{xy} - S_{yx}] &= G_Z(a,x)P(x,y) - G_Z(a,y)P(y,x) \\ &= V(x)c(x)P(x,y) - V(y)c(y)P(y,x) \\ &= (V(x) - V(y))c(x,y) = I(x,y), \end{aligned}$$

as required.

Exercise 71 Show that $(a,b) \to d(a,b) = \mathcal{R}(a \leftrightarrow b)$ defines a distance on E. For the triangle inequality, one shall use that the unit current from a to c can be seen as the sum of unit currents from a to b and from b to c.
Assume K is fixed in the following, we will not refer to it in our notation.
The fact that $d(a,b) \geq 0$ is obvious.
From the preceding exercise, using its notation, we find that

$$\mathcal{R}(a \leftrightarrow b) = V^{a,y}(a) = V^{y,a}(y) = \mathcal{R}(b \leftrightarrow a),$$

which ensures symmetry.
Moreover if $\mathcal{R}(a \leftrightarrow b) = 0$ then

$$\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = 0$$

hence $a = b$.
Finally fix $a,b,c \in E$. Let $I^{a,b}$ (resp. $I^{b,c}$, $I^{a,c}$) the unit current from a to b (resp. from b to c and from a to c). These three currents are respectively associated with $V^{a,b}, V^{b,c}, V^{a,c}$.
Observe that, as $I^{a,c}$, $I^{a,b} + I^{b,c}$ is antisymmetric, satisfies the cycle law because both $I^{a,b}$ and $I^{b,c}$ do, and finally

$$\begin{aligned} \mathrm{div}_x(I^{a,b} + I^{b,c}) &= \mathrm{div}_x(I^{a,b}) + \mathrm{div}_x(I_{b,c}) \\ &= \mathbb{1}_{\{x=a\}} - \mathbb{1}_{\{x=b\}} + \mathbb{1}_{\{x=b\}} - \mathbb{1}_{\{x=c\}} \\ &= \mathbb{1}_{\{x=a\}} - \mathbb{1}_{\{x=c\}}. \end{aligned}$$

It follows that indeed $I^{a,b} + I^{b,c} = I^{a,c}$, and the potential associated with $I^{a,c}$ is $W := V^{a,b} + V^{b,c}$. It follows that

$$\mathcal{R}(a \leftrightarrow c) = W(a) - W(c) = V^{a,b}(a) - V^{a,b}(c) + V^{b,c}(a) - V^{b,c}(c).$$

By maximum principle $V^{a,b}$ reaches its maximum at a and its minimum at b so $V^{a,b}(c) \geq 0$ and similarly $V^{b,c}(a) \leq V^{b,c}(b) = \mathcal{R}(b \leftrightarrow c)$, we conclude that

$$\mathcal{R}(a \leftrightarrow c) \leq \mathcal{R}(a \leftrightarrow b) + \mathcal{R}(b \leftrightarrow c).$$

Exercise 72 Let $\mathcal{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a finite oriented graph. We assume each edge to be present in both its orientations.
Extremities of an oriented edge $\vec{e}$ are denoted e^-, e^+.
We let $\mathcal{E}_{1/2}$ a set containing exactly one element out of each $\{e, -e\}$.
Denote by $\ell^2(\mathcal{V})$ the space of functions from $\mathcal{V} \to \mathbb{R}$ equipped with the scalar product

$$\langle f_1, f_2 \rangle_{\ell^2(\mathcal{V})} = \sum_{x \in \mathcal{V}} f_1(x) f_2(x).$$

Denote by $\ell^2_-(\vec{\mathcal{E}})$ the set antisymmetric functions on oriented edges, equipped with the scalar product

$$\langle \theta_1, \theta_2 \rangle_{\ell^2_-(\vec{\mathcal{E}})} = \sum_{e \in \mathcal{E}_{1/2}} \theta_1(e) \theta_2(e).$$

The following applications map one of these two spaces to the other :

$$d : \begin{cases} \ell^2(E) \to \ell^2_-(\vec{\mathcal{E}}) \\ f \to df : df(e) = f(e^-) - f(e^+), \end{cases} \qquad d^* : \begin{cases} \ell^2_-(\vec{\mathcal{E}}) \to \ell^2(E) \\ \theta \to d^*\theta : d^*\theta(x) = \sum_{e \in \vec{\mathcal{E}} : e^- = x} \theta(e). \end{cases}$$

1. Check that
$$\langle \theta_1, \theta_2 \rangle_{\ell^2_-(\vec{\mathcal{E}})} = \frac{1}{2} \sum_{e \in \vec{\mathcal{E}}} \theta_1(e) \theta_2(e)$$

2. Establish that d, d^* are adjoint operators, i.e. for any $f \in \ell^2(\mathcal{V}), g \in \ell^2_-(\vec{\mathcal{E}})$,
$$\langle df, g \rangle_{\ell^2_-(\vec{\mathcal{E}})} = \langle f, d^* g \rangle_{\ell^2(\mathcal{V})}.$$

3. Check that if v is a potential fixed on a, Z and if i is the corresponding current from a to Z then Ohm's law can be rewritten $dv = ri$, and the node law becomes $d^*i(x) = 0 \; \forall x \notin (\{a\} \cup Z)$. How can the cycle law be rewritten in this setting ?

1. The equality follows from the definition of $\mathcal{E}_{1/2}$ and the fact that antisymmetry of θ_1, θ_2 ensures
$$\theta_1(-e)\theta_2(-e) = \theta(e)\theta(e).$$

2. Letting $\varepsilon = -e$ and using antisymmetry of g,

$$\begin{aligned}
\langle df, g\rangle_{\ell^2_-(\vec{\mathcal{E}})} &= \frac{1}{2}\sum_{e\in\vec{\mathcal{E}}} f(e^-) - f(e^+)g(e) \\
&= \frac{1}{2}\sum_{e\in\vec{\mathcal{E}}} f(e^-)g(e) - \frac{1}{2}\sum_{e\in\vec{\mathcal{E}}} f(e^+)g(e) \\
&= \frac{1}{2}\sum_{e\in\vec{\mathcal{E}}} f(e^-)g(e) - \frac{1}{2}\sum_{\varepsilon\in\vec{\mathcal{E}}} f(\varepsilon^-)g(-\varepsilon) \\
&= \sum_{e\in\vec{\mathcal{E}}} f(e^-)g(e).
\end{aligned}$$

Moreover

$$\begin{aligned}
\langle f, d^*g\rangle_{\ell^2(\mathcal{V})} &= \sum_{x\in\mathcal{V}} f(x) \sum_{e\in\vec{\mathcal{E}}:e^-=x} g(e) \\
&= \sum_{x\in\mathcal{V}} \sum_{e\in\vec{\mathcal{E}}:e^-=x} f(e^-)g(e) \\
&= \sum_{e\in\vec{\mathcal{E}}} f(e^-)g(e)
\end{aligned}$$

which completes the proof.

3. For $e \in \vec{\mathcal{E}}$, if we set $x = e^-, y = e^+$, we have

$$dv(e) = v(x) - v(y); \quad i(e) = i(x,y); \quad c(e) = c(x,y),$$

so Ohm's law indeed reads $c \times dv = i$, or $dv = ri$.
Moreover for $x \in \mathcal{V}$

$$d^*i(x) = \sum_{y\in\mathcal{V}:(x,y)\in\vec{\mathcal{E}}} i(x,y) = \mathrm{div}_x(i),$$

so that node law now reads $d^*i(x) = 0,\ \forall x \notin \{a\} \cup Z$.
Finally cycle law ensures that if $e_1, ..., e_n$ is a cycle, then

$$\sum_{k=1}^{n} dv(e_k) = \sum_{k=1}^{n} i(e_k)r(e_k) = 0.$$

Exercise 73 We make similar assumption and use the same notation as in the previous exercise. Although here, we assume that the initial edges of our graph are unoriented and equipped with conductances $\{c(e), e \in \mathcal{E}\}$. We then simply extend the definition of conductance to oriented edges : $e \in \vec{\mathcal{E}}$ has the same conductance as the unoriented edge. We introduce a new scalar product on $\ell^2_-(\vec{\mathcal{E}})$:

$$\langle \theta, \theta' \rangle_r = \frac{1}{2}\sum_{e\in\vec{\mathcal{E}}} r(e)\theta(e)\theta'(e).$$

For $e \in \overrightarrow{\mathcal{E}}$ we let $\chi^e = \mathbb{1}_{\{e\}} - \mathbb{1}_{\{-e\}}$. Finally we introduce two subspaces of $\ell^2_-(\overrightarrow{\mathcal{E}})$:

$$\star = \text{Vect}\left\{ \sum_{e \in \overrightarrow{\mathcal{E}} : e^- = x} c(e)\chi^e, x \in \mathcal{V} \right\}, \qquad \Diamond = \text{Vect}\left\{ \sum_{k=1}^{n} \chi^{e_k}, e_1, ..., e_n \text{ cycle } \right\}$$

Show that $\star = \Diamond^\perp$, and recover Thomson's principle.
It is easily checked that

$$\langle \chi^e, \chi^{e'} \rangle = \begin{cases} 1 & \text{if } e = e' \\ -1 & \text{si } e = -e' \\ 0 & \text{otherwise .} \end{cases} .$$

Fix $x \in \mathcal{V}$ and $e_1, ..., e_n$ cycle, and set $\theta_1 = \sum_{e \in \overrightarrow{\mathcal{E}} : e^- = x} c(e)\chi^e, \theta_2 = \sum_{k=1}^n \chi^{e_k}$.
If e_k is such that $e_k^- \neq x, e_k^+ \neq x$, then clearly

$$\langle \theta_1, \chi^{e_k} \rangle_r = 0.$$

For any $k \in \{2, ..., n\}$, such that $x = e_k^-$, then $x = e_{k-1}^+$ (and so $x = (-e_{k-1})^-$). But then

$$\langle \theta_1, \chi^{e_{k-1}} + \chi^{e_k} \rangle >_r = -c(e_{k-1})r(e_{k-1}) + c(e_k)r(e_k) = 1 - 1 = 0.$$

Finally if $x = e_1^-$, $x = e_n^+$ since we are dealing with a cycle, and the above reasoning again applies. This proves that $\star, \Diamond$ are orthogonal spaces.
Let us now show their intersection is reduced to $\{0\}$. A θ in the intersection must be orthogonal to both $\star$ and $\Diamond$. Since $\theta \in \star^\perp$ we have for any $x \in \mathcal{V}$,

$$\langle \sum_{e \in \overrightarrow{\mathcal{E}} : e^- = x} c(e)\chi^e, \theta \rangle_r = \sum_{e \in \overrightarrow{\mathcal{E}} : e^- = x} c(e)\theta(e)r(e) = \text{div}_x(\theta) = 0,$$

so θ satisfies the node law everywhere.
On the other hand since $\theta \in \Diamond^\perp$, we have for any cycle $e_1, ..., e_n$,

$$\langle \sum_{k=1}^n \chi^{e_k}, \theta \rangle_r = \sum_{k=1}^n \theta(e_k)r(e_k) = 0,$$

so θ satisfies the cycle law.
Since θ checks both node and cycle laws and the space is finite, it must be that there exists v such that $\theta = dv$. The node law for θ means v is harmonic on the whole of $\mathcal{V}$ entier, it must be constant which implies $\theta = 0$.
We have therefore shown that $\star = \Diamond^\perp$, in other words $\ell^2_-(\overrightarrow{\mathcal{E}}) = \star \oplus \Diamond$.
Let the unit current from a to Z be denoted by I. It satisfies cycle law everywhere so $I \in \Diamond^\perp = \star$. It also satisfies node law at any $x \notin \{a\} \cup Z$, so that $d^*I(x) = 0 \forall x \notin \{a\} \cup Z$.
If $\theta \in \ell^2_-(\overrightarrow{\mathcal{E}})$ satisfies $d^*\theta = d^*I$ (i.e. if it is a unit flow from a to Z), then $d^*(\theta - I) = 0$, i.e. $\theta - I \in \star^\perp$, therefore $\theta = I + \theta - I$ is the orthogonal decomposition of θ on $\star \oplus \Diamond$.
Bu then

$$||\theta||_r^2 = ||I||_r^2 + ||\theta - I||_r^2,$$

and we conclude that I minimizes the energy of unit flows from a to Z.
It remains to see that $I = I^{a,Z} = dV^{a,Z}$ where $V^{a,Z}$ is a potential, that without loss of generality can be assumed to vanish on Z, so it must take value $\mathcal{R}(a \leftrightarrow Z)$ at a. Then

$$\begin{aligned} ||I||_r^2 = \langle I, dV\rangle_r &= \frac{1}{2}\sum_{e\in\vec{\mathcal{E}}} I(e)(V(e^-)-V(e^+)) \\ &= \frac{1}{2}\sum_{x,y\in\mathcal{V}:(x,y)\in\vec{\mathcal{E}}} I(x,y)(V(x)-V(y)) \\ &= \sum_{x\in\mathcal{V}} V(x)\sum_{y\in\mathcal{V}} I(x,y) \\ &= V(a)\mathrm{div}_a(I) = \mathcal{R}(a\leftrightarrow Z). \end{aligned}$$

We deduce the following rewriting of Thomson's principle

$$\mathcal{R}(a\leftrightarrow Z) = \inf\{||\theta||_r^2 : \theta\in\ell_-^2(\vec{\mathcal{E}}) \quad d^*\theta(a) = d^*I^{a,Z}\}$$

and this infimum is reached at $\theta = I^{a,Z}$.

Exercise 74 Show that if we modify a graph by identifying two of its nodes, then the effective resistance between a and Z in the new graph is bouded by that in the old graph. What happens when the transformation consists in removing an edge between two points?
Identifying two points is equivalent to put an infinite conductance between the two vertices leaving other conductances unchanged. We conclude thanks to Rayleigh's principle.
On the other hand, removing an edge is equivalent to setting its conductance to zero, leaving other conductances unchanged. Rayleigh's principle again applies.

Exercise 75 Recover ruin's probability $\mathbb{P}_x(\tau_0 < \tau_N)$ (for SRW going up with probability p) using the analogy with electric networks.
Fix $p\in(0,1)$. Let c^i conductance of edge $(i,i+1)$ for $i=0,...,N-1$, so for any $i\in\{0,...,N-1\}$,

$$P(i,i+1) = \frac{c^i}{c^i+c^{i+1}} = \frac{1}{1+c},$$

and if we set $c = \frac{1}{p}-1 = \frac{q}{p}$ we recover SRW going up with probability p, and down with probability $q = 1-p$.
Now fix $x\in\{1,...,N-1\}$. If $p=1/2$, $c=1$, and equivalent resistance between 0 and x is x, that between x and N is $N-x$, so equivalent conductances are $1/x, 1/(N-x)$, hence

$$\mathbb{P}_x(\tau_N<\tau_0) = \frac{\frac{1}{N-x}}{\frac{1}{x}+\frac{1}{N-x}} = \frac{x}{N}.$$

If $p\neq 1/2$ equivalent resistance between 0 and x is $\frac{1-c^k}{1-c} = \frac{1-\left(\frac{q}{p}\right)^k}{1-\frac{q}{p}}$, that between x and N

equals $c^k\frac{1-c^{N-k}}{1-c} = \frac{\left(\frac{q}{p}\right)^k-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}}$. We deduce

$$\mathbb{P}(\tau_N < \tau_0) = \frac{\frac{1-\frac{q}{p}}{\left(\frac{q}{p}\right)^k-\left(\frac{q}{p}\right)^N}}{\frac{1-\frac{q}{p}}{\left(\frac{q}{p}\right)^k-\left(\frac{q}{p}\right)^N}+\frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^k}} = \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N}.$$

Exercise 76 Consider a square split into 3×3 squares ; and the RW whose every step is to a uniformly random neighbouring or diagonally neighbouring square. We write x for the bottom left square, y the top right square. Compute $\mathbb{P}_x(\tau_y < \tau_x^+)$, then $\mathbb{E}_x[\tau_y]$.
Let V be the unique function harmonic outside x, y, and such that $V(x) = 1, V(y) = 0$. A priori determining V amounts to solve a linear system of 7 equations with 7 variables, but this is a bit cumbersome. Instead note that a symmetry argument allows to see that squares on each antidiagonal must be at same potential. Again by symmetry, squares on the central antidiagonal must all be at potential $1/2$. It only remains two equations and two unknowns, and it is then easily checked that the two direct neighbours of x are at potential $3/5$, and squares directly neighbouring y are at potential $2/5$.
It is then easy to find the force of the correponding current, it is $2*2/5+1/2 = 13/10$, and in the end $\mathcal{R}(x\leftrightarrow y) = \frac{10}{13}$.
Now since $c(x) = 3$

$$\mathbb{P}(\tau_y < \tau_x^+) = \frac{1}{3\mathcal{R}(x\leftrightarrow y)} = \frac{13}{30}.$$

For $\mathbb{E}_x[\tau_y]$ we use formula of exercise 5, which reads

$$\mathbb{E}_x[\tau_y] = \sum_{v\in E} c(v)V(v),$$

with V the unique function harmonic outside x, y, and such that
$V(x) = \mathcal{R}(x\leftrightarrow y) = 10/13, V(y) = 0$. This potential is only $10/13$ times that which we just computed, it follows that

$$\mathbb{E}_x[\tau_y] = \mathcal{R}(x\leftrightarrow y)*\left(3+(5+5)*\frac{3}{5}+(3+8+3)*\frac{1}{2}+(5+5)*\frac{2}{5}\right) = \frac{200}{13}.$$

Remark : One could also have used triangle-star equivalence in order to compute the effective resistance between x and y.

Exercise 77 Let $\mathcal{G} = (\mathcal{V},\mathcal{E})$ be a countably infinite graph equipped with conductances $(c(e), e\in\mathcal{E})$, and satisfying

$$\forall x\in\mathcal{V},\quad \sum_{(x,y)\in\mathcal{E}} c(x,y) < \infty.$$

Let $(\mathcal{G}_n), (\mathcal{H}_n)$ two nondecreasing sequence of subgraphs of $\mathcal{G}$ s.t. for any n, $a\in\mathcal{G}_n\cap\mathcal{H}_n$, and s.t. $\bigcup\mathcal{G}_n = \bigcup\mathcal{H}_n = \mathcal{G}$. We let $Z_n = \mathcal{G}\setminus\mathcal{G}_n$, (resp. $Y_n = \mathcal{H}\setminus\mathcal{H}_n$), and denote by $\mathcal{G}_n^*$ (resp. $\mathcal{H}_n^*$) the graph deduced from $\mathcal{G}$ by identifying all vertices in Z_n as a single node z_n (resp. all vertices in Y_n as a single node y_n). Let $\mathcal{R}(a\leftrightarrow Z_n;\mathcal{G}_n^*)$ denote the effective resistance from a to z_n in $\mathcal{G}_n^*$, (resp. $\mathcal{R}(a\leftrightarrow Y_n;\mathcal{H}_n^*)$ the effective resistance from a to y_n in $\mathcal{H}_n^*$).

1. Show that

$$\lim_{n\to\infty} \mathcal{R}(a \leftrightarrow Z_n; \mathcal{G}_n^*), \lim_{n\to\infty} \mathcal{R}(a \leftrightarrow Y_n; \mathcal{H}_n^*)$$

both exist in $\overline{\mathbb{R}_+}$, and that they must be equal.

2. Show that the chain on $\mathcal{G}$ defined by this conductance model is recurrent iff this common limit is finite.

1. By Rayleigh $\mathcal{R}(a \leftrightarrow Z_n; \mathcal{G}_n^*)_{n\geq 0}$ is nondecreasing, so its limit ℓ_1 exists in $\overline{\mathbb{R}_+}$. Similarly we let $\ell_2 := \lim_{n\to\infty} \mathcal{R}(a \leftrightarrow Y_n; \mathcal{H}_n^*)$.
If $\ell_1 = +\infty$, fix $A > 0$, there must exist n_1 s.t. $\mathcal{R}(a \leftrightarrow Z_n; \mathcal{G}_n^*) \geq A$ for any $n \geq n_1$. Since $(\mathcal{H}_n)$ is nondecreasing and s.t. $\bigcup \mathcal{H}_n = \mathcal{G}$, there must exist n_2 s.t. $\mathcal{H}_n \supset \mathcal{G}_{n_1}$ for any $n \geq n_2$. But then again by Rayleigh, for any $n \geq n_2$ we have

$$\mathcal{R}(a \leftrightarrow Y_n; \mathcal{H}_n^*) \geq \mathcal{R}(a \leftrightarrow Z_{n_1}; \mathcal{G}_{n_1}) \geq A,$$

and since this hold for any A we conclude that $\ell_2 = \ell_1 = +\infty$.
Otherwise both our limits are finite. Fix $\varepsilon > 0$, there exists n_1 s.t. $\mathcal{R}(a \leftrightarrow Z_n; \mathcal{G}_n^*) \geq \ell_1 - \varepsilon$ for any $n \geq n_1$. By the same proof as above, one can find n_2 s.t. for any $n \geq n_2$,

$$\mathcal{R}(a \leftrightarrow Y_n; \mathcal{H}_n^*) \geq \mathcal{R}(a \leftrightarrow Z_{n_1}; \mathcal{G}_{n_1}) \geq \ell - \varepsilon.$$

Since this hold for any ε we conclude that $\ell_2 \geq \ell_1$. By a symmetric argument $\ell_1 \geq \ell_2$, and we conclude that $\ell_1 = \ell_2$. In what follows we let $\ell = \mathcal{R}(a \leftrightarrow \infty)$ denote the common limit.

2. From the previous question we can always choose without affecting the limit

$$\mathcal{G}_n = \{y \in \mathcal{G} : d_{\mathcal{G}}(a, y) \leq n\} = B_{\mathcal{G}}(a, n),$$

the ball of radius n centered at a, (with $d_{\mathcal{G}}$ the graph distance).
Let us remind that

$$\mathbb{P}_a(\tau_{Z_n} < \tau_a^+) = \frac{1}{c(a)\mathcal{R}(a \leftrightarrow Z_n)},$$

so that

$$\mathbb{P}_a(\tau_{Z_n} < \tau_a^+) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } \ell = +\infty \\ \frac{1}{c(a)\ell} & \text{otherwise.} \end{cases},$$

using the assumption $c(a) < \infty$.
Under $\mathbb{P}_a$, $\tau_{Z_n} \geq n$, so that $\tau_{Z_n} \to \infty$ p.s. We deduce that $\lim_{n\to\infty} \mathbb{P}_a(\tau_{Z_n} < \tau_a^+) = \mathbb{P}_a(\tau_a^+ = +\infty)$, which allows to conclude that

$$\mathbb{P}_a(\tau_a^+ < +\infty) = \begin{cases} 1 & \text{if } \mathcal{R}(a \leftrightarrow \infty) = +\infty \\ 1 - \frac{1}{c(a)\ell} < 1 & \text{otherwise.} \end{cases}$$

Exercise 78 Use Nash-Williams' inequality to show that symmetric SRW on $\mathbb{Z}^d, d \leq 2$ is recurrent.
We set $c(e) = 1$ for any $e \in \mathbb{Z}^d$.

We use the disjoint cutting (between 0 and infinity) sets

$$\Pi_k := \{(x,y) \in \mathcal{E} : ||x||_\infty = k, ||y||_\infty = k+1\},$$

and we notice that $\sum_{e \in \Pi_k} c(e) = |\Pi_k| = 2d(2k+1)^{d-1}, k \geq 1$. By Nash-Williams,

$$\mathcal{R}(0 \leftrightarrow \infty) \geq \sum_{k \geq 1} \frac{1}{2d(2k+1)^{d-1}}.$$

It follows that $\mathcal{R}(a \leftrightarrow \infty) = +\infty$ provided $d \leq 2$, which by the preceding exercise, ensures that symmetric SRW on $\mathbb{Z}^d, d = 1,2$ is recurrent.

Exercise 79 Consider an infinite rooted d-regular tree (where the root has degree d and all other vertices have degree $d+1$), and the λ-biased walk on that tree which is such that, when not at the root, the walk moves towards the root with probability $\frac{\lambda}{\lambda+d}$, and otherwise, it moves from its location to one of its d descendants choosen uniformly at random.

1. Explain what is the corresponding electric network, and compute the effective resistance between the root and depth n of the tree as a function of n, d, λ.
2. What is the limit of these effective resistances when $n \to \infty$ as a function of d, λ? Find a recurrence criterion for our walk in terms of d, λ.
3. What can be said for the λ-biased RW on a rooted tree where any vertex at depth k has exactly $d_k \in \mathbb{N}^*$ descendants

1. In order to get a λ-biased walk, we need, at each given vertex v other than the root, that the conductance of the edge from v leading to the root is λ times the conductance of any other edge from v. We can therefore set $c(e) = \lambda^{-\ell}$ for any edge $e \in \mathcal{E}$ linking a vertex of depth ℓ to one of depth $\ell+1$ in $\mathcal{T}$.
   We aim at computing $\mathcal{R}(0 \leftrightarrow n)$, the effective resistance between the root and vertices at depth at least n, $B(0, n-1)^c$.
   If we impose potential $V_0 > 0$ at the root and potential 0 at every vertex of depth at least n, we find by a symmetry argument that all nodes at same depth must have same potential.
   For any $k \in \{0, ..., n\}$, we may therefore identify as a single node v_k every vertex at depth k. The resulting electric network possesses d^{k+1} parallel edges between v_k and v_{k+1}, of same conductance λ^{-k}, which is equivalent to a single edge with conductance $d^{k+1}\lambda^{-k}$.
   It remains to sum the series resistance between v_0 (the root) and v_n (vertices of depth n) :
   $$\mathcal{R}(0 \leftrightarrow n) = \sum_{k=0}^{n-1} \frac{1}{d^{k+1}\lambda^{-k}} = \begin{cases} \frac{1}{d}\frac{1-\left(\frac{\lambda}{d}\right)^n}{1-\frac{\lambda}{d}} \text{ if } \lambda \neq d \\ \frac{n}{d} \text{ if } \lambda = d. \end{cases}$$
2. As $n \to \infty$, we deduce that $\mathcal{R}(0 \leftrightarrow n) \to \infty$ iff $\lambda \geq d$. From exercise 12, it follows that λ-biased on the rooted infinite d-regular tree enraciné, is recurrent iff $\lambda \geq d$.

3. We only have used the fact that the number of descendants of a given vertex only depends on its depth.
   More precisely, if vertices at generation k have d_k descendants, the number of edges between v_k and v_{k+1} is equal to $d_0 d_1 \dots d_k$, and one finds that
   $$\mathcal{R}(0 \leftrightarrow n) = \sum_{k=0}^{n-1} \frac{\lambda^k}{\prod_{i=0}^{k} d_i}.$$
   The walk on such tree is recurrent iff the above sum diverges.

Exercise 80 A professor owns a total of n umbrellas, some at home and some left at his office. The professor walks from home to office in the morning and from office to home in the evening. He takes one umbrella with him if it rains and if he can (i.e. if there is at least one umbrella at his starting location). In addition we assume that independently at each travel, it rains with probability p.

1. Find a reversible Markov chain which models the problem and compute its invariant probability. Asymptotically, what is the proportion of walks during which the professor gets wet ?
2. Find the expectation of the number of walks before n umbrellas are found in the same place.
3. Find the expectation of the number of walks before (not including) the first one during which the professor gets wet.

Set $q = 1 - p$ in what follows.

1. Let us say 1 denotes the house and 2 the office. Our Markov chain will take values in $\{(i,k), i \in \{1,2\}, k \in \{0,...,n\}$, state (i,k) meaning that the professor is in location i with k umbrellas. For example, state $(1,4)$ would mean that the professor is at home with 4 umbrellas at his disposal ; the next state visited by the chain will be $(2, n-3)$ with probability p (if it rains the next morning), else $(2, n-4)$ (if it does not). It is then easily seen that the chain is irreducible, positive recurrent, and that it corresponds to the following conductance model (so it is reversible)

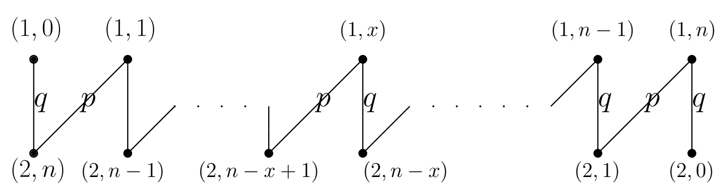

FIGURE 1. Le modèle de conductances

   The initial location is irrelevant for the questions we want to address, which is why w.l.o.g we may assume the chain is started at $(1,x)$, for some $x \in \{0,...,n\}$ (if you would rather have him start from his office, simply permute the labels for locations). The function $c : \mathcal{V} \to \mathbb{R}_+^*$ (recall $c(x) = \sum_{y \sim x} c(x,y)$) is everywhere 1, except at the boundary $\{(1,0),(2,0)\}$ where it takes value q. Hence $c_{\mathcal{G}} = 2n + 2q$, and
   $$\pi(i,0) = \frac{q}{2q+2n}, i = 1,2, \qquad \pi(i,k) = \frac{1}{2q+2n}, i = 1,2, k = 1,...,n.$$

The asymptotic proportion of time spent at $(1,0)$ or $(2,0)$ (i.e. at a place without any umbrella) is therefore

$$\pi(1,0)+\pi(2,0)=\frac{2q}{2q+2n}.$$

Walks under the rain are necessarily from one of these two states, and the proportion of walks from these two states that are indeed made under the rain is p we conclude that the asymptotic proportion of walks during which the professor gets wet is $\frac{2pq}{2q+2n}$.

2. We are going to make use of formulaes obtained in exercise 5. First let us compute the effective resistance between $a=(1,x)$ and $Z=(1,n)\cup(2,n)\cup(1,0)\cup(2,0)$ (so Z exactly corresponds to states of the chain for which umbrellas are all located at the same location). Add series resistance :

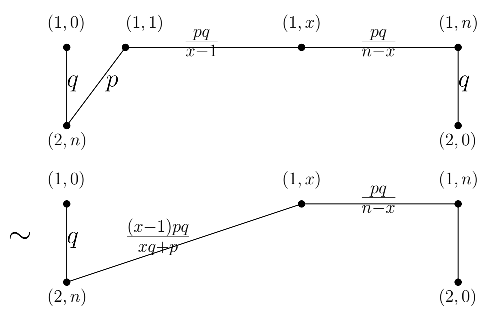

FIGURE 2. Equivalent networks : given values are resistances

Vertices of Z have the same null potential, they can be identified. This leads to summing parallel conductances between a and Z and obtain

$$\mathcal{R}(a\leftrightarrow Z)=\frac{1}{\frac{pq}{x-1+q}+\frac{pq}{n-x}}=\frac{(x-p)(n-x)}{pq(n-p)}.$$

Note that when $n\to\infty$, this is maximal for $x\sim n/2$ and then $\mathcal{R}(a\leftrightarrow Z)\sim\frac{n}{4pq}$. Impose potential $\mathcal{R}=\mathcal{R}(a\leftrightarrow Z)$ at $a=(1,x)$, and null potential at Z, it is then easy to see that for some constant W, we must have

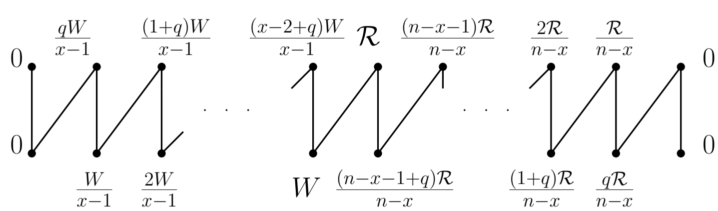

FIGURE 3. Potential at each point when it is fixed to $\mathcal{R}$ at a and to 0 at Z.

Of course $W=0$ if $x=1$, but if $x\geq 2$, use the potential is harmonic at $(2,n-x+1)$ to find that

$$W=q\mathcal{R}+p\frac{x-2+q}{x-1}W,$$

so

$$W = \frac{q}{q + \frac{p^2}{x-1}} \mathcal{R} = \mathcal{R} - \frac{p^2 \mathcal{R}}{q(x-1) + p^2}.$$

It only remains to compute

$$\begin{aligned}
\mathbb{E}_a[\tau_Z] &= \sum_{y \in E} c(y) V(y) = \sum_{y \in E} V(y) \\
&= \sum_{k=1}^{x-1} \frac{kW}{x-1} + \sum_{k=1}^{x-1} \frac{(k+q)W}{x-1} + \sum_{k=1}^{n-x} \frac{k\mathcal{R}}{n-x} + \sum_{k=1}^{n-x} \frac{(k+q)\mathcal{R}}{n-x} \\
&= (x+q)W + (n-x+q)\mathcal{R} = \left(n + 2q - \frac{p^2}{q(x-1)+p^2} \right) \mathcal{R}.
\end{aligned}$$

As $n \to \infty$, notice that the above is maximal for $x \sim n/2$, and then $\mathbb{E}_a[\tau_Z] \sim \frac{n^2}{4pq}$.

3. Let $f(x) := \mathbb{E}_{(1,x)}[\tau_Z]$ which we compute in the previous question, and $\mathcal{T}$ the number of walks before (not including) the first one during which the professor gets wet. Evidently $\mathcal{T} \geq \tau_Z$. But in fact, notice that $\mathcal{T}$ has same distribution under $\mathbb{P}_{(1,n)}$ and under $\mathbb{P}_{(2,n)}$, so that $\mathcal{T} - \tau_Z$ under $\mathbb{P}_a$ has same distribution as $\mathcal{T}$ under $\mathbb{P}_{(2,n)}$. Using Markov property

$$\begin{aligned}
\mathbb{E}_{(2,n)}(\mathcal{T}) &= 1 + pf(1) + q\mathbb{E}_{(1,0)}(\mathcal{T}) \\
&= 1 + pf(1) + qp + q^2(1 + \mathbb{E}_{(2,n)}(\mathcal{T})),
\end{aligned}$$

thus

$$\mathbb{E}_{(2,n)}(\mathcal{T}) = \frac{1}{1-q^2} \left(1 + pf(1) + q^2\right)$$

In the end

$$\mathbb{E}_a[\mathcal{T}] = f(a) + \frac{1}{1-q^2} \left(1 + pf(1) + q^2\right).$$

In particular, when $n \to \infty$ and $x \sim n/2$, since $f(1) \sim \frac{n(1-p)}{pq} << f(x)$, we find that $\mathbb{E}_a[\mathcal{T}] \sim \frac{n^2}{4pq}$.

Exercise 81 (Pólya's urn)
Consider (Pólya's) urn with d balls of distinct colors. At time $t - 1/2, t \geq 1$, one draws uniformly and independently of previous steps a ball from the urn, and then replace that ball along with another one of the same color.
Let $X_t(j)$ the number of balls of color j which have been drawn up to time t.

1. Show that for any $(n_1, ..., n_d)$ such that $n_1 + ... + n_d = n$, we have

$$\mathbb{P}(X_n = (n_1, ..., n_d)) = \frac{\prod_{i=1}^d n_i!}{d(d+1)...(d+n-1)} \binom{n}{n_1, ..., n_j} = \frac{(d-1)!n!}{(d+n-1)!}.$$

How does the above simplify when $d = 2$?

2. Deduce that $\frac{X_t}{t}$ converges in distribution towards a Dirichlet variable with parameters $(1, ..., 1)$ (that is, of uniform density on the simplex $\{(x_1, ..., x_d) \in \mathbb{R}_+^d : x_1 + ...x_d = 1\}$).

1. For any $t \geq 0$, at time t there are exactly $d+t$ balls in the urn. Conditionally given X_t, the probability to draw a ball of color j at time $t+1/2$ is $\frac{X_t(j)+1}{d+t}$, for any $j \in \{1, ..., d\}$. Observe there are $\binom{n}{n_1,...,n_j}$ ways to draw n_1 balls of color 1, ..., n_d balls of color d up to time n, and the desired formula follows.
   When $d = 2$, we thus have $\mathbb{P}(X_t = (k, n-k)) = \frac{1}{n+1}$ for any $k \in \{0, ..., n\}$, so that the number of balls of color 1 drawn up to time n has a uniform distribution on $\{0, ..., n\}$.
2. From the above X_n follows a uniform distribution on $\{(n_1, ..., n_d) : n_1 + ... + n_d = n\}$. It is then straightforward to establish that X_n/n converges in distribution towards a variable uniform on $\{(x_1, ..., x_d) \in \mathbb{R}_+^d : x_1 + ...x_d = 1\}$.

Exercise 82 Assume $\mathcal{G}_n$ has vertices $\mathcal{V}_n = \{x \in \mathbb{Z}^d : \forall i \in \{1, ..., d\}\ 1 \leq x_i \leq n\}$ and edges $\mathcal{E}_n$ are between vertices of $\mathcal{V}_n$ which are nearest-neighbours in $\mathbb{Z}^d$ (so we are dealing with a $n \times ... \times n$ slab of $\mathbb{Z}_+^d$).
Set all conductances of edges of $\mathcal{E}_n$ to 1, and set $a = (1, ..., 1), z = (n, ..., n)$.

1. Assume $\Pi_k = \{(v, w) \in \mathcal{E}_n : ||v||_\infty = k, ||w||_\infty = k+1\}, k = 1, ..., n-1$. Show that for any $k = 1, ..., n-1$, Π_k is a cutting set, and that
$$\sum_{e \in \Pi_k} c(e) = |\Pi_k| = dk^{d-1}.$$
   Deduce that
$$\mathcal{R}(a \leftrightarrow z) \geq \begin{cases} n-1 & \text{if } d = 1 \\ \frac{1}{2}\log(n-1) & \text{if } d = 2 \\ C_d := \frac{1}{d}\sum_{k \geq 1} k^{1-d} & \text{if } d \geq 3 \end{cases}.$$
2. Consider a Polya's urn with initially one ball of each color $1, ..., d$. Introduce as in the previous exercise the process $(X_t, t \geq 0)$, and set $\widetilde{X}_t = X_t + (1, ..., 1)$ so that $\widetilde{X}_t(i)$ is the number of balls of color i in the urn at time t. Introduce I a unit flow from a to z. On edges $\{(x, y) \in \mathcal{E}_n : \sum_{i=1}^d y_i \leq n+d\}$ set
$$I(x, y) = \mathbb{P}(\exists t \in \{0, ..., n-1\} : \widetilde{X}_t = x, \widetilde{X}_{t+1} = y),$$
   and on edges $\{(x, y) \in \mathcal{E}_n : \sum_{i=1}^d y_i > n+d\}$ set
$$I(x, y) = I((n+1, ..., n+1) - y, (n+1, ..., n+1) - x).$$
   For $k \in \{1, ..., n\}$, show that for any $y \in B_n$ such that $\sum_{i=1}^d y_i = k+d$, the flow entering y is
$$\sum_{\{x : \sum_{i=1}^d x_i = k+d-1, x \sim y\}} I(x, y) = \frac{1}{\binom{k+d-1}{d-1}}.$$
   Deduce that the energy $E(I)$ of this flow satisfies
$$E(I) \leq \begin{cases} n & \text{if } d = 1 \\ 2\log(n) & \text{if } d = 2 \\ 2\ (d-1)!\ \sum_{k=1}^n \frac{1}{k^{d-1}} & \text{if } d \geq 3 \end{cases}.$$

3. For what values of d do we have $\mathcal{R}(a \leftrightarrow z) \to \infty$ as $n \to \infty$? For what values of d does this effective resistance converges to a finite limit as $n \to \infty$?
4. Use the above to recover Pólya's theorem : symmetric SRW on $\mathbb{Z}^d$ is recurrent iff $d \leq 2$.

1. Let $k \in \{1, ..., n-1\}$ be fixed. Each vertex v of $\mathcal{G}_n$ s.t. $||v||_\infty = k+1$ which possesses a unique coordinate equal to $k+1$ is linked with exactly one vertex w s.t. $||w||_\infty = k$. There are exactly dk^{d-1} tels noeuds, and thus there are dk^{d-1} elements in Π_k.
By Nash-Williams,
$$\mathcal{R}(a \leftrightarrow z) \geq \sum_{k=1}^{n-1} \frac{1}{\sum_{e \in \Pi_k} c(e)} = \sum_{k=1}^{n-1} \frac{1}{dk^{d-1}},$$
which yields the desired inequality.
2. Let us write $S_k = \{x \in \mathcal{G}_n : \sum_{i=1}^d x_i = k\}$ (this is the intersection of the sphere of radius 1 for ℓ^1-norm with vertices in $\mathcal{G}_n$). For $d \leq k \leq n+d$, observe the incident flow into S_k is 1, and it is uniformly split between the distinct vertices of S_k. Since S_k precisely has $\binom{k+d-1}{d-1}$ vertices, we deduce that for $k \geq 2$ and $e^+ \in S_k$, we have
$$I(e) \leq \frac{k!(d-1)!}{(k+d-1)!} \leq \frac{(d-1)!}{k^{d-1}}$$
From the definition of $E(I)$ (reminding all edges have conductance 1), we find that
$$E(I) \leq 2 \sum_{k=2}^{n} \frac{(d-1)!}{k^{d-1}},$$
yielding the desired inequalities. As $\mathcal{R}(a \leftrightarrow z)$ minimizes the energy of all unit flows from a to Z, we conclude to similar bounds for $\mathcal{R}(a \leftrightarrow z)$.
3. It follows that $\mathcal{R}(a \leftrightarrow z) \to \ell < \infty$ as long as $d \geq 3$.
4. Une légère modification du raisonnement de la première question (en remplaçant $\mathcal{G}_n$ par la boule de rayon n pour la norme infinie dans $\mathbb{Z}^d$) permet d'assurer que $\mathcal{R}(0 \leftrightarrow \infty) = +\infty$ lorsque $d = 1, 2$, ce qui permet d'assurer que la marche simple sur $\mathbb{Z}^d, d = 1, 2$ est récurrente.
Une légère modification de l'argument de la deuxième question (quitte à répartir le courant d'intensité 1 de façon équitable entre les $2d$ cadrans, et à considérer le courant entre 0 et la sphère de rayon n pour la norme 1) permet quant à lui de démontrer que $\mathcal{R}(0 \leftrightarrow \infty)$ reste bornée pour tout $d \geq 3$, de sorte que la marche simple est transiente lorsque $d \geq 3$.

17 Martingales for continuous-time chains

Exercise 83 Assume Q is the generator on a discrete space E, and that the forward equation $P'(t) = P(t)Q, P(0) = I$ has a unique nonnegative solution.
Show that the two following assertions are equivalent for a continuous-time process X taking values in E :

(i) X is a continuous-time Markov chain with generator Q.

(ii) For any $f : E \to \mathbb{R}$ bounded

$$\left(M_t^f := f(X_t) - f(X_0) - \int_0^t Qf(X_s)ds \right)_{t\geq 0}$$

is a continuous-time martingale w.r.t. the natural filtration of X.

Assume (i), and consider a bounded $f : \mathbb{R} \to \mathbb{R}$, so integrability conditions are easily satisfied. Recall $p_{xy}(t) = \mathbb{P}_x(X_t = y) = \mathbb{P}(X_{t+s} = y \mid X_s = x)$, so we have,

$$\begin{aligned} \mathbb{E}[f(X_{t+s}) \mid \mathcal{F}_s] &= \sum_{x\in E, y\in E} \mathbb{E}[f(X_{t+s})\mathbb{1}_{\{X_s=x\}}\mathbb{1}_{\{X_{t+s}=y\}} \mid \mathcal{F}_s] \\ &= \sum_{x\in E, y\in E} \mathbb{1}_{\{X_s=x\}} \sum_{y\in E} p_{xy}(t)f(y) = P(t)f(X_s). \end{aligned}$$

We also remind that $(P(t), t \geq 0)$ is the unique solution to the forward equation $P(0) = I, P'(t) = QP(t)$. Thus

$$\begin{aligned} \mathbb{E}\left[\int_0^{t+s} Qf(X_u)du \mid \mathcal{F}_s\right] &= \int_0^s Qf(X_u)du + \int_s^{t+s} Q\mathbb{E}\left[f(X_u) \mid \mathcal{F}_s\right]du \\ &= \int_0^s Qf(X_u)du + \int_s^{t+s} QP(u-s)f(X_s)du \\ &= \int_0^s Qf(X_u)du + \left(\int_s^{t+s} P'(u-s)du\right) f(X_s) \\ &= \int_0^s Qf(X_u)du + P(t)f(X_s) - f(X_s) \end{aligned}$$

It follows that

$$\mathbb{E}[M_{t+s}^f \mid \mathcal{F}_s] = P(t)f(X_s) - f(X_0) - \int_0^s Qf(X_u)du - P(t)f(X_s) + f(X_s) = M_s^f,$$

as required.

Conversely assume (ii) holds. Fix $0 \leq t_0 \leq ... \leq t_n$, $x_0, ..., x_n \in E$ and set $B = \{X_{t_0} = x_0, ..., X_{t_n} = x_n\}$, and for $y \in E$, $t \geq t_n$, $g_B(y,t) = \mathbb{P}(X_{t_n+t} = y \mid B)$. Setting $f = \mathbb{1}_{\{y\}}$ and using martingale property for $(M_t^f)_{t\geq 0}$, for any $t \geq 0$, yields

$$\mathbb{E}[\mathbb{1}_{\{X_{t_n+t}=y\}} \mid \mathcal{F}_{t_n}] = \mathbb{1}_{\{X_{t_n}=y\}} + \sum_{z\in E}\int_{t_n}^{t_n+t} q_{zy}\mathbb{P}(X_u = z \mid \mathcal{F}_{t_n})du.$$

Since $B = \{X_{t_0} = x_0, ...X_{t_n} = x_n\} \in \mathcal{F}_{t_n}$, we deduce

$$g_B(t+h, y) = \mathbb{P}(\{X_{t_n+t+h} = y\} \mid B) = \mathbb{1}_{\{x_n=y\}} + \sum_{z\in E}\int_{t_n}^{t_n+t+h} q_{zy}g_B(u,z)du.$$

Since g_B is bounded, it follows that it is differentiable w.r.t. its first variable, and that

$$g_B(t_n, y) = \mathbb{1}_{\{x_n=y\}}, \qquad \frac{\partial g_B}{\partial t}(t,y) = \sum_{z\in E} q_{zy}g_B(t,z), t \geq 0$$

It is therefore clear that $g_B(t,y), t \geq t_n, y \in E$ only depends on B through the value x_n. In particular, given $\{X_{t_n} = x_n\}$, for any $t \geq 0$, X_{t_n+t} is independent of B. Since events such as B generate $\mathcal{F}_{t_n}$, it follows by standard arguments that given $\{X_{t_n} = x_n\}$, $(X_{t_n+t})_{t\geq 0}$ is independent of $\mathcal{F}_{t_n}$ and has the same law as $(X_t)_{t\geq 0}$ under $\mathbb{P}_{x_n}$, so we have establish the Markov property for $(X_t, t \geq 0)$.
Again with the same argument $\mathbb{P}_x(X_t = y) = P_{xy}(t)$ satisfies $P_{xy}(0) = \mathbb{1}_{\{x=y\}}$, $\frac{\partial P_{xy}(t)}{\partial t}(t,y) = \sum_{z\in E} q_{xz}P_{zy}(t)$, so that $P(t)$ is the unique solution to the forward equation, and we are done.

Exercise 84 Consider a continuous-time Markov chain X with generator Q, and assume $f : E \to \mathbb{R}$ is such that $Qf = \alpha f$ for some $\alpha \in \mathbb{R}$. Show that, under suitable integrability conditions, $(\exp(-\alpha t)f(X_t))_{t\geq 0}$ is an $(\mathcal{F}_t)$-martingale.
Thanks to the forward equation, if we set $g(t,x) = P(t)f(x)$, we find that

$$\frac{\partial g}{\partial t}(t,x) = P'(t)f(x) = P(t)Qf(x) = \alpha P(t)f(x) = \alpha g(t,x),$$

and since $g(0,x) = P(0)f(x) = f(x)$ it follows easily that $g(t,x) = P(t)f(x) = \exp(\alpha t)f(x)$ for any $x \in E$. Now

$$\begin{aligned} \mathbb{E}\left[\exp(-\alpha(t+s))f(X_{t+s}) \mid \mathcal{F}_s\right] &= \exp(-\alpha(t+s))P(t)f(X_s) \\ &= \exp(-\alpha(t+s))\exp(\alpha t)f(X_s) = \exp(-\alpha s)f(X_s) \end{aligned}$$

and we are done.

18 Potentials for continuous-time chains

Exercise 85 Assume X is a continuous-time irreducible chain on E, $D \subsetneq E$, and $T = \inf\{t \geq 0 : X_t \notin D\}$. Fix a cost function $c : D \to \mathbb{R}_+$ and a boundary condition $\phi : D^c \to \mathbb{R}_+$. Introduce the potential

$$V(x) = \mathbb{E}_x\left[\int_0^T c(X_s)ds + \phi(X_T)\mathbb{1}_{\{T<\infty\}}\right].$$

Show that V is the nonnegative minimal solution to

$$\begin{cases} -QV(x) = c(x), \forall x \in D \\ V(x) = \phi(x), x \notin D \end{cases},$$

and that it is the unique bounded solution provided $T < \infty$ a.s. whatever the starting point. One may first establish that if J denotes the random number of jumps before hitting D^c, we have

$$\int_0^T c(X_s)ds + \phi(X_T)\mathbb{1}_{\{T<\infty\}} = \sum_{n=1}^{J} c(Y_{n-1})S_n + \phi(Y_J)\mathbb{1}_{\{J<\infty\}},$$

and use the corresponding result for discrete-time chains.
The suggested formula is obvious. Since the chain is assumed irreducible, $q_x > 0$ for any $x \in E$.

Since conditionally given Y_n, S_n is independent of $\mathcal{F}_{S_1+\ldots+S_n}$ and is exponentially distributed with parameter q_{Y_n}, we deduce that

$$V(x) = \mathbb{E}_x\left[\int_0^T c(X_s)ds + \phi(X_T)\mathbb{1}_{\{T<\infty\}}\right] = \mathbb{E}_x\left[\sum_{n=1}^{J}\frac{c(Y_n)}{q_{Y_n}} + \phi(Y_J)\mathbb{1}_{\{J<\infty\}}\right],$$

Using the discrete time corresponding result it follows that V is the nonnegative minimal solution (and the unique bounded solution provided $T_{D^c} < \infty$ a.s.) to

$$\begin{cases} V(x) = \Pi V(x) + \frac{c(x)}{q_x}, x \in D \\ V(x) = \phi(x), x \notin D \end{cases}.$$

Now observe that $\Pi V(x) = \sum_{y\neq x}\frac{q_{xy}}{q_x}V(y)$ so that $\Pi V(x) = \frac{QV(x)}{q_x} + V(x)$, thus

$$\begin{cases} V(x) = \Pi V(x) + \frac{c(x)}{q_x}, x \in D \\ V(x) = \phi(x), x \notin D \end{cases} \Leftrightarrow \begin{cases} -QV(x) = c(x), x \in D \\ V(x) = \phi(x), x \notin D \end{cases},$$

and the desired result follows.

Exercise 86 Assume X is non explosive, and that the function $f : E \to \mathbb{R}$ is bounded. Establish that

$$G(x) := \mathbb{E}_x\left[\int_0^\infty \exp(-\lambda t)f(X_t)dt\right]$$

is the unique bounded solution to $(\lambda - Q)G(x) = f(x), x \in E$. One may introduce $T \sim \exp(\lambda)$ independent of X, the chain $\tilde{X}$ on $E \cup \{\dagger\}$ defined as X killed at time T, and use the result of the previous exercise.

Introduce the chain $\tilde{X}$ as suggested, its generator $\tilde{Q}$ is such that for any $x \in E$,

$$\tilde{Q}(x,x) = Q(x,x) - \lambda, \quad \tilde{Q}(x,y) = q_{xy}, x \neq y \in E, \quad \tilde{Q}(x,\dagger) = \lambda.$$

For any $\phi : \tilde{E} \to \mathbb{R}$ such that $\phi(\dagger) = 0$, we find that for any $x \in E$,

$$\tilde{Q}\phi(x) = Q\phi(x) - \lambda\phi(x),$$

hence for $x \in E$, $(\lambda - Q)\phi(x) = f(x)$ is equivalent to $-\tilde{Q}(x) = f(x)$. Since $\{\dagger\}$ is an absorbing state, note further that $-\tilde{Q}\phi(\dagger) = 0$ whatever the function ϕ. In the end as long as $f(\dagger) = 0$,

$$(\lambda - Q)G(x) = f(x), \forall x \in E \Leftrightarrow -\tilde{Q}G(x) = f(x), x \in \tilde{E}.$$

Moreover, since T is independent of X we have

$$G(x) = \mathbb{E}_x\left[\int_0^\infty f(X_t)\mathbb{1}_{\{T>t\}}dt\right]$$

so that

$$G(x) = \mathbb{E}_x\left[\int_0^T f(\tilde{X}_t)dt\right], x \in E.$$

Of course G is bounded since f is. By the previous exercise, we find that G it is the only bounded solution to $-\tilde{Q}G(x) = f(x), x \in \tilde{E}$, and we are done.