Occupation time limit for super-brownian motion in dimension 4

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ABSTRACT: We derive the asymptotic behavior of the occupation measure $\mathcal{Z}(B_1)$ of the unit ball for super-Brownian motion started from the Dirac measure at a distant point x and conditioned to hit the unit ball. In the critical dimension $d = 4$, we obtain a limiting exponential distribution for the ratio $\mathcal{Z}(B_1)/\log|x|$.

1. Introduction

The results of the present work are motivated by the following simple problem about branching random walk in $\mathbb{Z}^d$. Consider a population of branching particles in $\mathbb{Z}^d$, such that individuals move independently in discrete time according to a random walk with zero mean and finite second moments, and at each integer time individuals die and give rise independently to a random number of offspring according to a critical offspring distribution. Suppose that the population starts with a single individual sitting at a point $x \in \mathbb{Z}^d$ located far away from the origin, and condition on the event that the population will eventually hit the origin. Then what will be the typical number of individuals that visit the origin, and is there a limiting distribution for this number?

In the present work, we address a continuous version of the previous problem, and so we deal with super-Brownian motion in $\mathbb{R}^d$. We denote by $M_F(\mathbb{R}^d)$ the space of all finite measures in $\mathbb{R}^d$. We also denote by $X = (X_t)_{t\geq 0}$ a d-dimensional super-Brownian motion with branching rate γ, that starts from μ under the probability measure $\mathbb{P}_\mu$, for every $\mu \in M_F(\mathbb{R}^d)$. We refer to Perkins [Pe] for a detailed presentation of super-Brownian motion. For every $x \in \mathbb{R}^d$, we also denote by $\mathbb{N}_x$ the excursion measure of super-Brownian motion from x. We may and will assume that both $\mathbb{P}_\mu$ and $\mathbb{N}_x$ are defined on the canonical space $C(\mathbb{R}_+, M_F(\mathbb{R}^d))$ of continuous functions from $\mathbb{R}_+$ into $M_F(\mathbb{R}^d)$ and that $(X_t)_{t\geq 0}$ is the canonical process on this space. Recall from Theorem II.7.3 in Perkins [Pe] that X started at the Dirac measure δ_x can be constructed from the atoms of a Poisson measure with intensity $\mathbb{N}_x$.

The total occupation measure of X is the finite random measure on $\mathbb{R}^d$ defined by

$$\mathcal{Z}(A) = \int_0^\infty X_t(A)\, dt,$$

for every Borel subset A of $\mathbb{R}^d$. We set $\mathcal{R} = \mathrm{supp}(\mathcal{Z})$, where $\mathrm{supp}(\mu)$ denotes the topological support of the measure μ. Equivalently,

$$\mathcal{R} = \mathrm{cl}\Big(\bigcup_{t\geq 0} \mathrm{supp}(X_t)\Big), \qquad (1)$$

where $\mathrm{cl}(A)$ denotes the closure of the set A. In dimension $d \geq 4$, points are polar, meaning that $\mathbb{N}_x(0 \in \mathcal{R}) = 0$ if $x \neq 0$, or equivalently $\mathbb{P}_\mu(0 \in \mathcal{R}) = 0$ if 0 does not belong to the closed support of μ. In dimension $d \leq 3$, we have if $x \neq 0$,

$$\mathbb{N}_x(0 \in \mathcal{R}) = \frac{8-2d}{\gamma}\,|x|^{-2} \tag{2}$$

(see Theorem 1.3 in [DIP] or Chapter VI in [LG]). It follows from the results in Sugitani [Su 89] that, again in dimension $d \leq 3$, the measure $\mathcal{Z}$ has a continuous density under $\mathbb{P}_{\delta_x}$ or under $\mathbb{N}_x$, for any $x \in \mathbb{R}^d$. We write $(\ell^y, y \in \mathbb{R}^d)$ for this continuous density.

For every $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ denotes the open ball centered at x with radius r. To simplify notation, we write $B_r = B(0,r)$ for the ball centered at 0 with radius r. By analogy with the discrete problem mentioned above, we are interested in the conditional distribution of $\mathcal{Z}(B_1)$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ when $|x|$ is large. As a simple consequence of (2) and scaling, we have when $d \leq 3$,

$$\mathbb{P}_{\delta_x}(\mathcal{Z}(B_1) > 0) \sim \mathbb{N}_x(\mathcal{Z}(B_1) > 0) \sim \frac{8-2d}{\gamma}\,|x|^{-2} \qquad \text{as } |x| \to \infty. \tag{3}$$

Here and later the notation $f(x) \sim g(x)$ as $|x| \to \infty$ means that the ratio $f(x)/g(x)$ tends to 1 as $|x| \to \infty$. On the other hand, when $d \geq 4$, it is proved in [DIP] that, as $|x| \to \infty$,

$$\mathbb{P}_{\delta_x}(\mathcal{Z}(B_1) > 0) \sim \mathbb{N}_x(\mathcal{Z}(B_1) > 0) \sim \begin{cases} \dfrac{2}{\gamma}\,|x|^{-2}(\log|x|)^{-1} & \text{if } d = 4, \\[2mm] \dfrac{\kappa_d}{\gamma}\,|x|^{2-d} & \text{if } d \geq 5, \end{cases} \tag{4}$$

where $\kappa_d > 0$ is a constant depending only on d.

For $d \geq 3$, the Green function of d-dimensional Brownian motion is

$$G(x,y) = c_d|x-y|^{2-d},$$

where $c_d = (2\pi^{d/2})^{-1}\Gamma(\frac{d}{2}-1)$. If $\mu \in M_F(\mathbb{R}^d)$ and φ is a nonnegative measurable function on $\mathbb{R}^d$, we use the notation $\langle \mu, \varphi \rangle = \int \varphi\, d\mu$. We can now state our main result.

Theorem 1. *Let φ be a bounded nonnegative measurable function supported on B_1, and set $\overline{\varphi} = \int \varphi(y)dy$.*

- (i) *If $d \leq 3$, the law of $|x|^{d-4}\langle \mathcal{Z}, \varphi \rangle$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ towards the distribution of $\overline{\varphi}\,\ell^0$ under $\mathbb{N}_{x_0}(\cdot \mid 0 \in \mathcal{R})$, where x_0 is an arbitrary point in $\mathbb{R}^d$ such that $|x_0| = 1$.*
- (ii) *If $d = 4$, the law of $(\log|x|)^{-1}\langle \mathcal{Z}, \varphi \rangle$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ to an exponential distribution with mean $\gamma\overline{\varphi}/(4\pi^2)$.*
- (iii) *If $d \geq 5$, the law of $\langle \mathcal{Z}, \varphi \rangle$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ to the probability measure μ_φ on $\mathbb{R}_+$ with moments $m_{p,\varphi} = \int r^p\, \mu_\varphi(dr)$ given by*

$$m_{1,\varphi} = \frac{c_d}{\kappa_d}\,\gamma\,\overline{\varphi},$$

and for every $p \geq 2$,

$$m_{p,\varphi} = \frac{c_d}{\kappa_d}\,\frac{\gamma^2}{2}\sum_{j=1}^{p-1}\binom{p}{j}\int \mathbb{N}_z(\langle \mathcal{Z}, \varphi \rangle^j)\mathbb{N}_z(\langle \mathcal{Z}, \varphi \rangle^{p-j})dz.$$

The scaling invariance properties of super-Brownian motion allow us to restate Theorem 1 in terms of super-Brownian motion started with a fixed initial value and the occupation measure of a small ball with radius ε tending to 0. Part (i) of Theorem 1 then becomes a straightforward consequence of the fact that the measure $\mathcal{Z}$ has a continuous density in dimension $d \leq 3$: See Lee [Le] and Merle [Me] for more precise results along these lines. On the other hand, the proof of part (iii) is relatively easy from the method of moments and known recursive formulas for the moments of the random measure $\mathcal{Z}$ under $\mathbb{N}_x$. For the sake of completeness, we include proofs of the three cases in Theorem 1, but the most interesting part is really the critical dimension $d = 4$, where it is remarkable that an explicit limiting distribution can be obtained.

Notice that dimension 4 is critical with respect to the polarity of points for super-Brownian motion. Part (ii) of the theorem should therefore be compared with classical limit theorems for additive functionals of planar Brownian (note that $d = 2$ is the critical dimension for polarity of points for ordinary Brownian motion). The celebrated Kallianpur-Robbins law states that the time spent by planar Brownian motion in a bounded set before time t behaves as $t \to \infty$ like $\log t$ times an exponential variable (see e.g. section 7.17 in Itô and McKean [IM]). The Kallianpur-Robbins law can be derived by "conceptual proofs" which explain the occurrence of the exponential distribution. Our initial approach to part (ii) was based on a similar conceptual argument based on the Brownian snake approach to super-Brownian motion. Informally, if $a > 0$ is fixed, we may apply the strong Markov property of the Brownian snake at the first time when the occupation time of the unit ball by the "tip" of the snake exceeds a, and infer that any limiting distribution for the occupation time of the ball must satisfy the lack of memory property which characterizes exponential distributions. Since it seems delicate to make this argument completely rigorous, we rely below on a careful analysis of the moments of $\langle \mathcal{Z}, \varphi \rangle$.

Let us finally comment on the branching random walk problem discussed at the beginning of this introduction. Although we do not consider this problem here, it is very likely that a result analogous to Theorem 1 holds in this discrete setting, just replacing $\langle \mathcal{Z}, \varphi \rangle$ with the number of particles that hit the origin. In particular, the limiting distributions obtained in (i) and (ii) of Theorem 1 should also appear in the discrete setting.

2. Preliminary remarks

Let us briefly recall some basic facts about super-Brownian motion and its excursion measures. If $x \in \mathbb{R}^d$ and if

$$\mathcal{N} = \sum_{i \in I} \delta_{\omega_i}$$

is a Poisson point measure on $C(\mathbb{R}_+, M_F(\mathbb{R}^d))$ with intensity $\mathbb{N}_x(\cdot)$, then the measure-valued process Y defined by

$$\begin{aligned} Y_0 &= \delta_x, \\ Y_t &= \sum_{i \in I} X_t(\omega_i)\ ,\ \text{for every } t > 0, \end{aligned}$$

has law $\mathbb{P}_{\delta_x}$ (see Theorem II.7.3 in [Pe]).

We can use this Poisson decomposition to observe that it is enough to prove Theorem 1 with the conditional measure $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ replaced by $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$. Indeed, write $M = \#\{i \in I : \mathcal{R}(\omega_i) \cap B_1 \neq \emptyset\}$ (where $\mathcal{R}(\omega_i)$ is defined as in (1)). Then, M is Poisson with parameter $\mathbb{N}_x(\mathcal{Z}(B_1) > 0)$, and $\{M \geq 1\}$ is the event that the range of Y hits B_1. Furthermore, the preceding Poisson decomposition just shows that the law of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{P}_{\delta_x}$ coincides with the law of $Z^1 + \cdots + Z^M$, where conditionally given M, the variables $Z^1, Z^2, \ldots$ are independent and distributed according to the law of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$. Since $P(M = 1 \mid M \geq 1)$ tends to 1 as $|x| \to \infty$ (by the estimates (3) and (4)), we see that the law of $\langle \mathcal{Z}, \varphi\rangle$ (or the law of $f(x)\langle \mathcal{Z}, \varphi\rangle$ for any deterministic function f) under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ will be arbitrarily close to the law of the same variable under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ when $|x|$ is large, which is what we wanted. Note that this argument is valid in any dimension.

Let us also discuss the dependence of our results on the branching rate γ. If $(Y_t)_{t\geq 0}$ is a super-Brownian motion with branching rate γ started at μ, and $\lambda > 0$, then $(\lambda Y_t)_{t\geq 0}$ is a super-Brownian motion with branching rate $\lambda\gamma$ started at $\lambda\mu$. A similar property then holds for excursion measures. Write $\mathbb{N}_x^{(\gamma)}$ instead of $\mathbb{N}_x$ to emphasize the dependence on γ. Then the "law" of $(\lambda X_t)_{t\geq 0}$ under $\mathbb{N}_x^{(\gamma)}$ is $\lambda^{-1}\mathbb{N}_x^{(\lambda\gamma)}$. Thanks to these observations, it will be enough to prove Theorem 1 for one particular value of γ.

In what follows, we take $\gamma = 2$, as this will simplify certain formulas. For any nonnegative measurable function φ on $\mathbb{R}^d$, the moments of $\langle \mathcal{Z}, \varphi\rangle$ are determined by induction by the formulas

$$\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle) = \int_{\mathbb{R}^d} G(x,y)\varphi(y)\,dy \tag{5}$$

and, for every $p \geq 2$,

$$\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p) = \sum_{j=1}^{p-1} \binom{p}{j} \int_{\mathbb{R}^d} G(x,z)\,\mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^j)\mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^{p-j})dz. \tag{6}$$

See e.g. formula (16.2.3) in [LG2], and note that the extra factor 2 there is due to the fact that the the Brownian snake approach gives $\gamma = 4$.

3. Low dimensions

In this section, we prove part (i) of Theorem 1. Let $\varepsilon > 0$, and set $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$. By scaling, the law of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ coincides with the law of $\varepsilon^{-4}\langle \mathcal{Z}, \varphi_\varepsilon\rangle$ under $\mathbb{N}_{\varepsilon x}(\cdot \mid \mathcal{Z}(B_\varepsilon) > 0)$. Taking $\varepsilon = |x|^{-1}$, we see that the proof of part (i) reduces to checking that the law of $|x|^d\langle \mathcal{Z}, \varphi_{|x|^{-1}}\rangle$ under $\mathbb{N}_{x/|x|}(\cdot \mid \mathcal{Z}(B_{|x|^{-1}}) > 0)$ converges to the distribution of $\overline{\varphi}\ell^0$ under $\mathbb{N}_{x_0}(\cdot \mid 0 \in \mathcal{R})$.

However, as $|x| \to \infty$,

$$\mathbb{N}_{x/|x|}(\mathcal{Z}(B_{|x|^{-1}}) > 0) = \mathbb{N}_{x_0}(\mathcal{Z}(B_{|x|^{-1}}) > 0) \longrightarrow \mathbb{N}_{x_0}(0 \in \mathcal{R}) = 4 - d.$$

On the other hand, since

$$|x|^d\langle \mathcal{Z}, \varphi_{|x|^{-1}}\rangle = |x|^d \int dy\, \ell^y\, \varphi_{|x|^{-1}}(y) = \int dy\, \ell^{y/|x|}\, \varphi(y)$$

the continuity of the local times ℓ^y implies that, for every $\delta > 0$,

$$\mathbb{N}_{x/|x|}\Big(\Big||x|^d\langle \mathcal{Z}, \varphi_{|x|^{-1}}\rangle - \overline{\varphi}\,\ell^0\Big| > \delta\Big) \le \mathbb{N}_{x/|x|}\Big(\sup_{y\in B_{|x|^{-1}}} |\ell^y - \ell^0| > \frac{\delta}{\overline{\varphi}}\Big) \longrightarrow 0$$

as $|x| \to \infty$. By rotational invariance, the law of ℓ^0 under $\mathbb{N}_{x/|x|}$ coincides with the law of the same variable under $\mathbb{N}_{x_0}$. Part (i) of Theorem 1 now follows from the preceding observations.

4. High dimensions

We now turn to part (iii) of Theorem 1 and so we suppose that $d \ge 5$. As noticed earlier, we may replace $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ by $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$.

Without loss of generality, we assume in this part that $\varphi \le 1$.

Lemma 1. *There exists a finite constant K_d depending only on d, such that, for every $x \in \mathbb{R}^d$ and $p \ge 1$,*

$$\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p) \le K_d^p\, p!\,(|x|^{2-d} \wedge 1).$$

Proof. Obviously, it is enough to consider the case when $\varphi = \mathbf{1}_{B_1}$. From (5), one immediately verifies that

$$\mathbb{N}_x(\mathcal{Z}(B_1)) \le C_{1,d}\,(|x|^{2-d} \wedge 1)$$

for some constant $C_{1,d}$ depending only on d. Straightforward estimates give the existence of a constant a_d such that, for every $x \in \mathbb{R}^d$,

$$\int G(x,z)\,(|z|^{2-d} \wedge 1)^2\,dz \le a_d(|x|^{2-d} \wedge 1).$$

We then claim that for every integer $p \ge 1$,

$$\mathbb{N}_x(\mathcal{Z}(B_1)^p) \le C_{p,d}\,p!\,(|x|^{2-d} \wedge 1) \tag{7}$$

where the constants $C_{p,d}$, $p \ge 2$ are determined by induction by

$$C_{p,d} = a_d \sum_{j=1}^{p-1} C_{j,d}C_{p-j,d}. \tag{8}$$

Indeed, let $k \ge 2$ and suppose that (7) holds for every $p \in \{1, \dots, k-1\}$. From (6), we get

$$\mathbb{N}_x(\mathcal{Z}(B_1)^k) \le k! \sum_{j=1}^{k-1} C_{j,d}C_{k-j,d} \int G(x,z)\,(|z|^{2-d} \wedge 1)^2\,dz,$$

and our choice of a_d shows that (7) also holds for $p = k$. We have thus proved our claim (7) for every $p \ge 1$.

From (8) it is an elementary exercise to verify that $C_{p,d} \le K_d^p$ for some constant K_d depending only on d. This completes the proof. □

Let us now prove that for every $p \ge 1$, $\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ to $m_{p,\varphi}$. If $p = 1$, this is an immediate consequence of (4), (5) and dominated convergence. If $p \ge 2$, we write

$$\frac{\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p)}{|x|^{2-d}} = \sum_{j=1}^{p-1} \binom{p}{j} \int \frac{G(x,z)}{|x|^{2-d}}\,\mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^j)\mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^{p-j})dz.$$

In each of the integrals appearing in the previous display, the contribution of the set $\{z \in \mathbb{R}^d : |z-x| \leq |x|/2\}$ goes to 0 as $|x| \to \infty$, by an easy application of the bounds of Lemma 1. On the other hand, if we restrict our attention to the set $\{z \in \mathbb{R}^d : |z-x| > |x|/2\}$, we can again use the bounds of Lemma 1 together with the property $\int (|z|^{2-d} \wedge 1)^2 dz < \infty$, in order to justify dominated convergence and to get

$$\lim_{|x|\to\infty} \frac{\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p)}{|x|^{2-d}} = c_d \sum_{j=1}^{p-1} \binom{p}{j} \int \mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^j) \mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^{p-j}) dz.$$

The convergence of $\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p \mid \mathcal{Z}(B_1) > 0)$ towards $m_{p,\varphi}$ now follows from (4).

Finally, Lemma 1 and (4) also imply that any limit distribution of the laws of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ is characterized by its moments. Part (iii) of Theorem 1 now follows as a standard application of the method of moments.

5. The critical dimension

In this section, we consider the critical dimension $d = 4$. Recall that in that case $G(x, y) = (2\pi^2)^{-1}|y-x|^{-2}$. As in the previous sections, we take $\gamma = 2$. We start by stating two lemmas.

Lemma 2. *Let $x \in \mathbb{R}^4 \setminus \{0\}$. Then,*

$$\mathbb{N}_x[\mathcal{Z}(B_\varepsilon) > 0] \underset{\varepsilon\to 0}{\sim} |x|^{-2} \left(\log \frac{1}{\varepsilon}\right)^{-1}.$$

Lemma 3. *Let $x \in \mathbb{R}^4 \setminus \{0\}$, and $p \geq 1$. Let φ be a bounded nonnegative measurable function on B_1, and for every $\varepsilon > 0$, put $\varphi_\varepsilon(y) = \varphi(y/\varepsilon)$. Then,*

$$\mathbb{N}_x[\langle \mathcal{Z}, \varphi_\varepsilon\rangle)^p] \underset{\varepsilon\to 0}{\sim} p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p |x|^{-2}\, \varepsilon^{4p} \left(\log \frac{1}{\varepsilon}\right)^{p-1},$$

uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$.

Let us explain how part (ii) of Theorem 1 follows from these two lemmas. Notice that the estimate of Lemma 2 also holds uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$, by scaling and rotational invariance. Combining the results of the lemmas gives

$$\mathbb{N}_x\left[\left(\frac{\langle \mathcal{Z}, \varphi_\varepsilon\rangle}{\varepsilon^4 \log(\frac{1}{\varepsilon})}\right)^p \middle| \mathcal{Z}(B_\varepsilon) > 0\right] \underset{\varepsilon\to 0}{\sim} p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p,$$

uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$. By scaling, for any $x \in \mathbb{R}^4$ with $|x| > 1$ the law of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ coincides with the law of $|x|^4 \langle \mathcal{Z}, \varphi_{1/|x|}\rangle$ under $\mathbb{N}_{x/|x|}(\cdot \mid \mathcal{Z}(B_{1/|x|}) > 0)$. Hence, we deduce from the preceding display that we have also

$$\mathbb{N}_x\left[\left(\frac{\langle \mathcal{Z}, \varphi\rangle}{\log|x|}\right)^p \middle| \mathcal{Z}(B_1) > 0\right] \underset{|x|\to\infty}{\sim} p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p.$$

The statement in part (ii) of Theorem 1 now follows from an application of the method of moments.

It remains to prove Lemma 2 and Lemma 3.

5.1. **Proof of Lemma 2.** From well-known connections between super-Brownian motion and partial differential equations (see e.g. Chapter VI in [LG]), the function $u_\varepsilon(x) = \mathbb{N}_x[\mathcal{Z}(B_\varepsilon) > 0]$ defined for $|x| > \varepsilon$ solves the singular boundary problem

$$\begin{aligned} &\Delta u = 2\,u^2, &&\text{in the domain } \{|x| > \varepsilon\} \\ &u(x) \longrightarrow \infty, &&\text{as } |x| \to \varepsilon^+ \\ &u(x) \longrightarrow 0, &&\text{as } |x| \to \infty. \end{aligned}$$

As a consequence of a lemma due to Iscoe (see Lemma 3.4 in [DIP]), for every $x \in \mathbb{R}^4\backslash\{0\}$, we have $u_\varepsilon(x) \sim |x|^{-2}(\log(1/\varepsilon))^{-1}$ as $\varepsilon \to 0$. Lemma 2 follows.

5.2. **Proof of Lemma 3.**

5.2.1. *Lower bound.* Let us introduce the space of functions

$$\mathcal{F} := \left\{ f : \mathbb{R}_+ \to \mathbb{R} : \lim_{\varepsilon \to 0} f(\varepsilon) = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon} = \infty \right\}.$$

Claim. *For every integer $p \geq 1$, for every $f \in \mathcal{F}$, for every $\beta > 0$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,*

$$\inf_{y \notin B_{f(\varepsilon)}} |y|^2 \left(\log\left(\frac{|y|}{\varepsilon} \right) \right)^{1-p} \mathbb{N}_y \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^p \right] \geq (1-\beta)\, p! \left(\frac{\overline{\varphi}}{2\pi^2} \right)^p \varepsilon^{4p}. \tag{9}$$

We prove the claim by induction on p. Let us first consider the case $p = 1$. We fix $f \in \mathcal{F}$. Using (5), for $\varepsilon > 0$ and $y \in \mathbb{R}^4$ such that $|y| > f(\varepsilon)$, we have

$$\mathbb{N}_y \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle \right] = \int_{B_\varepsilon} dz \varphi_\varepsilon(z) G(y,z) \geq \varepsilon^4 \overline{\varphi} \inf_{z \in B_\varepsilon} G(y,z). \tag{10}$$

Since $\lim_{\varepsilon \to 0}(f(\varepsilon)/\varepsilon) = \infty$, we see that

$$\inf_{y \notin B_{f(\varepsilon)}} \left(|y|^2 \inf \left\{ G(y,z), z \in B_\varepsilon \right\} \right) \underset{\varepsilon \to 0}{\to} \frac{1}{2\pi^2}.$$

We thus deduce from (10) that for ε small enough,

$$\inf_{y \notin B_{f(\varepsilon)}} |y|^2 \mathbb{N}_y \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle \right] \geq \frac{1-\beta}{2\pi^2}\, \overline{\varphi}\, \varepsilon^4,$$

which gives our claim for $p = 1$.

Let $p \geq 2$ and suppose that the claim holds up to order $p - 1$. Fix $f \in \mathcal{F}$ and $\beta \in (0,1)$. Let $\beta' \in (0,1)$ be such that $(1-\beta')^4 = 1 - \beta$, and let $C > 0$ be such that $(1 + C^{-1})^{-2} = 1 - \beta'$. Introduce the function $\hat{f}$ defined by

$$\hat{f}(\varepsilon) = \varepsilon \log\left(\frac{f(\varepsilon)}{\varepsilon} \right). \tag{11}$$

Clearly, $\hat{f} \in \mathcal{F}$. Furthermore, we have

$$\lim_{\varepsilon \to 0} \frac{\log(\hat{f}(\varepsilon)/\varepsilon)}{\log(f(\varepsilon)/\varepsilon)} = 0. \tag{12}$$

Using (6), we obtain, for any $y \notin B_{f(\varepsilon)}$

$$\mathbb{N}_y \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^p \right] \geq \sum_{j=1}^{p-1} \binom{p}{j} \int_{B_{|y|/C} \backslash B_{\hat{f}(\varepsilon)}} dz\, G(y,z)\, \mathbb{N}_z \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^j \right] \mathbb{N}_z \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^{p-j} \right].$$

Using the induction hypothesis, we get, provided ε is small enough,

$$\begin{aligned}&\mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right]\\ &\geq (1-\beta')^2 p!\left(\frac{\overline{\varphi}}{2\pi^2}\right)^p (p-1)\varepsilon^{4p}\int_{B_{|y|/C}\setminus B_{\hat{f}(\varepsilon)}} dz G(y,z)\frac{1}{|z|^4}\left(\log\frac{|z|}{\varepsilon}\right)^{p-2}.\end{aligned}$$

From the definition of C, for any $z\in B_{|y|/C}$, we have $G(y,z)\geq(1-\beta')G(0,y)$. Using the fact that the area of the unit sphere S^3 is $2\pi^2$, we obtain

$$\begin{aligned}(p-1)&\int_{B_{|y|/C}\setminus B_{\hat{f}(\varepsilon)}} dz G(y,z)\frac{1}{|z|^4}\left(\log\frac{|z|}{\varepsilon}\right)^{p-2}\\ &\geq 2\pi^2(1-\beta')\,G(0,y)(p-1)\int_{\hat{f}(\varepsilon)}^{|y|/C}\frac{dr}{r}\left(\log\frac{r}{\varepsilon}\right)^{p-2}\\ &=(1-\beta')|y|^{-2}\left(\left(\log\frac{|y|}{C\varepsilon}\right)^{p-1}-\left(\log\frac{\hat{f}(\varepsilon)}{\varepsilon}\right)^{p-1}\right).\end{aligned}$$

Moreover, using the property $f\in\mathcal{F}$ and (12), we see that, if ε is sufficiently small, for any $y\notin B_{f(\varepsilon)}$

$$\left(\log\frac{|y|}{C\varepsilon}\right)^{p-1}-\left(\log\frac{\hat{f}(\varepsilon)}{\varepsilon}\right)^{p-1}\geq(1-\beta')\left(\log\frac{|y|}{\varepsilon}\right)^{p-1}.$$

From the preceding bounds, we get that, if ε is sufficiently small,

$$\inf_{y\notin B_{f(\varepsilon)}}|y|^2\left(\log\frac{|y|}{\varepsilon}\right)^{1-p}\mathbb{N}_y\left[\langle\mathcal{Z},\varphi_\varepsilon\rangle^p\right]\geq(1-\beta')^4 p!\left(\frac{\overline{\varphi}}{2\pi^2}\right)^p\varepsilon^{4p},$$

which is our claim at order p.

5.2.2. *Upper bound.* Without loss of generality we assume that $\varphi\leq 1$. We need to get upper bounds on $\mathbb{N}_y\left[\langle\mathcal{Z},\varphi_\varepsilon\rangle^p\right]$ for y belonging to different subsets of $\mathbb{R}^4$.

We will prove that, for every $p\geq 1$, for every $f\in\mathcal{F}$ and every $\beta\in(0,1)$ the following bounds hold for $\varepsilon>0$ sufficiently small:

$$(\mathfrak{H}_p^1)\qquad \sup_{|y|\leq 4\varepsilon}\mathbb{N}_y\left[\langle\mathcal{Z},\varphi_\varepsilon\rangle^p\right]\leq p!\,\varepsilon^{4p-2}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-1},$$

$$(\mathfrak{H}_p^2)\qquad \sup_{4\varepsilon\leq|y|\leq f(\varepsilon)}|y|^2\mathbb{N}_y\left[\langle\mathcal{Z},\varphi_\varepsilon\rangle^p\right]\leq p!\,\varepsilon^{4p}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-1},$$

$$(\mathfrak{H}_p^3)\qquad \sup_{|y|\geq f(\varepsilon)}|y|^2\left(\log\frac{|y|}{\varepsilon}\right)^{1-p}\mathbb{N}_y\left[\langle\mathcal{Z},\varphi_\varepsilon\rangle^p\right]\leq\left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p p!\,\varepsilon^{4p}.$$

Only $(\mathfrak{H}_p^3)$ is needed in our proof of Lemma 3. However, we will proceed by induction on p to get $(\mathfrak{H}_p^3)$, and we will use $(\mathfrak{H}_p^1)$ and $(\mathfrak{H}_p^2)$ in our induction argument. The bounds $(\mathfrak{H}_p^1)$ and $(\mathfrak{H}_p^2)$ are not sharp, but they will be sufficient for our purposes. Notice that $(\overline{\varphi}+\beta)/(2\pi^2)<1/3$ because $\varphi\leq 1$ and $\beta<1$.

We first note that when $p=1$ the bounds $(\mathfrak{H}_1^1)$, $(\mathfrak{H}_1^2)$ and $(\mathfrak{H}_1^3)$ are easy consequences of (5). Let $p\geq 2$ and assume that $(\mathfrak{H}_k^1)$, $(\mathfrak{H}_k^2)$ and $(\mathfrak{H}_k^3)$ hold for every $1\leq k\leq p-1$, for any choice of β and f. Let us fix $f\in\mathcal{F}$ and $\beta_0\in(0,1)$. In our induction argument we will use $(\mathfrak{H}_k^3)$, for $1 \le k \le p-1$, with $\beta \in (0,1)$ chosen small enough so that

$$(1+\beta)^2(\overline{\varphi}+\beta)^p < (\overline{\varphi}+\beta_0)^p.$$

For every $j \in \{1, \dots, p-1\}$, every $y \in \mathbb{R}^4$ and every Borel subset A of $\mathbb{R}^4$, we set

$$I_{p,j}^{\varepsilon}(A,y) := \frac{1}{j!(p-j)!} \int_A dz G(y,z) \mathbb{N}_z \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^j\right] \mathbb{N}_z \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^{p-j}\right].$$

We also set $I_{p,j}^{\varepsilon}(y) = I_{p,j}^{\varepsilon}(\mathbb{R}^4, y)$.

We first verify $(\mathfrak{H}_p^1)$, and so we assume that $|y| \le 4\varepsilon$. We fix $j \in \{1, \dots, p-1\}$ and we split the the integral in $I_{p,j}^{\varepsilon}(y)$ into three parts corresponding to the sets

$$A_1^{(1)} = B_{8\varepsilon},\ A_2^{(1)} = B_{f(\varepsilon)} \setminus B_{8\varepsilon},\ A_3^{(1)} = \mathbb{R}^4 \setminus B_{f(\varepsilon)}.$$

From (6), we have

$$\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^p\right] = p! \sum_{j=1}^{p-1} \left(I_{p,j}^{\varepsilon}(A_1^{(1)}, y) + I_{p,j}^{\varepsilon}(A_2^{(1)}, y) + I_{p,j}^{\varepsilon}(A_3^{(1)}, y) \right). \tag{13}$$

If ε is small enough, we deduce from the bounds $(\mathfrak{H}_k^1)$ and $(\mathfrak{H}_k^2)$, with $1 \le k \le p-1$, that

$$I_{p,j}^{\varepsilon}(A_1^{(1)}, y) \le \varepsilon^{4p-4} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \int_{B_{8\varepsilon}} \frac{dz}{2\pi^2 |z-y|^2}.$$

It follows that

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \le 4\varepsilon} \varepsilon^{2-4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_1^{(1)}, y) \right) = 0.$$

If $z \in A_2^{(1)} \cup A_3^{(1)}$, we have $G(y,z) \le 4G(0,z)$. Using $(\mathfrak{H}_k^2)$ with $1 \le k \le p-1$, we obtain, if ε is small enough,

$$I_{p,j}^{\varepsilon}(A_2^{(1)}, y) \quad \le \quad 4\,\varepsilon^{4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \int_{8\varepsilon}^{f(\varepsilon)} \frac{dr}{r^3},$$

and thus

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \le 4\varepsilon} \varepsilon^{2-4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_2^{(1)}, y) \right) = 0.$$

Using finally $(\mathfrak{H}_k^3)$ with $1 \le k \le p-1$, we obtain, for ε sufficiently small,

$$I_{p,j}^{\varepsilon}(A_3^{(1)}, y) \le 4\,\varepsilon^{4p} \int_{f(\varepsilon)}^{\infty} \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2}.$$

Since

$$\int_{f(\varepsilon)}^{\infty} \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2} \underset{\varepsilon \to 0}{\sim} \frac{1}{2f(\varepsilon)^2} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2}$$

we get

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \le 4\varepsilon} \varepsilon^{2-4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_3^{(1)}, y) \right) = 0.$$

Combining the estimates we obtained for $I_{p,j}^{\varepsilon}(A_1^{(1)},y), I_{p,j}^{\varepsilon}(A_2^{(1)},y)$ and $I_{p,j}^{\varepsilon}(A_3^{(1)},y)$, we arrive at

$$\lim_{\varepsilon\to 0}\left(\sup_{|y|\le 4\varepsilon}\varepsilon^{2-4p}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(y)\right)=0.$$

From (13), we obtain that $(\mathfrak{H}_p^1)$ holds.

We now turn to the proof of $(\mathfrak{H}_p^2)$, and so we assume that $4\varepsilon \le |y| \le f(\varepsilon)$. Again we fix $j\in\{1,\dots,p-1\}$. We split the integral in $I_{p,j}^{\varepsilon}(y)$ into five parts corresponding to the sets

- $A_1^{(2)} = B_{2\varepsilon}$,
- $A_2^{(2)} = B(y,|y|/2)$,
- $A_3^{(2)} = B_{\hat f(\varepsilon)} \setminus (B_{2\varepsilon}\cup B(y,|y|/2))$,
- $A_4^{(2)} = B_{2f(\varepsilon)} \setminus \left(B_{\hat f(\varepsilon)}\cup B(y,|y|/2)\right)$,
- $A_5^{(2)} = \mathbb{R}^4 \setminus B_{2f(\varepsilon)}$,

where $\hat f(\varepsilon)=\varepsilon\log(f(\varepsilon)/\varepsilon)$ as in (11). We have thus

$$\mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right] = p!\sum_{j=1}^{p-1}\sum_{i=1}^{5} I_{p,j}^{\varepsilon}(A_i^{(2)},y). \tag{14}$$

Notice first that if $z\in A_1^{(2)}$, we have $|z|\le |y|/2$ so that $G(z-y)\le 4G(y)$. Using $(\mathfrak{H}_k^1)$ with $1\le k\le p-1$, we obtain, provided ε is small enough

$$I_{p,j}^{\varepsilon}(A_1^{(2)},y)\le \varepsilon^{4p-4}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-2}\frac{2}{\pi^2|y|^2}\int_{\{|z|\le 2\varepsilon\}} dz,$$

so that

$$\lim_{\varepsilon\to 0}\left(\sup_{4\varepsilon<|y|<f(\varepsilon)}\varepsilon^{-4p}|y|^2\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_1^{(2)},y)\right)=0.$$

If $z\in A_2^{(2)}$, using the bound $|z|^{-2}\le 4|y|^{-2}$, we deduce from $(\mathfrak{H}_k^1)$ and $(\mathfrak{H}_k^2)$ for $1\le k\le p-1$ that for sufficiently small ε,

$$I_{p,j}^{\varepsilon}(A_2^{(2)},y)\le \varepsilon^{4p}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-2}\frac{16^3}{|y|^4}\int_{B(y,|y|/2)} G(y,z)dz.$$

It follows that

$$\lim_{\varepsilon\to 0}\left(\sup_{4\varepsilon<|y|<f(\varepsilon)}\varepsilon^{-4p}|y|^2\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_2^{(2)},y)\right)=0.$$

If $z\in A_3^{(2)}$, $G(y,z)\le 4G(0,y)$. Since $\hat f\in\mathcal{F}$, we can use $(\mathfrak{H}_k^1)$ and $(\mathfrak{H}_k^2)$ with $1\le k\le p-1$ to get that, for ε small enough,

$$I_{p,j}^{\varepsilon}(A_3^{(2)},y)\le \varepsilon^{4p}\left(\log\frac{\hat f(\varepsilon)}{\varepsilon}\right)^{p-2}\frac{4\times 16^2}{|y|^2}\int_{\varepsilon}^{\hat f(\varepsilon)} r^{-1}dr.$$

It then follows from (12) that

$$\lim_{\varepsilon\to 0}\left(\sup_{4\varepsilon<|y|<f(\varepsilon)}\varepsilon^{-4p}|y|^2\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_3^{(2)},y)\right)=0.$$

If $z \in A_4^{(3)}$, we still have $G(y,z) \leq 4G(0,y)$. Again, $\hat{f} \in \mathcal{F}$, and we can use $(\mathfrak{H}_k^3)$ with $1 \leq k \leq p-1$, recalling that $(\overline{\varphi}+\beta)/(2\pi^2) < 1/3$, to obtain for ε small

$$\begin{aligned} I_{p,j}^{\varepsilon}(A_4^{(2)}, y) &\leq 3^{-p}\,\varepsilon^{4p}\frac{4}{|y|^2}\int_{\hat{f}(\varepsilon)}^{2f(\varepsilon)} \frac{dr}{r}\left(\log\frac{r}{\varepsilon}\right)^{p-2} \\ &= \frac{4\times 3^{-p}}{p-1}\,\varepsilon^{4p}\frac{1}{|y|^2}\left(\log\frac{2f(\varepsilon)}{\hat{f}(\varepsilon)}\right)^{p-1}. \end{aligned}$$

It follows that

$$\limsup_{\varepsilon\to 0}\left(\sup_{4\varepsilon<|y|<f(\varepsilon)} \varepsilon^{-4p}|y|^2\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_4^{(2)}, y)\right) \leq \frac{4\times 3^{-p}}{p-1} < \frac{1}{p-1}.$$

Finally, if $|z| \geq 2f(\varepsilon)$, we have $G(y,z) \leq 4(2\pi^2)^{-1}|z|^{-2}$ and using again $(\mathfrak{H}_k^3)$ with $1 \leq k \leq p-1$, we get for ε sufficiently small,

$$I_{p,j}^{\varepsilon}(A_5^{(2)}, y) \leq 4\,\varepsilon^{4p}\int_{2f(\varepsilon)}^{\infty}\frac{dr}{r^3}\left(\log\frac{r}{\varepsilon}\right)^{p-2},$$

and as before in the estimate for $I_{p,j}^{\varepsilon}(A_3^{(1)}, y)$, it follows that

$$\lim_{\varepsilon\to 0}\left(\sup_{4\varepsilon<|y|<f(\varepsilon)} \varepsilon^{-4p}|y|^2\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_5^{(2)}, y)\right) = 0.$$

We get $(\mathfrak{H}_p^2)$ by combining the preceding estimates on $I_{p,j}^{\varepsilon}(A_i^{(2)}, y)$ for $1 \leq i \leq 5$, and using (14).

We now prove $(\mathfrak{H}_p^3)$, and we thus assume that $|y| \geq f(\varepsilon)$. Let $C' > 1$ be such that $1-(C')^{-1} = (1+\beta)^{-1}$. For $\varepsilon > 0$ sufficiently small, we can split the integral in $I_{p,j}^{\varepsilon}(y)$ into five parts corresponding to the sets

- $A_1^{(3)} = B_{4\varepsilon}$,
- $A_2^{(3)} = B_{\hat{f}(\varepsilon)} \setminus B_{4\varepsilon}$,
- $A_3^{(3)} = B_{|y|/C'} \setminus B_{\hat{f}(\varepsilon)}$,
- $A_4^{(3)} = B_{2|y|} \setminus B_{|y|/C'}$,
- $A_5^{(3)} = \mathbb{R}^4 \setminus B_{2|y|}$.

We have then

$$\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^p\right] = p!\sum_{j=1}^{p-1}\sum_{i=1}^{5} I_{p,j}^{\varepsilon}(A_i^{(3)}, y). \tag{15}$$

Using $(\mathfrak{H}_k^1)$ with $1 \leq k \leq p-1$, we get for ε small that

$$I_{p,j}^{\varepsilon}(A_1^{(3)}, y) \leq \varepsilon^{4p-4}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-2}\frac{4}{|y|^2}\int_0^{4\varepsilon} r^3\,dr,$$

so that

$$\lim_{\varepsilon\to 0}\left(\sup_{|y|\geq f(\varepsilon)} \varepsilon^{-4p}|y|^2\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_1^{(3)}, y)\right) = 0.$$

Then, using $(\mathfrak{H}_k^2)$ with $1 \le k \le p-1$, we obtain for ε small enough,

$$I_{p,j}^{\varepsilon}(A_2^{(3)}, y) \le \varepsilon^{4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \frac{4}{|y|^2} \int_{4\varepsilon}^{\hat{f}(\varepsilon)} r^{-1} dr.$$

Thus, using (12), we have

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \ge f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_2^{(3)}, y) \right) = 0.$$

If $z \in A_3^{(3)}$, we have $G(y,z) \le (1-(C')^{-1})^{-2} G(0,y) = (1+\beta)^2 G(0,y)$. Since $\hat{f} \in \mathcal{F}$, we can use $(\mathfrak{H}_k^3)$ with $1 \le k \le p-1$ to get for ε sufficiently small,

$$\begin{aligned} I_{p,j}^{\varepsilon}(A_3^{(3)}, y) &\le (1+\beta)^2 \left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p \varepsilon^{4p} \frac{1}{|y|^2} \int_{\hat{f}(\varepsilon)}^{|y|/C'} \frac{dr}{r} \left(\log \frac{r}{\varepsilon}\right)^{p-2} \\ &= \frac{(1+\beta)^2}{p-1} \left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p \varepsilon^{4p} \frac{1}{|y|^2} \left(\log \frac{|y|}{C' \hat{f}(\varepsilon)}\right)^{p-1}, \end{aligned}$$

Recalling (12), we obtain that

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \left(\sup_{|y| \ge f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{|y|}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_3^{(3)}, y) \right) \\ &\quad \le \frac{(1+\beta)^2}{p-1} \left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p < \frac{1}{p-1} \left(\frac{\overline{\varphi}+\beta_0}{2\pi^2}\right)^p, \end{aligned}$$

using our choice of β. If $z \in A_4^{(3)}$, we have $|z|^{-2} \le {C'}^2 |y|^{-2}$. Since $\hat{f} \in \mathcal{F}$, we obtain from $(\mathfrak{H}_k^3)$ with $1 \le k \le p-1$ that for sufficiently small ε,

$$\begin{aligned} I_{p,j}^{\varepsilon}(A_4^{(3)}, y) &\le \varepsilon^{4p} \frac{{C'}^4}{|y|^4} \int_{\{|y|/C' \le |z| \le 2|y|\}} dz G(y,z) \left(\log \frac{|z|}{\varepsilon}\right)^{p-2} \\ &\le \varepsilon^{4p} \frac{{C'}^4}{|y|^4} \int_{\{|z| \le 3|y|\}} \frac{dz}{2\pi^2 |z|^2} \left(\log_+ \left(\frac{|z+y|}{\varepsilon}\right)\right)^{p-2} \\ &\le \varepsilon^{4p} \frac{9{C'}^4}{2|y|^2} \left(\log \frac{4|y|}{\varepsilon}\right)^{p-2}. \end{aligned}$$

It follows that

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \ge f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{|y|}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_4^{(3)}, y) \right) = 0.$$

Finally, using $(\mathfrak{H}_k^3)$ with $1 \le k \le p-1$, we get for ε small,

$$\begin{aligned} I_{p,j}^{\varepsilon}(A_5^{(3)}, y) &\le 4\, \varepsilon^{4p} \int_{2|y|}^{\infty} \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2} \\ &\le K\, \varepsilon^{4p} \frac{1}{|y|^2} \left(\log \frac{2|y|}{\varepsilon}\right)^{p-2}, \end{aligned}$$

for some constant K depending only on p. Thus,

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \ge f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{|y|}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_5^{(3)}, y) \right) = 0.$$

Combining our estimates on $I^{\varepsilon}_{p,j}(A^{(3)}_i, y)$ for $1 \leq i \leq 5$ and then summing over j using (15), we get that $(\mathfrak{H}^3_p)$ holds for the given f and β_0. This completes the proof of the bounds $(\mathfrak{H}^1_p)$, $(\mathfrak{H}^2_p)$ and $(\mathfrak{H}^3_p)$, for every $p \geq 1$.

It now follows from (9) and $(\mathfrak{H}^3_p)$ that, for every $p \geq 1$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-4p} \left(\log \frac{|x|}{\varepsilon} \right)^{1-p} \mathbb{N}_x \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^p \right] = p! \left(\frac{\overline{\varphi}}{2\pi^2} \right)^p |x|^{-2},$$

uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$. This completes the proof of Lemma 3. □

References

[DIP] Dawson D.A., Iscoe I., Perkins E.A., Super-Brownian Motion: Path Properties and hitting probabilities *Probab. Th. Rel. Fields* , **83**(5), (1989), pp. 135-205.

[IM] Itô, K., McKean, H.P., *Diffusion Processes and their Sample Paths*. Springer (1965).

[Le] Lee T.Y., Asymptotic Results for Super-Brownian Motions and Semilinear Differential Equations, *The Annals of Probability*, **29** (2001), pp. 1047-1060.

[LG] Le Gall J.-F. *Spatial Branching Processes, The Brownian Snake and Partial Differential Equations*, Birkhäuser, Boston, (1999).

[LG1] Le Gall J.F., *Spatial Branching Processes, Random Snakes and Partial Differential Equations* Lectures in Mathematics. Birkhuser, 1999.

[LG2] Le Gall J.F., The Hausdorff Measure of the Range of Super-Brownian Motion LMENS-98-44

[LG3] Le Gall J.F., The Hausdorff Measure of the Range of Super-Brownian Motion LMENS-98-44

[Me] Merle, M., Local behaviour of local times of super-Brownian motion. *Ann. Institut H. Poincaré Probab. Stat.*, to appear.

[Pe] Perkins E.A., *Dawson-Watanabe Superprocesses and Measure-valued Diffusions*, Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités de Saint-Flour XXIX, Springer LNM **1781**, (1999).

[Su 89] Sugitani S., Some properties for the measure-valued branching diffusion processes *J. Math. Soc. Japan*, Vol.**41**(3), (1989), pp. 437-462