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Résultats asymptotiques pour le super-mouvement brownien et le modèle du votant

Asymptotic Results for Super-Brownian Motion and the Voter Model

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CHAPITRE I

Introduction

Dans de très nombreux contextes, tels que la physique statistique, la génétique, l'épidémiologie, ou encore la sociologie, on souhaite modéliser l'évolution d'un système de particules ou d'une population d'individus. A cette motivation correspondent des modèles mathématiques discrets, comme le *modèle du votant*, et continus, comme le *super-mouvement brownien*. Ces deux objets ont un intérêt mathématique propre que l'on va tenter de mettre en évidence dans cette introduction.

Cette thèse est tout d'abord consacrée à l'étude de théorèmes limites concernant la mesure d'occupation et les temps locaux du super-mouvement brownien. Dans un second temps, elle vise à approfondir les liens existant entre le modèle du votant et le super-mouvement brownien, à travers le calcul des asymptotiques de la probabilité pour le modèle du votant d'atteindre un point éloigné. Nos travaux traitent donc de questions relatives à deux objets principaux :

- le *super-mouvement brownien*, processus de Markov à valeurs dans un espace de mesures. Il appartient à la classe plus large des *superprocessus*, qui apparaissent comme limites continues de modèles discrets où les individus (ou particules) constituant une population sont soumis à la fois à un mécanisme de branchement et à un déplacement spatial. Ces objets, et leurs liens avec une classe d'équations aux dérivées partielles, ont été étudiés de façon approfondie dans les années 80, entre autres par Dawson, Perkins, Dynkin (cf [**D 93**], [**DIP 89**], [**Pe 99**]). Une autre construction des superprocessus à l'aide du *serpent brownien* est due à Le Gall [**LG 99**]. Cette approche permet de mieux séparer le mécanisme de branchement du mouvement spatial des particules, et s'avère un outil puissant dans l'étude des superprocessus.
- le *modèle du votant*, qui peut être interprété comme un modèle de compétition entre différentes opinions dans une population (cf [**HL 75**]), ou également comme un modèle de compétition entre espèces (cf [**CS 73**]). Le modèle du votant peut en outre être considéré comme le processus dual d'un système de marches aléatoires coalescentes. Récemment, Cox, Durrett et Perkins [**CDP 00**], puis Bramson, Cox et Le Gall [**BCLG 01**] ont établi qu'une version rééchelonnée du modèle du votant converge vers le super-mouvement brownien. Ce résultat est à rapprocher de théorèmes similaires faisant apparaître le super-mouvement brownien à la limite de changement d'échelle d'une classe plus large de systèmes de particules en interaction, comme par exemple la percolation orientée et une version du processus de contact (cf [**vdHS 00**]), les "arbres sur treillis"(lattice trees, cf [**DS 98**]), et des modèles de génétique des populations (cf [**CP 06**]).

Nos résultats sont exposés dans les chapitres III, IV et V, qui correspondent respectivement à :

- un article intitulé *Comportement local des temps locaux du super-mouvement brownien*, à paraître dans *Annales de l'Institut Henri Poincaré*, 2005.

– un article intitulé *Sur la mesure d'occupation du super-mouvement brownien*, écrit en collaboration avec Jean-François Le Gall.
– un article intitulé *Probabilité d'atteinte d'un point éloigné pour le modèle du votant*.

Ces trois derniers chapitres peuvent être lus séparément, ils possèdent chacun les rappels et leurs notations propres qui assurent leur cohérence.

Dans le chapitre III, nous établissons, en dimension $d = 2$ et 3, des théorèmes limites concernant des fonctionnelles additives du super-mouvement brownien. On obtient tout d'abord des informations précises sur le comportement local des temps locaux du super-mouvement brownien en dimensions 2 et 3. En particulier, on obtient le résultat suivant comme un cas particulier du théorème III.1.

Une version simplifiée du résultat principal du chapitre III *: Soit $d = 2$ ou 3, et deux points $x_0 \in \mathbb{R}^d \backslash \{0\}$, $x \in \mathbb{R}^d$ fixés. On désigne par X un super-mouvement brownien initialement en δ_{x_0}, et par L_t^y le temps local de X au temps $t \geq 0$ et au point $y \in \mathbb{R}^d$. On pose d'autre part $\psi(c) = \sqrt{c}$ lorsque $d = 3$, $\psi(c) = \frac{c}{\sqrt{\ln(c)}}$ lorsque $d = 2$. Enfin on note $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ l'espace des trajectoires continues réelles, muni de la topologie de la convergence uniforme.*

Alors, au sens de la convergence faible dans l'espace $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$,

$$(\psi(c)(L_t^{x/c} - L_t^0))_{t \geq 0} \underset{c \to \infty}{\longrightarrow} (\beta_{a(x)L_t^0})_{t \geq 0},$$

où β est un mouvement brownien standard indépendant de X, et $a(x)$ est une constante qui ne dépend que de x.

En fait, le théorème III.1 énonce une convergence plus détaillée, faisant intervenir un nombre fini, quelconque, de valeurs de x.

Au chapitre III, on déduit ensuite du théorème III.1 des résultats, en dimension $d \leq 3$, concernant le comportement local de la mesure d'occupation de X (voir proposition III.2) sous $\mathbf{P}_{\delta_{x_0}}$ (voir proposition III.2), puis sous sa mesure d'excursion et conditionné à toucher 0 (voir proposition III.5).

En dimension 4, la mesure d'occupation du super-mouvement brownien n'a pas de densité (autrement dit, le super-mouvement brownien dans $\mathbb{R}^4$ n'a pas de temps locaux). On peut tout de même obtenir en dimension 4 des informations sur le comportement local de la mesure d'occupation totale. On démontre en effet au chapitre IV le résultat suivant (corollaire IV.1).

Résultat principal du chapitre IV *: On note $Y := \int_0^\infty X_s ds$ la mesure d'occupation totale d'un super mouvement brownien X en dimension 4, x un point de $\mathbb{R}^4 \setminus \{0\}$. La loi de $\frac{2Y(B(0,\varepsilon))}{\varepsilon^4 \ln(\frac{|x|}{\varepsilon})}$ sous $\mathbf{P}_{\delta_x}\left[\ .\ \middle| Y(B(0,\epsilon)) > 0\right]$ converge lorsque $\varepsilon \to 0$ vers une loi exponentielle.*

On démontre le même résultat pour la mesure d'occupation d'un super-mouvement brownien sous sa mesure d'excursion, et conditionné à toucher $B(0, \varepsilon)$ (voir théorème IV.1).

On renforce au chapitre V les liens entre super-mouvement brownien et modèle du votant. On considère, en dimension $d \geq 2$, le modèle du votant sur le réseau $\mathbb{Z}^d$, partant avec l'opinion 1 à l'origine, et l'opinion 0 partout ailleurs. En utilisant la convergence d'une version rééchelonnée et à valeurs mesures de ce modèle vers le super-mouvement brownien sous sa mesure d'excursion, démontrée par [**BCLG 01**], on détermine l'ordre de grandeur de la probabilité pour l'opinion 1 d'atteindre un point éloigné de l'origine.

Avant d'énoncer le résultat principal du chapitre V, on a besoin d'introduire quelques notations. On considère un noyau p sur $\mathbb{Z}^d \times \mathbb{Z}^d$ qui vérifie les hypothèses suivantes.

$$\mathcal{H}(\text{noyau}) \begin{cases} p \text{ irréductible, centré, } p(0,0)=0, \\ p \text{ symétrique, invariant par translation :} \\ \quad p(x,y)=p(y,x)=p(0,y-x), \\ p \text{ isotrope :} \\ \quad \sum_{y\in\mathbb{Z}^d} p(0,y)y^i y^j = \sigma^2\delta_{ij} \text{ pour un } 0<\sigma^2<\infty, \\ p \text{ possède des moments exponentiels :} \\ \quad \exists C>0 \ : \ \sum_{y\in\mathbb{Z}^d} p(0,y)\exp(C|y|)<\infty. \end{cases}$$

On note alors $\beta_2 = 2\pi\sigma^2$, et $\beta_d, d \geq 3$ la probabilité qu'une marche aléatoire à taux 1 sur $\mathbb{Z}^d$, de noyau p, partie de l'origine, n'y retourne jamais. D'autre part, si $x \in \mathbb{R}^d$ et $c > 0$, on note $[x]_c$ le point de $c^{-1}\mathbb{Z}^d$ le plus proche de x (au cas où il y en a plusieurs, on choisit celui qui est le plus proche de l'origine).

Résultat principal du chapitre V *: Soit $x \in \mathbb{R}^d \setminus \{0\}$ fixé. On note $(\xi_t^0, t \geq 0)$ le modèle du votant sur le réseau $\mathbb{Z}^d$, de noyau p. Pour $c > 0$, on introduit*

$$\phi_d(c) = \begin{cases} \frac{c^2}{2\ln(c)} & \text{si } d=2, \\ c^2 & \text{si } d=3, \\ c^{d-2} & \text{si } d\geq 5, \end{cases}$$

Alors, si $d = 2$ ou 3,

$$\lim_{c\to\infty} \phi_d(c) P(\exists t \geq 0 : c[x]_c \in \xi_t^0) = \frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}.$$

Si $d \geq 5$, il existe des constantes positives a_d, b_d dépendant seulement de x qui vérifient

$$a_d \leq \liminf_{c\to\infty} \phi_d(c) P(\exists t\geq 0 : c[x]_c \in \xi_t^0) \leq \limsup_{c\to\infty} \phi_d(c)P(\exists t \geq 0 : c[x]_c \in \xi_t^0) \leq b_d.$$

On conjecture que pour $d = 4$ et pour $\phi_4(c) = c^2\ln(c)$, on a un résultat similaire à celui du cas $d \geq 5$, mais nous n'avons montré, pour l'instant, que la borne pour la limite inférieure.

Dans le chapitre II, on expose plus en détail tous ces résultats, leurs motivations et conséquences, ainsi que les moyens employés pour les obtenir. Avant cela, on prend le parti de faire dans cette partie une introduction relativement détaillée des deux objets étudiés et de leurs propriétés (super-mouvement brownien, §I.1, et modèle du votant, §I.2).

1. Super-mouvement brownien

Dans cette partie, on définit le super-mouvement brownien, tout d'abord en tant que limite en loi de systèmes de particules soumis à un mécanisme de branchement rééchelonnés (§I.1.1), puis à l'aide du serpent brownien (§I.1.2). On en décrit ensuite certaines propriétés (§I.1.3).

1.1. Une première définition du super-mouvement brownien. Pour rendre la définition du super-mouvement brownien plus intuitive et motiver notre étude, on commence par décrire des modèles discrets d'évolution de population.

1.1.1. *Processus, arbres et forêts de Galton-Watson.* Avant de définir le modèle, on introduit quelques notations. On considère l'ensemble de mots $\mathcal{U} := \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$, où $(\mathbb{N}^*)^0 = \{\emptyset\}$. Pour $n \in \mathbb{N}^*$, un élément $u \in (\mathbb{N}^*)^n$ est noté $(u_1, ..., u_n)$, et on écrit dans ce cas $|u| := n$. Un *arbre ordonné enraciné* θ est la donnée d'une sous-ensemble de $\bigcup_{n=0}^{\infty} \mathbb{N}^n$ tel que

- La racine $\emptyset \in \mathbb{N}^0$ est un élément de θ.
- Pour tout $n \in \mathbb{N}^*$, si $(u_1, ..., u_n, u_{n+1}) \in \theta$, alors $(u_1, ..., u_n) \in \theta$.
- un élément $(u_1, ..., u_n) \in \theta$ possède un nombre de descendants $k_u(\theta) \in \mathbb{N}$, et on a $(u_1, ..., u_n, j) \in \theta \Leftrightarrow 1 \leq j \leq k_u(\theta)$.

La n-ième génération de θ est l'ensemble $\{u \in \theta : |u| = n\}$.

On décrit maintenant l'évolution en temps discret d'une population d'individus qui se reproduisent suivant une loi de descendance donnée, de façon indépendante. Cette loi de descendance est décrite par une variable aléatoire D à valeurs dans $\mathbb{N}$, telle que

$$E[D] = 1, \qquad \mathrm{var}[D] = \gamma > 0.$$

On note $D_u, u \in \mathcal{U}$ des copies indépendantes de D.

Définition I.1. *Un* arbre de Galton-Watson critique *est un arbre aléatoire ordonné et enraciné de même loi que*

$$\theta := \left\{u = (u^1, ..., u^n) : \quad \textit{pour tout } 1 \leq j \leq n \quad u^j \leq D_{(u^1,...,u^{j-1})}\right\}.$$

Ce modèle permet de décrire la généalogie d'une population descendante d'un ancêtre original $\emptyset$. La taille de la n-ième génération d'un arbre de Galton-Watson θ est la variable $T_n := \#\{u \in \theta : |u| = n\}$.

Définition I.2. *On appelle* processus de Galton-Watson *de valeur initiale* 1 *un processus à valeurs dans* $\mathbb{N}$*, de même loi que la suite* $T := (T_n, n \geq 0)$ *des tailles des générations d'un arbre de Galton-Watson. Un* processus de Galton-Watson de valeur initiale p*, noté* $T^{(p)}$*, est un processus de même loi que la somme de* p *copies indépendantes de* T*.*

Ce modèle très simple permet de décrire l'évolution de la taille de la population. Le processus $T^{(p)}$ décrit la suite des tailles de l'ensemble des n-ièmes générations des p premiers arbres d'une *forêt de Galton-Watson*, laquelle est donnée par une suite d'arbres de Galton-Watson critiques $\theta_1, \theta_2, ...$ indépendants.

En outre, remarquons que $T^{(p)}$ a même loi que le processus $\tilde{T}^{(p)}$, défini par la relation de récurrence

$$\tilde{T}_0^{(p)} = p, \quad \tilde{T}_{n+1}^{(p)} = \sum_{i=1}^{\tilde{T}_n^{(p)}} D_{n,i},$$

où les $D_{n,i}, n \in \mathbb{N}, i \in \mathbb{N}^*$ sont des copies indépendantes de la variable D. Ainsi, d'après nos hypothèses sur la loi de D, $(T_n^{(p)}, n \geq 0)$ est une martingale, et donc, $\max\{n > 0 : T_n^{(p)} > 0\}$ est fini presque sûrement. Ceci permet de vérifier en particulier qu'un arbre de Galton-Watson est également fini presque sûrement.

1.1.2. *Systèmes de particules soumises à un mécanisme de branchement.* Le modèle d'arbres que l'on vient de décrire permet uniquement de modéliser une généalogie. Pour affiner notre modèle de populations, on va en outre supposer que les individus sont soumis à un déplacement spatial. On peut penser à ce déplacement comme décrivant l'évolution dans le temps d'une caractéristique donnée des individus (leur génotype par exemple, ou tout simplement leur position dans l'espace). On considère un espace polonais (E, d) et ξ un processus de Markov à trajectoires càdlàg dans E. On part d'un nombre fini d'individus, et le mécanisme d'évolution est décrit de la façon suivante. De façon indépendante :

– Durant son temps de vie, chaque individu se déplace dans E suivant la loi de ξ .
– Les temps de vie de chaque individu sont des variables exponentielles indépendantes[1] de paramètre 1, indépendantes des déplacements spatiaux.
– À sa mort, une particule donne naissance (à l'endroit où elle meurt) à un nombre aléatoire de descendants suivant la loi de D. Ces nombres d'enfants sont également indépendants, et indépendants des temps de vie et des déplacements spatiaux.

Remarquons que si l'on fait abstraction du déplacement spatial, l'arbre de descendance d'un individu donné est un arbre de Galton-Watson[2]. En conséquence, la population que nous décrivons s'éteint presque sûrement au bout d'un certain temps.

On désigne par $M_F(E)$ l'espace des mesures finies sur E, muni de la topologie de la convergence faible (ce qui en fait un espace polonais). Introduisons maintenant le processus à valeurs dans l'espace $M_F(E)$, qui décrit le mécanisme d'évolution ci-dessus. On indexe par I_t les particules en vie au temps t et on note $Z_t^i, i \in I_t$ leurs positions respectives au temps t.

Définition I.3. *On appelle système de particules soumises à un mécanisme de branchement (*branching particle sytem*) le processus $(BPS_t^\xi, t \geq 0)$ défini par*

$$\mathrm{BPS}_t^\xi := \sum_{I_t} \delta_{Z_t^i}.$$

Si ν est une mesure purement atomique on note $\mathbf{Q}_\nu^\xi$ la mesure de probabilité sous laquelle $\mathrm{BPS}_0^\xi = \nu$.

On note $D(\mathbb{R}_+, M_F(E))$ l'espace des trajectoires càdlàg dans $M_F(E)$. D'après nos hypothèses sur ξ, $(\mathrm{BPS}_t^\xi, t \geq 0)$ est à valeurs dans $D(\mathbb{R}_+, M_F(E))$. Notons d'autre part que l'absence de mémoire de la loi exponentielle, et le caractère markovien de ξ, entraînent facilement que BPS^ξ est un processus de Markov.

On va essentiellement considérer deux cas particuliers pour le processus de Markov ξ introduit plus haut. Un exemple fondamental est constitué par le mouvement brownien $(B_t, t \geq 0)$ à valeurs dans $\mathbb{R}^d$. Un deuxième exemple fondamental est celui d'une marche aléatoire sur le réseau entier. On introduit pour cela $p : \mathbb{Z}^d \times \mathbb{Z}^d$ un noyau de probabilité qui satisfait les hypothèses $\mathcal{H}_{\text{noyau}}$ introduites dans le premier

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[1] Il est également possible de faire l'hypothèse que les temps de vie sont déterministes, par exemple tous égaux à 1 (voir par exemple [**Pe 99**], chapitre II).

[2] Remarquons également que lorsque les temps de vie sont tous égaux à 1 et le système compte p particules au temps 0, le processus qui décrit la taille de la population est un processus de Galton-Watson de valeur initiale p.

paragraphe[3]. On note alors $(Z_t, t \geq 0)$ un processus qui est une marche aléatoire à temps continu sur $\mathbb{Z}^d$, à taux r, de noyau p, issue de x, sous la probabilité $\mathbb{Q}_x^{r,\sigma^2}$.

Dans le cas particulier où E est l'espace $\mathbb{R}^d$ et $\xi = B$, BBM $:= \mathrm{BPS}^B$ est appelé "mouvement brownien branchant" (*branching Brownian motion*).

Dans le cas où ξ est une marche aléatoire sur E, BPS^ξ est appelé "marche aléatoire branchante" (*branching random walk*). Dans le cas particulier où $E = \mathbb{Z}^d$, muni de sa distance usuelle, et $\xi = Z$ est la marche aléatoire à taux $r = 1$ définie plus haut, on note BRW la marche aléatoire branchante obtenue, et $\mathbf{Q}_\nu^{\sigma^2} := \mathbf{Q}_\nu^Z$.

1.1.3. *Une définition alternative des marches aléatoires branchantes.* On propose dans ce paragraphe un autre modèle discret de population. Quoique différent, on le nomme également *marche aléatoire branchante*. On considère le noyau de probabilité $p : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}_+$, décrit plus haut. Ici, de façon indépendante,

- les temps de vie sont des variables exponentielles de paramètre $\gamma' > 0$,
- pendant son temps de vie, une particule née au point x reste en ce point, et, pour tout y tel que $p(x, y) > 0$, cette particule donne naissance à taux $\gamma' p(x, y)$ à une nouvelle particule au point y.

Les déplacements spatiaux, contrairement au paragraphe précedent, n'ont pas lieu pendant les temps de vie des particules, mais à l'instant de leur naissance. D'autre part, il est facile de se rendre compte que l'arbre de descendance sous-jacent d'une particule donnée est ici un arbre binaire de Galton-Watson (c'est-à-dire associé à la loi de descendance $P(D = 0) = P(D = 2) = 1/2$). Une nouvelle fois, la population que nous décrivons s'éteint presque sûrement.

Pour $t \geq 0, x \in \mathbb{Z}^d$, on note $\zeta_t(x)$ le nombre de particules présentes au site x au temps t. Le processus à valeurs mesures associé au processus des configurations $\{\zeta_t(x), x \in \mathbb{Z}^d\}, t \geq 0$, s'écrit

$$\mathrm{MAB}_t := \sum_{x \in \mathbb{Z}^d} \zeta_t(x) \delta_x.$$

Ici encore, $\mathrm{MAB}_t, t \geq 0$ est un processus de Markov. Pour une mesure ν purement atomique, on note $\mathfrak{Q}_\nu^{\gamma',\sigma^2}$ la mesure de probabilité sous laquelle il est défini lorsque $\mathrm{MAB}_0 = \nu$.

Comme on le verra plus loin, on peut exprimer le modèle du votant en ces termes, à la différence que les taux de naissance et de mort vont cette fois dépendre de la configuration. On pourrait également décrire en termes similaires des modèles de compétition d'espèces plus élaborés, tels que le *modèle de Lotka-Volterra* (voir [**CP 06**]).

1.1.4. *Super-mouvement brownien en tant que limite de marches aléatoires branchantes.* On peut se demander comment se comportent asymptotiquement les modèles de populations que nous venons d'introduire. Rappelons que dans tous ces modèles, la population s'éteint presque sûrement, c'est-à-dire

$$\mathbf{Q}_\nu^\xi(\exists\, T > 0 : \mathrm{BPS}_t^\xi = 0\ \forall t \geq T) = 1, \quad \mathfrak{Q}_\nu^{\gamma',\sigma^2}(\exists\, T > 0 : \mathrm{MAB}_t = 0\ \forall t \geq T) = 1.$$

Malgré tout, en supposant qu'on a initialement affaire à une population de très grande taille, il semble légitime de tenter de faire une approximation continue de

[3]Ces hypothèses sur le noyau p sont relativement restrictives, sans être d'ailleurs toutes indispensables dans un premier temps. Cependant, elles nous permettent une relative cohérence avec la partie concernant le modèle du votant. Il est parfois utile de penser au cas simple où $p(0, .)$ charge de façon uniforme les plus proches voisins de l'origine. $p_0(x, y) = p_0(y - x) = 1/(2d)$ si et seulement si $|y - x| = 1$.

notre système discret. Que se passe-t-il si on considère une suite de modèles dont les populations initiales respectives sont d'ordre N ?

Un résultat dû à Feller affirme qu'il faut alors rééchelonner la taille de la population de la façon suivante. On considère des processus de Galton-Watson $T^{(N)}$ de valeurs initiales respectives p_N.

PROPOSITION I.1. *Supposons que* $\frac{p_N}{N} \to x_0 \in \mathbb{R}_+^*$ *lorsque* $N \to \infty$. *Alors, la suite de processus* $\left(\frac{1}{N} T^{(N)}_{\lfloor Nt \rfloor}, t \geq 0\right)_{N \in \mathbb{N}^*}$ *converge en loi (au sens de la convergence faible dans l'espace de Skorokhod* $D(\mathbb{R}_+, \mathbb{R}_+)$*) vers un processus* $(F_t, t \geq 0)$, *unique solution de l'équation différentielle stochastique :*

$$F_t = x_0 + \int_0^t \sqrt{\gamma F_s} d\beta_s, \tag{1}$$

où $(\beta_t, t \geq 0)$ *désigne un mouvement brownien standard.*

En réalité (cf §I.1.2), non seulement la taille de la population, mais également sa structure généalogique possèdent une approximation continue.

Nous souhaitons à présent rééchelonner des systèmes de particules soumises à un mécanisme de branchement, lorsque la population initiale est d'ordre N. Considérons l'exemple simple des marches aléatoires branchantes. Heuristiquement, le théorème de Donsker suggère que lorsqu'on accélère le temps par un facteur N, il faut diviser les distances par $\sqrt{N}$. D'après la proposition I.1 il est alors naturel d'attribuer la masse $1/N$ à chaque particule vivante. Plus précisément, pour $N \in \mathbb{N}^*$, on considère une mesure $\nu^{(N)}$ purement atomique sur $\mathbb{Z}^d$, et on définit sous $\mathbf{Q}^{\sigma^2}_{\nu^{(N)}}$ le processus

$$\left(\mathrm{BRW}^{(N)}_t(.) := \frac{1}{N}\mathrm{BRW}_{Nt}(\sqrt{N}.), t \geq 0\right),$$

Pour espérer une limite non dégénérée à la suite de processus $(\mathrm{BRW}^N)_{N \in \mathbf{N}^*}$, il faut que les mesures initiales convergent, c'est-à-dire qu'il existe une mesure $\nu \in M_F(\mathbb{R}^d)$ telle que, au sens de la convergence faible dans $M_F(\mathbb{R}^d)$, $\frac{1}{N}\nu^{(N)}(\sqrt{N}.) \underset{N\to\infty}{\longrightarrow} \nu$. On va voir (Théorème I.1 ci dessous) que cette hypothèse suffit en fait à trouver une limite non dégénérée à la suite des marches aléatoires branchantes rééchelonnées.

De manière similaire, on définit, sous $\mathfrak{Q}^{\gamma', \sigma^2}_{\nu^{(N)}}$, le processus

$$\left(\mathrm{MAB}^{(N)}_t(.) := \frac{1}{N}\mathrm{MAB}^{\nu^{(N)}}_{Nt}(\sqrt{N}.), t \geq 0\right),$$

ainsi que, sous $\mathbf{Q}^{B}_{\nu^{(N)}}$, le processus

$$\left(\mathrm{BBM}^{(N)}_t(.) := \frac{1}{N}\mathrm{BBM}^{\nu^{(N)}}_{Nt}(\sqrt{N}.), t \geq 0\right).$$

La remarque suivante donne une interprétation plus intuitive de ce changement d'échelle.

<u>Remarque : un changement d'échelle équivalent pour les BRW</u> :

Fixons $N \in \mathbb{N}^*$. Notons $S_N := N^{-1/2}\mathbb{Z}^d$, et $p_N : S_N \times S_N \to \mathbb{R}_+$ le noyau de saut défini par $p_N(x, y) := p(\sqrt{N}x, \sqrt{N}y)$. Rappelons que $\nu^{(N)}$ est une mesure finie purement atomique sur S_N, et s'écrit donc $\sum_{y \in \mathbb{Z}^d} k_y \delta_y$, où, pour tout $y \in \mathbb{Z}^d$, $k_y \in \mathbb{N}$. On considère alors, la configuration initiale sur S_N qui, pour tout $y \in S_N$, compte $k_{\sqrt{N}y}$ particules au point y, et le mécanisme d'évolution suivant. De façon indépendante,

– durant son temps de vie, une particule se déplace suivant la trajectoire d'une marche aléatoire sur S_N, à taux N, de noyau p_N.
– Les temps de vie de chaque particule sont des exponentielles indépendantes de paramètre N.
– La loi de descendance est la même que précedemment.

Le processus à valeurs mesures, qui décrit ce mécanisme d'évolution en attribuant la masse $1/N$ à chaque particule vivante, possède alors la même loi que le processus $\mathrm{BRW}^{(N)}$ sous $\mathbf{Q}^{\sigma^2}_{\nu^{(N)}}$.

On peut faire une remarque similaire concernant les marches aléatoires branchantes rééchelonnées $\mathrm{MAB}^{(N)}$ et pour les mouvements browniens branchants rééchelonnés $\mathrm{MAB}^{(N)}, \mathrm{BBM}^{(N)}$. Dans ce dernier cas cependant, d'après les propriétés d'invariance du mouvement brownien, il est également possible de ne pas changer l'échelle spatiale.

Watanabe [**W 68**] a, le premier, établi qu'une suite de mouvements browniens rééchelonnés possède une limite en loi, appelée *super-mouvement brownien* (on peut également trouver une démonstration plus concise de ce fait dans [**Eth 00**], parties 1.4 et 1.5). La preuve fut ensuite généralisée aux BPS, dont les limites respectives constituent la classe plus large des *superprocessus*, par Dawson [**D 75**] (voir également [**Pe 99**], parties II.4 et III.1).

Avant de donner la définition du super-mouvement brownien, nous avons besoin d'introduire les principales notations que nous utilisons par la suite. On va voir que le super-mouvement brownien est à valeurs dans l'espace des trajectoires continues dans $M_F(\mathbb{R}^d)$, noté $\Omega_{M_F}(\mathbb{R}^d)$. De façon générale, lorsque E' est un espace polonais, on introduit deux espaces de trajectoires à valeurs dans E', à savoir $\Omega_{E'} := \mathcal{C}(\mathbb{R}_+, E')$ l'espace des trajectoires continues dans E', muni de la topologie de la convergence uniforme sur les compacts, et $D(\mathbb{R}_+, E')$ l'espace des trajectoires càdlàg dans E', muni de la topologie J_1 de Skorokhod. On désigne systématiquement par $\mathcal{F}$ la tribu des boréliens de $\Omega_{E'}$.

D'autre part, pour une mesure $\mu \in M_F(\mathbb{R}^d)$, et une fonction $f : \mathbb{R}^d \to \mathbb{R}$ μ-intégrable, on écrit $< \mu, f >:= \int_{\mathbb{R}^d} f(x)\mu(dx)$. On note $\mathcal{B}(\mathbb{R}^d)$, resp. $\mathcal{B}_b(\mathbb{R}^d)$, resp. $\mathcal{B}_b^+(\mathbb{R}^d)$ l'espace des fonctions de $\mathbb{R}^d$ dans $\mathbb{R}$, boréliennes, resp. boréliennes et bornées, resp. boréliennes, bornées et positives. De la même manière, on note $\mathcal{C}(\mathbb{R}^d)$ resp. $\mathcal{C}_b(\mathbb{R}^d)$, resp. $\mathcal{C}_b^+(\mathbb{R}^d)$ l'espace des fonctions de $\mathbb{R}^d$ dans $\mathbb{R}$ continues, resp. continues bornées, resp. continues bornées positives. De plus, $\mathcal{C}_b^n(\mathbb{R}^d)$ désigne les fonctions de $\mathcal{C}_b(\mathbb{R}^d)$ qui sont n fois différentiables. Enfin, pour $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$, le laplacien de ϕ est noté $\Delta\phi$.

Définition I.4. *Soit $\nu \in M_F(\mathbb{R}^d)$, $\gamma > 0, \sigma > 0$. On note $(\mathcal{O}, \mathfrak{F}, \mathbf{P})$ un espace de probabilité muni d'une filtration complète et continue à droite $(\mathfrak{F}_t)_{t\geq 0}$. Un processus $(X_t, t \geq 0)$ sur $(\mathcal{O}, \mathfrak{F}, (\mathfrak{F}_t), \mathbf{P})$, adapté, continu, à valeurs dans $M_F(\mathbb{R}^d)$ est un* super-mouvement brownien *de valeur initiale ν, avec coefficient de branchement γ et coefficient de diffusion σ^2 (noté en abrégé $\mathrm{SBM}_\nu^{\gamma,\sigma^2}$), si et seulement si il est* *solution du problème de martingales* $(MP)^{\gamma,\sigma^2}_\nu$ *ci-dessous.*

$$(MP)^{\gamma,\sigma^2}_\nu \begin{cases} \text{Pour toute fonction } \phi \in \mathcal{C}^2_b(\mathbb{R}^d), \\ < X_t, \phi > = < \nu, \phi > + M_t(\phi) + \frac{1}{2}\int_0^t < X_s, \sigma^2 \Delta\phi > ds, \\ \text{où } M_t(\phi) \text{ est une } \mathcal{F}_t\text{-martingale continue telle que} \\ M_0(\phi) = 0, \text{ et la variation quadratique de } M(\phi) \text{ est} \\ < M(\phi) >_t = \int_0^t < X_s, \gamma\phi^2 > ds. \end{cases}$$

Le théorème II.5.1 de [**Pe 99**] vérifie que la définition ci-dessus a un sens, en montrant qu'il existe une solution sur l'espace $\Omega_{M_F(\mathbb{R}^d)}$ au problème $(MP)^{\gamma,\sigma^2}_\nu$, puis qu'une telle solution est unique en loi. Ce théorème affirme également que le processus canonique sur $(\Omega_{M_F(\mathbb{R}^d)}, \mathcal{F})$ solution de $(MP)^{\gamma,\sigma^2}_\nu$ est un processus de Markov, défini sous une mesure de probabilité que l'on note $\mathbf{P}^{\gamma,\sigma^2}_\nu$.

Lorsque le contexte est clair, on se permettra parfois d'oublier l'exposant dans les notations $\mathbf{P}^{\gamma,\sigma^2}_\nu, \mathrm{SBM}^{\gamma,\sigma^2}_\nu, (MP)^{\gamma,\sigma^2}_\nu$. On est désormais en mesure de reformuler partiellement le résultat de [**D 75**][4].

THÉORÈME I.1. *Pour* $N \in \mathbb{N}^*$*, on considère une mesure* $\nu^{(N)}$ *sur* S_N*. On suppose qu'il existe* $\nu \in M_F(\mathbb{R}^d)$ *telle que*

$$\frac{1}{N}\nu^{(N)}(.\sqrt{N}) \underset{N\to\infty}{\longrightarrow} \nu.$$

Alors, lorsque $N \to \infty$*, au sens de la convergence faible dans l'espace de Skorokhod* $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$*,*
– *la suite de processus* $(\mathrm{BBM}^{(N)})$ *sous* $\mathbf{Q}^B_{\nu^{(N)}}$ *converge vers un* $\mathrm{SBM}^{\gamma,1}_\nu$*,*
– *la suite de processus* $(\mathrm{BRW}^{(N)})$ *sous* $\mathbf{Q}^{\sigma^2}_{\nu^{(N)}}$ *converge vers un* $\mathrm{SBM}^{\gamma,\sigma^2}_\nu$*,*
– *la suite de processus* $(\mathrm{MAB}^{(N)})$ *sous* $\mathfrak{Q}^{\gamma',\sigma^2}_{\nu^{(N)}}$ *converge vers un* $\mathrm{SBM}^{2\gamma',\gamma'\sigma^2}_\nu$*.*

Le super-mouvement brownien est un processus à valeurs mesures, on peut en outre définir sa *mesure d'occupation.*

DÉFINITION I.5. *La mesure d'occupation du super-mouvement brownien* X *est le processus* $(Y_t, t \geq 0)$ *défini par la relation*

$$\forall\psi \in \mathcal{C}_b(\mathbb{R}^d) \qquad < Y_t, \psi > = \int_0^t < X_s, \psi > ds.$$

L'existence d'une densité pour la mesure d'occupation correspond à l'existence de *temps locaux* pour le super-mouvement brownien.

1.1.5. *Equation de Laplace du super-mouvement brownien.* Il est possible (voir [**Pe 99**], Théorème II.5.9), de déduire du problème de martingale $(MP)^{\gamma,\sigma^2}_\nu$ l'expression de la transformée de Laplace du super-mouvement brownien et de sa mesure d'occupation à l'aide des solutions d'une classe d'équations aux dérivées partielles.

[4]Celui-ci se situe dans le cadre plus général d'un BPS^ξ. La limite obtenue dans le cas général est un *superprocessus*, qui vérifie également un problème de martingale. Il faut alors, dans $(MP)^{\gamma,\sigma^2}_\nu$, remplacer $\sigma^2\Delta/2$ (générateur du mouvement brownien σ-dilaté) par le générateur de ξ.

THÉORÈME I.2. *Soit* $(X_t, t \geq 0)$ *un* $\mathrm{SBM}_\nu^{\gamma,\sigma^2}$.
Pour toutes fonctions $\phi \in \mathcal{C}_b^+(\mathbb{R}^d) \cap \mathcal{C}_b^2(\mathbb{R}^d), \psi \in \mathcal{C}_b^+(\mathbb{R}^d)$, *on a*

$$\mathbf{E}_\nu \left(\exp\{-<X_t,\phi> - <Y_t,\psi> ds\}\right) = \exp\left(-<\nu, v_{\phi,\psi}(t,.)>\right). \tag{2}$$

où la fonction $(t,x) \to v_{\phi,\psi}(t,x)$ *est solution sur* $]0,\infty[\times\mathbb{R}^d$ *de l'équation*

$$v_{\phi,\psi}(0,x) = \phi, \quad \frac{\partial v_{\phi,\psi}(s,x)}{\partial s} = \frac{\sigma^2}{2}\Delta v_{\phi,\psi}(s,x) - \frac{\gamma}{2}(v_{\phi,\psi}(s,x))^2 + \psi. \tag{3}$$

L'équation (2) est souvent appelée *équation de Laplace* du super-mouvement brownien. La quantité $\frac{\sigma^2}{2}\Delta v$ est un terme de diffusion, et $-\frac{\gamma}{2}v^2$ un terme de branchement[5].

Il est à noter qu'on préfère souvent travailler avec la forme intégrée de (3). Soit P_x la mesure de probabilité sous laquelle le mouvement brownien $(B_t, t \geq 0)$ est défini lorsque $B_0 = x$. On note $(P_t, t \geq 0)$ le semigroupe de B, et $(p_t, t \geq 0)$ le noyau de la chaleur. Plus précisément, pour $\phi \in \mathcal{B}_b(\mathbb{R}^d, \mathbb{R})$,

$$P_t(\phi)(x) = \mathbb{E}_x[\phi(B_t)] = \int_{\mathbb{R}^d} p_t(y-x)\phi(y)dy, \quad \text{où } p_t(z) = \frac{1}{(2\pi t)^{d/2}}\exp\left(-\frac{|z|^2}{2t}\right).$$

Notons également $Q_t := \int_0^t P_s ds$. On a alors

$$v_{\phi,\psi}(0,x) = P_t(\phi)(x) + Q_t(\psi)(x) - \frac{\gamma}{2}\int_0^t P_s(v_{\phi,\psi}(t-s,.)^2)(x)ds. \tag{4}$$

Le théorème I.2 s'avère extrêmement utile pour établir les propriétés principales du super-mouvement brownien, que l'on discutera dans la partie I.1.3. Inversement, les propriétés du super-mouvement brownien ont permis d'établir des réultats analytiques concernant les solutions de l'équation (3).

Le cas particulier $\psi = 0$ dans le théorème I.2 permet d'accéder, comme on le verra plus loin, à l'expression des moments de $<X_t, \phi>$. Le cas $\phi = 0$ va nous permettre de décrire les propriétés de la mesure d'occupation du super-mouvement brownien. En particulier, l équation de Laplace du super-mouvement brownien permet, comme on le verra en I.1.3, d'établir l'existence de temps locaux en dimension $d \leq 3$.

Notre définition du super-mouvement brownien nous a quelque peu fait perdre de vue les structures de branchement et de déplacement spatial de ce processus. On va pour cela rappeler dans le paragraphe suivant l'approche en termes de serpent brownien due à Le Gall.

1.2. Serpent brownien et super-mouvement brownien. L'approche que nous allons décrire permet une meilleure compréhension du super-mouvement brownien, et en particulier de sa *mesure d'excursion* (qui représente la contribution de la descendance d'un individu de la génération initiale). Le serpent brownien s'est également avéré être un outil très puissant dans l'étude des superprocessus et de la classe d'équations aux dérivées partielles reliées aux superprocessus (voir par exemple [**LG 99**], Chapitre VI, [**DLG 02**], chapitre IV). Les démonstrations des résultats cités dans cette partie peuvent être trouvés dans [**LG 05**], ainsi que dans le chapitre IV de [**LG 99**].

[5]La raison de ces appellations est rendue très claire dans [**Eth 00**], chapitre I, à travers la preuve de l'expression analogue que l'on peut obtenir lorsqu'on exprime la transformée de Laplace d'un mouvement brownien branchant.

1.2.1. *Fonctions de contour.* On a besoin de détailler un peu le formalisme d'arbre introduit au paragraphe précédent. À un arbre ordonné enraciné $\{v : v \in \theta\}$, on peut associer des nombres réels positifs $\{h_v, v \in \theta\}$. Un arbre *marqué* est la donnée du couple $(\theta, \{h_v, v \in \theta\})$. Pour $v \in \theta \setminus \emptyset$, on peut interpréter le nombre h_v comme la longueur de la branche reliant v à son ancêtre[6]. On choisit d'identifier l'arbre ordonné enraciné θ à l'arbre marqué $(\theta, \{h_\emptyset = 0, h_v = 1 \; \forall v \in \theta \setminus \emptyset\})$.

On adopte alors les conventions suivantes pour dessiner un arbre ordonné enraciné. On place sa racine au niveau 0, la n-ième génération au niveau n (au dessus de 0). On range de plus les descendants de chaque noeud de gauche à droite par ordre lexicographique. La *fonction de contour* d'un arbre ordonné enraciné décrit la hauteur d'un mobile lancé à vitesse verticale 1 et parcourant les branches de l'arbre en le contournant dans le sens des aiguilles d'une montre. On choisit en outre de prolonger cette fonction à l'intervalle $[0, 2\#(\theta)-1]$ en lui donnant simplement la valeur 0 sur l'intervalle $[2(\#(\theta)-1), 2\#(\theta)-1]$. Ceci permet en particulier que la fonction de contour de l'arbre $\emptyset$ ne soit pas dégénérée. On note $C_t(\theta), t \in [[0, 2\#(\theta)-1]$ la fonction de contour de l'arbre θ. Il est clair que cette fonction caractérise l'arbre θ. La fonction de contour $C_t, t \geq 0$ d'une forêt $\theta_1, \theta_2, ...$ correspond tout simplement à concaténer les fonctions $C_t(\theta_1), t \in [[0, 2\#(\theta)-1], C_t(\theta_2), t \in [0, 2\#(\theta)-1], ...$

Notons alors Λ_t le nombre de racines déjà “explorées” par la fonction de contour C dans l'intervalle $[0,t]$ c'est-à-dire $\Lambda_t := \#\{u \in [1,t] \; : \; C(u) = 0 \text{ sur } [u-1, u]\}$. D'après la partie 1.6 de [**LG 05**], on obtient la proposition suivante.

Proposition I.2. *Soit C la fonction de contour d'une forêt de Galton-Watson. La suite $\left(\frac{1}{p}C_{2p^2t}, \frac{1}{p}\Lambda_{2p^2t}; t \geq 0\right)_{p \in \mathbb{N}^*}$ converge en loi (i.e. au sens de la convergence faible dans l'espace $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$) lorsque $p \to \infty$, vers $\left(\frac{2}{\sqrt{\gamma}}|\beta_t|, \sqrt{\gamma}\ell_t^0; t \geq 0\right)$, où β est un mouvement brownien standard, et $(\ell_t^0, t \geq 0)$ désigne le temps local de β en 0 jusqu'au temps t.*

Si on définit $\tau_1^{(p)} = \inf\{t \geq 0 : \Lambda_t = p\}$, alors, la fonction $t \to C_t \mathbf{1}_{\{t \leq \tau_1(p)\}}$ correspond sur l'intervalle $[0, \tau_1^{(p)}]$ à la fonction de contour des p premiers arbres d'une forêt de Galton-Watson. On définit la version rééchelonnée $C^{(p)}$ de cette fonction par $C^{(p)}(t) := \frac{1}{p}C_{2p^2t}\mathbf{1}_{\{\Lambda_{2p^2t} \leq p\}}$. Définissons d'autre part $\tau_r := \inf\{t \geq 0 : \ell_t^0 > r\}$. D'après la proposition I.2, $C^{(p)}$ converge lorsque $p \to \infty$ vers

$$\left(\frac{2}{\sqrt{\gamma}}|\beta_t|\mathbf{1}_{\{t \leq \tau_{1/\sqrt{\gamma}}\}}, t \geq 0\right).$$

Il existe donc une limite pour la fonction de contour rééchelonnée des p premiers arbres d'une forêt de Galton-Watson [7]. Cette limite est une trajectoire brownienne réfléchie, donc un élément de $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$. L'idée est maintenant d'associer

[6] On peut en outre imaginer une nouvelle racine, reliée à son unique descendant $\emptyset$ par une branche de longueur $h_\emptyset$.

[7] On ne peut pas ici parler de limite en termes d'arbres. Ceci est cependant possible, et rendu rigoureux par la notion d'arbre réel. Ce formalisme permet de considérer toute fonction $f : \mathbb{R}_+ \to \mathbb{R}_+$ à support compact, et vérifiant $f(0) = 0$ comme la fonction de codage d'un arbre continu. Elle est l'analogue de la fonction de contour dans le cas discret. Quand la fonction de codage est aléatoire, on a affaire à un arbre réel aléatoire. L'exemple le plus connu d'un tel arbre est l'arbre continu d'Aldous, dont la fonction de codage est une excursion brownienne normalisée. Cet arbre apparaît en outre comme la limite d'un arbre de Galton-Watson conditionné à avoir n éléments. On ne décrit pas ici le formalisme d'arbre réel, car il n'est pas indispensable pour

un déplacement spatial à une telle trajectoire. On commence par le faire pour un élément déterministe de $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$.

1.2.2. *Serpent à déplacement spatial brownien conduit par une fonction* $f \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$. Il nous faut introduire quelques notations supplémentaires. Soit $\mathcal{W}$ l'espace des trajectoires finies de $\mathbb{R}_+$ dans $\mathbb{R}^d$, et $\mathcal{W}_x = \{w \in \mathcal{W} : w(0) = x\}$. Pour $w \in \mathcal{W}$, on note ζ_w le temps de vie de w, et $\hat{w}$ sa valeur terminale, c'est-à-dire $w(\zeta_w)$. Notons que l'espace $\mathcal{W}$ est polonais quand on le munit de la distance

$$d(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{r \geq 0} |w(r \wedge \zeta_w) - w'(r \wedge \zeta_{w'})|.$$

Notons $\Omega := \Omega_{\mathcal{W}} = \mathcal{C}(\mathbb{R}_+, \mathcal{W})$, et rappelons que $\mathcal{F}$ est la tribu borélienne de cet espace. Le processus canonique sur cet espace est noté $(W_s, s \geq 0)$, et on définit pour $s \geq 0$, $\zeta_s := \zeta(W_s)$, et $\sigma(\zeta) := \inf\{s > 0 : \zeta_s = 0\}$. On définit également $(\mathcal{F}_t)_{t \geq 0}$ la filtration canonique. Enfin, pour $w \in \mathcal{W}$, $a \in [0, \zeta_w]$ et $b \geq a$ on note $R_{a,b}(w, dw')$ l'unique mesure de probabilité qui vérifie

– $\zeta_{w'} = b$, $R_{a,b}(w, dw')$-presque sûrement.
– $w'(t) = w(t)$ pour tout $t \leq a$, $R_{a,b}(w, dw')$-presque sûrement.
– Sous $R_{a,b}(w, dw')$, $(w'(a+t), 0 \leq t \leq b-a)$ a la loi d'un mouvement brownien d-dimensionel[8] issu de $w(a)$ et stoppé au temps $b - a$.

Définition I.6. *Soit* $f \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ *localement hölderienne d'exposant* η *pour tout* $\eta \in (0, 1/2)$, *et* $w_0 \in \mathcal{W}$ *tel que* $f(0) = \zeta_{w_0}$. *Il existe une unique mesure de probabilité* $\Gamma^f_{w_0}$ *sur* Ω *telle que le processus canonique* $(W_s, s \geq 0)$ *sous* Γ^f_w *est décrit de la façon suivante.*

– $W_0 = w_0$, *presque sûrement,*
– $(W_s, s \geq 0)$ *est un processus de Markov dont le noyau de transition entre les temps* s *et* s' *est* $R_{m_f(s,s'), f(s')}(w, dw')$.

Le processus canonique sous Γ^f_w *est appelé* serpent (à déplacement spatial brownien) conduit par la fonction f

Pour une preuve de l'existence et de l'unicité de Γ^f_w, on peut voir par exemple [**LG 05**], partie 3.1. Si W_0 est une trajectoire de ξ arrêtée en $f(0)$, alors il est clair que pour tout $s \geq 0$, W_s est également une trajectoire de ξ dont le temps de vie est $f(s)$. D'autre part, de façon informelle, lorsque s varie, on supprime le bout de cette trajectoire si f diminue ; on y ajoute un bout de trajectoire brownienne indépendant si f augmente.

Si f est la fonction de contour d'un arbre ordonné enraciné θ, il est facile de voir que le serpent conduit par f décrit des trajectoires browniennes le long des branches de l'arbre [9] θ.

1.2.3. *Serpent brownien.* On est désormais en mesure d'introduire le *serpent brownien*[10] .

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introduire le serpent brownien. Néanmoins, on ne peut qu'encourager le lecteur à trouver des explications détaillées de ces objets et de ces résultats dans [**LG 05**] ou dans [**DLG 02**].

[8] On pourrait ici considérer un processus de Markov plus général, ce qui permettrait par la suite d'obtenir la construction d'un superprocessus.

[9] Cette correspondance persiste lorsque f est la fonction de contour d'un arbre marqué, et même en fait quand elle est la fonction de codage d'un arbre réel.

[10] L'adjectif “brownien” ne réfère en aucun cas au déplacement spatial, mais au processus temps de vie, qui aura la loi d'un mouvement brownien réfléchi. Le cadre plus général des serpents de Lévy, pour lesquels le processus temps de vie a la loi d'un processus de Lévy, est traité dans [**DLG 02**].

Définition I.7. *Pour* $r \geq 0$*, notons* Refl_r *la loi d'un mouvement brownien réfléchi parti de* r*. Pour tout* $w \in \mathcal{W}$*, on note* Π_w *la mesure de probabilité sur* Ω *définie par*

$$\Pi_w(d\omega) = \int_\Omega \mathrm{Refl}_{\zeta_w}(df)\Gamma_w^f(d\omega).$$

Le processus canonique W *sous* Π_w *est appelé* serpent brownien (à déplacement spatial brownien)*.*

D'après les théorèmes 3.2.1 et 3.2.2 de [**LG 05**], ce processus est un processus de Markov homogène dont on peut exprimer le noyau de transition ; de plus ce processus vérifie la propriété de Markov forte pour la filtration $(\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s)_{t\geq 0}$.

1.2.4. *Serpent brownien sous sa mesure d'excursion.* Soit n la mesure d'Itô des excursions browniennes. La durée d'une excursion e est notée $\sigma(e)$, son temps local au niveau a jusqu'en t est noté $\ell_t^a(e)$.

Définition I.8. *La* mesure d'excursion *du serpent brownien est la mesure* σ*-finie sur* Ω*, définie par*

$$\mathbb{N}_x(d\omega) = \int_\Omega n(de)\Gamma_{\{x\}}^e(d\omega),$$

où $\{x\}$ *désigne la trajectoire triviale, réduite au point* $x \in \mathbb{R}^d$*.*

La mesure $\mathbb{N}_x$ apparaît facilement comme la mesure d'excursion du processus $(W_s, s \geq 0)$ sous $\Pi_{\{x\}}$ au sens de la théorie des excursions des processus de Markov (voir par exemple [**Bl 92**]). La propriété de Markov de W sous $\mathbb{N}_x$ peut être exprimée de la façon suivante (cf théorème 3.3.1 de [**LG 05**]). On note Π_w^* la loi sous Π_w de $(W_{s\wedge\sigma(\zeta)}, s \geq 0)$, et θ_t l'opérateur de translation usuel sur Ω. Soit T un temps d'arrêt pour la filtration $(\mathcal{F}_t)_{t\geq 0}$ qui vérifie $T > 0$ $\mathbb{N}_x$-presque partout. Alors, pour toute fonction positive F $\mathcal{F}_T$-mesurable, et pour toute fonction positive G $\mathcal{F}$-mesurable, on a

$$\mathbb{N}_x\left(\mathbf{1}_{\{T<\infty\}} F \times G \circ \theta_T\right) = \mathbb{N}_x\left(\mathbf{1}_{\{T<\infty\}} F \times \Pi_{W_T}^*(G)\right).$$

1.2.5. *Une propriété remarquable du serpent brownien.* Pour $t > 0$, le lemme V.5 de [**LG 99**] permet de décomposer la trajectoire de W choisi selon Π_w^* suivant les excursions du temps de vie ζ au dessus de son minimum. On considère donc un serpent brownien W sous Π_w^*, où $w \in \mathcal{W}_x$ et on note $[\alpha_i, \beta_i], i \in I$ les intervalles d'excursion de $\zeta_s - \inf_{r\in[0,s]} \zeta_r$ en dehors de 0. Pour $i \in I$, on définit $W^i \in \Omega$ par

$$W_s^i(t) = W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i} + t), \quad 0 \leq t \leq \zeta_s^i := \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i}.$$

Proposition I.3. *Sous* Π_w^**, la mesure*

$$\Lambda := \sum_{i\in I} \delta_{(\zeta_{\alpha_i}, W^i)}$$

est une mesure de Poisson ponctuelle sur $[0, \zeta_w] \times \Omega$ *d'intensité* $2dt\mathbb{N}_{w(t)}(dW')$*.*

1.2.6. *Lien entre serpent brownien et super-mouvement brownien.* D'après le théorème IV.4 de [**LG 99**], on peut, à partir du serpent brownien (avec déplacement spatial brownien) sous sa mesure d'excursion, retrouver le super-mouvement brownien sous $\mathbf{P}_\mu$.

THÉORÈME I.3. *Soit $\mu \in M_F(\mathbb{R}^d)$, et $\sum_{i\in I} \delta_{(x_i,W^i)}$ une mesure de Poisson ponctuelle d'intensité $\mu(dx)\mathbb{N}_x(dW)$. Pour $i \in I$ on note $\zeta^i = \zeta(W^i)$ (à qui on associe $\sigma_i := \sigma(\zeta^i)$ et $\ell_s^t(\zeta^i)$.) Le processus $(X_t, t \geq 0)$ défini par $X_0 = \mu$, et, pour tout $t > 0$, pour toute fonction $\phi \in \mathcal{B}_b^+(\mathbb{R}^d, \mathbb{R})$, par la relation*

$$< X_t, \phi >= \sum_{i\in I} \int_0^{\sigma_i} \phi(\hat{W}_s^i) d\ell_s^t(\zeta^i),$$

est un super-mouvement brownien initialement en μ, avec coefficient de branchement 4 et coefficient[11] *de diffusion 1.*

De ce théorème et de la théorie des excursions browniennes, on peut déduire le lien existant entre serpent brownien sous Π_w et super-mouvement brownien.

PROPOSITION I.4. *Soit $(W_s, s \geq 0)$ le serpent brownien avec déplacement spatial brownien sous $\Pi_{\{x\}}$, et soit ℓ_s^t le temps local en t jusqu'au temps s du processus temps de vie $(\zeta_s, s \geq 0)$. Enfin, on rappelle que, pour $r \geq 0$, $\tau_r = \inf\{t \geq 0 : \ell_t^0 > r\}$. Alors, pour $r > 0$, le processus $(X_t, t \geq 0)$ défini par la relation*

$$\forall \phi \in \mathcal{C}_b(\mathbb{R}^d) \qquad < X_t, \phi >= \int_0^{\tau_r} \phi(\hat{W}_s) d\ell_s^t$$

est un super-mouvement brownien sous $\mathbf{P}_{r\delta_x}$ avec coefficient de branchement 4 et coefficient de diffusion 1.

Les remarques suivantes premettent de donner l'intuition de cette correspondance.

En utilisant la proposition I.1, on peut prouver que la masse totale d'un super-mouvement brownien a la loi du processus de Feller solution de l'équation 1. On peut en outre, évidemment, vérifier ce fait directement grâce à la définition I.4, ou encore utiliser l'équation de Laplace (voir pour cela le premier paragraphe de la partie I.1.3). On retrouve une nouvelle fois cette propriété dans la formule ci-dessus en posant $\phi = 1$ et en utilisant un théorème de Ray-Knight.

D'autre part, d'après la proposition I.2, pour $\gamma = 4$, $C^{(p)}$ converge lorsque $p \to \infty$ vers la trajectoire d'un mouvement brownien réfléchi arrêté lorsque son temps local en 0 dépasse[12] la valeur $1/2$.

Enfin, introduisons $W^{C^{(p)}}$ un serpent de déplacement spatial brownien conduit par la fonction $C^{(p)}$, et $\mathrm{Term}_t^{(p)} := \{\hat{W}_s^{C^{(p)}} : C^{(p)}(s) = t\}$. Notons que $\mathrm{supp}(X_t^{(p)})$ décrit l'ensemble des positions des individus au temps t pour le processus $X^{(p)}$. D'après les définitions du BBM et du serpent conduit par $C^{(p)}$, $\mathrm{supp}(\mathrm{BBM}_t^{(p)})$ et $\mathrm{Term}_t^{(p)}$ ont même loi.

[11] Pour trouver un coefficient de diffusion quelconque, il suffit de travailler avec un serpent dont le déplacement spatial est σ-dilaté. D'autre part, d'après la définition du super-mouvement brownien, si $(X_t, t \geq 0)$ est un $\mathrm{SBM}_\mu^{\gamma,\sigma^2}$, alors $(X'_t := \lambda X_t, t \geq 0)$ est un $\mathrm{SBM}_{\lambda\mu}^{\lambda\gamma,\sigma^2}$, ce qui permet facilement d'obtenir un coefficient de branchement quelconque.

[12] Il est surprenant de constater que, pour retrouver le $\mathrm{SBM}_{\delta_x}(4,1)$, il nous faut intégrer jusqu'à τ_1 et non $\tau_{1/2}$. Ceci peut se comprendre en remarquant qu'au temps 0, on a, $\Pi_{\{x\}}$ presque sûrement, $\int_0^{\tau_{1/2}} \phi(\hat{W}_s) d\ell_s^0 = \phi(x)/2$. Si on pense donc à notre modèle discret initial, le processus $\int_0^{\tau_{1/2}} \mathbf{1}_{\{\hat{W}_s \in .\}} d\ell_s^t, t \geq 0$ ne permet de compter les individus de la génération initiale "qu'à moitié".

1.2.7. *Mesure d'excursion du super-mouvement brownien.* L'une des contributions de la somme poissonienne qui apparaît dans le théorème I.3 est communément appelée super-mouvement brownien sous sa mesure d'excursion. Plus précisément, introduisons le processus $(X_t(W), t > 0)$ défini par la relation

$$\forall \phi \in \mathcal{C}_b(\mathbb{R}^d) \qquad < X_t(W), \phi >= \int_0^\sigma \phi(\hat{W}_s) d\ell_s^t(\zeta).$$

DÉFINITION I.9. *On considère W un serpent brownien sous sa mesure d'excursion $\mathbb{N}_x$. On appelle* mesure d'excursion du super-mouvement brownien *et on note $\tilde{\mathbb{N}}_x$ la mesure sur l'espace $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ qui est la "loi" du processus $(X_t := X_t(W), t \geq 0)$ sous $\mathbb{N}_x$.*

D'après le théorème I.3 et les propriétés des mesures de Poisson ponctuelles, on peut retrouver les mesures de probabilité $\tilde{\mathbb{N}}_x(.|X_\alpha \neq 0)$ à partir du super-mouvement brownien défini dans la première partie. Pour tout $\alpha > 0$ fixé, pour toute fonction $F \in \mathcal{C}_b(\mathbb{R}_+, M_F(\mathbb{R}^d))$,

$$\lim_{\epsilon \to 0} \epsilon^{-1} \mathbf{P}_{\epsilon\delta_x} \left(F((X_t, t \geq 0)) \middle| X_\alpha \neq 0 \right) = \tilde{\mathbb{N}}_x \left(F(X_t, t \geq 0) \middle| X_\alpha \neq 0 \right).$$

Lorsque W est choisi selon Π_w^*, on peut, en réécrivant la proposition I.3, décomposer le processus $X_{\zeta_w}(W)$ de la façon suivante. Rappelons que Λ est la mesure de Poisson ponctuelle décrite à la proposition I.3. On a, Π_w^*-presque sûrement,

$$X_{\zeta_w}(W) = \int_0^{\zeta_w} \int_\Omega X_{\zeta_w - t}(W') \Lambda(dt, dW') = \sum_{i \in I} X_{\zeta_w - \zeta_{\alpha_i}}(W^i).$$

On fera usage de cette représentation dans le chapitre IV.

1.2.8. *Mesures canoniques associées au super-mouvement brownien.* Le théorème I.3 et la formule exponentielle pour une mesure de Poisson impliquent que pour tout $t > 0$, pour toute fonction $\phi \in \mathcal{C}_b^+(\mathbb{R}^d)$, on a

$$\begin{aligned} &\mathbf{E}_\mu(\exp(- < X_t, \phi >)) \\ &= \exp\left(- \int_{\mathbb{R}^d} \mu(dx) \mathbb{N}_x \Big(1 - \exp(- < X_t, \phi >) \Big) \right). \end{aligned}$$

Cette formule fait apparaître pour tout $t > 0$ le caractère infiniment divisible de la mesure X_t sous $\mathbf{P}_\mu$, et les mesures $(R_t(x, .))_{t>0, x \in \mathbb{R}^d}$ définies par

$$R_t(x, .) = \tilde{\mathbb{N}}_x(X_t \in .)$$

sont appelées les *mesures canoniques* associées au super-mouvement brownien[13]. D'après ce qui précède, la mesure canonique, au temps t, représente la contribution au super-mouvement brownien des descendants d'un individu de la génération initiale, conditionnellement à l'existence de tels descendants.

[13]Notons que si $X^i, i \in \{1, ..., n\}$ sont des copies indépendantes d'un super-mouvement brownien de valeur initiale μ/n, alors $\sum_{i=1}^n X^i$ a la loi d'un super-mouvement brownien de valeur initiale μ. Le fait que pour tout $t > 0$, la mesure X_t sous $\mathbf{P}_\mu$ est infiniment divisible, peut également être vu comme conséquence de cette remarque (voir par exemple [**Pe 99**], partie II.7 pour une construction des mesures $R_{t,.}$ qui n'utilise pas $\tilde{\mathbb{N}}_x$).

1.3. Propriétés du super-mouvement brownien. On dresse ici une liste de résultats concernant le super-mouvement brownien. Cette liste est bien entendu loin d'être exhaustive. On s'intéresse en particulier à l'existence et aux propriétés des temps locaux en dimension $d \leq 3$, et aux propriétés de la mesure d'occupation du super-mouvement brownien, en vue des chapitres III et IV.

Dans cette partie, sauf mention contraire, on fixe $\gamma = 4, \sigma = 1$. Comme on l'a vu précédemment, il est facile d'adapter au cas général.

1.3.1. *Propriété de changement d'échelle du super-mouvement brownien.* Cette propriété découle directement de la convergence énoncée au théorème I.1. On peut cependant la retouver facilement grâce au théorème I.2 (voir pour cela la partie III.1.3). Soit $c > 0$, $\mu \in M(\mathbb{R}^d)$. On désigne par $\mathscr{B}(\mathbb{R}^d)$ l'ensemble des boréliens de $\mathbb{R}^d$. Pour $t \geq 0$, on définit les mesure $\mu^{(c)}, X_t^{(c)}$ par les relations

$$\forall A \in \mathscr{B}(\mathbb{R}^d) \qquad \mu^{(c)}(A) = c^2 \mu(c^{-1}A), \qquad X_t^{(c)}(A) = c^{-2} X_{c^2 t}(cA).$$

Alors, le processus $(X_t^{(c)},\ t \geq 0)$ sous $\mathbf{P}_{\mu^{(c)}}$ a même loi que le processus $(X_t, t \geq 0)$ sous $\mathbf{P}_\mu$.

Traduisons cette propriété d'invariance en termes de mesure d'occupation. Pour $\phi \in C_b(\mathbb{R}^d)$ on introduit ϕ_c la fonction définie par $\phi_c(x) = \phi(cx)$. Alors, le processus $\big(c^{-4} < Y_{c^2 t}, \phi >,\ t \geq 0\big)$ sous $\mathbf{P}_{\mu^{(c)}}$ a même loi que $(< Y_t, \phi_c >,\ t \geq 0)$ sous $\mathbf{P}_\mu$.

1.3.2. *Masse totale du super-mouvement brownien, extinction.* Dans le but de mettre en évidence l'utilité du problème de martingales et de l'équation de Laplace du super-mouvement brownien, ainsi que leur correspondance, on étudie ici sous $\mathbf{P}_\mu$ le processus $(< X_t, 1 >, t \geq 0)$ de la masse totale. Il est tout d'abord facile de vérifier que d'après la définition I.4, ce processus vérifie l'équation (I.1). D'autre part, d'après le théorème I.2, on a, pour $\lambda > 0$,

$$\mathbf{E}_\mu \left(\exp(-\lambda < X_t, 1 >)\right) = \exp(- < \mu, v(t, .) >),$$

où v est solution de l'équation

$$v(0, x) = \lambda, \qquad \frac{\partial v}{\partial t} = \frac{1}{2}(\Delta v - 4v^2).$$

On voit alors que v ne dépend pas de x et que $v(t) = \frac{2\lambda}{2+4\lambda t}$. On a donc établi que

$$\mathbf{E}_\mu \left(\exp(-\lambda < X_t, 1 >)\right) = \exp\left(-\frac{2\lambda < \mu, 1 >}{2 + 4\lambda t}\right),$$

ce qui confirme que $(< X_t, 1 >, t \geq 0)$ vérifie (I.1). En faisant tendre $\lambda \to \infty$, on obtient d'autre part que si $t > 0$,

$$\mathbf{P}_\mu(X_t = 0) = \mathbf{P}_\mu(< X_t, 1 >= 0) = \exp\left(-\frac{< \mu, 1 >}{2t}\right),$$

et donc que X s'éteint, $\mathbf{P}_\mu$-presque sûrement.

En particulier, la mesure d'occupation $(Y_t, t \geq 0)$ est presque sûrement constante à partir d'un certain temps.

Définition I.10. *Soit X un super-mouvement brownien. On appelle mesure d'occupation totale du super-mouvement brownien et on note Y la variable aléatoire à valeurs dans $M_F(\mathbb{R}^d)$ définie sous $\mathbf{P}_\mu$ par*

$$Y(A) = \int_0^\infty X_t(A)dt, \qquad \forall A \in \mathscr{B}(\mathbb{R}^d).$$

1.3.3. *Mesure d'occupation du super-mouvement brownien sous sa mesure d'excursion.* On note $\mathcal{Z}$ la mesure d'occupation de X sous sa mesure d'excursion $\tilde{\mathbb{N}}_x$. Il est facile de voir, d'après, la définition I.9, qu'il s'agit de la mesure aléatoire définie directement sous la mesure $\mathbb{N}_x$, à partir du serpent brownien $(W_s, s \geq 0)$, par la relation

$$\mathcal{Z}(A) = \int_0^\sigma ds \mathbf{1}_A(\hat{W}_s) \quad \forall A \in \mathscr{B}(\mathbb{R}^d).$$

En utilisant la relation entre le serpent brownien et le super-mouvement brownien, Le Gall [**LG 94**] a relié la mesure d'occupation totale sous $\mathbf{P}_{\delta_x}$ à la mesure d'occupation sous $\mathbb{N}_x$. En effet, d'après le théorème I.3, un super-mouvement brownien X issu de δ_x est obtenu par la relation

$$\forall h \in \mathcal{B}_b^+(\mathbb{R}^d, \mathbb{R}), \quad \int_0^\infty h(t) < X_t, \phi > dt = \sum_{i \in I} \int_0^{\sigma_i} h(\zeta_s^i)\phi(\hat{W}_s^i) ds,$$

où $\sum_{i\in I} \delta_{W^i}$ est une mesure de Poisson ponctuelle d'intensité $\mathbb{N}_x(dW)$. En particulier, pour tout $A \in \mathscr{B}(\mathbb{R}^d)$, si on pose $h = 1$, $\phi = \mathbf{1}_A$, on obtient

$$Y(A) = \int_0^\infty < X_t, \phi > dt = \sum_{i \in I} \int_0^{\sigma_i} \phi(\hat{W}_s^i) ds = \sum_{i \in I} \mathcal{Z}_i(A),$$

où les $\mathcal{Z}_i, i \in I$ sont des copies indépendantes de $\mathcal{Z}$. D'après la formule exponentielle des mesures de Poisson, on en déduit que

$$\mathbb{N}_x \left[1 - \exp(-\lambda \mathcal{Z}(A))\right] = -\log \mathbf{E}_{\delta_x} \left[\exp(-\lambda Y(A))\right]. \tag{5}$$

D'après le théorème I.2, la fonction $u_\lambda(.) := \mathbb{N}_. \left[1 - \exp(-\lambda \mathcal{Z}(A))\right]$ est donc solution de l'équation intégrale

$$u_\lambda(x) + 2 \int_{\mathbb{R}^d} dy G(x,y) u_\lambda^2(y) = \lambda \int_A dy G(x,y).$$

On déduit de cette dernière équation une relation de récurrence pour les moments de la mesure d'occupation du super-mouvement brownien sous sa mesure d'excursion,

$$\mathbb{N}_x \left[\mathcal{Z}(A)\right] = \int_A dy \ G(x,y). \tag{6}$$

et, pour tout $p \geq 2$,

$$\mathbb{N}_x \left[\mathcal{Z}(A)^p\right] = 2 \sum_{j=1}^{p-1} \binom{p}{j} \int_{\mathbb{R}^d} dz \ G(x,z) \ \mathbb{N}_z \left[\mathcal{Z}(A)^j\right] \mathbb{N}_z \left[\mathcal{Z}(A)^{p-j}\right]. \tag{7}$$

Cette formule de récurrence est, au chapitre IV, un outil principal de l'étude de la mesure d'occupation du super-mouvement brownien en dimension 4.

Remarquons que pour exprimer les moments de $\mathcal{Z}$, on a donc utilisé une forme intégrée de l'équation aux dérivées partielles reliée au super-mouvement brownien d'après le théorème I.2. On va voir dans les paragraphes suivants qu'une méthode analogue permet d'exprimer les moments de $< X_t, \phi >$ et ceux de $< Y_t, \psi >$.

1.3.4. *Moments exponentiels pour* $< X_t, \phi >$. On considère $\mu \in M_F(\mathbb{R}^d)$, $\phi \in \mathcal{C}_b^+(\mathbb{R}^d) \cap \mathcal{C}_b^2(\mathbb{R}^d)$, et $t > 0$ fixés. D'après l'équation de Laplace du super-mouvement brownien, $\mathbf{E}_\mu(\exp(-\theta < X_t, \phi >))$ s'exprime à l'aide d'une solution $u_{\theta\phi,0}$ de l'équation (4) :

$$v_{\theta\phi,0}(t,x) = P_t(\theta\phi) - 2\int_0^t P_s(v_{\theta\phi,0}(t-s,.)^2)(x)ds.$$

On développe alors v_θ suivant la série formelle

$$v_{\theta\phi,0}(t,x) = \sum_{n\geq 1} a_n^\phi(t,x)\theta^n.$$

Lorsque cette série a, uniformément en $x \in \mathbb{R}^d$, un rayon de convergence strictement positif, on a l'existence de moments exponentiels[14] pour $< X_t, \phi >$. On peut alors de plus exprimer les moments de $< X_t, \phi >$ en fonction des coefficients a_n^ϕ (cf [**Dy 91**]).

On énonce ici un critère simple d'existence de moments exponentiels pour $< X_t, \phi >$:

$$(M1) \qquad ||ta_1^\phi(t,.)||_\infty := \sup_{x\in\mathbb{R}^d} |ta_1^\phi(t,x)| < \infty,$$

$$(M2) \qquad ||a_2^\phi(t,.)||_\infty := \sup_{x\in\mathbb{R}^d} |a_2^\phi(t,x)| < \infty.$$

Ce critère va s'avérer utile dans la preuve de l'existence d'une densité dans le cas $d = 1$, et dans la preuve analogue de l'existence de temps locaux en dimension $d \leq 3$.

Donnons rapidement la preuve de la validité de ce critère. On obtient tout d'abord facilement que les coefficients vérifient la relation de récurrence :

$$\begin{cases} a_1^\phi(t,x) & = P_t(\phi)(x) \\ a_n^\phi(t,x) & = -2\sum_{k=1}^{n-1}\int_0^t dsP_s\left(a_k^\phi(t-s,.)a_{n-k}^\phi(t-s,.)\right)(x). \end{cases}$$

On peut faire une interprétation plus intuitive de ces quantités. On associe pour cela à la structure de branchement d'un arbre binaire ordonné enraciné τ un déplacement spatial brownien. On considère donc la quantité $U_{\tau,\phi}(t,x)$ définie par récurrence de la façon suivante.

Si $|\tau| = 1$, $U_{\tau,\phi}(t,x) := E_x[\phi(B_t)]$. Si $|\tau| \geq 2$, les deux sous-arbres τ_1, τ_2 issus de la racine de τ vérifient $|\tau_1| + |\tau_2| = n$, et on peut alors définir $U_{\tau,\phi}(t,x) := \int_0^t dsE_x[U_{\tau_1,\phi}(t-s,B_s)U_{\tau_2,\phi}(t-s,B_s)]$. On voit alors que pour tout $n \geq 1$, si on note $\mathcal{T}_n$ l'ensemble des arbres binaires ordonnés enracinés à n feuilles (une feuille est un noeud de l'arbre qui n'a aucun descendant),

$$a_n^\phi(t,x) = -2\sum_{\tau\in\mathcal{T}_n} U_{\tau,\phi}(t,x).$$

Lorsque les conditions $(M1), (M2)$ sont vérifiées, on pose

$$C := 1 \vee t \vee ||ta_1^\phi(t,.)||_\infty \vee ||a_2^\phi(t,.)||_\infty.$$

[14]Ceci permet d'étendre l'équation de Laplace à une classe de fonctions _non nécessairement positives_ inférieurement bornées (voir la partie III.1.3 pour une formulation précise dans le cas des fonctions à support compact).

Une récurrence facile permet alors de montrer que

$$\sup_{x\in\mathbb{R}^d} |U_{\tau,\phi}(t,x)| \leq C^{2|\tau|-1}.$$

Les propriétés bien connues du nombre d'arbres ordonnés enracinés à n feuilles (il s'agit du n-ième nombre de Catalan) permettent ensuite de conclure que la série $\sum_{n\geq 1}\theta^n \int_{\mathbb{R}^d} a_n^\phi(t,x)\mu(dx)$ converge, uniformément en $x \in \mathbb{R}^d$, pour $|\theta| < \frac{1}{8C^2<\mu,1>}$, ce qui garantit, comme on l'a expliqué plus haut, l'existence de moments exponentiels [15] pour $< X_t, \phi >$.

1.3.5. *Existence d'une densité dans le cas unidimensionnel.* La question de l'existence d'une densité pour une classe plus large de superprocessus fut résolue par [**DH 79**]. Une conséquence de [**DH 79**] est que, pour $t > 0$ la mesure X_t du super-mouvement brownien est absolument continue par rapport à la mesure de Lebesgue si et seulement si $d = 1$. On va ici donner une idée de preuve de l'existence de cette densité dans le cas $d = 1$.

Fixons $t > 0$. Une condition suffisante du fait que la mesure X_t sous $\mathbf{P}_\mu$ est absolument continue par rapport à la mesure de Lebesgue est l'existence, pour un certain $\theta > 0$ et pour tous $t > 0, x \in \mathbb{R}^d$, de la limite $\lim_{\varepsilon\to 0}\left[\mathbf{E}_\mu \exp\left(- < X_t, \theta\phi_x^\varepsilon >\right)\right]$, où les fonctions $\phi_x^{(\varepsilon)} \in \mathcal{C}_b^+(\mathbb{R}^d)$ convergent faiblement vers δ_x, la masse de Dirac au point x.

Sans perte de généralité, on peut se restreindre au cas $x = 0$. D'après le lemme 3.1 de [**Su 89**], l'existence de la limite correspond à montrer pour θ suffisamment petit l'existence de $u_{\theta\delta_0,0}(t,x)$. D'après la discussion du paragraphe précédent, il faut et il suffit que

$$||tP_t(\delta_0)(.)||_\infty < \infty, \quad ||a_2^{\delta_0}(t,.)||_\infty < \infty.$$

La deuxième condition correspond à

$$\int_0^t \int_{\mathbb{R}^d} (p_s(y))^2 p_{t-s}(y)dyds < \infty.$$

[15] On a repris ici le raisonnement de [**Eth 00**], partie 2.1. Le lemme III.3.7 de [**Pe 99**] fournit également un critère analogue pour l'existence de moments exponentiels pour $< X_t, \phi >$. Dynkin [**Dy 91**] donne une formule explicite des moments de $< X_t, \phi >$ sous $\mathbf{P}_\mu$, et, de façon plus générale, de quantités telles que $\mathbf{E}_\mu[< X_{t_1}, \phi_1 > ... < X_{t_n}, \phi_n >]$.

Enfin, ces résultats sont, quoique distincts, à mettre en relation avec la formule des moments du serpent brownien due à Le Gall. On introduit Λ_p la mesure uniforme sur les arbres binaires, ordonnés, enracinés, marqués, à p feuilles :

$$\int \Lambda_p(d\theta)F(\theta) = \sum_{\tau\in\mathcal{T}_p} \int \prod_{v\in\tau} dh_v F(\tau, \{h_v, v \in \tau\}).$$

La fonction de contour f^T d'un arbre marqué $T \in \mathcal{T}_p$ décrit la hauteur d'une particule lancée à vitesse 1 autour de T (i.e. elle un temps h_v pour parcourir la branche reliant v à son ancêtre). On définit $s_1, ..., s_p$ les p temps auxquels la particule se trouve en l'une des feuilles de T. Pour une fonction $F \in \mathcal{B}(\mathcal{W}_x^p)$, on note alors

$$\mathbb{P}_x^\theta(F) = \Gamma_{\{x\}}^{f_\theta}(F(W_{s_1}, ..., W_{s_p}))$$

La proposition IV.2 de [**LG 99**] affirme alors que, si $F \in \mathcal{B}(\mathcal{W}_x^p)$ est positive, alors

$$\mathbb{N}_x\left[\int_{0\leq t_1\leq ...\leq t_p} dt_1...dt_p F(W_{t_1}, ..., W_{t_p})\right] = 2^{p-1}\int \Lambda_p(d\theta)\mathbb{P}_x^\theta(F).$$

Il est alors facile de voir que les deux conditions sont vérifiées si et seulement si $d = 1$. Dans ce cas, on peut alors montrer que la densité obtenue est continue en (t, x) et qu'elle est solution d'une équation aux dérivées partielles stochastique.

1.3.6. *Existence de temps locaux pour* $d \leq 3$. Rappelons que les fonctions ϕ_x^ε constituent une approximation de la masse de Dirac au point x.

DÉFINITION I.11. *Sous réserve d'existence, on appelle temps local du super-mouvement brownien au point x et au temps t la quantité*

$$L_t^x := \lim_{\varepsilon \to 0} \int_0^t < X_s, \phi_x^\varepsilon > ds.$$

L'existence de temps locaux du super-mouvement brownien en dimension $d \leq 3$ a été établie par Sugitani [**Su 89**]. La généralisation à une classe plus large de superprocessus fut effectuée par Dynkin [**Dy 91**].

Ici encore, on peut se restreindre au cas $x = 0$ sans perte de généralité. D'après la lemme 3.1 de [**Su 89**], une condition suffisante de l'existence de temps locaux correspond à montrer pour θ suffisamment petit l'existence de $u_{0,\theta\delta_0}(t, x)$. On développe de la même manière que précedemment une telle solution d'après une série formelle $u_{0,\theta\delta_0}(t, x) = \sum_{n \geq 1} a_n(t, x)\theta^n$. On obtient cette fois que les coefficients a_n vérifient la relation de récurrence

$$\begin{cases} a_1(t, x) = & \int_0^t P_s(\delta_0)(x) ds, \\ a_n(t, x) = & -\frac{\gamma}{2} \sum_{k=1}^{n-1} \int_0^t P_s\left(a_k(t - s, .)a_{n-k}(t - s, .)\right)(x). \end{cases}$$

Par un raisonnement analogue à celui du paragraphe précédent, la série $\sum_{n \geq 1} a_n(t, x)\theta^n$ a un rayon de convergence strictement positif si et seulement si

$$||t \int_0^t P_s(\delta_0)(.) ds||_\infty < \infty, \quad ||a_2(t, .)||_\infty < \infty,$$

ce qui équivaut à $d \leq 3$.

Les temps locaux permettent en dimension $d \leq 3$ une nouvelle expression de la mesure d'occupation :

$$\forall A \in \mathscr{B}(\mathbb{R}^d) \qquad Y_t(A) = \int_A L_t^x dx.$$

On étudie au chapitre III les propriétés locales des temps locaux, ainsi que de la mesure d'occupation du super-mouvement brownien, en dimension $d \leq 3$. Comme on le verra à travers les résultats de [**DIP 89**] sur les probabilités d'atteinte de petites boules et de points par le super-mouvement-brownien, il n'existe pas de temps locaux en dimension $d \geq 4$.

1.3.7. *Propriétés des temps locaux du super-mouvement brownien.* Il est à noter que la méthode pour prouver l'existence de temps locaux en dimension $d \leq 3$ implique de plus l'existence de moments exponentiels pour les temps locaux du super-mouvement brownien. Une discussion plus approfondie à ce sujet, qui fait en particulier intervenir des conditions sur la mesure initiale μ, se trouve au début du chapitre III.

Dans le cas $d = 1$, Sugitani [**Su 89**] montre que $(x, t) \to L_t^x$ est toujours continue. De plus, $(x, t) \to L_t^x$ est différentiable par rapport à la variable d'espace aux points $x \in \mathbb{R}^d$ tels que $\mu(\{x\}) = 0$. Ceci permet à Sugitani d'établir les propriétés locales des temps locaux du super-mouvement brownien en dimension 1. On discute ces propriétés dans la partie III.4.

Dans le cas $d = 2$ ou 3, il est important de noter que la fonction $(x, t) \to L_t^x$ n'est pas forcément continue aux points $(0, x), x \in \mathbb{R}^d$. Cependant, Sugitani [**Su 89**] a obtenu pour $d \geq 2$ que L_t^x est continue en (x, t) sur l'ensemble des points de continuité de $\int_{\mathbb{R}^d} \int_0^t dsp_s(x - y)\mu(dy)$.

Barlow, Evans et Perkins [**BEP 91**] ont établi une formule de Tanaka pour les temps locaux du super-mouvement brownien. Intuitivement, celle-ci s'obtient en montrant que le problème de martingales qui définit le super-mouvement brownien s'étend à des fonctions g_α^x dont le laplacien au sens des distributions vaut δ_x. Pour $\alpha \geq 0$ ($\alpha > 0$ si $d = 2$) on pose ainsi $g_\alpha(z) = \int_0^\infty \exp(-\alpha t)p_t(z)dt$, et pour $x \in \mathbb{R}^d$, $g_\alpha^x(z) = g_\alpha(z - x)$. Si $\mu \in M_F(\mathbb{R}^d)$, et si on suppose $< \mu, g_\alpha^y > < \infty$, alors, $\mathbf{P}_\mu$-presque sûrement,

$$< X_t, g_\alpha^y > = < \mu, g_\alpha^y > + M_t(g_\alpha^y) + \alpha \int_0^t < X_s, g_\alpha^y > ds - L_t^y \tag{8}$$

où $M_t(g_\alpha^y)$ est une martingale dans la filtration canonique de X. En particulier, elle vérifie $M_0(g_\alpha^y) = 0$, et a pour variation quadratique

$$< M(g_\alpha^y) >_t = 4 \int_0^t < X_s, (g_\alpha^y)^2 > ds.$$

Cette formule est un outil important dans notre étude au chapitre III.

1.3.8. *Probabilités d'atteinte.* Nous faisons, dans les chapitres III,IV et V, maintes fois usage des propriétés que nous décrivons ci-dessous.

On note $\mathcal{R}_t := \text{supp}(X_t)$ le support topologique de la mesure X_t, et

$$\mathcal{R} := \bigcup_{\varepsilon > 0} \left(\overline{\bigcup_{t \geq \varepsilon} \mathcal{R}_t} \right),$$

le *support* de X (notons que celui-ci coïncide avec le support de la mesure d'occupation totale éventuellement privé d'une partie du support de la mesure initiale). Dawson, Iscoe et Perkins [**DIP 89**] ont obtenu les ordres des probabilité d'atteinte de boules par le super-mouvement brownien et par son support. Ils utilisent pour cela le théorème I.2 et des arguments analytiques. Leurs résultats impliquent en particulier que pour tout $\mu \in M_F(\mathbb{R}^d)$ tel que $0 \notin \text{supp}(\mu)$,

– pour $d \leq 3$,

$$\mathbf{P}_\mu(0 \in \mathcal{R}) = 1 - \exp\left(-\frac{4 - d}{2} \int_{\mathbb{R}^d} |y|^{-2} \mu(dy) \right),$$

– pour $d \geq 4$, $\mathbf{P}_\mu(0 \in \mathcal{R}) = 0$,
– pour $d \geq 3$, $\mathbf{P}_\mu(X_t(B(0, \varepsilon) > 0))$ est d'ordre ε^{d-2} (voir le théorème 3.1 de [**DIP 89**]),
– pour $d = 4$, et pour tout $\delta > 0$, $\mathbf{P}_\mu(\mathcal{R} \cap B(0, \varepsilon) \neq \emptyset)$ est d'ordre $\left(\log\left(\varepsilon^{-1}\right)\right)^{-1}$ (voir le théorème 3.2 (b) et (d), combinés avec le théorème 3.3 de [**DIP 89**]),
– pour $d \geq 5$, et pour tout $\delta > 0$, $\mathbf{P}_\mu(\mathcal{R} \cap B(0, \varepsilon) \neq \emptyset)$ est d'ordre ε^{d-4} (voir le théorème 3.2 (a) et (c) combinés avec le théorème 3.3 de [**DIP 89**]).
– pour $d \geq 1$, la probabilité d'échapper avant t à $\{x + y, x \in B(0, 2), y \in \text{supp}(\mu)\}$ est, lorsque $t \to 0$, un $O(t^{-d/2-1} \exp(-t^{-1/2}))$ (voir le théorème 3.3 (a) de [**DIP 89**])).

Le deuxième résultat ci-dessus interdit l'existence d'une densité pour la mesure d'occupation et donc l'existence de temps locaux en dimension $d \geq 4$. Il est à noter que certains de ces résultats utilisent le fait que le processus des supports $(\mathcal{R}_t)_{t\geq 0}$ possède un module de continuité (cf théorème 1.1 de [**DIP 89**]). Intuitivement, ce résultat exprime que le super-mouvement brownien se déplace à vitesse finie. En particulier, il implique que le processus $(\mathcal{R}_t, t \geq 0)$ est continu à droite et à valeurs dans les compacts de $\mathbb{R}^d$.

Pour $d \geq 4$, des résultats analogues pour le support du super-mouvement brownien sous sa mesure d'excursion peuvent être obtenus en utilisant le lien entre serpent brownien et super-mouvement brownien. Ces résultats peuvent également être démontrés directement grâce à l'approche en termes de serpent brownien en utilisant le lien entre serpent et équations aux dérivées partielles (voir [**LG 99**], proposition V.9).

En particulier (voir [**LG 99**], partie VI.1), les deux approches permettent d'obtenir que

$$\tilde{\mathbb{N}}_x(0 \in \mathcal{R}) = \left(2 - \frac{d}{2}\right)^+ |x|^{-2}.$$

1.3.9. *Propriétés du support en termes de h-mesure et dimension de Hausdorff.* Ces propriétés ont été étudiées par de nombreux auteurs, en particulier par Dawson, Perkins et Le Gall (cf [**DH 79**], [**Pe 89**], [**DIP 89**], [**LGP 95**]). Nous allons simplement citer ces propriétés. Nous faisons usage de certaines d'entre elles au chapitre IV dans le cas des dimensions 2 et 3. Une formulation plus précise ainsi qu'une démonstration dans le cas $d \geq 3$ peuvent être trouvées dans [**Pe 99**], chapitre III. Certains de ces résultats sont redemontrés à l'aide du serpent brownien dans [**LG 99**], partie IV.5, et dans [**LG 98**]. Le cas $d = 2$ est démontré dans [**LGP 95**]. Enfin, rappelons qu'en dimension 1, X_t possède une densité bicontinue par rapport à la mesure de Lebesgue, $\mathbf{P}_\mu$-presque-sûrement.

On considère $h : \mathbb{R}_+ \to \mathbb{R}_+$ une fonction strictement croissante, continue en 0. La h-mesure de $\mathcal{A} \subset \mathbb{R}^d$ est

$$h{-}m(\mathcal{A}) = \lim_{\varepsilon \to 0} \left(\inf \left[\sum_{i=0}^{\infty} h(\varepsilon_i) \ : \ \mathcal{A} \subset \bigcup_{i=0}^{\infty} B_i,\ B_i \text{ boréliens de diamètre } \varepsilon_i \leq \varepsilon \right] \right).$$

La dimension de Hausdorff de $\mathcal{A}$ est

$$\dim_H(\mathcal{A}) = \inf \{\alpha > 0 : x^\alpha - m(\mathcal{A}) < \infty\}.$$

La dimension de Hausdorff de $\mathcal{R}_t$ est $d \wedge 2$, $\mathbf{P}_\mu$-presque sûrement. Le support du super-mouvement brownien $\mathcal{R}$ a quant à lui pour dimension de Hausdorff $d \wedge 4$, $\mathbf{P}_\mu$-presque sûrement.

De plus, pour $d \geq 2, X_t$ distribue sa masse sur $\mathcal{R}_t$ de façon déterministe et extrêmement régulière. Posons

$$h_d(r) = \begin{cases} r^2 \log\log(r^{-1} \vee e) & \text{si } d \geq 3 \\ r^2 \log(r^{-1} \vee 1) \log\log\log(r^{-1} \vee e^e) & \text{si } d = 2. \end{cases}$$

On peut alors trouver une constante universelle $c(d)$ telle que pour tout $\mu \in M_F(\mathbb{R}^d)$, pour tout $t > 0$,

$$X_t(\mathcal{A}) = \gamma c(d) h_d{-}m(\mathcal{A} \cap \mathcal{R}_t) \ \ \forall \mathcal{A} \in \mathscr{B}(\mathbb{R}^d), \quad \mathbf{P}_\mu - \text{presque sûrement}.$$

Le théorème III.3.1 de [**Pe 99**] donne, de plus, des résultats correspondants lorsque t est variable.

2. Modèle du votant

Le modèle du votant est un système de particules en interaction, qui fut initialement introduit par Clifford et Sudburry [**CS 73**] en 1973 pour modéliser une compétition entre deux espèces. Il fut réintroduit en 1975 de façon indépendante par Holley et Liggett [**HL 75**]. Nous allons tout d'abord décrire ce modèle (§I.2.1). Les propriétés du modèle du votant ont été étudiées de façon approfondie par de nombreux auteurs, en particulier par Liggett, Bramson, Cox et Griffeath (voir §I.2.2). Plus récemment, Cox, Durrett et Perkins [**CDP 00**], puis Bramson, Cox et Le Gall [**BCLG 01**] ont établi un principe d'invariance pour le modèle du votant, en démontrant qu'une version à valeurs mesures correctement rééchelonnée de ce processus converge vers le super-mouvement brownien (voir §I.2.3).

2.1. Description du modèle. Notons p un noyau de probabilité sur $\mathbb{Z}^d \times \mathbb{Z}^d$ qui satisfait l'ensemble d'hypothèses $\mathcal{H}$(noyau) introduit au chapitre I.

En chaque point de $\mathbb{Z}^d$ se trouve un électeur. Dans le *modèle du votant à deux opinions*, chaque électeur possède initialement une parmi deux opinions possibles, notées 0 et 1. Le mécanisme d'évolution est décrit de la façon suivante. Chaque électeur, à des temps espacés par des variables exponentielles indépendantes de paramètre 1, choisit l'un de ses voisins suivant le noyau de probabilité p et adopte alors l'opinion de ce voisin. Ces temps et les choix de voisins sont également supposés indépendants.

On note $\xi_t \in \{0,1\}^{\mathbb{Z}^d}$ la configuration des opinions au temps t. Ainsi, $\xi_t(x)$ est l'opinion au temps t de l'électeur situé au point $x \in \mathbb{Z}^d$. Il est facile de voir que le processus ξ est un processus de Markov.

Comme on l'a mentionné en 1.1, on peut interpréter ce modèle comme un marche aléatoire branchante (MAB) pour laquelle les taux de naissance et de mort dépendent de la configuration. Pour établir cette correspondance, on interprète les sites hébergeant un électeur d'opinion 1 comme occupés par une particule, et les sites hébergeant un électeur d'opinion 0 comme libres. Alors
- la particule située en x meurt au taux $\sum_{y\in\mathbb{Z}^d} p(x,y)\mathbf{1}_{\eta_t(y)=0}$.
- la particule située en x donne naissance à une nouvelle particule en y au taux $p(x,y)\mathbf{1}_{\eta_t(y)=0}$.

Dans le modèle de Lotka-Volterra, les taux de naissance et de mort dépendent également de la densité "locale" de 1.

On préfère en fait travailler avec un processus à valeurs dans les sous-ensembles de $\mathbb{Z}^d$, et on identifie alors ξ à $\xi_t := \{x \in S : \eta_t(x) = 1\}$, l'ensemble des sites hébergeant un électeur d'opinion 1. Lorsque $\xi_0 = A \subset \mathbb{Z}^d$, on note P_A la mesure de probabilité sous laquelle ξ est défini. Lorsque ξ_0 est fini, on parle de modèle du votant fini. Notons que dans ce cas, il est facile de voir que $|\xi_t|, t \geq 0$ est une martingale, et d'en déduire que $\inf\{t \geq 0 : \xi_t = \emptyset\}$ est fini, P_{ξ_0}-presque sûrement. On préfère alors parfois travailler avec la version à valeurs mesures de ξ :

$$X_t := \sum_{x\in\xi_t} \delta_x,$$

et on note P_{X_0} la mesure de probabilité sur $D(\mathbb{R}_+, M_F(\mathbb{Z}^d))$ sous laquelle X est défini. Le cas particulier du modèle du votant parti d'une unique opinion 1 à l'origine nous intéressera particulièrement. Dans ce cas, $\xi_0 = \{0\}$, et on note ξ^0, X^0 les processus correspondants. Enfin, avec un léger abus de notation, P désigne à la fois $P_{\{0\}}$ et P_{δ_0}.

Remarques :

- L'hypothèse qu'il n'y a que deux opinions possibles n'est pas aussi restrictive qu'il y paraît. En effet, dans un modèle où il y a un ensemble Op d'opinions, l'évolution d'une opinion $o \in Op$ donnée peut être décrite grâce au modèle à deux opinions, en associant "l'opinion 1" à o et en associant "l'opinion 0" à tous les autres éléments de Op. Par exemple, on peut considérer le modèle du votant à opinions multiples, décrit par le même mécanisme d'évolution, et où chaque électeur possède initialement son opinion propre (dans ce cas, on peut donc identifier Op à $\mathbb{Z}^d$).
- Dans l'interprétation de Clifford et Sudburry, les ensembles $\{x \in \mathbb{Z}^d : \xi_t(x) = 0\}, \{x \in \mathbb{Z}^d : \xi_t(x) = 1\}$ représentent les territoires occupés par deux espèces en compétition. La site x est envahi par l'espèce concurrente à un taux
$$c_t(x) = \sum_{y \neq x} p(x,y) \mathbf{1}_{\{\eta_t(x) \neq \eta_t(y)\}}.$$
- De façon plus générale, on peut également étudier le modèle du votant sur un graphe S fini ou dénombrable, et on peut faire des hypothèses moins restrictives sur le noyau de saut (voir [**Li 85**], chapitre V). Durrett [**Du 04**] s'est en outre récemment intéressé au modèle du votant sur un graphe S aléatoire.

Le paragraphe suivant décrit une représentation graphique souvent très utile.

2.1.1. *Représentation graphique du modèle du votant.* L'espace $\mathbb{Z}^d$ est représenté par l'axe des abscisses, et le temps par l'axe des ordonnées. En chaque temps où l'électeur en x adopte l'opinion de l'électeur en y, on dessine une flèche de y vers x. Grâce à cette représentation, il est facile de trouver l'unique origine au temps s de l'opinion de x au temps $t > s$. Pour cela, partant de (x,t), on descend verticalement et on emprunte horizontalement chacune des flèches rencontrées *en sens inverse.* Si au temps s, on se trouve en y, on dit qu'il existe *un chemin descendant* de (x,t) vers (y,s), et on note $(x,t) \searrow (y,s)$ (voir V.2.3 pour plus de détails).

Il est facile de voir qu'un chemin descendant issu de (x,t) suit la trajectoire sur l'intervalle de temps $[0,t]$ d'une marche aléatoire à taux 1, de noyau p, partie de x. Deux chemins descendants issus de $(x,t),(y,t)$ suivent les trajectoires de marches indépendantes issues de x et y jusqu'à leur éventuelle rencontre, après quoi les deux chemins coïncident. Ceci va nous permettre de relier le modèle du votant à un système de marches aléatoires coalescentes.

2.2. Propriétés du modèle du votant.

2.2.1. *Système dual : marches aléatoires coalescentes.* Notons $(Z_s^{x,t})_{0 \le s \le t}$ le processus qui vérifie $Z_0^{x,t} = x$ et
$$\text{pour tout } 0 < s \le t, Z_s^{x,t} = y \iff (x,t) \searrow (y, t-s).$$
D'après les remarques du paragraphe précédent, $(Z_s^{x,t})_{0 \le s \le t}$ est une marche aléatoire à taux 1, de noyau p, partie de x, et a donc pour loi $\mathbb{Q}_x^{1,\sigma^2}$, $(Z_s^{x,t})_{0 \le s \le t, x \in \mathbb{Z}^d}$ est un *système de marches aléatoires coalescentes*, à taux 1, de noyau p (voir V.2.3 pour une formulation plus précise). La relation entre les deux processus est désormais facile à écrire
$$\xi_t^0 = \{y \in \mathbb{Z}^d : Z_t^{y,t} = 0\}.$$

Cette dualité entre modèle du votant et système de marches coalescentes est à l'origine de nombreuses propriétés du modèle du votant. D'autre part, les résultats concernant le modèle du votant ont également une traduction en termes de marches aléatoires coalescentes.

2.2.2. *Mesures invariantes, premières propriétés asymptotiques.*
Liggett [**Li 85**], chapitre V, étudie les mesures invariantes du modèle du votant ξ sur $\mathscr{P}(\mathbb{Z}^d)$, les parties de $\mathbb{Z}^d$. Le modèle du votant possède deux mesures invariantes triviales, $\delta_{\{\emptyset\}}$ et $\delta_{\{\mathbb{Z}^d\}}$. Il n'y en a pas d'autres lorsque $d \leq 2$. En revanche, si $d \geq 3$, il existe une famille $\{\mu_\alpha, \alpha \in [0,1]\}$ de mesures invariantes.

Faisons l'hypothèse que la configuration initiale ξ_0 est choisie suivant une loi de probabilité μ sur $\mathscr{P}(\mathbb{Z}^d)$, que l'on suppose invariante par translation. Liggett [**Li 85**] montre alors que, lorsque $t \to \infty$,

– Si $d \leq 2$, un consensus est atteint presque sûrement. Les électeurs s'accordent alors sur l'opinion 1 avec probabilité $\mu(A \in \mathscr{P}(\mathbb{Z}^d) : x \in A)$ (cette quantité, d'après l'hypothèse d'invariance par translation, ne dépend pas de x, et correspond à la proportion initiale d'opinion 1).
– Si $d \geq 3$, et que l'on suppose de plus que μ est extrêmale, et que α, la proportion initiale d'opinion 1, appartient à $(0,1)$, alors, le désaccord persiste, et la loi de ξ_t converge faiblement vers la mesure invariante μ_α.

Un exemple de mesures initiales invariantes par translation et extrêmales est constitué par la famille de mesures $(\nu_\alpha)_{\alpha\in[0,1]}$. Il s'agit de la mesure produit sur $\mathscr{P}(\mathbb{Z}^d)$, dont les marginales sont données par $\nu_\alpha(A \in \mathscr{P}(\mathbb{Z}^d) : x \in A) = \alpha$. D'après le résultat que nous venons de citer, au sens de la convergence faible dans l'espace des mesures de probabilité sur $\mathscr{P}(\mathbb{Z}^d)$,

$$P_{\nu_\alpha}(\xi_t \in .) \underset{t\to\infty}{\longrightarrow} \begin{cases} (1-\alpha)\delta_\emptyset + \alpha\delta_{\mathbb{Z}^d} & \text{si } d \leq 2, \\ \mu_\alpha & \text{si } d \geq 3. \end{cases}$$

Cox et Griffeath [**CG 83**] ont d'autre part étudié, sous P_{ν_α} le comportement asymptotique du temps d'occupation en x du modèle du votant $T_t^x = \int_0^t \mathbf{1}_{\{x\in\xi_s\}} ds$. On a clairement pour toute dimension et pour tout $x \in \mathbb{Z}^d$, $E_{\nu_\alpha}(T_t^x) = \alpha t$. En revanche, l'écart-type $\sigma(t)$ de T_t^x dépend quant à lui de la dimension. Il est d'ordre t si $d = 1$, $t/\sqrt{\log(t)}$ si $d = 2$, $t^{3/4}$ si $d = 3$, $\sqrt{t\log(t)}$ si $d = 4$, et $\sqrt{t}$ si $d = 5$. De plus, Cox et Griffeath [**CG 83**] établissent que

$$\left(\frac{T_t^x - \alpha t}{\sigma(t)}\right)_{x\in\mathbb{Z}^d}$$

converge en loi lorsque $t \to \infty$ vers une limite non dégénérée. Cette limite est un champ gaussien centré et invariant par translation lorsque $d \geq 2$, dont on peut exprimer les covariances (voir [**CG 83**]). Ce résultat est à rapprocher de notre théorème principal du chapitre III, concernant le comportement local des temps locaux du super-mouvement brownien.

2.2.3. *Le cas particulier* $\xi_0 = \{0\}$. On s'intéresse ici au cas particulier où initialement, seul l'électeur placé à l'origine possède l'opinion 1. D'après Liggett [**Li 85**] (partie V.4) en dimension 1, et d'après Bramson et Griffeath [**BG 80**] en dimension $d \geq 2$, on connaît la vitesse à laquelle $P(\xi_t^0 \neq \emptyset)$ tend vers 0. Notons $\beta_1 = \sqrt{\pi}\sigma^2$, et rappelons que $\beta_2 = 2\pi\sigma^2$, et pour $d \geq 3$, β_d la probabilité qu'une marche aléatoire de noyau p partie de 0 n'y retourne jamais.

Proposition I.5. *Soit $(\xi_t^0, t \geq 0)$ le modèle du votant de noyau p, parti d'une unique opinion 1 à l'origine. Alors, si on note $p_t := P(\xi_t^0 \neq \emptyset)$, on a*

$$p_t \sim \begin{cases} \frac{1}{\beta_1 \sqrt{t}} & \textit{si } d = 1, \\ \log(t)/(\beta_2 t) & \textit{si } d = 2, \\ 1/(\beta_d t) & \textit{si } d \geq 3. \end{cases}$$

De plus, si on définit $P_t^*(.) = P(.|\xi_t^0 \neq \emptyset)$, alors sous P_t^*,

$$p_t |\xi_t^0| \xrightarrow[t\to\infty]{(loi)} \begin{cases} |\mathcal{N}| & \textit{si } d = 1, \\ \mathbf{e} & \textit{si } d \geq 2, \end{cases}$$

où $\mathcal{N}$ est une variable gaussienne standard, $\mathbf{e}$ une variable exponentielle standard.

La dichotomie entre les cas $d = 1$ et $d \geq 2$ persiste lorsque l'on s'intéresse aux limites de changement d'echelle du modèle du votant.

2.3. Limite par changement d'échelle. Des informations plus précises sur le comportement asymptotique du modèle du votant en dimension $d \geq 2$ ont été obtenues par Cox, Durrett et Perkins [**CDP 00**]. L'étude des asymptotiques du modèle du votant parti d'une unique opinion 1 à l'origine (cf. paragraphe précédent) suggère que la masse totale des opinions 1 au temps t doit ici être renormalisée par

$$m_t = \begin{cases} \sqrt{t} & \text{si } d = 1, \\ \frac{t}{\ln(t)} & \text{si } d = 2, \\ t & \text{si } d \geq 3. \end{cases}$$

D'après notre comparaison avec les marches aléatoires branchantes, on peut s'attendre intuitivement à trouver une limite par changement d'échelle super-brownienne, pourvu que l'interaction entre voisins ne soit pas trop importante. C'est en effet le cas (pour toute dimension) si on fait en sorte que les noyaux de transition successifs soient de plus en plus étalés (voir théorèmes 1.1 et 1.3 de [**CDP 00**]).

Revenons au cas où on fixe le noyau p qui détermine les changements d'opinion. Pour un changement d'échelle analogue à celui que nous avons fait en I.1 pour une marche aléatoire branchante, on ne trouve de limite super-brownienne que lorsque $d \geq 2$. On rééchelonne alors notre modèle de la façon suivante. Le réseau est désormais S_N, les individus changent d'opinion au taux N au lieu de 1, et le noyau de saut devient p_N. On note alors ξ_t^N l'ensemble, au temps t, des sites de S_N où l'électeur présent à ce site possède l'opinion 1. On associe le processus à valeurs mesures

$$X_t^N := \frac{1}{m_N} \sum_{x \in \xi_t^N} \delta_x,$$

et la mesure de probabilité associée P^N. On rappelle enfin que $\beta_2 = 2\pi\sigma^2$ et que β_d est pour $d \geq 3$ la probabilité qu'une marche aléatoire de noyau p ne retourne jamais à son point de départ.

Théorème I.4. *Soit $d \geq 2$. On suppose que $X_0^N \to \mu \in M_F(\mathbb{R}^d)$ lorsque $N \to \infty$. Alors, au sens de la convergence faible dans l'espace $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$, la suite de processus X^N sous P^N converge vers le super-mouvement brownien $(X_t, t \geq 0)$ sous $P_\mu^{2\beta_d, \sigma^2}$.*

Remarque : En dimension 1, dans le cas de la marche aléatoire simple (i.e. de noyau p tel que $p(1) = p(-1) = 1/2$), la limite existe, mais n'est pas super-brownienne. Elle est décrite dans [**A 79**] à l'aide de mouvements browniens annihilants.

2.3.1. *Le cas particulier du modèle du votant parti d'une unique opinion* 1 *à l'origine.* Bramson, Cox et Le Gall [**BCLG 01**] ont établi un théorème correspondant pour le modèle du votant parti de $\xi_0 = \{0\}$. D'après le dernier paragraphe de la partie I.2.2, on rééchelonne exactement de la même manière que dans le paragraphe précédent, on définit ainsi $\xi_t^{N,0}$, $X_t^{N,0}$. Pour obtenir une limite non dégénérée, on va cette fois fixer $\alpha > 0$ et travailler conditionnellement au fait que $X_\alpha^{N,0} \neq 0$.

THÉORÈME I.5. *Soit $d \geq 2$. Au sens de la convergence faible dans l'espace $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$, la suite de processus $X^{N,0}$ sous $P(.\big|X_\alpha^{N,0} \neq 0)$ converge vers le super-mouvement brownien (avec coefficient de branchement $2\beta_d$ et coefficient de diffusion σ^2) $(X_t, t \geq 0)$, sous la mesure $\tilde{\mathbb{N}}_0(.|X_\alpha \neq 0)$.*

D'autre part, Bramson, Cox et Le Gall [**BCLG 01**] obtiennent également pour $t > 0$ fixé la convergence de $\xi_t^{N,0}$ vers le support de X_t sous sa mesure canonique au temps t.

Les théorèmes I.4 et I.5 nous apprennent que le comportement asymptotique du modèle du votant fini en dimension $d \geq 2$ peut se déduire des propriétés du super-mouvement brownien. On renforce cette idée au chapitre V, en étudiant la probabilité d'atteinte d'un point éloigné par le modèle du votant en dimension $d \geq 2$ grâce au théorème I.5.

CHAPITRE II

Exposé des résultats

1. Comportement local des temps locaux en dimension $d \leq 3$ et de la mesure d'occupation du super-mouvement brownien en dimension $d \leq 4$

1.1. Le cas $d \leq 3$. On a vu dans le chapitre précédent l'existence de temps locaux $(L_t^x)_{t \geq 0, x \in \mathbb{R}^d}$ du super-mouvement brownien en dimension $d \leq 3$. La fonction $x \to L_t^x$ constitue une densité pour la mesure d'occupation Y_t du super-mouvement brownien.

En dimension 1, le résultat, dû à Sugitani [**Su 89**], de différentiabilité de ces temps locaux par rapport à la variable d'espace permet de comprendre leur comportement local (voir partie III.4).

En dimensions 2 et 3, les temps locaux ne sont plus différentiables par rapport à la variable d'espace, mais on obtient au chapitre III le résultat suivant.

Théorème (III.1) *Soit $d = 2$ or 3, et $\nu \in M_F(\mathbb{R}^d)$, telle que $\nu(B(0, \rho)) = 0$ pour un certain $\rho > 0$. On considère X un super-mouvement brownien dans $\mathbb{R}^d$ sous $\mathbf{P}_\nu$, et L_t^x son temps local au temps t et au niveau x. On note enfin*

$$\begin{aligned} \psi(c) &= \sqrt{c} \quad si \;\; d = 3 \\ &= \frac{c}{\sqrt{\ln(c)}} \quad si \;\; d = 2, \end{aligned}$$

et on fixe $x_1, ..., x_k$ des points de $\mathbb{R}^d \setminus \{0\}$.

Alors, au sens de la convergence faible de l'espace $C(\mathbb{R}^+, M_F(\mathbb{R}^d) \times \mathbb{R}^k)$, le processus

$$\left(X_t, \psi(c)(L_t^{x_1/c} - L_t^0), ..., \psi(c)(L_t^{x_k/c} - L_t^0) \right)_{t \geq 0}$$

converge lorsque $c \to \infty$, vers le processus limite

$$\left(X_t, \beta^{x_1}_{L_t^0}, ..., \beta^{x_k}_{L_t^0} \right)_{t \geq 0}.$$

Ici, $(\beta_t^x)_{t \geq 0, x \in \mathbb{R}^d}$ est un processus gaussien centré independant de X tel que

$$\mathrm{cov}(\beta_t^x, \beta_s^y) = a(x, y)(t \wedge s),$$

et $a(x, y)$ est donné par

$$a(x, y) = \begin{cases} \frac{1}{4\pi^2} \int_{\mathbb{R}^3} dz \left(\frac{1}{|z-x|} - \frac{1}{|z|} \right) \times \left(\frac{1}{|z-y|} - \frac{1}{|z|} \right) & si \;\; d = 3, \\ \frac{1}{\pi} x.y & si \;\; d = 2. \end{cases}$$

Ce résultat rappelle celui de Cox et Griffeath [**CG 83**], cité en I.2.2, concernant les temps d'occupation du modèle du votant sous P_{ν_α}. Il est d'autre part à mettre en relation avec un résultat de Yor [**Y 83**] concernant les temps locaux du mouvement brownien standard. Celui-ci obtient en effet que si ℓ_t^x désigne le temps local au temps t et au niveau x d'un mouvement brownien standard, alors $(\sqrt{c}(\ell_t^{x/c} - \ell_t^0)/2)_{t\geq 0, x\geq 0}$ converge en distribution vers $\left(\mathcal{B}_{\ell_t^0, x}\right)_{t\geq 0, x\geq 0}$, où $\mathcal{B}$ est un drap brownien issu de 0, indépendant de β.

On renvoie le lecteur à la partie III.1.6 pour un résumé de la preuve du théorème III.1.

Ce théorème s'applique à l'étude du comportement local de la mesure d'occupation du super-mouvement brownien. En particulier, on retrouve un résultat de Lee [**Le 01**], qui fut l'une des motivations originales de notre travail. Par des méthodes analytiques, celui-ci obtient pour $d = 3$ le comportement dans l'ensemble $K^{(c)} := \{x : cx \in K\}$, où K est un compact de $\mathbb{R}^d$, de la mesure d'occupation d'un super-mouvement brownien issu de $c^{-2}\delta_{x_0}$ (où $x_0 \in \mathbb{R}^d \setminus \{0\}$), et conditionné à toucher $K^{(c)}$.

Soyons plus précis. On considére des fonctions $\phi, \xi \in \mathcal{C}_b(\mathbb{R}^d)$ à support compact K, qui satisfont respectivement $\int_K \phi(x)dx \neq 0, \int_K \xi(x)dx = 0$, et les fonctions $\phi_c(.) = \phi(c.)$, $\xi_c(.) = \xi(c.)$ à support dans le compact $K^{(c)} = \{x : cx \in K\}$[1] En utilisant que les temps locaux sont la densité de la mesure d'occupation, et que $\int \xi(x)dx = 0$, on obtient la relation

$$\begin{aligned}(D_1^{(c)}, D_2^{(c)}) &:= (c^d < Y_t, \phi_c >, c^d\psi(c) < Y_t, \xi_c >)\\ &= \left(\int_{\mathbb{R}^d} L_t^{x/c}\phi(x)dx, \int_{\mathbb{R}^3} \psi(c)(L_t^{x/c} - L_t^0)\xi(x)dx\right).\end{aligned}$$

La continuité des temps locaux implique facilement que $D_1^{(c)}$ converge vers $L_t^0 \int_K \phi(x)dx$. Intuitivement, le théorème III.1 permet d'établir la convergence de $D_2^{(c)}$ vers $\int_K \xi(x)\beta^x_{L_t^0}dx$. Ceci est rendu rigoureux dans la partie III.3.2 où on établit en particulier

Proposition (III.2) *Sous $P_{\delta_{x_0}}$, on a la convergence en distribution*

$$(D_1^{(c)}, D_2^{(c)}) \underset{c\to\infty}{\longrightarrow} \left(L_t^0 \int_K \phi(x)dx, U_t\right),$$

où, conditionnellement à L_t^0, la variable U_t est gaussienne, centrée, de variance $a_\xi L_t^0$, avec

$$a_\xi := \int_K \int_K \xi(y)\xi(z)a(y,z)dydz.$$

On obtient en outre le même résultat pour un super-mouvement brownien sous $\tilde{\mathbb{N}}_{x_0}(.|0 \in \mathcal{R})$ (voir proposition III.5) puis pour un super-mouvement brownien sous $\mathbf{P}_{\delta_{x_0}/c^2}(.|\mathcal{R} \cap K^{(c)} \neq \emptyset)$ (voir proposition III.4), ce qui nous permet de retrouver le résultat de Lee [**Le 01**].

L'étude locale des temps locaux du super-mouvement brownien en dimension $d \leq 3$ permet donc d'obtenir des informations précises sur le comportement local de sa mesure d'occupation.

[1] On peut penser à l'exemple simple où $\phi = \mathbf{1}_{B(0,1)}$, et $\xi = \mathbf{1}_{\{x\in B(0,1), x_1\geq 0\}} - \mathbf{1}_{\{x\in B(0,1), x_1<0\}}$.

1.2. Le cas $d = 4$. On étudie au chapitre IV le comportement local de la mesure d'occupation du super-mouvement brownien. Le résultat véritablement intéressant concerne le cas de la dimension critique $d = 4$. Dans ce cas, la mesure d'occupation n'a pas de densité, et on ne peut donc pas employer la même méthode que précedemment pour étudier son comportement local.

On considère X un super-mouvement brownien sous sa mesure d'excursion, où x est un point fixé de $\mathbb{R}^4$, $x \neq 0$. Rappelons que $\mathcal{Z}$ désigne la mesure d'occupation de X sous sa mesure d'excursion, définie sous $\mathbb{N}_x$ par la relation

$$\forall A \in \mathscr{B}(\mathbb{R}^d) \quad \mathcal{Z}(A) := \int_0^\sigma \mathbf{1}_A(\hat{W}_s)ds.$$

Dans le chapitre IV, on étudie le comportement asymptotique de la mesure d'occupation d'une petite boule, c'est-à-dire qu'on cherche à évaluer $\mathcal{Z}(B(0,\varepsilon))$ lorsque $\varepsilon \to 0$.

Théorème (IV.1 (ii)) *La loi sous* $\mathbb{N}_x \left(.\middle|\mathcal{Z}(B(0,\varepsilon)) > 0\right)$ *de la variable*

$$\frac{2\mathcal{Z}(B(0,\varepsilon))}{\varepsilon^4 \ln\left(\frac{|x|}{\varepsilon}\right)}$$

converge lorsque $\varepsilon \to 0$ *vers une loi exponentielle de paramètre* 1.

On utilise une méthode de moments pour établir le théorème IV.1. Les résultats de Dawson, Iscoe et Perkins [**DIP 89**], sur la probabilité d'atteinte d'une boule (cf I.1.3), permettent tout d'abord d'évaluer $\tilde{\mathbb{N}}_x(\mathcal{Z}(B(0,\varepsilon)) > 0)$. D'autre part, on se sert de la formule de récurrence (6) (qui concerne les moments de la mesure d'occupation du super-mouvement brownien sous sa mesure d'excursion) pour évaluer $\tilde{\mathbb{N}}_x(\mathcal{Z}(B(0,\varepsilon))^p)$, pour $p \in \mathbb{N}^*$.

Enfin, la relation (5) permet de déduire un résultat similaire pour la mesure d'occupation totale du super-mouvement brownien sous $\mathbf{P}_{\delta_x}(.|Y(B(0,\varepsilon)) > 0)$ (voir corollaire IV.1).

2. Probabilité d'atteinte d'un point éloigné pour le modèle du votant

2.1. Enoncé du résultat principal. On considère le modèle du votant $(\xi_t^0, t \geq 0)$, sur $\mathbb{Z}^d$, $d \geq 2$, de noyau p satisfaisant $\mathcal{H}$(noyau), partant avec l'opinion 1 à l'origine, et l'opinion 0 partout ailleurs, que l'on a introduit en I.2. On a vu en I.2.3 d'après [**BCLG 01**] qu'une version rééchelonnée, à valeurs mesures, et conditionnée à survivre, de ce modèle, converge vers le super-mouvement brownien sous sa mesure d'excursion. Le but principal du chapitre V est d'exploiter cette convergence, et des propriétés connues du super-mouvement brownien, afin d'établir des résultats asymptotiques pour le modèle du votant.

On obtient ainsi, dans le cas $d \geq 2$, les asymptotiques de la probabilité d'atteinte d'un point éloigné de l'origine par $(\xi_t^0, t \geq 0)$. Plus précisément, on fixe $x \in \mathbb{R}^d \setminus \{0\}$ fixé, et pour $c > 0$, on note $[x]_c$ le point de $\mathbb{Z}^d/c$ le plus proche de x (au cas où il y en a plusieurs on choisit celui qui est le plus proche de l'origine). On définit de plus $T_{c[x]_c} := \inf\{t \geq 0 : c[x]_c \in \xi_t^0\}$. Le théorème principal du chapitre V exprime les asymptotiques de $P(T_{c[x]_c} < \infty)$.

Théorème (V.1) *Introduisons*

$$\phi_d(c) = \begin{cases} \frac{c^2}{2\ln(c)} & si\ d=2, \\ c^2 & si\ d=3, \\ c^{d-2} & si\ d \geq 5, \end{cases}$$

Soient $\beta_2 = 2\pi\sigma^2$*, et, pour* $d \geq 3$*,* β_d *la probabilité qu'une marche aléatoire de noyau* p*, partie de l'origine, n'y retourne jamais. Alors, si* $d = 2$ *ou* 3*,*

$$\lim_{c\to\infty} \phi_d(c) P(T_{c[x]_c} < \infty) = \frac{2\sigma^2}{\beta_d}\left(2 - \frac{d}{2}\right)|x|^{-2}.$$

Si $d \geq 5$*, il existe des constantes strictement positives* a_d, b_d *dependant seulement de* x *qui vérifient*

$$a_d \leq \liminf_{c\to\infty} \phi_d(c) P(T_{c[x]_c} < \infty) \leq \limsup_{c\to\infty} \phi_d(c) P(T_{c[x]_c} < \infty) \leq b_d.$$

Le cas $d = 4$ est plus délicat, et n'est que partiellement traité. On est en mesure d'établir l'existence d'une constante $a_4 > 0$ telle que

$$a_4 \leq \liminf_{c\to\infty} c^2 \ln(c) P(T_{c[x]_c} < \infty).$$

De plus on a

$$\limsup_{c\to\infty} c^2 P(T_{c[x]_c} < \infty) = 0.$$

Au vu de la correspondance entre super-mouvement brownien et modèle du votant, il semble par ailleurs naturel de faire la *conjecture* de l'existence d'une constante $b_4 > 0$ telle que $\limsup_{c\to\infty} c^2 \ln(c) P(T_{c[x]_c} < \infty) \leq b_4$.

2.2. Raisonnement heuristique. On va tenter ici de donner l'intuition du théorème V.1, ainsi que de la conjecture formulée en dimension 4.

Rappelons que le modèle du votant est dual d'un système de marches aléatoires coalescentes. En particulier, pour tout $t > 0$, le chemin descendant parti de $(c[x]_c, t)$ suit la trajectoire d'une marche aléatoire sur $\mathbb{Z}^d$ de noyau p. De façon intuitive, et d'après les propriétés bien connues des marches aléatoires, le temps d'atteinte du point $c[x]_c$, lorsqu'il est fini, est d'ordre c^2. On a donc, de façon heuristique, pour $T > 0$ suffisamment petit et lorsque $c \to \infty$,

$$P(T_{c[x]_c} < \infty) \approx p_{c^2T} P^*_{c^2T}(T_{c[x]_c} < \infty), \tag{9}$$

où on rappelle que $p_t = P(\xi^0_t \neq \emptyset)$.

Par ailleurs, en rééchelonnant, on obtient

$$\begin{aligned} P^*_{c^2T}(T_{c[x]_c} < \infty) &= P^*_{c^2T}(\exists t \geq 0 : X^0_t(B(c[x]_c, 1)) > 0) \\ &= P\left(\exists t \geq 0 : X^{c^2,0}_t(B([x]_c, 1/c)) > 0 \middle| X^{c^2,0}_T \neq 0\right). \end{aligned}$$

Evidemment, une application rigoureuse du théorème I.5 pour passer à la limite dans l'expression ci-dessus ne donne aucune information intéressante. Cependant, on peut remarquer, en utilisant pour cela les résultats sur la probabilité d'atteinte d'un point par le super-mouvement brownien cités au paragraphe I.1.3, que le théorème V.1 montre précisément pour $d = 2$ ou 3 que

$$P\left(\exists t \geq 0 : X^{c^2,0}_t(B([x]_c, 1/c)) > 0 \middle| X^{c^2,0}_T \neq 0\right) \underset{c\to\infty}{\to} \tilde{\mathbb{N}}^{(T)}_x(0 \in \mathcal{R}).$$

Il montre d'autre part qu'en dimension $d \geq 5$, la quantité

$$P\left(\exists t \geq 0 : X_t^{c^2,0}(B([x]_c, 1/c)) > 0 \middle| X_T^{c^2,0} \neq 0\right)$$

est du même ordre que $\tilde{\mathbb{N}}_x^{(T)}(\mathcal{R} \cap B(0,1/c) \neq \emptyset)$.

Enfin, en dimension 4, $\tilde{\mathbb{N}}_x^{(T)}(\mathcal{R} \cap B(0,1/c) \neq \emptyset)$ est d'ordre $\ln(c)^{-1}$. C'est la raison de notre conjecture.

2.3. Aperçu de la preuve. En réalité, dans le cas $d \geq 5$, on peut en fait se contenter d'utiliser la dualité entre modèle du votant et système de marches coalescentes, et démontrer directement le théorème V.1 a partir de résultats concernant ces marches (voir parties V.3.1 et V.4.1).

Le cas des dimensions 2 et 3 est plus intéressant, car il nécessite l'utilisation du théorème I.5. On établit en premier lieu que le modèle du votant ne se déplace rapidement que de façon exceptionnelle.

Lemme (V.4) *Il existe des constantes $K_1 > 0, K_2 > 0$ telles que pour tout $\alpha > 1$, pour tout $A > 0$,*

$$P_\alpha^*\left(\sup_{t \leq 2\alpha} \sup_{x \in \xi_t^0} |x| > A\sqrt{\alpha}\right) \leq K_1 \exp(-K_2 A).$$

Ceci permet en particulier de rendre notre argument heuristique (9) rigoureux. Ecrivons en effet

$$P(T_{c[x]_c} < \infty) = p_{c^2T} P_{c^2T}^*(T_{c[x]_c} < \infty) + P(T_{c[x]_c} < \infty, \xi_{c^2T}^0 = \emptyset).$$

D'après le lemme V.4, pour $T > 0$ suffisamment petit, le deuxième terme de la somme ci-dessus est, lorsque $c \to \infty$, négligeable en comparaison du premier.

Nous tentons à présent de donner l'intuition du fait que $P_{c^2T}^*(T_{c[x]_c} < \infty) \to \tilde{\mathbb{N}}_0^{(T)}(x \in \mathcal{R})$ lorsque $c \to \infty$.

2.3.1. *Borne supérieure pour $P_{c^2T}^*(T_{c[x]_c} < \infty)$.* L'idée est de montrer qu'avec une probabilité arbitrairement proche de 1, lorsque $c[x]_c$ est touché, la version rééchelonnée de la mesure d'occupation de $B(cx, \eta c)$ possède une limite non dégénérée lorsque $c \to \infty$. Rappelons que m_t est défini au paragraphe I.2.3.

Lemme (V.3) *Fixons $T > 0, \rho > 0$, $\eta > 0$. On peut trouver $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0]$, il existe $\delta > 0$ tel que, lorsque c est suffisamment grand,*

$$P_{c^2T}^*\left(\exists t \geq 4\varepsilon c^2 \ \exists x \in \xi_t^0 : \inf_{s \in [t-3\varepsilon c^2, t-2\varepsilon c^2]} |\xi_s^0 \cap \overline{B}(x, \eta c)| < \delta m_{c^2}\right) \leq \rho.$$

Après rééchelonnement, ce résultat entraîne que

$$P_{c^2T}^*(T_{c[x]_c} < \infty) \leq \frac{1}{1+\rho} P_{c^2T}^*\left[\int_0^\infty \mathbf{1}_{\left\{X_s^{c^2,0}(\overline{B}(x,\eta)) \geq \delta\right\}} \geq \varepsilon\right].$$

On est alors en mesure d'utiliser le théorème I.5 pour assurer que

$$P_{c^2T}^*(T_{c[x]_c} < \infty) \leq \tilde{\mathbb{N}}_0^{(T)}(\mathcal{R} \cap B(x, \eta) \neq \emptyset).$$

Il est à noter que ce raisonnement reste valable pour toute dimension $d \geq 2$.

2.3.2. *Borne inférieure pour* $P^*_{c^2T}(T_{c[x]_c} < \infty)$. L'idée est ici d'exploiter la régularité avec laquelle le super-mouvement brownien charge son support. D'autre part, heuristiquement, lorsque le super-mouvement brownien X sous sa mesure d'excursion $\tilde{\mathbb{N}}_x$ touche l'origine, son support intersecte de façon conséquente la couronne centrée en 0 et de rayons $\varepsilon/2, \varepsilon$, pourvu que ε soit suffisamment petit. En conséquence, au vu des propriétés de h-mesure du support du super-mouvement brownien (cf I.1.3), il n'est pas surprenant d'obtenir le résultat suivant.

Lemme (V.8) *Soit $d = 2$ or 3. Pour $0 < r < r'$, on note $\mathcal{C}(0, r, r') := \{x \in \mathbb{R}^d : r < |x| < r'\}$, et on définit*

$$g_d(r) = \begin{cases} 2^{-\left(\frac{\ln(r)}{\ln(2)}\right)^4} & si\ d = 2, \\ r^{16} & si\ d = 3. \end{cases}$$

On peut choisir $\alpha > 0$ tel que, pour tout $\delta > 0$, il existe $\varepsilon_1 \in \left(0, 1 \wedge \frac{|x|}{2}\right)$ tel que pour tout $\varepsilon_0 \in (0, \varepsilon_1)$,

$$\mathbb{N}_0 \left[\exists s \geq 0 \ \exists\ \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : Y_s \left(\mathcal{C}\left(x, \frac{\varepsilon}{2}, \varepsilon\right)\right) > \alpha h_d(\varepsilon) \ \middle|\ x \in \mathcal{R} \right] \geq 1 - \delta.$$

Ce lemme est démontré au paragraphe V.4.4.

Grâce au théorème I.5, le lemme V.8 se traduit par une propriété asymptotique pour le modèle du votant. Avec au moins une probabilité arbitrairement proche de $\mathbb{N}_x^{(T)}(0 \in \mathcal{R})$, pourvu que c soit suffisamment grand, le modèle du votant sous $P^*_{c^2T}$ touche au moins $\alpha m_{c^2} h_d(\varepsilon)$ points dans la couronne $\mathcal{C}(c[x]_c, \varepsilon c/4, 2\varepsilon c)$, pour une valeur $\varepsilon \in [g_d(\varepsilon_0), \varepsilon_0]$.

Il suffit alors d'établir que, lorsque c'est le cas, le modèle du votant touche également $c[x]_c$ avec une probabilité arbitrairement proche de 1 (voir le Lemme V.9). Ce fait est une conséquence du résultat suivant (cf partie V.4.3).

Lemma (V.10) *Soit $d = 2$ ou 3. Rappelons que le modèle du votant parti de $A \subset \mathbb{Z}^d$ est défini sous la mesure de probabilité P_A. Pour tout $\gamma > 0$, il existe $M > 0$ et $U > 0$ tels que pour $u \geq U$ et pour tout A sous-ensemble de $\mathbb{Z}^d \cap \mathcal{C}(0, \frac{u}{4}, 2u)$ de cardinal $|A| \geq M\phi_d(u)$, on a*

$$P_A(\exists t \geq 0 : 0 \in \xi_t) \geq 1 - \gamma.$$

2.3.3. *Résultats intermédiaires.* Il est à noter que les lemmes V.3, V.4 et V.10 fournissent des informations intéressantes sur le modèle du votant.

Le lemme V.3 exprime ainsi que sous P^*_α, et pour tout $\varepsilon > 0$, la mesure d'occupation de l'opinion 1 sur l'intervalle de temps $[\varepsilon\alpha, \infty)$,

$$\sum_{t \geq \alpha\varepsilon} \sum_{x \in \xi^0_t} \delta_x,$$

ne peut pas charger un point sans charger aussi "beaucoup" de ses voisins. Le lemme V.3 est démontré au paragraphe V.3.2.2, à l'aide du lemme V.4.

Celui-ci exprime quant à lui que, sous P_{α^*}, la probabilité pour le modèle du votant de sortir de $B(0, A)$ avant le temps 2α décroît exponentiellement avec A. Il est démontré au paragraphe V.3.2.1.

Enfin, le lemme V.10 affirme qu'en dimension $d = 2$ ou 3, lorsque l'opinion 1 part de sites distants de l'origine (situés dans un couronne de rayons d'ordre u), elle touche tout de même celle-ci avec une grande probabilité, pourvu que ces sites originaux soient suffisamment nombreux (plus de $M\phi_d(u)$). On démontre ce résultat au paragraphe V.4.3, après l'avoir reformulé en termes de marches aléatoires coalescentes (voir également V.5).

CHAPITRE III

Local behaviour of local times of super-Brownian motion

ABSTRACT : For $x_0 \in \mathbb{R}^d \setminus \{0\}$, $d \leq 3$, we study the local behaviour near 0 of the local times (L_t^y) of a super-Brownian motion X initially in δ_{x_0}. More precisely, if $\psi(c)$ is a suitable normalization, our main result implies that the process $\left(\psi(c)(L_t^{x/c} - L_t^0), x \in \mathbb{R}^d, t \geq 0\right)$ converges in distribution to a non-degenerate limit as $c \to \infty$. This allows us to study the local behaviour of the occupation measure of X, then to recover and to generalize a result of Lee concerning the occupation measure of three-dimensional super-Brownian motion conditioned to hit a distant ball.

1. Introduction and statement of results

1.1. Introduction. The main goal of this work is to study the local behaviour of the occupation measure of super-Brownian motion in dimension $d \leq 3$.

Super-Brownian motion is a model for spatial populations undergoing a continuous branching phenomenon, which arises in a variety of different contexts. It was introduced independently by Watanabe (69) and Dawson (75) as the weak limit of branching particle systems. Connections between the wider class of superprocesses and partial differential equations have helped derive some of the basic properties of super-Brownian motion and have also allowed to prove analytic results using probabilistic arguments. More recently, it has been shown that super-Brownian motion appears in scaling limit of a wide range of lattice systems such as the voter model, the contact process, lattice trees or oriented percolation. Therefore, in a way similar to standard Brownian motion, super-Brownian motion can be thought of as a universal object providing information on the asymptotics of many interacting particle systems or statistical mechanics models.

Local times of superprocesses have been studied by many authors (cf Sugitani [**Su 89**], Adler and Lewin [**AL 92**], Krone [**K 93**]). Our main result Theorem 1 gives precise information about the local behaviour of local times of super-Brownian motion, in dimension $d = 2$ or 3. Let us be more precise. Let $x_0 \in \mathbb{R}^d \setminus \{0\}$, X a super-Brownian motion initially in δ_{x_0}, and L_t^x the local time of X at time t and point x. As a direct consequence of Theorem 1, if $x \in \mathbb{R}^d$ and if we set $\psi(c) = \sqrt{c}$ when $d = 3$, $\psi(c) = c(\ln(c))^{-1/2}$ when $d = 2$, we will obtain that $(\psi(c)(L_t^{x/c} - L_t^0))_{t\geq 0}$ converges in distribution as $c \to \infty$ to $(\beta_{a(x)L_t^0})_{t\geq 0}$ where β is a one-dimensional Brownian motion which is independent of X and $a(x)$ is a constant depending only on x. Theorem 1 in fact gives a more general statement involving finitely many different values of x. This allows us to study the local behaviour of the occupation measure of X (Proposition III.2).

Our results are related to a recent paper of Lee [**Le 01**]. Lee considers a super-Brownian motion started at δ_{cx_0} with c large and conditioned on hitting the unit ball. Using analytic methods, he obtains limit theorems for the occupation measure of this ball by super-Brownian motion (see Proposition III.1 below). We will show how to recover and to generalize Lee's results from our main theorem. To do this, we will need to study the local behaviour of the occupation measure of X under its canonical measure $\mathbb{N}_{x_0}$ (Proposition III.5).

Our results on the local behaviour of local times of super-Brownian motion are analogous to the ones obtained by Yor for local times of standard Brownian motion. Let B be a linear Brownian motion started at the origin and let ℓ_t^x denote the local time of B at level x and time t. Then Yor [**Y 83**], proved that $(\sqrt{c}(\ell_t^{x/c}-\ell_t^0))_{t\geq 0, x\geq 0}$ converges in distribution to a Brownian sheet as $c \to \infty$. This should be compared with Theorem 1 below.

After introducing the basic notation, Section 1 first summarizes known results which motivate our study, mainly found in [**Le 01**], [**Su 89**], and [**Pe 99**], then states our results. It also provides an outline of the proof of our main result (Theorem 1) about the local behaviour of local times. Section 2 is devoted to the proof of Theorem 1. In Section 3, we apply Theorem 1 to prove a non-conditional equivalent form of Lee's result (Proposition III.2), then we recover and extend Lee's result (Proposition III.1). Section 4 is devoted to the one-dimensional case.

1.2. Notation. Let $M_F(\mathbb{R}^d)$ be the space of finite measures on $\mathbb{R}^d$. For $\mu \in M_F(\mathbb{R}^d)$, $f : \mathbb{R}^d \to \mathbb{R}$, $< \mu, f >$ is shorthand for $\int_{\mathbb{R}^d} f(x)\mu(dx)$.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra in $\mathbb{R}^d$. The notation $B(x, r)$ stands for the open ball of radius r centered at $x \in \mathbb{R}^d$.

We denote by $C_b(\mathbb{R}^d)$ the space of bounded continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$, and by $C_b(\mathbb{R}^d)_+$ the space of nonnegative functions in $C_b(\mathbb{R}^d)$. If K is a compact subset of $\mathbb{R}^d$, $C_K(\mathbb{R}^d)$ is the subset of $C_b(\mathbb{R}^d)$ consisting of functions supported on K. If $n \in \mathbb{N}$, we let $C_b^n(\mathbb{R}^d)$ be the set of all n times continuously differentiable functions on $\mathbb{R}^d$ with bounded derivatives of order less than n, and $C_b^\infty(\mathbb{R}^d) := \bigcap_{n\geq 0} C_b^n(\mathbb{R}^d)$.

We denote by p_t the transition density of d-dimensional Brownian motion, that is for $t \geq 0$, $z \in \mathbb{R}^d$,

$$p_t(z) = (2\pi t)^{-d/2} \exp\left(-\frac{|z|^2}{2t}\right).$$

Set $q_t(x) = \int_0^t p_s(x)ds$, and if $\mu \in M_F(\mathbb{R}^d)$,

$$\mu q_t(z) = \int_{\mathbb{R}^d} q_t(z-y)\mu(dy).$$

We consider a super-Brownian motion $(X_t, t \geq 0)$ in the space $M_F(\mathbb{R}^d)$. We will write P_μ for the probability measure under which $X_0 = \mu$. When there is no confusion we will write P for P_μ, E for E_μ. We denote by $\left(\mathcal{F}_t^X\right)_{t\geq 0}$ the right-continuous filtration generated by X.

1.3. Basic properties of super-Brownian motion and its local times. In this section we consider a super-Brownian motion X under P_μ, for $\mu \in M_F(\mathbb{R}^d)$.

It is well known (see for example [**Pe 99**], Theorem II.5.9) that for any function $\phi \in C_b(\mathbb{R}^d)_+$ we can express $E_\mu\left[\exp\left(-\int_0^t < X_s, \phi > ds\right)\right]$ in terms of the solution of a partial differential equation :

$$E_\mu\left(\exp\left\{-\int_0^t <X_s,\phi> ds\right\}\right) = \exp\left(-\int v(t,x,\phi)\mu(dx)\right). \tag{10}$$

where $(t,x) \to v(t,x,\phi)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^d$ and solves the partial differential equation

$$\frac{\partial v(s,x)}{\partial s} = \frac{1}{2}\Delta v(s,x) - (v(s,x))^2 + \phi, \tag{11}$$

on $]0,\infty[\times\mathbb{R}^d$, with initial value $v(0,x) = 0$.

(10) is a particular case of the *Laplace equation* for the super-Brownian motion X. Note that formula (10) remains valid if $\phi \in C_K(\mathbb{R}^d)$ is not necessarily nonnegative, but only then if t is less than a certain explosion time $t^* > 0$. More precisely, for a fixed $T > 0$, (10) will remain valid for $t \le T$ for any compactly supported function that is bounded from below by a constant depending on T.

As shown in Chapter IV of [**LG 99**] by the Brownian snake approach (see also Section II.7 of [**Pe 99**]), for any $x \in \mathbb{R}^d$, there exists a σ-finite measure $\mathbb{N}_x$ on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ called the excursion measure of super-Brownian motion such that the law of $(X_t)_{t>0}$ under P_μ is the same as the law of $\left(\sum X_t^i\right)_{t>0}$, where $\sum \delta_{X^i}$ is a Poisson point process with intensity $\int \mathbb{N}_x d\mu(x)$. This fact is called the *canonical decomposition of super-Brownian motion*.

Intuitively, if one thinks of super-Brownian motion as the scaling limit of critical branching random walks as it is introduced in [**Pe 99**], the measure X_t under $\mathbb{N}_x$ for $t > 0$ represents the contribution to the population of the descendants at time t of one single individual alive at time 0 at the point x. The canonical decomposition expresses that the super-Brownian motion at time t is obtained by superimposing a Poisson number of such contributions.

It is also well known that the process X solves a *martingale problem* (see [**Pe 99**], Theorem II.5.1). For any $\phi \in C_b^2(\mathbb{R}^d)$,

$$<X_t,\phi> = <X_0,\phi> + M_t(\phi) + \frac{1}{2}\int_0^t <X_s,\Delta\phi> ds, \tag{12}$$

where $M_t(\phi)$ is an $\mathcal{F}_t^X$- martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$<M(\phi)>_t = \int_0^t <X_s,\phi^2> ds.$$

Sugitani [**Su 89**] proved that for $d \le 3$, there exists a random continuous function $(t,x) \to L_t^x$ from $(0,\infty) \times \mathbb{R}^d$ into $\mathbb{R}_+$ such that for any $\Psi \in C_b(\mathbb{R}^d)$,

$$\int_0^t <X_s,\Psi> ds = \int_{\mathbb{R}^d} \Psi(x) L_t^x dx. \tag{13}$$

L_t^x is called the *local time* of X at point $x \in \mathbb{R}^d$ and time $t > 0$.

Take $L_0^x = 0$ for every $x \in \mathbb{R}^d$. The function $(L_t^x)_{t\ge 0, x\in\mathbb{R}^d}$ needs not being continuous at points of the form $(0,x), x \in \mathbb{R}^d$. However Sugitani established for $d \ge 2$ that L_t^x is continuous in the pair (x,t) on the set of continuity points for $\mu q_t(x)$. For example, note that this set contains $\mathbb{R}_+ \times B(0,r)$ whenever $\mu(B(0,r)) = 0$.

Sugitani also proved that for $d = 1$, L_t^x is always continuous in the pair (x, t) and even differentiable with respect to the space variable at points $x \in \mathbb{R}^d$ where $\mu(\{x\}) = 0$.

Under the assumption that $(t, x) \to \mu q_t(x)$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^d$, Sugitani [**Su 89**] obtained the existence of *exponential moments for local times* :
for any $T > 0$, there exists a constant $\mathcal{K}(\mu, T) > 0$ such that

$$E_\mu \left[\exp(rL_t^x)\right] < \infty \tag{14}$$

holds for any $t \le T$, $x \in \mathbb{R}^d$ and $r < \mathcal{K}(\mu, T)$. We shall denote by
$G_\mu^{t,x} : \{z \in \mathbb{C}, |z| < \mathcal{K}(\mu, t)\} \to \mathbb{C}$ the function such that for any $z \in \mathbb{C}, |z| < \mathcal{K}(\mu, t)$,

$$E_\mu[\exp(zL_t^x)] = \exp(G_\mu^{t,x}(z)). \tag{15}$$

For convenience we will write $G_{\delta_{x_0}}^{t,x} =: G_{x_0}^{t,x}$.

If K is a compact set in $\mathbb{R}^d$, and if we only know that $(t, x) \to \mu q_t(x)$ is continuous in $\mathbb{R}_+ \times K$ (for example if $\mu(K) = 0$), it is easy to adapt the proof of (14) in [**Su 89**] to obtain for any $T > 0$ the existence of constants $\mathcal{K}(\mu, T, K) > 0$, $\mathcal{C}(\mu, T, K) > 0$ such that

$$E_\mu \left[\exp(rL_t^x)\right] \le \mathcal{C}(\mu, T, K) \tag{16}$$

holds for any $t \le T$, $x \in K$ and $r < \mathcal{K}(\mu, T, K)$. In this case, this clearly allows us to define the functions $G_\mu^{t,x}$ for $t \ge 0, x \in K$ in a way such that (15) remains valid for $z \in \mathbb{C}, |z| < \mathcal{K}(\mu, t, K)$.

Let us now discuss the *scaling properties of super-Brownian motion.* Let $c > 0$, and if $f \in C_b(\mathbb{R}^d)_+$ let f_c be the function $f_c(x) = f(cx)$. From (11) we get

$$v(t, x, f_c) = c^2 v(c^2 t, cx, c^{-4} f).$$

The scaling properties of super-Brownian motion easily follow from (10) and that observation. Let $\mu \in M(\mathbb{R}^d)$. For any $A \in \mathcal{B}(\mathbb{R}^d), t \ge 0$, let

$$X_t^{(c)}(A) = c^{-2} X_{c^2 t}(cA), \qquad\qquad \mu^{(c)}(A) = c^2 \mu(c^{-1} A).$$

Then, the law of the process $(X_t^{(c)},\ t \ge 0)$ under $P_{\mu^{(c)}}$ is equal to the law of the process $(X_t,\ t \ge 0)$ under P_μ.

Consequently, for any $\phi \in C_K(\mathbb{R}^d)$, the law of the process

$$\left(c^{-4} \int_0^{c^2 t} < X_u, \phi > du,\ t \ge 0 \right)$$

under $P_{\mu^{(c)}}$ is the same as the law of the process

$$\left(\int_0^t < X_u, \phi_c > du,\ t \ge 0 \right)$$

under P_μ.

We are now in position to state our main result concerning the local behaviour of local times of super-Brownian motion.

1.4. The main result.

THEOREM III.1. *Assume $d = 2$ or 3. Let*

$$\begin{aligned}\psi(c) &= \sqrt{c} \quad \textit{if } d=3 \\ &= \frac{c}{\sqrt{\ln(c)}} \quad \textit{if } d=2.\end{aligned}$$

Let $\nu \in M_F(\mathbb{R}^d)$ such that $\nu(B(0,\rho)) = 0$ for some $\rho > 0$, and let X be a super-Brownian motion in $\mathbb{R}^d$ with initial value ν. If $x_1, ..., x_k$ are fixed points in $\mathbb{R}^d \setminus \{0\}$, the process

$$\left(X_t, \psi(c)(L_t^{x_1/c} - L_t^0), ..., \psi(c)(L_t^{x_k/c} - L_t^0)\right)_{t\geq 0}$$

converges as $c \to \infty$ in the sense of weak convergence in the space $C(\mathbb{R}^+, M(\mathbb{R}^d) \times \mathbb{R}^k)$ to a limiting process which can be written in the form

$$\left(X_t, \beta^{x_1}_{L_t^0}, ..., \beta^{x_k}_{L_t^0}\right)_{t\geq 0}.$$

Here, $(\beta_t^x)_{t\geq 0, x\in\mathbb{R}^d}$ is a centered Gaussian process independent of X such that

$$\operatorname{cov}(\beta_t^x, \beta_s^y) = a(x,y)(t \wedge s),$$

and $a(x,y)$ is given by

$$a(x,y) = \begin{cases} \frac{1}{4\pi^2}\int_{\mathbb{R}^3} dz \left(\frac{1}{|z-x|} - \frac{1}{|z|}\right) \times \left(\frac{1}{|z-y|} - \frac{1}{|z|}\right) & \textit{if } d=3, \\ \frac{1}{\pi} x.y & \textit{if } d=2. \end{cases}$$

For convenience we will write $a(x,x) =: a(x)$.

Part of the motivation for Theorem 1 came from a recent paper of Lee [**Le 01**] dealing with asymptotics for the occupation measure of super-Brownian motion.

1.5. Lee's result and extensions. Lee [**Le 01**] only considers the three-dimensional case, but we will extend his result to the case of dimension 2.

Let $d = 2$ or 3, X a super-Brownian motion in $\mathbb{R}^d$, with initial value $\mu = \delta_{cx_0}$, where $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$ and $c > 0$.

Let K be a compact set in $\mathbb{R}^d$, and let ϕ, ξ be integrable functions on $\mathbb{R}^d$ supported on K satisfying $\int_K \xi(y)dy = 0, \int_K \phi(y)dy \neq 0$. We also define ϕ_c, ξ_c the functions such that for any $x \in \mathbb{R}^d$,

$$\phi_c(x) = \phi(cx), \quad \xi_c(x) = \xi(cx).$$

Note that ϕ_c, ξ_c are supported on $K^{(c)} := c^{-1}K = \{y \ : \ cy \in K\}$.

One may think of the particular example where $K = \overline{B}(0,1)$, and where we consider the functions

$$\phi^0 = \mathbf{1}_{\overline{B}(0,1)}, \qquad \xi^0 = \mathbf{1}_{\overline{B}(0,1)\cap\{x:x_d\geq 0\}} - \mathbf{1}_{\overline{B}(0,1)\cap\{x:x_d<0\}}.$$

For $T > 0$, let us introduce the quantities

$$D_{\phi,T} = \int_0^T < X_t, \phi > dt, \quad D_{\xi,T} = \int_0^T < X_t, \xi > dt, \quad T \geq 0.$$

We also set

$$a_\xi : = \int_K \int_K \xi(y)\xi(z)a(y,z)dydz.$$

The following result, is proved by Lee in [**Le 01**] (for the case $d = 3$), by analysing the behaviour of $v(t, x_0, ac^3\phi_c + bc^{7/2}\xi_c)$ as $c \to \infty$ (cf (11)).

Proposition III.1. *Let $d = 2$ or 3, $t > 0$, $x_0 \in \mathbb{R}^d \setminus \{0\}$ and let K be a compact subset of $\mathbb{R}^d$. Let ϕ and ξ be integrable functions on $\mathbb{R}^d$ supported on K satisfying $\int_K \phi(y)dy \neq 0, \int_K \xi(y)dy = 0$. Under $P_{\delta_{cx_0}}(.|X$ hits $K)$ the random vector*

$$\left(c^{d-4}D_{\phi,c^2t}, c^{d-4}\psi(c)D_{\xi,c^2t}\right)$$

converges in distribution as $c \to \infty$ to a non-degenerate limit (D_1, D_2). Furthermore, if $|a|$ and $|b|$ are small enough the Fourier transform of the limit is

$$E\bigg[\exp(iaD_1 + ibD_2)\bigg] = 1 + \frac{2|x_0|^2}{4-d} G_{x_0}^{T,0}\bigg(ia\int_K \phi(y)dy - \frac{b^2 a_\xi}{2}\bigg),$$

where the function $G_{x_0}^{T,0}$ is determined by (15).

The case $d = 3$ stands as Corollary 1.2 in [**Le 01**]. The form of the Fourier transform of (D_1, D_2) implies that the conditional law of D_2 knowing D_1 is Gaussian with mean 0 and variance $a_\xi D_1 \left(\int \phi(y)dy\right)^{-1}$.

In the example $\phi = \phi^0, \xi = \xi^0$, Proposition III.1 has the following interpretation : conditionally on the event that a super-Brownian motion X started at δ_{cx_0} hits the unit ball, the occupation time of the unit ball up to time c^2t (that is D_{ϕ^0,c^2t}) is of order c^{4-d}, while the difference between the occupation time of the top half of the unit ball and the occupation time of the bottom half (that is D_{ξ^0,c^2t}) is of order $\sqrt{c}$ when $d = 3$, and $c(\ln(c))^{1/2}$ when $d = 2$.

In Section 3.2, we will derive Proposition III.1 from the following Proposition, which is itself a consequence of Theorem 1, as we will see in Section 3.1.

Proposition III.2. *Let $d = 2$ or 3. Let t, K, ϕ, ξ be as in Proposition III.1, and let ν be as in Theorem III.1. Under P_ν, the random vector*

$$(c^d D_{\phi_c,t}, c^d\psi(c)D_{\xi_c,t})$$

converges in distribution as $c \to \infty$ to

$$\left(L_t^0 \int_K \phi(y)dy, U_t\right).$$

where conditionally on L_t^0, U_t is Gaussian with variance $a_\xi L_t^0$.

As we will explain below, the first component $c^d D_{\phi_c,t}$ indeed converges almost surely. Also note that, from (15), the Fourier transform of $(\int \phi(y)dy L_t^0, U_t)$ is simply

$$E_\nu\left[\exp\left(iaL_t^0\int_K \phi(y)dy + ibU_t\right)\right] = \exp\left(G_\nu^{T,0}\left(ia\int_K \phi(y)dy - \frac{b^2a_\xi}{2}\right)\right).$$

Let us explain how Proposition III.2 follows from Theorem 1. The following approach is also valid in the one-dimensional case, and so we consider the general case $d \leq 3$. Let us first give a simple argument showing that $c^d D_{\phi_c,t}$ converges P_ν-almost surely as $c \to \infty$. By (13), and our assumption on the support of ϕ,

$$c^d D_{\phi_c,t} = c^d \int_{\mathbb{R}^d} \phi(cx) L_t^x dx = \int_K \phi(y) L_t^{y/c} dy.$$

Since $(t, x) \to L_t^x$ is continuous on $\mathbb{R}_+ \times B(0, \rho)$,

$$c^d D_{\phi_c,t} \underset{c\to\infty}{\longrightarrow} L_t^0 \int_K \phi(y)dy$$

almost surely under P_ν by dominated convergence. See Theorem 5 in [**Su 89**] for a related statement ([**Su 89**] only states the convergence in distribution of $c^d D_{\phi_c,t}$).

If ϕ is replaced by ξ, the preceeding argument is not sufficient. Indeed, since $\int \xi(x)dx = 0$, the last step yields $c^d D_{\xi_c,t} \to 0$ as $c \to \infty$. Still, we can write

$$\int_{\mathbb{R}^d} \xi(y) L_t^{y/c} dy = \int_{\mathbb{R}^d} \xi(y)(L_t^{y/c} - L_t^0)dy,$$

which suggests to focus on the convergence of $\psi(c)(L_t^{y/c} - L_t^0)$, where $\psi(c)$ is a suitable normalization.This approach is similar to the work of Stroock - Varadhan - Papanicolaou [**SVP 77**] and Yor [**Y 83**] concerning limit theorems for additive functionals of standard Brownian motion.

The one-dimensional case follows almost immediately from [**Su 89**] and will be treated briefly in Section 4. *Until then we set* $d = 2$ *or* 3.

1.6. Outline of the proof of Theorem 1. We will first consider the case $k = 1$. **We fix** $x \in \mathbb{R}^d \setminus \{0\}$ and a measure ν satisfying the assumption of Theorem III.1.

For $\alpha \geq 0$ ($\alpha > 0$ if $d = 2$) we set $g_\alpha(z) = \int_0^\infty \exp(-\alpha t) p_t(z) dt$, and for $y \in \mathbb{R}^d$, $g_\alpha^y(z) = g_\alpha(z - y)$. If $d = 3$, note that we have $g_0(z) = \frac{1}{2\pi|z|}$.

The key idea is to use the Tanaka formula for local times of super-Brownian motion in dimension less than 3 (see [**BEP 91**], Theorem 6.1). Let $y \in \mathbb{R}^d$, $d \leq 3$, $\mu \in M_F(\mathbb{R}^d)$, and let $\alpha \geq 0$ if $d = 3$ or $\alpha > 0$ if $d = 2$. Under the assumption $< \mu, g_\alpha^y > < \infty$, we have P_μ-almost surely,

$$< X_t, g_\alpha^y > = < \mu, g_\alpha^y > + M_t(g_\alpha^y) + \alpha \int_0^t < X_s, g_\alpha^y > ds - L_t^y \tag{17}$$

where $M_t(g_\alpha^y)$ is an $\mathcal{F}_t^X$- martingale which is defined in terms of the martingale measure associated with super-Brownian motion. In particular, $M_0(g_\alpha^y) = 0$ and $M_t(g_\alpha^y)$ has quadratic variation

$$< M(g_\alpha^y) >_t = \int_0^t < X_s, (g_\alpha^y)^2 > ds.$$

If c is large enough so that $x/c \in B(0, \rho)$, the conditions $< \nu, g_\alpha^0 > < \infty$, $< \nu, g_\alpha^{x/c} > < \infty$, will clearly be satisfied. Thus we can use (17) for $y = x/c$ and $y = 0$ to obtain, P_ν-almost surely,

$$\begin{aligned} L_t^{x/c} - L_t^0 = &< X_0 - X_t, g_\alpha^{x/c} - g_\alpha^0 > + M_t(g_\alpha^{x/c}) - M_t(g_\alpha^0) \\ &+ \alpha \int_0^t < X_s, g_\alpha^{x/c} - g_\alpha^0 > ds. \end{aligned} \tag{18}$$

In what follows we will take $\alpha = 0$ when $d = 3$. Note that the last term in the right hand side of the previous formula then vanishes.

In Section 2.1 we will prove the following lemmas :

LEMMA III.1. *Let* $d = 3$, $T > 0$. *Then* P_ν*-almost surely*

$$\sup_{t \leq T} \left| \sqrt{c} < X_0 - X_t, g_0^{x/c} - g_0^0 > \right| \underset{c \to \infty}{\longrightarrow} 0.$$

LEMMA III.2. *Let $d = 2,\ T > 0$. Then P_ν-almost surely*

$$(\mathbf{a}) \qquad \sup_{t\le T}\left| < X_0 - X_t, \frac{c}{\sqrt{\ln(c)}}(g_\alpha^{x/c} - g_\alpha^0) > \right| \underset{c\to\infty}{\longrightarrow} 0,$$

$$(\mathbf{b}) \qquad \sup_{t\le T}\left|\int_0^t < X_s, \frac{c}{\sqrt{\ln(c)}}(g_\alpha^{x/c} - g_\alpha^0) > ds\right| \underset{c\to\infty}{\longrightarrow} 0.$$

From (18), Lemma 1 and Lemma 2 we see that the the convergence of $(\psi(c)(L_t^{x/c} - L_t^0))_{t\ge 0}$ follows from that of $(\psi(c)M_t(g_\alpha^{x/c} - g_\alpha^0))_{t\ge 0}$.

All that remains to do is thus to study the convergence of the martingales

$$M_t^{x,c} := \begin{cases} \sqrt{c}\ M_t\left((g_0^{x/c} - g_0^0)\right) & \text{if } d = 3; \\ \frac{c}{\sqrt{\ln(c)}}\ M_t\left((g_\alpha^{x/c} - g_\alpha^0)\right) & \text{if } d = 2. \end{cases}$$

LEMMA III.3. *P_ν-almost surely,*

$$< M^{x,c} >_t \underset{c\to\infty}{\longrightarrow} a(x)L_t^0.$$

This lemma will be proven in Section 2.2.

We then have to discuss the convergence in distribution of the martingale $M_t^{x,c}$. Using the Dubins-Schwarz theorem (see [**RY 94**], Theorem V.1.6), we can write

$$M_t^{x,c} = \beta^{x,c}_{<M^{x,c}>_t}. \tag{19}$$

where $\beta_t^{x,c}$ is a standard Brownian motion. We may and will assume that for $s \ge u \ge < M^{x,c} >_\infty$, $\beta_s^{x,c} - \beta_u^{x,c} = \gamma_s^x - \gamma_u^x$, where γ^x is a one-dimensional Brownian motion independent of X. The collection of the laws of the family $(X, \beta^{x,c})_{c>0}$ is clearly tight.

In Section 2.3 we will prove

LEMMA III.4. *Suppose that along a subsequence $c_n \nearrow \infty$ we have*

$$(X, \beta^{x,c_n}) \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} (X, \beta^x).$$

Then β^x is independent of X.

From the tightness of the laws of $(X, \beta^{x,c})$ and Lemma III.4, it follows that

$$(X, \beta^{x,c}) \underset{c\to\infty}{\overset{(\mathrm{d})}{\longrightarrow}} (X, \beta^x) \tag{20}$$

with a Brownian motion β^x independent of X. We know from Lemma 3 that $< M^{x,c} >_t$ converges P_ν-almost surely to $a(x)L_t^0$, and the convergence is uniform on every compact time interval by Dini's theorem. It then follows from (19) and (20) that $(X_t, M_t^{x,c})_{t\ge 0}$ converges in distribution to $(X_t, \beta^x_{a(x)L_t^0})_{t\ge 0}$.

The case $k = 1$ of Theorem 1 now follows from this fact and Lemmas 1 and 2. We will then extend this argument to the general case in Section 2.4.

2. Proof of Theorem 1

2.1. Preliminary reduction. In this section we will prove Lemmas 1 and 2. Recall that the point $x \ne 0$ is fixed, and that we have also fixed a measure ν satisfying the assumption of Theorem III.1 : there exists $\rho > 0$ such that $\nu(B(0, \rho)) = 0$.

We start with a preliminary result providing a uniform bound for the measure of small balls. From [**BEP 91**] Corollary 4.8, we know the following

PROPOSITION III.3. *Let $\delta > 0$ be fixed. If $d = 2$ or 3 then, for any $\mu \in M_F(\mathbb{R}^d)$, P_μ-almost surely,*

$$\limsup_{r\searrow 0} \sup_{t\geq\delta} \sup_{y\in\mathbb{R}^d} X_t(B(y,r))\Phi(r)^{-1} \leq \kappa_\mu,$$

where $\Phi(r) = r^2(\ln(1/r))^{4-d}$ and κ_μ is a constant depending on μ.

An easy consequence is the following

COROLLARY 1. *P_ν-almost surely,*

$$\limsup_{r\searrow 0} \sup_{t\geq 0} \sup_{y\in B(0,\rho/2)} X_t(B(y,r))r^{-2}\left(\ln\left(\frac{1}{r}\right)\right)^{d-4} \leq \kappa_\nu.$$

Proof of Corollary 1 : Since $\nu(B(0,\rho)) = 0$, it is well known that P_ν-almost surely there exists $n_0(\omega) \in \mathbb{N}$ such that $\mathrm{supp}(X_t)$ does not intersect $B(0,\frac{3\rho}{4})$ for any $t \in [0, 2^{-n_0}]$.
Provided $r \leq \rho/4$, we then have

$$\sup_{t\leq 2^{-n_0(\omega)}} \sup_{y\in B(0,\rho/2)} X_t(B(y,r))r^{-2}\left(\ln\left(\frac{1}{r}\right)\right)^{d-4} = 0,$$

and thanks to Proposition III.3, for P_ν-almost all ω,

$$\limsup_{r\searrow 0} \sup_{t\geq 2^{-n_0(\omega)}} \sup_{y\in\mathbb{R}^d} X_t(B(y,r))r^{-2}\left(\ln\left(\frac{1}{r}\right)\right)^{d-4} \leq \kappa_\nu.$$

Corollary 1 follows. $\square$

In the following we write $\kappa_0 = \kappa_0(x), \kappa_1 = \kappa_1(x), \ldots$ for constants that depend on the point x which is fixed.

2.1.1. *The case $d = 3$.* For convenience we will use the notation

$$h^{x,c}(z) := \psi(c)(g_0^{x/c}(z) - g_0^0(z)) = \frac{\sqrt{c}}{2\pi}\left(\frac{1}{|z - x/c|} - \frac{1}{|z|}\right), \text{ for } z \neq 0, x/c.$$

We will use the following easy estimates on $h^{x,c}$:

($\mathbf{A_1}$): If $|z| \leq \frac{|x|}{2c}$, then $|h^{x,c}(z)| \leq \frac{\sqrt{c}}{|z|}$.

($\mathbf{A_2}$): Let $r \geq \frac{2|x|}{c}$. Then, the maximum of $|h^{x,c}|$ outside the ball $B(0,r)$ is attained at the point $\frac{rx}{|x|}$ and its value is $\left(2\pi\sqrt{c}r\left(\frac{r}{|x|} - \frac{1}{c}\right)\right)^{-1}$.

($\mathbf{A_3}$): If $|z| \wedge |z - x/c| \geq \frac{|x|}{2c}$ and $|z| \leq \frac{4|x|}{c}$, then $|h^{x,c}(z)| \leq \frac{2c\sqrt{c}}{\pi|x|}$.

Proof of Lemma 1 : If c is sufficiently large and $z \notin B(0,\rho)$, using ($\mathbf{A_2}$),

$$|h^{x,c}(z)| \leq \frac{1}{2\pi\sqrt{c}\rho(\frac{\rho}{|x|} - \frac{1}{c})} \underset{c\to\infty}{\longrightarrow} 0.$$

Thus $< X_0, h^{x,c} > = < \nu, h^{x,c} >$ clearly goes to 0 as $c \to \infty$.

Hence, to get Lemma 1, it is enough to prove that P_ν-almost surely,

$$\sup_{t\leq T} | < X_t, h^{x,c} > | = \sup_{t\leq T}\left|\int_{\mathbb{R}^3} h^{x,c}(z)X_t(dz)\right| \underset{c\to\infty}{\longrightarrow} 0. \tag{21}$$

If $t > 0$ is fixed we know that 0 does not belong to $\mathrm{supp}X_t$, P_ν-almost surely. Thus the same argument as when $t = 0$ gives $< X_t, h^{x,c} > \to 0$ as $c \to \infty$. The point is that we need this convergence uniformly in $t \leq T$, and we know that there may exist exceptional times $t \leq T$ such that $0 \in \text{supp} X_t$.

It will be convenient to cut the domain $\mathbb{R}^3$ of the integral in (21) into different areas where we will be able to estimate the integrand. First of all, if $r > 0$ is fixed, if $z \notin B(0,r)$ and c is large enough, using again ($\mathbf{A_2}$) we have $|h^{x,c}(z)| \leq \frac{\kappa_0}{\sqrt{c}}$ so that

$$\left| \int_{\mathbb{R}^3 \setminus B(0,r)} h^{x,c}(z) X_t(dz) \right| \leq \frac{\kappa_0}{\sqrt{c}} < X_t, 1 > .$$

Since $\sup_{t \geq 0} < X_t, 1 >$ is P_ν-almost surely bounded, we get

$$\sup_{t \leq T} \left| \int_{\mathbb{R}^3 \setminus B(0,r)} h^{x,c}(z) X_t(dz) \right| \underset{c \to \infty}{\longrightarrow} 0.$$

Thus, to get (21) it only remains to prove that for some $r_1 > 0$,

$$\sup_{t \leq T} \left| \int_{B(0,r_1)} h^{x,c}(z) X_t(dz) \right| \underset{c \to \infty}{\longrightarrow} 0. \tag{22}$$

From Corollary 1 it follows that P_ν-almost surely, there exists $r_1 > 0$ such that, for any $r \leq r_1$,

$$\sup_{t \geq 0} \sup_{y \in B(0,\rho/2)} X_t(B(y,r)) \leq 2\kappa_\nu r^2 \ln\left(\frac{1}{r}\right). \tag{23}$$

Clearly we can assume $r_1 \leq \rho/4$. Let c be large enough so that $B(0, 4|x|/c) \subset B(0, r_1)$. To get rid of the singularities of $h^{x,c}$, we will first deal with the integrals in neighborhoods of 0 and x/c. First, we have

$$\left| \int_{B(0,\frac{|x|}{2c})} h^{x,c}(z) X_t(dz) \right| \leq \sum_{p \geq 1} \left| \int_{B(0,\frac{|x|}{2^p c}) \setminus B(0,\frac{|x|}{2^{p+1} c})} h^{x,c}(z) X_t(dz) \right|.$$

Using ($\mathbf{A_1}$) and (23) we see that P_ν-almost surely

$$\sup_{t \leq T} \left| \int_{B(0,\frac{|x|}{2^p c}) \setminus B(0,\frac{|x|}{2^{p+1} c})} h^{x,c}(z) X_t(dz) \right| \leq 2\kappa_\nu \frac{2^{p+1} c\sqrt{c}}{|x|} \left(\frac{|x|}{2^p c}\right)^2 \ln\left(\frac{2^p c}{|x|}\right).$$

It is now clear that P_ν-almost surely

$$\begin{aligned} \sup_{t \leq T} \left| \int_{B(0,\frac{|x|}{2c})} h^{x,c}(z) X_t(dz) \right| &\leq \frac{4\kappa_\nu |x|}{c^{1/2}} \sum_{p \geq 1} (p \ln(2) + \ln(c) - \ln(|x|)) 2^{-p} \\ &\leq \kappa_1 \kappa_\nu c^{-1/2} \ln(c). \end{aligned}$$

Clearly we can obtain an analogous bound for the quantity

$$\sup_{t \leq T} \left| \int_{B(\frac{x}{c},\frac{|x|}{2c})} h^{x,c}(z) X_t(dz) \right|.$$

Now using ($\mathbf{A_3}$) and (23) we get

$$\sup_{t \leq T} \left| \int_{B(0,\frac{4|x|}{c}) \setminus (B(0,\frac{|x|}{2c}) \cup B(\frac{x}{c},\frac{|x|}{2c}))} h^{x,c}(z) X_t(dz) \right| \leq 2\kappa_\nu \frac{2c\sqrt{c}}{\pi|x|} \left(\frac{4|x|}{c}\right)^2 \ln\left(\frac{c}{4|x|}\right),$$

so we finally obtain that

$$\sup_{t\le T}\left|\int_{B(0,\frac{4|x|}{c})} h^{x,c}(z)X_t(dz)\right| \le \kappa_\nu\kappa_2\frac{\ln(c)}{\sqrt{c}}. \tag{24}$$

Let us now consider the integral on $B(0,r_1)\setminus B(0,\frac{4|x|}{c})$. Let N be such that

$$\frac{r_1}{2^N}\le\frac{4|x|}{c}\le\frac{r_1}{2^{N-1}}.$$

Note that, for $1\le p\le N$, $r_1/2^p\ge 2^{N-p+1}|x|/c$ while $r_1/2^{p-1}\le 2^{N-p+3}|x|/c$. Once again, using ($\mathbf{A_2}$) and (23), we obtain for $1\le p\le N$

$$\sup_{t\le T}\left|\int_{B(0,\frac{r_1}{2^{p-1}})\setminus B(0,\frac{r_1}{2^p})} h^{x,c}(z)X_t(dz)\right|$$
$$\le \frac{2\kappa_\nu c^{3/2}}{|x|(2^{N-p+1}-1)^2}\left(\frac{2^{N-p+3}|x|}{c}\right)^2\ln\left(\frac{c}{2^{N-p+3}|x|}\right)\le 2\kappa_\nu\frac{2^6|x|}{c^{1/2}}\ln\left(\frac{c}{|x|}\right).$$

Therefore,

$$\sup_{t\le T}\left|\int_{B(0,r_1)\setminus B(0,\frac{4|x|}{c})} h^{x,c}(z)X_t(dz)\right|\le N\times\frac{2^7\kappa_\nu|x|}{c^{1/2}}\ln\left(\frac{c}{|x|}\right)\le\kappa_\nu\kappa_3\frac{(\ln(c))^2}{c^{1/2}}.$$

The limit (22) now follows from the above and (24), which completes the proof of Lemma III.1. □

2.1.2. *The case d=2.* Now we consider $\alpha>0$ and the function

$$h^{x,c}(z):=\frac{c}{\sqrt{\ln(c)}}(g_\alpha^{x/c}-g_\alpha^0)(z),\ \text{ for } z\ne 0, x/c.$$

We will need the following estimates on $h^{x,c}$ which shall be proven in the Appendix of this chapter. When c is large enough,

($\mathbf{B_1}$): If $z\in B(0,\frac{|x|}{2c})$,

$$|h^{x,c}(z)|\le\kappa_4\frac{c}{\sqrt{\ln(c)}}\ln^+\left(\frac{1}{|z|}\right).$$

If $z\in B(\frac{x}{c},\frac{|x|}{2c})$,

$$|h^{x,c}(z)|\le\kappa_4\frac{c}{\sqrt{\ln(c)}}\ln^+\left(\frac{1}{|z-x/c|}\right).$$

($\mathbf{B_2}$): If $z\in B(0,c^{-3/4})\setminus(B(0,\frac{|x|}{2c})\cup B(\frac{x}{c},\frac{|x|}{2c}))$,

$$|h^{x,c}(z)|\le\kappa_5 c\sqrt{\ln(c)}.$$

($\mathbf{B_3}$): Let $r>c^{-7/8}$. Then if $z\notin B(0,r)$,

$$|h^{x,c}(z)|\le\kappa_6\frac{1}{r\sqrt{\ln(c)}}.$$

Proof of Lemma 2 : We will use a similar method as for proving Lemma 1.

Fix $T>0$. To obtain Lemma 2 (**a**), it is enough to establish that P_ν-almost surely,

$$\sup_{t\le T}|<X_t,h^{x,c}>|\underset{c\to\infty}{\longrightarrow}0. \tag{25}$$

If $r>0$ is fixed and c is sufficiently large we can use ($\mathbf{B_3}$) to get

$$\sup_{t\le T}\left|\int_{\mathbb{R}^3\setminus B(0,r)} h^{x,c}(z)X_t(dz)\right| \le \frac{\kappa_6}{r\sqrt{\ln(c)}}\sup_{t\le T} < X_t,1 > .$$

Since $\sup_{t\ge0}|<X_t,1>|$ is finite P_ν-almost surely, we have

$$\sup_{t\le T}\left|\int_{\mathbb{R}^2\setminus B(0,r)} h^{x,c}(z)X_t(dz)\right| \underset{c\to\infty}{\longrightarrow} 0. \tag{26}$$

Now, from Corollary 1 we know that P_ν-almost surely, there exists $r_2>0$ such that for any $r\le r_2$,

$$\sup_{t\ge0}\sup_{y\in B(0,\rho/2)} X_t(B(y,r)) \le 2\kappa_\nu r^2\left(\ln\left(\frac{1}{r}\right)\right)^2. \tag{27}$$

We can choose $r_2\le\rho/4$ and c large enough so that $B(0,c^{-3/4})\subset B(0,r_2)$. We will first deal with the neighbourhood of singularities 0 and x/c. Using (27) and the estimate ($\mathbf{B_1}$) we get for every $p\ge1$,

$$\sup_{t\le T}\left|\int_{B(0,\frac{|x|}{2^pc})\setminus B(0,\frac{|x|}{2^{p+1}c})} h^{x,c}(z)X_t(dz)\right| \le \kappa_4\frac{2\kappa_\nu c}{\sqrt{\ln(c)}}\left(\ln\left(\frac{2^{p+1}c}{|x|}\right)\right)^3\left(\frac{|x|}{2^pc}\right)^2,$$

and thus

$$\begin{aligned}\sup_{t\le T}\left|\int_{B(0,\frac{|x|}{2c})} h^{x,c}(z)X_t(dz)\right| &\le \sum_{p\ge1}\frac{2\kappa_4\kappa_\nu c}{\sqrt{\ln(c)}}\left(\ln\left(\frac{2^{p+1}c}{|x|}\right)\right)^3\left(\frac{|x|}{2^pc}\right)^2\\ &\le \kappa_7\frac{(\ln(c))^{5/2}}{c}\underset{c\to\infty}{\longrightarrow}0.\end{aligned}$$

We can bound the integral on $B(\frac{x}{c},\frac{|x|}{2c})$ by the same quantity. Furthermore using ($\mathbf{B_2}$) and (27) we have

$$\left|\int_{B(0,c^{-3/4})\setminus(B(0,\frac{|x|}{2c})\cup B(\frac{x}{2c},\frac{|x|}{2c}))} h^{x,c}(z)X_t(dz)\right| \le 2\kappa_\nu\kappa_5(\ln(c))^{5/2}c^{-1/2}\underset{c\to\infty}{\longrightarrow}0.$$

Let us now consider the integral on $D:=B(0,r_2)\setminus B(0,c^{-3/4})$. Let $N\in\mathbb{N}$ such that

$$\frac{r_2}{2^N}\le c^{-3/4}\le\frac{r_2}{2^{N-1}}.$$

We may assume that c is large enough so that $c^{-7/8}<2^{-N}r_2$. The domain D is contained in the union of the sets $B(0,\frac{r_2}{2^{p-1}})\setminus B(0,\frac{r_2}{2^p})$ for $1\le p\le N$.
Since $c^{-7/8}\le r_2/2^N$ we can use ($\mathbf{B_3}$). Together with (27) this leads to :

$$\begin{aligned}\left|\int_{B(0,r_2)\setminus B(0,c^{-3/4})} h^{x,c}X_t(dz)\right| &\le \sum_{p=1}^N\frac{2^{p+1}\kappa_\nu\kappa_6}{r_2\sqrt{\ln(c)}}\left(\frac{r_2}{2^{p-1}}\right)^2\left(\ln\left(\frac{2^{p-1}}{r_2}\right)\right)^2\\ &\le \sum_{p=1}^N\frac{8\kappa_\nu\kappa_6}{\sqrt{\ln(c)}}2^{-p}\left(\ln\left(\frac{2^p}{r_2}\right)\right)^2\\ &\le \frac{\kappa_7}{\sqrt{\ln(c)}}\underset{c\to\infty}{\longrightarrow}0.\end{aligned}$$

By considering the preceding estimates, we get

$$\sup_{t\le T}\left|\int_{B(0,r_2)} h^{x,c}(z)X_t(dz)\right| \underset{c\to\infty}{\longrightarrow} 0.$$

Together with (26) this proves (25) and thus Lemma 2 (**a**).

Lemma 2 (**b**) is also a consequence of (25). Simply notice that if $t\le T$,

$$\left|\int_0^t < X_s, h^{x,c} > ds\right| \le t\times \sup_{s\le T} |< X_s, h^{x,c} >| ,$$

which goes to 0 as $c\to\infty$ by (25). □

2.2. Convergence of the quadratic variation of $M_t^{x,c}$. From the martingale problem for X (see (12)) we know that $M_t^{x,c}$ is a local martingale whose quadratic variation is $\psi(c)^2\int_0^t < X_s, (g_\alpha^{x/c}-g_\alpha^0)^2 > ds$.
Let us now prove that this quantity converges to a non-degenerate limit as $c\to\infty$.

2.2.1. *Proof of Lemma 3, the case $d=3$.* Recall we set $\alpha=0$ for $d=3$. We simply have

$$\begin{aligned}
< M^{x,c} >_t &= \int_0^t c < X_s, (g^{x/c}-g^0)^2 > ds\\
&= \frac{c}{4\pi^2}\int_{\mathbb{R}^3} dz L_t^z\left(\frac{1}{|z-x/c|}-\frac{1}{|z|}\right)^2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^3} dz L_t^{z/c}\left(\frac{1}{|z-x|}-\frac{1}{|z|}\right)^2.
\end{aligned}$$

Note that the function $z\to(1/|z-x|-1/|z|)^2$ is integrable over $\mathbb{R}^3$. From Sugitani [**Su 89**] we know that the function $x\to L_t^x$ is continuous with compact support. Hence, by dominated convergence, the above quantity goes P_ν-almost surely to $a(x)L_t^0$ where

$$a(x)=\frac{1}{4\pi^2}\int_{\mathbb{R}^3} dz\left(\frac{1}{|z-x|}-\frac{1}{|z|}\right)^2.$$

2.2.2. *Proof of Lemma 3, the case $d=2$.* We now have

$$< M^{x,c} >_t = \int_0^t < X_s, (h^{x,c})^2 > ds = \int_{\mathbb{R}^2} L_t^z (h^{x,c}(z))^2 dz. \tag{28}$$

Changing into polar coordinates (r,θ) and then setting $r=c^\delta$ leads to

$$< M^{x,c} >_t = \int_{-\infty}^{\infty}\int_0^{2\pi} c^{2\delta}\ln(c) L_t^{(c^\delta,\theta)}(h^{x,c}(c^\delta,\theta))^2 d\theta d\delta,$$

where $L_t^{(r,\theta)}$ refers to the local time at time t and at the point with polar coordinates (r,θ).

We will then need some sharper estimates on $h^{x,c}(z)$. These will also be proven in the Appendix of this chapter.

For two positive functions f and g we will write $f(c)=o(g(c))$, if for any $\epsilon>0$ there is c_ϵ such that $f(c)\le\epsilon g(c)$ for any $c\ge c_\epsilon$.

In the following estimates, z and δ are linked by the relation $|z|=c^\delta$, and c is supposed to be large enough.

($\mathbf{C_1}$): If $z \in B(0, \frac{1}{c(\ln(c))^{1/8}})$ or equivalently if $\delta \le -1 - \frac{\ln(\ln(c))}{8\ln(c)}$,

$$|h^{x,c}(z)| \le \kappa_5 |1+\delta| c\sqrt{\ln(c)}.$$

($\mathbf{C_2}$): If $\frac{1}{c(\ln(c))^{1/8}} \le |z| \le \frac{(\ln(c))^{1/8}}{c}$ and $z \notin B(\frac{x}{c}, \frac{1}{c(\ln(c))^{1/8}})$,

$$|h^{x,c}(z)| \le \kappa_8 \frac{c}{(\ln(c))^{1/4}}.$$

($\mathbf{C_3}$): If $\frac{(\ln(c))^{1/8}}{c} \le |z| \le (\ln(c))^{-1/4}$ or equivalently if $-1 + \frac{\ln(\ln(c))}{8\ln(c)} \le \delta \le -\frac{\ln(\ln(c))}{4\ln(c)}$,

$$|h^{x,c}(z)| = \frac{|z.x|}{\pi\sqrt{\ln(c)}} c^{-2\delta} + o(c^{-\delta}(\ln(c))^{-1/2}),$$

where $z.x$ denotes the usual scalar product.

($\mathbf{C_4}$): If $(\ln(c))^{-1/4} \le |z|$ or equivalently if $-\frac{\ln(\ln(c))}{4\ln(c)} \le \delta$,

$$|h^{x,c}(z)| \le \kappa_9 (\ln(c))^{-1/4}.$$

Now, in order to use these estimates, we will split $\mathbb{R}^2$ into the following five sets :

$$\begin{aligned}
D_0^{(c)} &= B\left(0, \frac{1}{c(\ln(c))^{1/8}}\right), \\
D_1^{(c)} &= B\left(\frac{x}{c}, \frac{1}{c(\ln(c))^{1/8}}\right), \\
D_2^{(c)} &= B\left(0, \frac{(\ln(c))^{1/8}}{c}\right) \setminus (D_0^{(c)} \cup D_1^{(c)}), \\
D_3^{(c)} &= B\left(0, \frac{1}{(\ln(c))^{1/4}}\right) \setminus \left(D_0^{(c)} \cup D_1^{(c)} \cup D_2^{(c)}\right), \\
D_4^{(c)} &= \mathbb{R}^2 \setminus \left(D_0^{(c)} \cup D_1^{(c)} \cup D_2^{(c)} \cup D_3^{(c)}\right).
\end{aligned}$$

We suppose c is large enough so that the sets $D_0^{(c)}$ and $D_1^{(c)}$ do not intersect, so that we can write

$$< M^{x,c} >_t = \sum_{i=0}^{4} \int_{D_i^{(c)}} L_t^z (h^{x,c}(z))^2 dz.$$

Fix $T > 0$ and consider $t \in [0,T]$. First notice that P_ν-almost surely,

$$L_t^x \le \sup_{y \in \mathbb{R}^d} L_T^y := L_T^* < \infty. \tag{29}$$

Using ($\mathbf{C_4}$) on the domain $D_4^{(c)}$ we obtain

$$\begin{aligned}
\int_{D_4^{(c)}} L_t^z (h^{x,c}(z))^2 dz &\le \frac{(\kappa_9)^2}{\sqrt{\ln(c)}} \int_{D_4^{(c)}} L_t^z dz \\
&= \frac{(\kappa_9)^2}{\sqrt{\ln(c)}} \int_0^T < X_s, 1 > ds \\
&\le \frac{(\kappa_9)^2}{\sqrt{\ln(c)}} T \sup_{0 \le s \le T} < X_s, 1 >,
\end{aligned}$$

which goes P_ν-almost surely to 0 as $c \to \infty$.

The integrals over $D_0^{(c)}$, $D_1^{(c)}$ are treated in a symmetric way. We have

$$\begin{aligned}\int_{D_0^{(c)}} L_t^z(h^{x,c}(z))^2 dz &= \int_{-\infty}^{-1-\frac{\ln(\ln(c))}{8\ln(c)}} \int_0^{2\pi} \ln(c) c^{2\delta} L_t^{(c^\delta,\theta)} \left(h^{x,c}(c^\delta,\theta)\right)^2 d\theta d\delta \\ &\leq 2\pi L_T^*(\kappa_5)^2(\ln(c))^2 \int_{-\infty}^{-1-\frac{\ln(\ln(c))}{8\ln(c)}} (1+\delta)^2 c^{2+2\delta} d\delta\end{aligned}$$

where we used (29) and the estimate $(\mathbf{C_1})$. Then,

$$(\ln(c))^2 \int_{-\infty}^{-1-\frac{\ln(\ln(c))}{8\ln(c)}} (1+\delta)^2 c^{2+2\delta} d\delta \leq \kappa_{10} \frac{(\ln(\ln(c)))^2}{(\ln(c))^{5/4}}$$

so that $\int_{D_0^{(c)}} L_t^z(h^{x,c}(z))^2 dz$ goes almost surely to 0 as $c \to \infty$, and so does $\int_{D_1^{(c)}} L_t^z(h^{x,c}(z))^2 dz$.

Consider now the integral over $D_2^{(c)}$. Using (29) and the estimate $(\mathbf{C_2})$ we get

$$\begin{aligned}\int_{D_2^{(c)}} dz L_t^z(h^{x,c}(z))^2 &\leq 2\pi L_T^* \kappa_8^2 \frac{c^2}{\sqrt{\ln(c)}} \int_{-1-\frac{\ln(\ln(c))}{8\ln(c)}}^{-1+\frac{\ln(\ln(c))}{8\ln(c)}} c^{2\delta} \ln(c) d\delta \\ &\leq \pi L_T^* \kappa_8^2 (\ln(c))^{-1/4}\end{aligned}$$

which goes almost surely to 0 as $c \to \infty$.

It remains to compute the integral over $D_3^{(c)}$ which is the preponderant part. We use $(\mathbf{C_3})$, and the fact that P_ν-almost surely, $\sup\{|L_t^z - L_t^0| : z \in D_3^{(c)}\}$ tends to 0 as $c \to \infty$ to obtain

$$\begin{aligned}\int_{D_3^{(c)}} dz L_t^z(h^{x,c})^2(z) &= \int_{-1+\frac{\ln(\ln(c))}{8\ln(c)}}^{-\frac{\ln(\ln(c))}{4\ln(c)}} \int_0^{2\pi} c^{2\delta} \ln(c) L_t^{(c^\delta,\theta)} (h^{x,c})^2 (c^\delta,\theta) d\theta d\delta \\ &= \int_{-1+\frac{\ln(\ln(c))}{8\ln(c)}}^{-\frac{\ln(\ln(c))}{4\ln(c)}} \int_0^{2\pi} \frac{1}{\pi^2}(x_1^2\cos^2(\theta) + x_2^2\sin^2(\theta)) L_t^{(c^\delta,\theta)} d\theta d\delta + o(1) \\ &\underset{c\to\infty}{\longrightarrow} \frac{|x|^2}{\pi} L_t^0.\end{aligned}$$

This completes the proof of Lemma 3. □

As we explained in Section 1.6, to get the convergence of $(X, M^{x,c})$, and thus the case $k=1$ in Theorem 1, it only remains to prove Lemma 4.

2.3. Independence of β^x and X. Proof of Lemma 4 : In what follows, when there is no ambiguity we will omit the x exponent in the notation $M^{x,c}, \beta^{x,c}, \beta^x, \gamma^x$.

By assumption, along a subsequence $c_n \nearrow \infty$,

$$(X, \beta^{c_n}) \underset{c\to\infty}{\overset{(d)}{\longrightarrow}} (X, \beta). \tag{30}$$

Recall from (12) the definition of the martingales $M_t(\phi)$. The formula for the quadratic variation of $M_t(\phi)$ shows that the collection $(M_t(\phi))_{t\geq 0, \phi\in\mathcal{C}_b^2(\mathbb{R}^d)}$ generates $\mathcal{F}^X$. Hence it is enough to check that β is independent of $(M_t(\phi))_{t\geq 0,\phi\in\mathcal{F}}$, where $\mathcal{F}$ is dense in $\mathcal{C}_b^2(E)$ for the topology $\mathcal{T}$ induced by the norm

$$||f|| = \max\left(||f||_\infty, \max_{i\in\{1,\dots,d\}} ||\partial^i f||_\infty, \max_{i,j\in\{1,\dots,d\}} ||\partial^{i,j} f||_\infty\right)$$

for $f \in \mathcal{C}_b^2(\mathbb{R}^d)$. For instance, we let $\mathcal{F}$ be the space of all functions $\phi \in C_b^\infty(\mathbb{R}^d)$ such that there exists $A > 0$ such that for all $n \in \mathbb{N}$, for all $\alpha = (\alpha_1, ..., \alpha_n) \in \{1, .., d\}^n$,

$$\sup_{x \in \mathbb{R}^d} |\partial^\alpha \phi(x)| \leq A^{n+1}.$$

We will use the following notation : Let $0 \leq t_1 \leq ... \leq t_p$, $0 = s_0 \leq s_1 \leq ... \leq s_q$, $\phi_1, ..., \phi_p \in \mathcal{F}$, $\lambda_1, ..., \lambda_p \in \mathbb{R}$ and $\mu_1, ..., \mu_q \in \mathbb{R}$. We set

$$N_t = \sum_{j=1}^p \lambda_j M_{t \wedge t_j}(\phi_j),$$

and

$$f = \sum_{j=1}^q \mu_j \mathbf{1}_{(s_{j-1}, s_j]}.$$

We let $\Lambda := \max_{1 \leq i \leq p} |\lambda_i|$, and $K := \max_{1 \leq i \leq p} ||\phi_i||_\infty$. If W is a standard Brownian motion we also set

$$W_t(f) = \int_0^t f(s) dW_s = \sum_{j=1}^q \mu_j (W_{t \wedge s_j} - W_{t \wedge s_{j-1}}).$$

We finally set $B = \exp(i\beta_\infty(f)) = \exp\left(i \int_0^\infty f(s) d\beta_s\right)$.

In order to prove Lemma 4, it is enough to establish the following statement

Lemma III.5. *For any choice of $(\phi_1, ..., \phi_p) \in \mathcal{F}^p$, for any $0 \leq t_1 \leq ... \leq t_p$,*

$$E\left[B M_{t_1}(\phi_1) ... M_{t_p}(\phi_p)\right] = E[B] E\left[M_{t_1}(\phi_1) ... M_{t_p}(\phi_p)\right]. \tag{31}$$

The proof of Lemma III.5 is based on

Lemma III.6. *If $\Lambda = \max_{1 \leq i \leq p} |\lambda_i|$ is small enough,*

$$E\left[B \exp\left(i \sum_{j=1}^p \lambda_j M_{t_j}(\phi_j) + \frac{1}{2} \sum_{j,k=1}^p \lambda_j \lambda_k < M(\phi_j), M(\phi_k) >_{t_j \wedge t_k}\right)\right] = E[B]. \tag{32}$$

<u>Proof of Lemma III.6</u> : From the definition of the Brownian motion β^c (see (19)) we have

$$\begin{aligned} \beta^c_\infty(f) &= \sum_{j=1}^q \mu_j (\beta^c_{s_j} - \beta^c_{s_{j-1}}) \\ &= \int_0^\infty f(< M^c >_s) dM^c_s + \int_{< M^c >_\infty}^\infty f(s) d\gamma_s. \end{aligned} \tag{33}$$

Since γ is independent of X we also have

$$E\left[\exp\left(i \int_{< M^c >_\infty}^\infty f(s) d\gamma_s + \frac{1}{2} \int_{< M^c >_\infty}^\infty f^2(s) ds\right) \middle| X\right] = 1. \tag{34}$$

We use the notation $\mathcal{E}(M)_t = \exp(M_t - 1/2 < M >_t)$ for the exponential martingale of the martingale M.

Lemma III.7. *If Λ is small enough, the exponential martingale*

$$\mathcal{E}\left(i\left(\int_0^t f(< M^c >_s) dM^c_s + N_t\right)\right)$$

is uniformly integrable.

Proof of Lemma III.7 : It suffices to check that

$$E\left[\exp\left(\int_0^\infty f^2(< M^c >_s)d < M^c >_s + < N >_\infty\right)\right] < \infty.$$

Since

$$\int_0^\infty f^2(< M^c >_s)d < M^c >_s \leq \int_0^\infty f^2(s)ds < \infty,$$

we only have to prove that

$$E\left[\exp\left(< N >_\infty\right)\right] < \infty. \tag{35}$$

Note that

$$< N >_\infty = < N >_{t_p} = \sum_{i=1}^p \sum_{j=1}^p \lambda_i\lambda_j \int_0^{t_p} < X_s, \phi_i\phi_j > ds.$$

Recall the notation $K = \max_{1\leq i\leq p} ||\phi_i||_\infty$. We have

$$\begin{aligned} E\left[\exp\left(< N >_{t_p}\right)\right] &\leq E\left[\exp\left(p^2\Lambda^2K^2\int_0^{t_p} < X_s, 1 > ds\right)\right] \\ &\leq \frac{1}{t_p}\int_0^{t_p} E\left[\exp\left(p^2\Lambda^2K^2 t_p < X_s, 1 >\right)\right] ds, \end{aligned}$$

where in the previous line we used Jensen's inequality. We know (see for example [**Pe 99**] page 32) that

$$E\left[\exp\left(\lambda < X_s, 1 >\right)\right] = \exp\left(\frac{< \nu, 1 > \lambda}{1-\lambda s}\right) \quad \text{for } \lambda < \frac{1}{s}. \tag{36}$$

It follows that $\int_0^{t_p} E\left[\exp\left(p^2\Lambda^2K^2t_p < X_s, 1 >\right)\right] ds$ is finite as soon as we have $p^2\Lambda^2K^2t_p < (2t_p)^{-1}$, which is equivalent to $\Lambda < (\sqrt{2}Kpt_p)^{-1}$. Under this condition, (35) holds and the exponential martingale $\mathcal{E}\left(i\int_0^t f(< M^c >_s)dM_s^c + N_t\right)$ is uniformly integrable. □

Let us now get back to the proof of Lemma 6. Using Lemma 7, we now have for every $t \in [0,\infty]$,

$$\begin{aligned} E\Bigg[\exp\Bigg\{i\left(\int_0^t f(< M^c >_s)dM_s^c + N_t\right) + \frac{1}{2}\Bigg(\int_0^t f^2(< M^c >_s)d < M^c >_s \\ + < N >_t + 2\int_0^t f(< M^c >_s)d < M^c, N >_s\Bigg)\Bigg\}\Bigg] = 1. \end{aligned} \tag{37}$$

Define $H_t^c := \int_0^t f(< M^c >_s)d < M^c, N >_s$.

LEMMA III.8. $\sup_{t\in[0,\infty]} |H_t^c| \longrightarrow 0$ *almost surely as* $c \to \infty$.

Proof of lemma III.8 : From the definition of the martingales M^c and N we have

$$H_t^c = \sum_{j=1}^p \int_0^{t\wedge t_j} \lambda_j f(< M^c >_s) < X_s, h^{x,c}\phi_j > ds.$$

Thus, if $\mu = \max_{1\leq j\leq p} |\mu_j|$, for any $t \in [0,\infty]$,

$$|H_t^c| \leq p\ t_p\ \Lambda\mu \sup_{s\leq t_p} | < X_s, h^{x,c}\phi_j > |.$$

Since the functions ϕ_j are bounded, the same arguments as in the proofs of Lemmas 1 and 2 show that P_ν-almost surely the right hand side of the above display goes to 0 as $c \to \infty$, which completes the proof of Lemma III.8. $\square$

Let us now complete the proof of Lemma 6. We use (37) with $t = \infty$ and (34), together with (33) and the fact that

$$\int_0^\infty f(< M^c >_s) d < M^c >_s = \int_0^{<M^c>_\infty} f^2(s) ds$$

to get

$$(38)\quad E_\nu\left[\exp\left\{i\left(\beta^c_\infty(f) + N_\infty\right) + \frac{1}{2}\left(\int_0^\infty f^2(s)ds + < N >_\infty + H^c_\infty\right)\right\}\right] = 1.$$

By the Kunita-Watanabe inequality (see for example [**RY 94**], Corollary IV.1.16), for any $c > 0$, P_ν-almost surely

$$\begin{aligned}
|H^c_\infty| &\leq \left(\int_0^\infty f^2(< M^c >_s) d < M^c >_s\right)^{1/2} \left(\int_0^\infty d < N >_s\right)^{1/2} \\
&\leq \left(\int_0^\infty f^2(s)ds\right)^{1/2} (< N >_\infty)^{1/2} \\
&\leq \frac{1}{2}\int_0^\infty f^2(s)ds + \frac{1}{2} < N >_\infty,
\end{aligned}$$

so that

$$(39)\qquad \frac{1}{2}\left[\int_0^\infty f^2(s)ds + <N>_\infty + 2|H^c_\infty|\right] \leq \int_0^\infty f^2(s)ds + <N>_\infty .$$

From (30), and the fact that both $M_t(\phi_j)$ and $< M(\phi_j), M(\phi_k) >_t$ are continuous functions of X (cf (12)), we see that

$$\begin{aligned}
&\big(\beta^c_\infty(f), (M_{t_j}(\phi_j))_{1\leq j\leq p}, (< M(\phi_k), M(\phi_l) >_{t_k \wedge t_l})_{1\leq k,l\leq p}\big) \\
(40)\qquad &\xrightarrow[c\to\infty]{(d)} \big(\beta_\infty(f), (M_{t_j}(\phi_j))_{1\leq j\leq p}, (< M(\phi_k), M(\phi_l) >_{t_k\wedge t_l})_{1\leq k,l\leq p}\big).
\end{aligned}$$

Since $N_\infty = \sum_{j=1}^p \lambda_j M_{t_j}(\phi_j)$, $< N >_\infty = \sum_{j,k=1}^p \lambda_j\lambda_k < M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}$, we can use (40) and Lemma III.8 to pass to the limit $c \to \infty$ in the left-hand side of (38). Note that (39) and (35) provide the domination required to justify the passage to the limit. In this way we get

$$\begin{aligned}
E\Bigg[\exp\Bigg\{ & i\left(\beta_\infty(f) + \sum_{j=1}^p \lambda_j M_{t_j}(\phi_j)\right) \\
&+ \frac{1}{2}\left(\int_0^\infty f^2(s)ds + \sum_{j,k=1}^p \lambda_j\lambda_k < M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\right)\Bigg\}\Bigg] = 1,
\end{aligned}$$

where $\beta_\infty(f) = \sum_{j=1}^q \mu_j(\beta_{s_j} - \beta_{s_{j-1}})$. Since $E[B] = \exp\left(-\frac{1}{2}\int_0^\infty f^2(s)ds\right)$, we get Lemma III.6. $\square$

Proof of Lemma III.5 : For any $(y_1, ..., y_p) \in \mathbb{R}^p, (z_{j,k})_{1\leq j,k\leq p} \in \mathbb{R}^{p^2}$, let us write

$$\exp\left(i\sum_{j=1}^{p}\lambda_j y_j + \frac{1}{2}\sum_{j,k=1}^{p}\lambda_j\lambda_k z_{j,k}\right) - 1 = \sum_{\substack{n_1,...,n_p\in\mathbb{N}\\ n_1+...+n_p\geq 1}} \lambda_1^{n_1}...\lambda_p^{n_p} Q_{n_1,...,n_p}((y_j),(z_{j,k})),$$

where the series converges absolutely and for every choice of $n_1, ..., n_p$, $Q_{n_1,...,n_p}((y_j),(z_{j,k}))$ is a polynomial of the $p+p^2$ variables $(y_j),(z_{j,k})$. Furthermore, the highest degree term in $Q_{1,...,1}$ is clearly $i^p y_1...y_p$. Thus (32) can be rewritten as

$$E\left[B\sum_{\substack{n_1,...,n_p\in\mathbb{N}\\ n_1+...+n_p\geq 1}} \lambda_1^{n_1}...\lambda_p^{n_p} Q_{n_1,...,n_p}\Big(\big(M_{t_j}(\phi_j)\big),\big(< M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\big)\Big)\right] = 0. \tag{41}$$

We now observe that for Λ small enough,

$$E\left[B\sum_{\substack{n_1,...,n_p\in\mathbb{N}\\ n_1+...+n_p\geq 1}} \lambda_1^{n_1}...\lambda_p^{n_p} Q_{n_1,...,n_p}\Big(\big(M_{t_j}(\phi_j)\big),\big(< M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\big)\Big)\right] \tag{42}$$

$$= \sum_{\substack{n_1,...,n_p\in\mathbb{N}\\ n_1+...+n_p\geq 1}} \lambda_1^{n_1}...\lambda_p^{n_p} E\left[B Q_{n_1,...,n_p}\Big(\big(M_{t_j}(\phi_j)\big),\big(< M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\big)\Big)\right].$$

To justify the interchange of summation and expectation it is enough to verify that

$$E\left[|B|\sum_{\substack{n_1,...,n_p\in\mathbb{N}\\ n_1+...+n_p\geq 1}} |\lambda_1|^{n_1}...|\lambda_p|^{n_p}\left|Q_{n_1,...,n_p}\big(\big(M_{t_j}(\phi_j)\big),\big(< M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\big)\big)\right|\right]$$

is finite. Let us define new polynomials $\hat{Q}_{n_1,...,n_p}$ by

$$\exp\left(\sum_{j=1}^{p}\lambda_j |y_j| + \frac{1}{2}\sum_{j,k=1}^{p}\lambda_j\lambda_k |z_{j,k}|\right) - 1 = \sum_{\substack{n_1,...,n_p\in\mathbb{N}\\ n_1+...+n_p\geq 1}} \lambda_1^{n_1}...\lambda_p^{n_p} \hat{Q}_{n_1,...,n_p}\big((|y_j|),(|z_{j,k}|)\big),$$

and observe that we always have

$$\left|Q_{n_1,...,n_p}((y_j),(z_{j,k}))\right| \leq \hat{Q}_{n_1,...,n_p}((|y_j|),(|z_{j,k}|)).$$

Since $|B| = 1$, it is then enough to prove that

$$E\left[\sum_{\substack{n_1,\ldots,n_p\in\mathbb{N}\\ n_1+\ldots+n_p\geq 1}} |\lambda_1|^{n_1}\ldots|\lambda_p|^{n_p}\hat{Q}_{n_1,\ldots,n_p}\Big(\big(\big|M_{t_j}(\phi_j)\big|\big),\big(\big|<M(\phi_j),M(\phi_k)>_{t_j\wedge t_k}\big|\big)\Big)\right]$$

is finite, which from the definition of $\hat{Q}$ holds if

$$E\left[\exp\left\{\sum_{j=1}^{p}|\lambda_j M_{t_j}(\phi_j)| + \frac{1}{2}\sum_{j,k=1}^{p}\big|\lambda_j\lambda_k< M(\phi_j),M(\phi_k)>_{t_j\wedge t_k}\big|\right\}\right] < \infty.$$

By the Cauchy-Schwarz inequality, it is enough to check the finiteness of

$$A_p(\lambda_1,...,\lambda_p) = E\left[\exp\left(2\sum_{j=1}^{p}|\lambda_j M_{t_j}(\phi_j)|\right)\right]$$

and

$$B_p(\lambda_1,...,\lambda_p) = E\left[\exp\left(\sum_{j,k=1}^{p}\big|\lambda_j\lambda_k < M(\phi_j),M(\phi_k) >_{t_j\wedge t_k}\big|\right)\right],$$

provided Λ is small enough. The fact that both $A_p(\lambda_1,...,\lambda_p)$ and $B_p(\lambda_1,...,\lambda_p)$ are finite when Λ is sufficiently small follows from (36) by arguments similar to the proof of Lemma III.7. Thus, the interchange of summation and expectation in (42) is justified.

From (41) we now get

$$\sum_{\substack{n_1,\ldots,n_p\in\mathbb{N}\\ n_1+\ldots+n_p\geq 1}} \lambda_1^{n_1}\ldots\lambda_p^{n_p}E\left[BQ_{n_1,\ldots,n_p}\big((M_{t_j}(\phi_j)),(< M(\phi_j),M(\phi_k) >_{t_j\wedge t_k})\big)\right] = 0.$$

Since this is true for any $(\lambda_1,...,\lambda_p)$ such that Λ is sufficiently small we obtain that for any $n_1,...,n_p \in \mathbb{N}$ such that $n_1+...+n_p \geq 1$,

$$E\left[BQ_{n_1,\ldots,n_p}\big((M_{t_j}(\phi_j)),(< M(\phi_j),M(\phi_k) >_{t_j\wedge t_k})\big)\right] = 0.$$

Specializing to the case $f = 0$ we have also, for any $n_1,...,n_p \in \mathbb{N}$ such that $n_1+...+n_p \geq 1$,

$$E\left[Q_{n_1,\ldots,n_p}\big((M_{t_j}(\phi_j)),(< M(\phi_j),M(\phi_k) >_{t_j\wedge t_k})\big)\right] = 0.$$

We have finally proven that

$$\begin{aligned} 0 &= E\left[BQ_{n_1,\ldots,n_p}\big((M_{t_j}(\phi_j)),(< M(\phi_j),M(\phi_k) >_{t_j\wedge t_k})\big)\right] \\ &= E[B]E\left[Q_{n_1,\ldots,n_p}\big((M_{t_j}(\phi_j)),(< M(\phi_j),M(\phi_k) >_{t_j\wedge t_k})\big)\right]. \end{aligned} \tag{43}$$

We are now in a position to finish the proof of Lemma III.5. We prove (31) by induction on p. For $p = 1$, we simply use (43) with $p = 1, n_1 = 1$ and observe that $Q_1(y) = iy$ to get

$$E\left[BM_{t_1}(\phi_1)\right] = E[B]E\left[M_{t_1}(\phi_1)\right].$$

Now let $p \geq 2$ and let us assume that (31) holds up to the order $p-1$. Observe first that we can write

$$Q_{1,\dots,1}\big((y_j),(z_{j,k})\big) = i^p y_1 \dots y_p + \sum_{\substack{J\subset\{1,\dots,p\}\\ K\subset\{1,\dots,p\}^2}} \alpha_{J,K}\left(\prod_{i\in J} y_i\right)\left(\prod_{(j,k)\in K} z_{j,k}\right)$$

where the constants $\alpha_{J,K}$ may be nonzero only if $\mathrm{Card}J + \mathrm{Card}K < p$.

Using (43) with $n_1 = \dots = n_p = 1$, we see that (31) will follow if we can prove that for any choice of (J,K) such that $\mathrm{Card}J + \mathrm{Card}K < p$,

$$\begin{aligned} & E\left[B\left(\prod_{i\in J} M_{t_i}(\phi_i)\right)\left(\prod_{(j,k)\in K} < M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\right)\right] \\ (44) \quad & = E[B]E\left[\left(\prod_{i\in J} M_{t_i}(\phi_i)\right)\left(\prod_{(j,k)\in K} < M(\phi_j), M(\phi_k) >_{t_j\wedge t_k}\right)\right]. \end{aligned}$$

To get rid of the quadratic variation terms we write

$$< M(\phi_j), M(\phi_k) >_t = \int_0^t < X_s, \phi_j\phi_k > ds,$$

and from the martingale problem for X, using an easy induction on n,

$$\begin{aligned} < X_s, \phi_j\phi_k > &= < X_0, \phi_j\phi_k > + M_s(\phi_j\phi_k) + \frac{1}{2}\int_0^s du < X_u, \Delta(\phi_j\phi_k) > \\ &= < X_0, \phi_j\phi_k > + \frac{s}{2} < X_0, \Delta(\phi_j\phi_k) > + M_s(\phi_j\phi_k) \\ &\quad + \frac{1}{2}\int_0^s du M_u(\Delta(\phi_j\phi_k)) + \frac{1}{4}\int_0^s du \int_0^u dr < X_r, \Delta^2(\phi_j\phi_k) > \\ &= X_s^n(\phi_j\phi_k) + R_s^n(\phi_j\phi_k), \end{aligned}$$

where

$$\begin{aligned} X_s^n(\phi_j\phi_k) = {} & < X_0, \phi_j\phi_k > + \frac{s}{2} < X_0, \Delta(\phi_j\phi_k) > \\ & + \dots + \frac{s^{n-1}}{2^{n-1}(n-1)!} < X_0, \Delta^{n-1}(\phi_j\phi_k) > \\ & + M_s(\phi_j\phi_k) + \frac{1}{2}\int_0^s du M_u(\Delta(\phi_j\phi_k)) \\ & + \dots + \frac{1}{2^{n-1}}\int_0^s du_1 \int_0^{u_1} du_2 \dots \int_0^{u_{n-2}} du_{n-1} M_{u_{n-1}}(\Delta^{n-1}(\phi_j\phi_k)), \end{aligned}$$

and

$$R_s^n(\phi_j\phi_k) = \frac{1}{2^n}\int_0^s du_1 \int_0^{u_1} du_2 \dots \int_0^{u_{n-1}} du_n < X_{u_n}, \Delta^n(\phi_j\phi_k) > .$$

By assumption both ϕ_j and ϕ_k belong to $\mathcal{F}$, and it easily follows that $\phi_j\phi_k$ is also in $\mathcal{F}$. Let A be the constant associated with $\phi_j\phi_k$ in the definition of $\mathcal{F}$. From the formula for $R_s^n(\phi_j\phi_k)$ one easily gets

$$E_\nu[(R_s^n(\phi_j\phi_k))^p] \leq \frac{d^{pn}A^{p(2n+1)}s^{pn}}{(n!)^p}E_\nu\left[\left(\sup_{u\in[0,s]} < X_u, 1 >\right)^p\right].$$

Recall that $< X_t, 1 >$ is an $\mathcal{F}_t^X$-martingale (cf (12) with $\phi = 1$). Furthermore we know from (36) that this quantity has finite moments of any order. If $q > 0$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, using Doob's inequality, we see that for any $s \leq t$,

$$E_\nu[(R_s^n(\phi_j\phi_k))^p] \leq \frac{d^{pn}A^{p(2n+1)}t^{pn}}{(n!)^p}q^p E_\nu\left[(< X_t, 1 >)^p\right],$$

so that $R_s^n(\phi_j\phi_k) \to 0$ as $n \to \infty$, in L^p for every $p < \infty$, uniformly on $[0, t]$.

From the expression of $X_s^n(\phi_j\phi_k)$, we see that we can use the induction hypothesis (recall $\mathrm{Card}J + \mathrm{Card}K < p$) to get

$$\begin{aligned} &E\left[B\left(\prod_{i\in J}M_{t_i}(\phi_i)\right)\left(\prod_{(j,k)\in K}\int_0^{t_j\wedge t_k} dsX_s^n(\phi_j\phi_k)\right)\right] \\ &= E[B]E\left[\left(\prod_{i\in J}M_{t_i}(\phi_i)\right)\left(\prod_{(j,k)\in K}\int_0^{t_j\wedge t_k} dsX_s^n(\phi_j\phi_k)\right)\right]. \end{aligned}$$

and letting $n \to \infty$ leads to (44), which completes the proof of Lemma III.5, and thus the one of Lemma 4. □

We have proven that for every $x \neq 0$,

$$(X, M_t^{x,c}) \underset{c\to\infty}{\overset{\text{(d)}}{\longrightarrow}} (X, \beta^x_{a(x)L_t^0}),$$

with β^x a standard Brownian motion independent of X. As it was explained in Section 1.6, this gives the case $k = 1$ of Theorem III.1.

We now adress the problem of the convergence in distribution of

$$(X, M_t^{x_1,c}, M_t^{x_2,c}, ..., M_t^{x_k,c})$$

as $c \to \infty$, where $\mathcal{X} = (x_1, ..., x_k)$ is a k-tuple of non-zero points in $\mathbb{R}^d$.

2.4. Space dependence for the limit. We denote by $\overline{M}_t^{\mathcal{X},c}$ the k-tuple of martingales $(M_t^{x_1,c}, ..., M_t^{x_k,c})$, and by $< \overline{M}^{\mathcal{X},c}, \overline{M}^{\mathcal{X},c} >_t$ the matrix $(< M^{x_i,c}, M^{x_j,c} >)_{1\leq i,j\leq k}$. Recall that for a fixed $x \neq 0, < M^{x,c} >_t \to a(x)L_t^0$ as $c \to \infty$. By adapting the argument used in Section 2.2 to prove this convergence, it is easy to see that P_ν-almost surely,

$$< \overline{M}^{\mathcal{X},c}, \overline{M}^{\mathcal{X},c} >_t \underset{c\to\infty}{\longrightarrow} \left(a(x_i, x_j)L_t^0\right)_{1\leq i,j\leq k}. \tag{45}$$

By the Dini Theorem, the convergence of the diagonal terms in (45) is uniform in $t \geq 0$. Then, using the Kunita-Watanabe inequality, it is not hard to see that the convergence of the non-diagonal terms is also uniform in $t \geq 0$.

Let A be the matrix $\left(a(x_i, x_j)\right)_{1\leq i,j\leq k}$. Since A is symmetric, there exists an orthogonal matrix O such that $D := OA\ {}^tO$ is a diagonal matrix.

Now let

$$\overline{N}_t^c = (N_t^{1,c}, ..., N_t^{k,c}) := O\overline{M}_t^{\mathcal{X},c}$$

and $<\overline{N}^c, \overline{N}^c>_t$ be the matrix $\left(< N^{i,c}, N^{j,c} >_t\right)_{1\leq i,j\leq k}$.

We clearly have $<\overline{N}^c, \overline{N}^c>_t = O <\overline{M}^{\mathcal{X},c}, \overline{M}^{\mathcal{X},c}>_t \ {}^tO$, so that P_ν-almost surely,

$$< \overline{N}^c, \overline{N}^c >_t \underset{c\to\infty}{\longrightarrow} L_t^0 D. \tag{46}$$

uniformly in $t \geq 0$.

If for $s \geq 0$ we let $\tau_s^{j,c} := \inf\{t \geq 0 : < N^{j,c} >_t \geq s\}$, we thus have that for any $j \neq k$, for any $s \geq 0$, both $< N^{j,c}, N^{k,c} >_{\tau_s^{j,c}}$, and $< N^{j,c}, N^{k,c} >_{\tau_s^{k,c}}$ go to 0 as $c \to \infty$.

Using the Dubins-Schwarz theorem, for every $i \in \{0, ..., k\}$, we have

$$N_t^{i,c} = B^{i,c}_{<N^{i,c}>_t},$$

where $B^{i,c}$ is a linear Brownian motion. For every $i \in \{1, ...k\}$, we may and will assume that for $s \geq u \geq < N^{i,c} >_\infty$, we have $B_s^{i,c} - B_u^{i,c} = \gamma_s^{i,c} - \gamma_u^{i,c}$, where $(\gamma^{1,c}, ..., \gamma^{k,c})_{c>0}$ is a family of independent k-dimensional Brownian motions, independent of X.

By an evident adaptation of Theorem (2.3) of [**RY 94**], Chapter XIII, the convergence to 0 of $< N^{j,c}, N^{k,c} >_{\tau_s^{j,c}}$, and $< N^{j,c}, N^{k,c} >_{\tau_s^{k,c}}$ implies that

$$(B^{1,c}, ..., B^{k,c}) \underset{c\to\infty}{\overset{\text{(d)}}{\longrightarrow}} (B^1, ..., B^k)$$

where $(B^1, ..., B^k)$ is a k-dimensional Brownian motion. By adapting the arguments of Section 2.3, we can also verify that

$$(X, B^{1,c}, ..., B^{k,c}) \underset{c\to\infty}{\overset{\text{(d)}}{\longrightarrow}} (X, B^1, ..., B^k)$$

where the k-dimensional Brownian motion $B = (B^1, ..., B^k)$ is independent of X.

It follows that

$$(X_t, N_t^{1,c}, ..., N_t^{k,c})_{t\geq 0} \underset{c\to\infty}{\longrightarrow} (X_t, B^1_{D_{11}L_t^0}, ..., B^k_{D_{kk}L_t^0})_{t\geq 0} \ .$$

Now recall that $\overline{M}^c = O^{-1}\overline{N}^c$ so that

$$(X_t, \overline{M}_t^{\mathcal{X},c})_{t\geq 0} \underset{c\to\infty}{\overset{\text{(d)}}{\longrightarrow}} (X_t, \beta^{\mathcal{X},1}_{L_t^0}, ..., \beta^{\mathcal{X},k}_{L_t^0}) \tag{47}$$

where

$$\beta_t^X = (\beta_t^{\mathcal{X},1}, ..., \beta_t^{\mathcal{X},k}) := O^{-1}(B^1_{D_{11}t}, ..., B^k_{D_{kk}})$$

is a centered k-dimensional Gaussian process satisfying

$$\mathrm{cov}(\beta_t^{\mathcal{X}}, \beta_s^{\mathcal{X}}) = (t \wedge s) O^{-1} D \ {}^tO^{-1} = (t \wedge s)A.$$

It is immediate that $\beta^{\mathcal{X}} \overset{(d)}{=} (\beta^{x_1}, ..., \beta^{x_k})$, where the collection $(\beta^x)_{x\in\mathbb{R}^d}$ is as in Theorem 1. The same arguments as in the case $k = 1$, using Lemma 1 and Lemma 2, show that the general case of Theorem 1 follows from (47). $\square$

3. Applications of Theorem 1

3.1. A non-conditioned result. We now turn to the convergence of the vector $\Xi_c = (c^d D_{\phi_c,t}, c^d \psi(c) D_{\xi_c,t})$ under P_ν, ν being as in Theorem 1.

Proof of Proposition III.2 : Let $t > 0$. Recall from Section 1.6 that

$$\Xi_c = \left(\int_{\mathbb{R}^d} L_t^{y/c} \phi(y) dy, \ \psi(c) \int_{\mathbb{R}^d} \xi(y)(L_t^{y/c} - L_t^0) dy \right). \tag{48}$$

We already noticed that P_ν-almost surely,

$$\lim_{c\to\infty} \int_{\mathbb{R}^d} L_t^{y/c} \phi(y)dy = \left(\int_{\mathbb{R}^d} \phi(y)dy\right) L_t^0. \tag{49}$$

Furthermore it is easy to check that

$$\lim_{c\to\infty} E_\nu \left[\left|\psi(c)\int_{\mathbb{R}^d} \xi(y)(L_t^{y/c} - L_t^0)dy - \int_{\mathbb{R}^d} \xi(y) M_t^{y,c} dy\right|\right] = 0. \tag{50}$$

Indeed the Tanaka formula (17) shows that for every $y \in K$ and c sufficiently large,

$$\psi(c)(L_t^{y/c} - L_t^0) - M_t^{y,c} =< X_0 - X_t, h^{y,c} > +\alpha \int_0^t < X_s, h^{y,c} > ds.$$

Hence (50) follows from the convergence

$$\lim_{c\to\infty} \left(\sup_{y\in K} \sup_{s\in[0,t]} E_\nu \left[< X_s, |h^{y,c}| >\right]\right) = 0,$$

which is itself an easy consequence of the first moment formula for X (see Proposition 2.10 in [**LG 99**]). By (48), (49) and (50), Proposition 2 reduces to verifying that

$$\left(L_t^0, \int_{\mathbb{R}^d} \xi(y) M_t^{y,c} dy\right) \xrightarrow[c\to\infty]{(d)} \left(L_t^0, U_t\right), \tag{51}$$

where, conditionally given L_t^0, U_t is centered Gaussian with variance $a_\xi L_t^0$.

LEMMA III.9. *We can find $c_0 > 0$ such that for every integer $p \geq 1$,*

$$\sup_{y\in K} \sup_{c\geq c_0} E_\nu \left[|M_t^{y,c}|^p\right] < \infty. \tag{52}$$

Let us postpone the proof of Lemma III.9 and complete that of Proposition 2. From (16), the limiting law in (51) is determined by its moments. Hence to get the convergence (51) it is enough to prove that, for every integers k and $p \geq 1$,

$$\lim_{c\to\infty} E\left[\left(L_t^0\right)^k \Big(\int \xi(y) M_t^{y,c} dy\Big)^p\right] = E\left[\left(L_t^0\right)^k \left(U_t\right)^p\right]. \tag{53}$$

Note that

$$(L_t^0, U_t) \overset{(d)}{=} \left(L_t^0, \int \xi(y) \beta^y_{L_t^0}\right)$$

with the notation of Theorem 1. By the Fubini theorem, (53) follows from the fact that for every $y_1, ..., y_p \in K$,

$$\lim_{c\to\infty} E\left[\left(L_t^0\right)^k M_t^{y_1,c} ... M_t^{y_p,c}\right] = E\left[\left(L_t^0\right)^k \beta^{y_1}_{L_t^0} ... \beta^{y_p}_{L_t^0}\right]. \tag{54}$$

However we proved (Theorem 1) that the $(p+1)$-tuple $(L_t^0, M_t^{y_1,c}, ..., M_t^{y_k,c})$ converges in distribution to $(L_t^0, \beta^{x_1}_{L_t^0}, ..., \beta^{x_k}_{L_t^0})$, and the bound of Lemma 9 allows us to derive (54) from this convergence in distribution. This completes the proof of Proposition 2.

<u>Proof of Lemma III.9</u> : We will only give the proof in the three-dimensional case. In the two-dimensional case, there are a few technical differences as can be guessed by looking at Sections 2.1 and 2.2, but the ideas remain very similar, and we leave this case to the reader.

Let $d = 3$, $a := \sup\{|y|, y \in K\}, c_0 : = 4a\rho^{-1}, t > 0, p \geq 1$, and $c \geq c_0$. For $y \in K$, we first use the Burkholder-Davis-Gundy inequality to obtain

$$E\left[(|M_t(h^{y,c})|)^p\right] \leq c_p E\left[< M(h^{y,c}) >_t^{p/2}\right] \tag{55}$$

where c_p is a constant. Let $\eta^{y,c} := E\left[< M(h^{y,c}) >_t^{p/2}\right]$.

From (12), we have

$$\eta^{y,c} = E\left[\left(\int_0^t < X_s, (h^{y,c})^2 > ds\right)^{p/2}\right].$$

From the fact that $c \geq 4a\rho^{-1}$, we can split $\mathbb{R}^d$ into the domains $B(0, 2ac^{-1})$, $B(0, \rho/2) \setminus B(0, 2ac^{-1}), \mathbb{R}^d \setminus B(0, \rho/2)$ and introduce

$$h_a^{y,c} : = h^{y,c}\mathbf{1}_{B(0,\frac{2a}{c})}, \qquad \eta_a^{y,c} : = E\left[\left(\int_0^t < X_s, (h_a^{y,c})^2 > ds\right)^{p/2}\right],$$

$$h_\rho^{y,c} : = h^{y,c}\mathbf{1}_{B(0,\frac{\rho}{2})\setminus B(0,\frac{2a}{c})}, \qquad \eta_\rho^{y,c} : = E\left[\left(\int_0^t < X_s, (h_\rho^{y,c})^2 > ds\right)^{p/2}\right],$$

$$\overline{h}^{y,c} : = h^{y,c}\mathbf{1}_{\mathbb{R}^3\setminus B(0,\frac{\rho}{2})}, \qquad \overline{\eta}^{y,c} : = E\left[\left(\int_0^t < X_s, (\overline{h}_\rho^{y,c})^2 > ds\right)^{p/2}\right].$$

We have to verify that the quantities

$$\sup_{c\geq c_0, y\in K} \overline{\eta}^{y,c}, \quad \sup_{c\geq c_0, y\in K} \eta_\rho^{y,c}, \quad \sup_{c\geq c_0, y\in K} \eta_a^{y,c}$$

are finite.

From ($\mathbf{A_2}$), we get that $\overline{h}^{y,c}$ is bounded from above by a constant $\tilde{\kappa}$ neither depending on y nor on c. Thus using (36), we have

$$\sup_{c\geq c_0}\sup_{y\in K} \overline{\eta}_\rho^{y,c} \leq \tilde{\kappa} E\left[\left(\int_0^t < X_s, 1 > ds\right)^{p/2}\right] < \infty.$$

Let us turn to $\eta_\rho^{y,c}$. From (16), introducing for convenience the notation $\mathcal{C}_\rho := \mathcal{C}\left(\nu, t, \overline{B}(0, \frac{\rho}{2})\right)$, $\mathcal{K}_\rho := \mathcal{K}\left(\nu, t, \overline{B}(0, \frac{\rho}{2})\right)$, we have for any integer $n \geq 0$

$$\sup_{z\in B(0,\frac{c\rho}{2})} E[(L_t^{z/c})^n] \leq \frac{n!}{(\mathcal{K}_\rho)^n}\mathcal{C}_\rho. \tag{56}$$

Using the trivial inequality $a^{p/2} \leq 1 + a^p$ for $a \geq 0$, we see that

$$\begin{aligned} \eta_\rho^{y,c} &\leq 1 + E\left[\left(\int_0^t < X_s, (h_\rho^{y,c})^2 > ds\right)^p\right] \\ &\leq 1 + E\left[\left(\frac{1}{4\pi^2}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)} dz\left(\frac{1}{|z-y|} - \frac{1}{|z|}\right)^2 L_t^{z/c}\right)^p\right]. \end{aligned} \tag{57}$$

Set

$$\hat{\kappa}(y, c) = \frac{1}{4\pi^2}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)} dz\left(\frac{1}{|z-y|} - \frac{1}{|z|}\right)^2.$$

We clearly have $\sup_{c\geq c_0}\sup_{y\in K}\hat\kappa(y,c)<\infty$. Using the Jensen inequality in (57) we obtain

$$\eta_\rho^{y,c}\leq 1+\hat\kappa(y,c)^{p-1}E\left[\frac{1}{4\pi^2}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)}dz\left(\frac{1}{|z-y|}-\frac{1}{|z|}\right)^2\left(L_t^{z/c}\right)^p\right].$$

Thus, using (56), we get

$$\eta_\rho^{y,c}\leq 1+\hat\kappa(y,c)\frac{p!\mathcal{C}_\rho}{(\mathcal{K}_\rho)^p}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)}\left(\frac{1}{|z|}-\frac{1}{|z-y|}\right)^{2p},$$

so that $\sup_{c\geq c_0,y\in K}\eta_\rho^{y,c}$ is also finite.

It remains to bound $\eta_a^{y,c}$. Using the trivial inequality $(a+b)^{p/2}\leq 1+2^p(a^p+b^p)$ for $a,b\geq 0$, we obtain

$$\eta_a^{y,c}\leq\frac{1}{2\pi^2}+\frac{2^p}{2\pi^2}\left(E\left[\left(\int_{B(0,2a)}\frac{dz}{|z|^2}L_t^{z/c}\right)^p+\left(\int_{B(0,2a)}\frac{dz}{|z-y|^2}L_t^{z/c}\right)^p\right]\right).$$

From the Hölder inequality with conjugate exponents $\frac{5}{4}$ and 5, we then get

$$E\left[\left(\int_{B(0,2a)}\frac{dz}{|z|^2}L_t^{z/c}\right)^p\right]\leq E\left[\left(\int_{B(0,2a)}\frac{dz}{|z|^{\frac{5}{2}}}\right)^{\frac{4p}{5}}\left(\int_{B(0,2a)}dz(L_t^{z/c})^5\right)^{\frac{p}{5}}\right]$$

and also,

$$E\left[\left(\int_{B(0,2a)}\frac{dz}{|z-y|^2}L_t^{z/c}\right)^p\right]\leq E\left[\left(\int_{B(0,2a)}\frac{dz}{|z|^{\frac{5}{2}}}\right)^{\frac{4p}{5}}\left(\int_{B(0,2a)}dz(L_t^{z/c})^5\right)^{\frac{p}{5}}\right].$$

Using the inequality $a^{p/5}\leq 1+a^p$ for $a\geq 0$, and then the Jensen inequality, we then obtain

$$\eta_a^{y,c}\leq\frac{1}{2\pi^2}+\frac{2^p}{\pi^2}\left(\int_{B(0,2a)}\frac{dz}{|z|^{\frac{5}{2}}}\right)^{\frac{4p}{5}}\left(1+\left(\frac{32\pi a^3}{3}\right)^{\frac{p}{5}-1}E\left[\int_{B(0,2a)}(L_t^{z/c})^{5p}\right]\right).$$

Using (56) we now get that $\sup_{c\geq c_0,y\in K}\eta_a^{y,c}<\infty$. We thus have proven that $\sup_{c\geq c_0,y\in K}\eta^{y,c}$ is finite, and (52) now follows from (55).

We thus have finished the proof of Lemma III.9 and of the non-conditioned result Proposition III.2. We now get back to Lee's result.

3.2. Back to Lee's result. In this section we prove Proposition III.1 with the help of Proposition III.2.

Proof of Proposition III.1 : We know from the scaling properties of super-Brownian motion that the law of $c^{d-4}D_{\phi,c^2t}$ under $P_{\nu^{(c)}}$ is the same as that of $c^dD_{\phi_c,t}$ under P_ν. Proposition III.1 is thus equivalent to the following statement

PROPOSITION III.4. *Consider t,x_0,K,ϕ,ξ as in Proposition 1.*
Under $P_{\frac{1}{c^2}\delta_{x_0}}\left(.\middle|X\text{ hits }K^{(c)}\right)$, the random vector $\left(c^dD_{\phi_c,t},c^d\psi(c)D_{\xi_c,t}\right)$ converges in distribution as $c\to\infty$ to (D_1,D_2).

Rather than proving Proposition III.4 immediately, we will first establish an analogous statement under the excursion measure $\mathbb{N}_{x_0}$ of super-Brownian motion. Let us give an informal explanation for this intermediate step.

Let $q_c := P_{c^{-2}\delta_{x_0}}\left(X \text{ hits } K^{(c)}\right)$. From [**DIP 89**] or [**Le 01**] [1],

$$q_c \sim_{c\to\infty} \frac{4-d}{2c^2|x_0|^2}.$$

Whenever a super-Brownian motion started at $c^{-2}\delta_{x_0}$ hits $K^{(c)}$, the probability that this is done by a single excursion goes to one as $c \to \infty$. At least informally it follows that, when c is large enough, $P_{c^{-2}\delta_{x_0}}\left(.\middle|X \text{ hits } K^{(c)}\right)$ is in some sense close to $\mathbb{N}_{x_0}\left(.\middle|X \text{ hits } \{0\}\right)$. This idea will be used below to reduce the proof of Proposition 4 to the following statement.

Proposition III.5. *Let t, x_0, K, ϕ, ξ be as in Proposition III.1. Let l_t^0 denotes the local time of X under $\mathbb{N}_{x_0}\left(.\middle|X \text{ hits } \{0\}\right)$ at level 0 and time t.*

Under $\mathbb{N}_{x_0}\left(.\middle|X \text{ hits } \{0\}\right)$, the vector $\Xi_c := (c^d D_{\phi_c,t}, c^d\psi(c)D_{\xi_c,t})$ converges in distribution as $c \to \infty$ to

$$\left(l_t^0 \int_K \phi(y)dy, \tilde{U}_t\right),$$

where conditionally on l_t^0, $\tilde{U}_t$ is centered Gaussian with variance $a_\xi l_t^0$.

We now prove Proposition III.5 as a consequence of Proposition III.2.
Proof of Proposition III.5 : Fix $t > 0$ and let $\rho_0 := 2^{-1}|x_0|$. Using the notation of (16), we also set $r_0 := \mathcal{K}(\delta_{x_0}, t, \overline{B}(0,\rho_0))$. Note from the discussion following (15) and (16) that the function $G_{x_0}^{t,0}$ is well defined on $\{z \in \mathbb{C}, |z| < r_0\}$. Let us introduce the function $g_{x_0}^{t,0}$ such that for any $z \in \mathbb{C}, |z| < r_0$,

$$g_{x_0}^{t,0}(z) = \mathbb{N}_{x_0}\left(\exp(zl_t^0)\middle|X \text{ hits } \{0\}\right).$$

The canonical decomposition of super-Brownian motion ensures that $g_{x_0}^{t,0}$ is also well defined for $|z| < r_0$. It is proven in Chapter VI of [**LG 99**] that $\mathbb{N}_{x_0}\left(X \text{ hits } \{0\}\right) = \left(2 - \frac{d}{2}\right)|x_0|^{-2}$, so using the canonical decomposition under $P_{\delta_{x_0}}$ we can write $P_{\delta_{x_0}}$-almost surely :

$$L_t^0 = \sum_{i=1}^N l_t^{0,(i)}$$

where the random variables $l_t^{0,(i)}$ are independent and distributed as l_t^0 under $\mathbb{N}_{x_0}(.|X \text{ hits } \{0\})$, and N is an independent Poisson variable with parameter $(2-\frac{d}{2})|x_0|^{-2}$.

We thus have for $z \in \mathbb{C}, |z| \le r_0$,

$$\begin{aligned}
\exp\left(G_{x_0}^{t,0}(z)\right) &= E_{\delta_{x_0}}[\exp(zL_t^0)] \\
&= E_{\delta_{x_0}}\left[E_{\delta_{x_0}}\left[\exp(zL_t^0)\middle|N\right]\right] \\
&= \sum_{k=0}^\infty \left(\frac{4-d}{2|x_0|^2}\right)^k \frac{1}{k!}\exp\left(-\frac{4-d}{2|x_0|^2}\right) g_{x_0}^{t,0}(z)^k \\
&= \exp\left\{\frac{4-d}{2|x_0|^2}\left(g_{x_0}^{t,0}(z) - 1\right)\right\}.
\end{aligned}$$

[1]in [**Le 01**] as in [**BEP 91**], q_c is twice bigger. In [**Le 01**] this comes from the non-standard underlying Brownian motion, whereas in [**BEP 91**] it comes from the branching rate being 2 instead of 1.

From the fact that $G^{t,0}_{x_0}(0) = 0$ and $g^{t,0}_{x_0}(0) = 1$ and the continuity of the functions $g^{t,0}_{x_0}, G^{t,0}_{x_0}$ we then deduce that for any $z \in \mathbb{C}, |z| \le r_0$,

$$g^{t,0}_{x_0}(z) = 1 + \frac{2|x_0|^2}{4-d} G^{t,0}_{x_0}(z). \tag{58}$$

Let a and b two real numbers and $Z^c_{a,b} := ac^d D_{\phi_c,t} + bc^d \psi(c) D_{\xi_c,t}$. Let $H_c(a,b)$, respectively $h_c(a,b)$ be the Fourier transform of Ξ_c with respect to the measure $P_{\delta_{x_0}}$, respectively $\mathbb{N}_{x_0}\left(.\middle|X \text{ hits } \{0\}\right)$, that is

$$\begin{aligned} H_c(a,b) &= E_{\delta_{x_0}}\left[\exp(iZ^c_{a,b})\right], \\ h_c(a,b) &= \mathbb{N}_{x_0}\left[\exp(iZ^c_{a,b})\middle|X \text{ hits } \{0\}\right]. \end{aligned}$$

Recall that $\mathbb{N}_{x_0}(X \text{ hits } \{0\}) = \left(2 - \frac{d}{2}\right)|x_0|^{-2}$. Using the canonical decomposition of super-Brownian motion we obtain that $Z^c_{a,b}$ is distributed under $P_{\delta_{x_0}}$ as $Z^{c,(1)}_{a,b} + Z^{c,(2)}_{a,b} + \dots + Z^{c,(N)}_{a,b} + R^c_{a,b}$ where $Z^{c,(1)}_{a,b}, Z^{c,(2)}_{a,b}, \dots$ are independent and distributed as $Z^c_{a,b}$ under $\mathbb{N}_{x_0}\left(.\middle|X \text{ hits } \{0\}\right)$, and N is Poisson with parameter $\left(2 - \frac{d}{2}\right)|x_0|^{-2}$. Also, $R^c_{a,b}$ represents the contribution to $Z^c_{a,b}$ under $P_{\delta_{x_0}}$ of excursions that hit $K^{(c)}$ but do not hit 0. Since the compact sets $K^{(c)}$ converge to $\{0\}$, it is easy to verify that $P_{\delta_{x_0}}\left(R^c_{a,b} = 0\right) \underset{c\to\infty}{\to} 1$. It follows that, uniformly in $(a,b) \in \mathbb{R}^2$,

$$\left|E_{\delta_{x_0}}\left[\exp(iZ^c_{a,b})\right] - E\left[\exp\left(i\lambda(Z^{c,(1)}_{a,b} + \dots + Z^{c,(N)}_{a,b})\right)\right]\right| \underset{c\to\infty}{\longrightarrow} 0,$$

that is

$$\left|H_c(a,b) - \exp\left[\frac{4-d}{2|x_0|^2}[h_c(a,b) - 1]\right]\right| \underset{c\to\infty}{\longrightarrow} 0, \quad \text{uniformly in } (a,b) \in \mathbb{R}^2. \tag{59}$$

On the other hand we know from Proposition III.2

$$H_c(a,b) \underset{c\to\infty}{\longrightarrow} H(a,b) := E\left[\exp\left\{ia\left(\int \phi(x)dx\right)L^0_t + ibU_t\right\}\right] \tag{60}$$

uniformly when (a,b) varies over a compact subset of $\mathbb{R}^2$. We also have

$$\begin{aligned} H(a,b) &= E\left[E\left[\exp\left\{ia\left(\int \phi(x)dx\right)L^0_t + ibU_t\right\}\middle|L^0_t\right]\right] \\ &= E\left[\exp\left(ia\left(\int \phi(x)dx\right)L^0_t\right)\exp\left(-\frac{L^0_t a_\xi b^2}{2}\right)\right] \\ &= \exp\left\{G^{t,0}_{x_0}\left(ia\left(\int \phi(x)dx\right) - \frac{a_\xi b^2}{2}\right)\right\}, \end{aligned}$$

assuming that (a,b) belongs to a sufficiently small neighbourhood $\mathcal{V}$ of $(0,0)$ (see the discussion following (16)). We then get by (58)

$$H(a,b) = \exp\left\{\frac{4-d}{2|x_0|^2}\left(g^{t,0}_{x_0}\left(ia\left(\int \phi(x)dx\right) - \frac{a_\xi b^2}{2}\right) - 1\right)\right\}.$$

If we let

$$h(a,b) := g^{t,0}_{x_0}\left(ia\left(\int \phi(x)dx\right) - \frac{a_\xi b^2}{2}\right),$$

we thus have from (59) and (60)

$$\exp\left[\frac{4-d}{2|x_0|^2}[h_c(a,b) - 1]\right] \underset{c\to\infty}{\longrightarrow} \exp\left\{\frac{4-d}{2|x_0|^2}\left(h(a,b) - 1\right)\right\} \tag{61}$$

uniformly in $\mathcal{V}$. From the fact that for any (a,b), the function $c \to h_c(a,b)$ is continuous, it follows from (61) that for any $(a,b) \in \mathcal{V}$, there exists an integer $k(a,b)$ such that

$$\frac{4-d}{2|x_0|^2} h_c(a,b) \underset{c\to\infty}{\longrightarrow} \frac{4-d}{2|x_0|^2} h(a,b) + 2ik(a,b)\pi.$$

Now from the continuity of both functions $(a,b) \to h_c(a,b)$ and $(a,b) \to h(a,b)$ and the uniformity of the convergence in (61), it follows that $k(a,b)$ does not depend on a nor b. Since $h_c(0,0) \to h(0,0)$ as $c \to \infty$, $k(a,b) = 0$ for every $(a,b) \in \mathcal{V}$. We have thus proved that for $(a,b) \in \mathcal{V}$,

$$h_c(a,b) = \mathbb{N}_{x_0}\left(\exp(iZ^c_{a,b}) \middle| X \text{ hits } \{0\}\right) \underset{c\to\infty}{\longrightarrow} h(a,b),$$

and $h(a,b)$ can be interpreted as the Fourier transform at 1 of $al^0_t \int \phi(x)dx + b\tilde{U}_t$. The statement of Proposition III.5 follows. $\square$

Proof of Proposition III.4 : From the canonical decomposition of super-Brownian motion , the law under $P_{c^{-2}\delta_{x_0}}$ of Ξ^c coincides with the law of $\sum_{i=1}^{N_c}(U^c_i, V^c_i)$, where the variables (U^c_i, V^c_i) are independent and distributed as Ξ^c under $\mathbb{N}_{x_0}\left(. \middle| X \text{ hits } K^{(c)}\right)$, and N_c is an independent Poisson variable with parameter $c^{-2}\mathbb{N}_{x_0}\left(X \text{ hits } K^{(c)}\right)$. Clearly $\{N_c \geq 1\} = \{X \text{ hits } K^{(c)}\}$, and moreover

$$P_{c^{-2}\delta_{x_0}}(N_c = 1) \underset{c\to\infty}{\sim} \frac{4-d}{2c^2|x_0|^2}.$$

It is also clear that $P(N_c \geq 2) \leq \kappa(x_0)c^4$ where $\kappa(x_0)$ is a constant depending on x_0. Since the laws of Ξ under $E_{c^{-2}\delta_{x_0}}\left(. \middle| N_c = 1\right)$ and $\mathbb{N}_{x_0}\left(. \middle| X \text{ hits } K^{(c)}\right)$ coincide, we have

$$\begin{aligned}&\left| E_{c^{-2}\delta_{x_0}}\left(\exp(iZ^c_{a,b}) \middle| X \text{ hits } K^{(c)}\right) - \mathbb{N}_{x_0}\left(\exp(iZ^c_{a,b}) \middle| X \text{ hits } K^{(c)}\right)\right| \\ &\leq \frac{\left| E_{c^{-2}\delta_{x_0}}(\exp(iZ^c_{a,b}), N_c \geq 2)\right|}{P_{c^{-2}\delta_{x_0}}\left(X \text{ hits } K^{(c)}\right)} \leq 2|x_0|^2\kappa(x_0)c^{-2}.\end{aligned}$$

From the fact that $\mathbb{N}_{x_0}\left(X \text{ hits } K^{(c)}\right) \to \mathbb{N}_{x_0}(X \text{ hits } \{0\})$ as $c \to \infty$ we now have

$$\left| E_{c^{-2}\delta_{x_0}}\left(\exp(iZ^c_{a,b}) \middle| X \text{ hits } K^{(c)}\right) - h_c(a,b)\right| \underset{c\to\infty}{\longrightarrow} 0.$$

From the proof of Proposition 5 we know that $h_c(a,b) \to h(a,b)$ as $c \to \infty$, and from (58)

$$h(a,b) = \frac{2|x_0|^2}{4-d} G^{t,0}_{x_0}\left(ia\int_K \phi(y)dy - \frac{a_\xi b^2}{2}\right) + 1$$

is the Fourier transform of (D_1, D_2). This finishes the proof of Proposition III.4, and thus of its rescaled equivalent form Proposition III.1. $\square$

4. The case $d = 1$

Sugitani showed in [**Su 89**] (Theorems 1 and 4) that under the condition that ν does not charge points in $\mathbb{R}$, P_ν-almost surely, L^x_t is continuously differentiable with respect to both time and space variables on $(0,\infty) \times \mathbb{R}$. We will denote $D_x L^y_t$ its continuous derivative with respect to the space variable taken at point (t,y). It is not hard to deduce from [**Su 89**] the following extension of Theorem 1 to the one dimensional case

PROPOSITION III.6. *Suppose $\nu \in M_F(\mathbb{R})$ is atomless in a certain neighbourhood of 0.*
Let $T > 0, K > 0$ be fixed. Then P_ν-almost surely, uniformly in $y \in [-K, K]$, $t \in [0, T]$, the random variable $c(L_t^{y/c} - L_t^0)$ converges to $yD_xL_t^0$, where $D_xL_t^0$ denotes the derivative of L_t^x with respect to the x-variable taken at point $(t, 0)$.

Also, it is evident that uniformly in $y \in [-K, K], t \in [0, T]$, $L_t^{y/c}$ converges P_ν-almost surely to L_t^0. As a direct consequence of these results we obtain a statement analogous to Proposition III.2 in the one-dimensional case.

PROPOSITION III.7. *Fix $t > 0$ and let ν be as in Proposition 6, and let ϕ, ξ be integrable function with compact support on $\mathbb{R}$ such that $\int \phi(y)dy \neq 0$, $\int \xi(y)dy = 0$. Then, for every $t > 0$, we have P_ν-almost surely*

$$\left(cD_{\phi_c,t}, c^2 D_{\xi_c,t}\right) \underset{c\to\infty}{\longrightarrow} \left(\left(\int_{\mathbb{R}} \phi(y)dy\right) L_t^0, \left(\int_{\mathbb{R}} y\ \xi(y)dy\right) D_xL_t^0\right).$$

5. Appendix. Proof of the estimates on $h^{x,c}$

Recall

$$h^{x,c}(z) = \frac{c}{\sqrt{\ln(c)}} \int_0^\infty (2\pi t)^{-1} e^{-\alpha t} \left(e^{-|z-x/c|^2/2t} - e^{-|z|^2/2t}\right) dt. \tag{62}$$

We will still use the notation $c^\delta = |z|$, and introduce $\delta^{'}$ so that $c^{\delta^{'}} = |z - x/c|$. By a symmetry argument, without loss of generality we may and will always assume $\delta \leq \delta^{'}$. Note that we then have for every $t > 0$

$$\left|e^{-|z-x/c|^2/2t} - e^{-|z|^2/2t}\right| \leq e^{-|z|^2/2t}. \tag{63}$$

Proof of $(\mathbf{B_1})$, $(\mathbf{B_2})$: Notice that the maximum of the function $t \to t^{-1}e^{-|z|^2/2t}$ is attained at $t = |z|^2/2$. Its value at this point is $2e^{-1}/|z|^2$, so that

$$\begin{aligned}
\left|\int_0^\infty (2\pi t)^{-1} e^{-\alpha t} e^{-\frac{|z|^2}{2t}}\right| &\leq e^{-1} + \left|\int_{|z|^2/2}^\infty (2\pi t)^{-1} e^{-\alpha t}\right| \\
&\leq \kappa_{11}(1 + \ln^+(1/|z|)).
\end{aligned}$$

The above and (63) imply $(\mathbf{B_1}), (\mathbf{B_2})$. □

Let us now prove the remaining estimates. The change of variable $t = c^\beta$ in the integral (62) leads to

$$h^{x,c}(z) = \frac{c}{2\pi\sqrt{\ln(c)}} \int_{-\infty}^\infty e^{-\alpha c^\beta} \left(e^{-\frac{1}{2}c^{2\delta^{'}-\beta}} - e^{-\frac{1}{2}c^{2\delta-\beta}}\right) \ln(c) d\beta.$$

For convenience let us define

$$F(u) = \int_{-\infty}^u e^{-\alpha c^\beta} \left(e^{-\frac{1}{2}c^{2\delta^{'}-\beta}} - e^{-\frac{1}{2}c^{2\delta-\beta}}\right) \ln(c) d\beta,$$

and

$$\tilde{F}(u) = \int_u^\infty e^{-\alpha c^\beta} \left(e^{-\frac{1}{2}c^{2\delta^{'}-\beta}} - e^{-\frac{1}{2}c^{2\delta-\beta}}\right) \ln(c) d\beta,$$

so that for any real x, $h^{x,c}(z) = c\left(2\pi\sqrt{\ln(c)}\right)^{-1} (F(u) + \tilde{F}(u))$. Note that $F(u)$ and $\tilde{F}(u)$ still depend on c, x and z, even though this is not apparent in our notation.

Proof of $(\mathbf{C_2})$: On the domain of $(\mathbf{C_2})$, $-1-\frac{\ln(\ln(c))}{8\ln(c)} \le \delta \le -1+\frac{\ln(\ln(c))}{8\ln(c)}$. Using (63) we first obtain

$$\begin{aligned}|F(2\delta)| &\le \int_{-\infty}^{2\delta} e^{-\alpha c^\beta} e^{-\frac{|z|^2}{2c^\beta}} \ln(c) d\beta \le \int_{-\infty}^{2\delta} c^{2\delta-\beta} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta \\ &\le 2\left[e^{-\frac{1}{2}c^{2\delta-\beta}}\right]_{-\infty}^{2\delta} \le 2.\end{aligned}$$

Using a Taylor expansion and the fact $e^{-\alpha c^\beta} \le 1$, we also have

$$\begin{aligned}|\tilde{F}(2\delta)| &\le \left|\int_{2\delta}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n! c^{\beta n}}\left(c^{2\delta' n}-c^{2\delta n}\right)\ln(c)d\beta\right)\right| \\ &\le \left|\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n n!}\left(c^{2(\delta'-\delta)n}-1\right)\right|.\end{aligned}$$

Since the Taylor expansion of $\int_0^u \frac{e^{-y}-1}{y}dy$ is $\sum_{n\ge1}\frac{(-1)^n u^n}{n n!}$, this last quantity is equal to

$$\left|\int_{\frac{1}{2}}^{\frac{1}{2}|z-x/c|^2 c^{-2\delta}} \frac{e^{-y}-1}{y}dy\right| \le \left(\frac{|z.x|}{c}c^{-2\delta}+\frac{x^2}{2c^2}c^{-2\delta}\right)\sup_{y\in[1/2,\infty)}\left|\frac{e^{-y}-1}{y}\right|.$$

Using the fact that $\delta \ge -1-\frac{\ln(\ln(c))}{8\ln(c)}$ we thus obtain that $|\tilde{F}(2\delta)| \le \kappa_{12}(\ln(c))^{1/4}$. Combining the above inequalities for $|F(2\delta)|, |\tilde{F}(2\delta)|$, we obtain $(\mathbf{C_2})$. □

Proof of $(\mathbf{C_1})$: Here $\delta \le -1-\frac{\ln(\ln(c))}{8\ln(c)}$. As in the proof of $(\mathbf{C_2})$ we first have $|F(2\delta)| \le 2$. Furthermore, using (63) we get

$$\begin{aligned}|\tilde{F}(-4-2\delta)-\tilde{F}(2\delta)| &\le \int_{2\delta}^{-2+(-2-2\delta)} e^{-\alpha c^\beta} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta \\ &\le \frac{-4-4\delta}{2\pi}\ln(c) \le |1+\delta|\ln(c).\end{aligned}$$

Using the same Taylor expansion technique as in the proof of $(\mathbf{C_2})$ we also obtain

$$|\tilde{F}(-4-2\delta)| \le \left|\int_{\frac{c^{4+4\delta}}{2}}^{\frac{c^{4+4\delta}}{2}-(z.x)c^{3+2\delta}+|x|^2\frac{c^{2+2\delta}}{2}} \frac{e^{-y}-1}{y}dy\right| \le \kappa_{13}(\ln(c))^{-1/4}.$$

Combining the above inequalities leads to $(\mathbf{C_1})$. □

Proof of $(\mathbf{C_3})$, $(\mathbf{C_4})$, $(\mathbf{B_3})$: Here $\delta \ge -1+\frac{\ln(\ln(c))}{8\ln(c)}$. In particular $\frac{3}{2}\delta-\frac{1}{2} \le 2\delta$. Using (63) we first obtain

$$\left|F\left(\frac{3}{2}\delta-\frac{1}{2}\right)\right| \le \left|\int_{-\infty}^{\frac{3}{2}\delta-\frac{1}{2}} e^{-\frac{1}{2}c^{2\delta-\beta}}\ln(c)d\beta\right| \le e^{-c^{1/2\delta+1/2}} \tag{64}$$

which is $o(c^{-\delta-1})$ since $\delta \ge -1+\frac{\ln(\ln(c))}{8\ln(c)}$. Furthermore, when $\beta \ge \frac{3}{2}\delta-\frac{1}{2}$, we have

$$\left|\frac{|z|^2-|z-x/c|^2}{2c^\beta}\right| = \left|\left(\frac{z.x}{c}-\frac{|x|^2}{2c^2}\right)c^{-\beta}\right| \underset{c\to\infty}{\longrightarrow} 0,$$

so that

$$e^{-\frac{1}{2}c^{2\delta'-\beta}} - e^{-\frac{1}{2}c^{2\delta-\beta}} \underset{c\to\infty}{\sim} \frac{z.x}{c^{\beta+1}}e^{-\frac{1}{2}c^{2\delta-\beta}}.$$

By dominated convergence, we thus have

$$\tilde{F}\left(\frac{3}{2}\delta - \frac{1}{2}\right) \underset{c\to\infty}{\sim} \int_{\frac{3}{2}\delta-\frac{1}{2}}^{\infty} e^{-\alpha c^{\beta}} \frac{z.x}{c^{\beta+1}} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta. \tag{65}$$

Let us first prove ($\mathbf{C_3}$), for which $\delta \leq -\frac{\ln(\ln(c))}{\ln(c)}$, in particular $\delta < 0$. Let us split the integral in the righthand side of (65) into two parts :

$$\begin{aligned} \int_{\frac{3}{2}\delta-\frac{1}{2}}^{\delta} e^{-\alpha c^{\beta}} \frac{z.x}{c^{\beta+1}} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta &\underset{c\to\infty}{\sim} \int_{\frac{3}{2}\delta-\frac{1}{2}}^{\delta} \frac{z.x}{2c^{\beta+1}} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta \\ &\underset{c\to\infty}{\sim} 2(z.x)c^{-2\delta-1}, \end{aligned} \tag{66}$$

and

$$\int_{\delta}^{\infty} e^{-\alpha c^{\beta}} \frac{z.x}{2c^{\beta+1}} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta \leq (z.x)c^{-1-2\delta} \int_{\delta}^{\infty} c^{2\delta-\beta} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta,$$

which, since $\delta < 0$ is $o(c^{-\delta-1})$ as $c \to \infty$. This fact, (64) and (66) give us the estimate ($\mathbf{C_3}$).

Let us now prove ($\mathbf{C_4}$), for which $\delta \geq \frac{\ln(\ln(c))}{4\ln(c)}$. Using (65), we obtain

$$\left|\tilde{F}(\frac{3}{2}\delta - \frac{1}{2})\right| \leq \left|\int_{\frac{3}{2}\delta-\frac{1}{2}}^{\infty} \frac{z.x}{c^{\beta+1}} e^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c) d\beta\right| \leq \kappa_{14} c^{-\delta-1}. \tag{67}$$

The above and (64) give us ($\mathbf{C_4}$).

Finally ($\mathbf{B_3}$) is obtained as a combination of (64), (67) and ($\mathbf{C_3}$). □

CHAPITRE IV

On the occupation measure of super-Brownian motion

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ABSTRACT : We derive the asymptotic behavior of the occupation measure $\mathcal{Z}(B_1)$ of the unit ball for super-Brownian motion started from the Dirac measure at a distant point x and conditioned to hit the unit ball. In the critical dimension $d = 4$, we obtain a limiting exponential distribution for the ratio $\mathcal{Z}(B_1)/\log|x|$.

1. Introduction

The results of the present work are motivated by the following simple problem about branching random walk in $\mathbb{Z}^d$. Consider a population of branching particles in $\mathbb{Z}^d$, such that individuals move independently in discrete time according to a random walk with zero mean and finite second moments, and at each integer time individuals die and give rise independently to a random number of offspring according to a critical offspring distribution. Suppose that the population starts with a single individual sitting at a point $x \in \mathbb{Z}^d$ located far away from the origin, and condition on the event that the population will eventually hit the origin. Then what will be the typical number of individuals that visit the origin, and is there a limiting distribution for this number ?

In the present work, we address a continuous version of the previous problem, and so we deal with super-Brownian motion in $\mathbb{R}^d$. We denote by $M_F(\mathbb{R}^d)$ the space of all finite measures in $\mathbb{R}^d$. We also denote by $X = (X_t)_{t\geq 0}$ a d-dimensional super-Brownian motion with branching rate γ, that starts from μ under the probability measure $\mathbb{P}_\mu$, for every $\mu \in M_F(\mathbb{R}^d)$. We refer to Perkins [**Pe 99**] for a detailed presentation of super-Brownian motion. For every $x \in \mathbb{R}^d$, we also denote by $\mathbb{N}_x$ the excursion measure of super-Brownian motion from x. We may and will assume that both $\mathbb{P}_\mu$ and $\mathbb{N}_x$ are defined on the canonical space $C(\mathbb{R}_+, M_F(\mathbb{R}^d))$ of continuous functions from $\mathbb{R}_+$ into $M_F(\mathbb{R}^d)$ and that $(X_t)_{t\geq 0}$ is the canonical process on this space. Recall from Theorem II.7.3 in Perkins [**Pe 99**] that X started at the Dirac measure δ_x can be constructed from the atoms of a Poisson measure with intensity $\mathbb{N}_x$.

The total occupation measure of X is the finite random measure on $\mathbb{R}^d$ defined by

$$\mathcal{Z}(A) = \int_0^\infty X_t(A)\, dt,$$

for every Borel subset A of $\mathbb{R}^d$. We set $\mathcal{R} = \mathrm{supp}(\mathcal{Z})$, where $\mathrm{supp}(\mu)$ denotes the topological support of the measure μ. Equivalently,

$$\mathcal{R} = \mathrm{cl}\Big(\bigcup_{t\geq 0} \mathrm{supp}(X_t)\Big), \tag{68}$$

where $\mathrm{cl}(A)$ denotes the closure of the set A. In dimension $d \geq 4$, points are polar, meaning that $\mathbb{N}_x(0 \in \mathcal{R}) = 0$ if $x \neq 0$, or equivalently $\mathbb{P}_\mu(0 \in \mathcal{R}) = 0$ if 0 does not belong to the closed support of μ. In dimension $d \leq 3$, we have if $x \neq 0$,

$$\mathbb{N}_x(0 \in \mathcal{R}) = \frac{8-2d}{\gamma}\,|x|^{-2} \tag{69}$$

(see Theorem 1.3 in [**DIP 89**] or Chapter VI in [**LG 99**]). It follows from the results in Sugitani [**Su 89**] that, again in dimension $d \leq 3$, the measure $\mathcal{Z}$ has a continuous density under $\mathbb{P}_{\delta_x}$ or under $\mathbb{N}_x$, for any $x \in \mathbb{R}^d$. We write $(\ell^y, y \in \mathbb{R}^d)$ for this continuous density.

For every $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ denotes the open ball centered at x with radius r. To simplify notation, we write $B_r = B(0,r)$ for the ball centered at 0 with radius r. By analogy with the discrete problem mentioned above, we are interested in the conditional distribution of $\mathcal{Z}(B_1)$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ when $|x|$ is large. As a simple consequence of (69) and scaling, we have when $d \leq 3$,

$$\mathbb{P}_{\delta_x}(\mathcal{Z}(B_1) > 0) \sim \mathbb{N}_x(\mathcal{Z}(B_1) > 0) \sim \frac{8-2d}{\gamma}\,|x|^{-2} \qquad \text{as } |x| \to \infty. \tag{70}$$

Here and later the notation $f(x) \sim g(x)$ as $|x| \to \infty$ means that the ratio $f(x)/g(x)$ tends to 1 as $|x| \to \infty$. On the other hand, when $d \geq 4$, it is proved in [**DIP 89**] that, as $|x| \to \infty$,

$$\mathbb{P}_{\delta_x}(\mathcal{Z}(B_1) > 0) \sim \mathbb{N}_x(\mathcal{Z}(B_1) > 0) \sim \begin{cases} \dfrac{2}{\gamma}\,|x|^{-2}(\log|x|)^{-1} & \text{if } d = 4, \\ \dfrac{\kappa_d}{\gamma}\,|x|^{2-d} & \text{if } d \geq 5, \end{cases} \tag{71}$$

where $\kappa_d > 0$ is a constant depending only on d.

For $d \geq 3$, the Green function of d-dimensional Brownian motion is

$$G(x,y) = c_d |x-y|^{2-d},$$

where $c_d = (2\pi^{d/2})^{-1}\Gamma(\frac{d}{2} - 1)$. If $\mu \in M_F(\mathbb{R}^d)$ and φ is a nonnegative measurable function on $\mathbb{R}^d$, we use the notation $\langle \mu, \varphi\rangle = \int \varphi\, d\mu$. We can now state our main result.

THEOREM IV.1. *Let φ be a bounded nonnegative measurable function supported on B_1, and set $\overline{\varphi} = \int \varphi(y)dy$.*

(i) *If $d \leq 3$, the law of $|x|^{d-4}\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ towards the distribution of $\overline{\varphi}\,\ell^0$ under $\mathbb{N}_{x_0}(\cdot \mid 0 \in \mathcal{R})$, where x_0 is an arbitrary point in $\mathbb{R}^d$ such that $|x_0| = 1$.*

(ii) *If $d = 4$, the law of $(\log|x|)^{-1}\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ to an exponential distribution with mean $\gamma\overline{\varphi}/(4\pi^2)$.*

(iii) *If* $d \geq 5$, *the law of* $\langle \mathcal{Z}, \varphi \rangle$ *under* $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ *converges as* $|x| \to \infty$ *to the probability measure* μ_φ *on* $\mathbb{R}_+$ *with moments* $m_{p,\varphi} = \int r^p \, \mu_\varphi(dr)$ *given by*

$$m_{1,\varphi} = \frac{c_d}{\kappa_d} \, \gamma \, \overline{\varphi},$$

and for every $p \geq 2$,

$$m_{p,\varphi} = \frac{c_d}{\kappa_d} \frac{\gamma^2}{2} \sum_{j=1}^{p-1} \binom{p}{j} \int \mathbb{N}_z(\langle \mathcal{Z}, \varphi \rangle^j) \mathbb{N}_z(\langle \mathcal{Z}, \varphi \rangle^{p-j}) dz.$$

The scaling invariance properties of super-Brownian motion allow us to restate Theorem IV.1 in terms of super-Brownian motion started with a fixed initial value and the occupation measure of a small ball with radius ε tending to 0. Part (i) of Theorem IV.1 then becomes a straightforward consequence of the fact that the measure $\mathcal{Z}$ has a continuous density in dimension $d \leq 3$: See Lee [**Le 01**] and Merle [**Me 06**] for more precise results along these lines. On the other hand, the proof of part (iii) is relatively easy from the method of moments and known recursive formulas for the moments of the random measure $\mathcal{Z}$ under $\mathbb{N}_x$. For the sake of completeness, we include proofs of the three cases in Theorem IV.1, but the most interesting part is really the critical dimension $d = 4$, where it is remarkable that an explicit limiting distribution can be obtained.

Notice that dimension 4 is critical with respect to the polarity of points for super-Brownian motion. Part (ii) of the theorem should therefore be compared with classical limit theorems for additive functionals of planar Brownian (note that $d = 2$ is the critical dimension for polarity of points for ordinary Brownian motion). The celebrated Kallianpur-Robbins law states that the time spent by planar Brownian motion in a bounded set before time t behaves as $t \to \infty$ like $\log t$ times an exponential variable (see e.g. section 7.17 in Itô and McKean [**IM 65**]). The Kallianpur-Robbins law can be derived by "conceptual proofs" which explain the occurrence of the exponential distribution. Our initial approach to part (ii) was based on a similar conceptual argument based on the Brownian snake approach to super-Brownian motion. Informally, if $a > 0$ is fixed, we may apply the strong Markov property of the Brownian snake at the first time when the occupation time of the unit ball by the "tip" of the snake exceeds a, and infer that any limiting distribution for the occupation time of the ball must satisfy the lack of memory property which characterizes exponential distributions. Since it seems delicate to make this argument completely rigorous, we rely below on a careful analysis of the moments of $\langle \mathcal{Z}, \varphi \rangle$.

Let us finally comment on the branching random walk problem discussed at the beginning of this introduction. Although we do not consider this problem here, it is very likely that a result analogous to Theorem IV.1 holds in this discrete setting, just replacing $\langle \mathcal{Z}, \varphi \rangle$ with the number of particles that hit the origin. In particular, the limiting distributions obtained in (i) and (ii) of Theorem IV.1 should also appear in the discrete setting.

2. Preliminary remarks

Let us briefly recall some basic facts about super-Brownian motion and its excursion measures. If $x \in \mathbb{R}^d$ and if

$$\mathcal{N} = \sum_{i \in I} \delta_{\omega_i}$$

is a Poisson point measure on $C(\mathbb{R}_+, M_F(\mathbb{R}^d))$ with intensity $\mathbb{N}_x(\cdot)$, then the measure-valued process Y defined by

$$Y_0 = \delta_x,$$
$$Y_t = \sum_{i \in I} X_t(\omega_i) \ , \ \text{for every } t > 0,$$

has law $\mathbb{P}_{\delta_x}$ (see Theorem II.7.3 in [**Pe 99**]).

We can use this Poisson decomposition to observe that it is enough to prove Theorem IV.1 with the conditional measure $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ replaced by $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$. Indeed, write $M = \#\{i \in I : \mathcal{R}(\omega_i) \cap B_1 \neq \emptyset\}$ (where $\mathcal{R}(\omega_i)$ is defined as in (68)). Then, M is Poisson with parameter $\mathbb{N}_x(\mathcal{Z}(B_1) > 0)$, and $\{M \geq 1\}$ is the event that the range of Y hits B_1. Furthermore, the preceding Poisson decomposition just shows that the law of $\langle \mathcal{Z}, \varphi \rangle$ under $\mathbb{P}_{\delta_x}$ coincides with the law of $Z^1 + \cdots + Z^M$, where conditionally given M, the variables $Z^1, Z^2, \ldots$ are independent and distributed according to the law of $\langle \mathcal{Z}, \varphi \rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$. Since $P(M = 1 \mid M \geq 1)$ tends to 1 as $|x| \to \infty$ (by the estimates (70) and (71)), we see that the law of $\langle \mathcal{Z}, \varphi \rangle$ (or the law of $f(x)\langle \mathcal{Z}, \varphi \rangle$ for any deterministic function f) under $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ will be arbitrarily close to the law of the same variable under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ when $|x|$ is large, which is what we wanted. Note that this argument is valid in any dimension.

Let us also discuss the dependence of our results on the branching rate γ. If $(Y_t)_{t \geq 0}$ is a super-Brownian motion with branching rate γ started at μ, and $\lambda > 0$, then $(\lambda Y_t)_{t \geq 0}$ is a super-Brownian motion with branching rate $\lambda\gamma$ started at $\lambda\mu$. A similar property then holds for excursion measures. Write $\mathbb{N}_x^{(\gamma)}$ instead of $\mathbb{N}_x$ to emphasize the dependence on γ. Then the "law" of $(\lambda X_t)_{t \geq 0}$ under $\mathbb{N}_x^{(\gamma)}$ is $\lambda^{-1}\mathbb{N}_x^{(\lambda\gamma)}$. Thanks to these observations, it will be enough to prove Theorem IV.1 for one particular value of γ.

In what follows, we take $\gamma = 2$, as this will simplify certain formulas. For any nonnegative measurable function φ on $\mathbb{R}^d$, the moments of $\langle \mathcal{Z}, \varphi \rangle$ are determined by induction by the formulas

$$\mathbb{N}_x(\langle \mathcal{Z}, \varphi \rangle) = \int_{\mathbb{R}^d} G(x,y)\varphi(y)\, dy \tag{72}$$

and, for every $p \geq 2$,

$$\mathbb{N}_x(\langle \mathcal{Z}, \varphi \rangle^p) = \sum_{j=1}^{p-1} \binom{p}{j} \int_{\mathbb{R}^d} G(x,z)\, \mathbb{N}_z(\langle \mathcal{Z}, \varphi \rangle^j)\mathbb{N}_z(\langle \mathcal{Z}, \varphi \rangle^{p-j})dz. \tag{73}$$

See e.g. formula (16.2.3) in [**LG 94**], and note that the extra factor 2 there is due to the fact that the the Brownian snake approach gives $\gamma = 4$.

3. Low dimensions

In this section, we prove part (i) of Theorem IV.1. Let $\varepsilon > 0$, and set $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$. By scaling, the law of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ coincides with the law of $\varepsilon^{-4}\langle \mathcal{Z}, \varphi_\varepsilon\rangle$ under $\mathbb{N}_{\varepsilon x}(\cdot \mid \mathcal{Z}(B_\varepsilon) > 0)$. Taking $\varepsilon = |x|^{-1}$, we see that the proof of part (i) reduces to checking that the law of $|x|^d\langle \mathcal{Z}, \varphi_{|x|^{-1}}\rangle$ under $\mathbb{N}_{x/|x|}(\cdot \mid \mathcal{Z}(B_{|x|^{-1}}) > 0)$ converges to the distribution of $\overline{\varphi}\ell^0$ under $\mathbb{N}_{x_0}(\cdot \mid 0 \in \mathcal{R})$.

However, as $|x| \to \infty$,

$$\mathbb{N}_{x/|x|}(\mathcal{Z}(B_{|x|^{-1}}) > 0) = \mathbb{N}_{x_0}(\mathcal{Z}(B_{|x|^{-1}}) > 0) \longrightarrow \mathbb{N}_{x_0}(0 \in \mathcal{R}) = 4 - d.$$

On the other hand, since

$$|x|^d\langle \mathcal{Z}, \varphi_{|x|^{-1}}\rangle = |x|^d \int dy\, \ell^y\, \varphi_{|x|^{-1}}(y) = \int dy\, \ell^{y/|x|}\, \varphi(y)$$

the continuity of the local times ℓ^y implies that, for every $\delta > 0$,

$$\mathbb{N}_{x/|x|}\Big(\Big||x|^d\langle \mathcal{Z}, \varphi_{|x|^{-1}}\rangle - \overline{\varphi}\,\ell^0\Big| > \delta\Big) \leq \mathbb{N}_{x/|x|}\Big(\sup_{y\in B_{|x|^{-1}}} |\ell^y - \ell^0| > \frac{\delta}{\overline{\varphi}}\Big) \longrightarrow 0$$

as $|x| \to \infty$. By rotational invariance, the law of ℓ^0 under $\mathbb{N}_{x/|x|}$ coincides with the law of the same variable under $\mathbb{N}_{x_0}$. Part (i) of Theorem IV.1 now follows from the preceding observations.

4. High dimensions

We now turn to part (iii) of Theorem IV.1 and so we suppose that $d \geq 5$. As noticed earlier, we may replace $\mathbb{P}_{\delta_x}(\cdot \mid \mathcal{Z}(B_1) > 0)$ by $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$.

Without loss of generality, we assume in this part that $\varphi \leq 1$.

Lemma IV.1. *There exists a finite constant K_d depending only on d, such that, for every $x \in \mathbb{R}^d$ and $p \geq 1$,*

$$\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p) \leq K_d^p\, p!\,(|x|^{2-d} \wedge 1).$$

Proof. Obviously, it is enough to consider the case when $\varphi = \mathbf{1}_{B_1}$. From (72), one immediately verifies that

$$\mathbb{N}_x(\mathcal{Z}(B_1)) \leq C_{1,d}\,(|x|^{2-d} \wedge 1)$$

for some constant $C_{1,d}$ depending only on d. Straightforward estimates give the existence of a constant a_d such that, for every $x \in \mathbb{R}^d$,

$$\int G(x,z)\,(|z|^{2-d} \wedge 1)^2\, dz \leq a_d(|x|^{2-d} \wedge 1).$$

We then claim that for every integer $p \geq 1$,

$$\mathbb{N}_x(\mathcal{Z}(B_1)^p) \leq C_{p,d}\, p!\,(|x|^{2-d} \wedge 1) \tag{74}$$

where the constants $C_{p,d}$, $p \geq 2$ are determined by induction by

$$C_{p,d} = a_d \sum_{j=1}^{p-1} C_{j,d} C_{p-j,d}. \tag{75}$$

Indeed, let $k \geq 2$ and suppose that (74) holds for every $p \in \{1, \ldots, k-1\}$. From (73), we get

$$\mathbb{N}_x(\mathcal{Z}(B_1)^k) \leq k! \sum_{j=1}^{k-1} C_{j,d} C_{k-j,d} \int G(x,z)\,(|z|^{2-d} \wedge 1)^2\,dz,$$

and our choice of a_d shows that (74) also holds for $p = k$. We have thus proved our claim (74) for every $p \geq 1$.

From (75) it is an elementary exercise to verify that $C_{p,d} \leq K_d^p$ for some constant K_d depending only on d. This completes the proof. □

Let us now prove that for every $p \geq 1$, $\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p \mid \mathcal{Z}(B_1) > 0)$ converges as $|x| \to \infty$ to $m_{p,\varphi}$. If $p = 1$, this is an immediate consequence of (71), (72) and dominated convergence. If $p \geq 2$, we write

$$\frac{\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p)}{|x|^{2-d}} = \sum_{j=1}^{p-1} \binom{p}{j} \int \frac{G(x,z)}{|x|^{2-d}}\, \mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^j) \mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^{p-j}) dz.$$

In each of the integrals appearing in the previous display, the contribution of the set $\{z \in \mathbb{R}^d : |z - x| \leq |x|/2\}$ goes to 0 as $|x| \to \infty$, by an easy application of the bounds of Lemma IV.1. On the other hand, if we restrict our attention to the set $\{z \in \mathbb{R}^d : |z - x| > |x|/2\}$, we can again use the bounds of Lemma IV.1 together with the property $\int (|z|^{2-d} \wedge 1)^2 dz < \infty$, in order to justify dominated convergence and to get

$$\lim_{|x|\to\infty} \frac{\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p)}{|x|^{2-d}} = c_d \sum_{j=1}^{p-1} \binom{p}{j} \int \mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^j) \mathbb{N}_z(\langle \mathcal{Z}, \varphi\rangle^{p-j}) dz.$$

The convergence of $\mathbb{N}_x(\langle \mathcal{Z}, \varphi\rangle^p \mid \mathcal{Z}(B_1) > 0)$ towards $m_{p,\varphi}$ now follows from (71).

Finally, Lemma IV.1 and (71) also imply that any limit distribution of the laws of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ is characterized by its moments. Part (iii) of Theorem IV.1 now follows as a standard application of the method of moments.

5. The critical dimension

In this section, we consider the critical dimension $d = 4$. Recall that in that case $G(x,y) = (2\pi^2)^{-1}|y - x|^{-2}$. As in the previous sections, we take $\gamma = 2$. We start by stating two lemmas.

Lemma IV.2. *Let $x \in \mathbb{R}^4 \setminus \{0\}$. Then,*

$$\mathbb{N}_x[\mathcal{Z}(B_\varepsilon) > 0] \underset{\varepsilon \to 0}{\sim} |x|^{-2} \left(\log \frac{1}{\varepsilon}\right)^{-1}.$$

Lemma IV.3. *Let $x \in \mathbb{R}^4 \setminus \{0\}$, and $p \geq 1$. Let φ be a bounded nonnegative measurable function on B_1, and for every $\varepsilon > 0$, put $\varphi_\varepsilon(y) = \varphi(y/\varepsilon)$. Then,*

$$\mathbb{N}_x[\langle \mathcal{Z}, \varphi_\varepsilon\rangle)^p] \underset{\varepsilon \to 0}{\sim} p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p |x|^{-2}\, \varepsilon^{4p} \left(\log \frac{1}{\varepsilon}\right)^{p-1},$$

uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$.

Let us explain how part (ii) of Theorem IV.1 follows from these two lemmas. Notice that the estimate of Lemma IV.2 also holds uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$, by scaling and rotational invariance. Combining the results of the lemmas gives

$$\mathbb{N}_x\left[\left(\frac{\langle \mathcal{Z}, \varphi_\varepsilon\rangle}{\varepsilon^4\log(\frac{1}{\varepsilon})}\right)^p \middle| \mathcal{Z}(B_\varepsilon) > 0\right] \underset{\varepsilon\to 0}{\sim} p!\left(\frac{\overline{\varphi}}{2\pi^2}\right)^p,$$

uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$. By scaling, for any $x \in \mathbb{R}^4$ with $|x| > 1$ the law of $\langle \mathcal{Z}, \varphi\rangle$ under $\mathbb{N}_x(\cdot \mid \mathcal{Z}(B_1) > 0)$ coincides with the law of $|x|^4\langle \mathcal{Z}, \varphi_{1/|x|}\rangle$ under $\mathbb{N}_{x/|x|}(\cdot \mid \mathcal{Z}(B_{1/|x|}) > 0)$. Hence, we deduce from the preceding display that we have also

$$\mathbb{N}_x\left[\left(\frac{\langle \mathcal{Z}, \varphi\rangle}{\log|x|}\right)^p \middle| \mathcal{Z}(B_1) > 0\right] \underset{|x|\to\infty}{\sim} p!\left(\frac{\overline{\varphi}}{2\pi^2}\right)^p.$$

The statement in part (ii) of Theorem IV.1 now follows from an application of the method of moments.

It remains to prove Lemma IV.2 and Lemma IV.3.

5.1. Proof of Lemma IV.2. From well-known connections between super-Brownian motion and partial differential equations (see e.g. Chapter VI in [**LG 99**]), the function $u_\varepsilon(x) = \mathbb{N}_x[\mathcal{Z}(B_\varepsilon) > 0]$ defined for $|x| > \varepsilon$ solves the singular boundary problem

$$\begin{aligned} &\Delta u = 2\,u^2, &&\text{in the domain } \{|x| > \varepsilon\}\\ &u(x) \longrightarrow \infty, &&\text{as } |x| \to \varepsilon^+\\ &u(x) \longrightarrow 0, &&\text{as } |x| \to \infty. \end{aligned}$$

As a consequence of a lemma due to Iscoe (see Lemma 3.4 in [**DIP 89**]), for every $x \in \mathbb{R}^4\setminus\{0\}$, we have $u_\varepsilon(x) \sim |x|^{-2}(\log(1/\varepsilon))^{-1}$ as $\varepsilon \to 0$. Lemma IV.2 follows.

5.2. Proof of Lemma IV.3.

5.2.1. *Lower bound.* Let us introduce the space of functions

$$\mathcal{F} := \Big\{f : \mathbb{R}_+ \to \mathbb{R} : \lim_{\varepsilon\to 0} f(\varepsilon) = 0 \text{ and } \lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\varepsilon} = \infty\Big\}.$$

Claim. *For every integer $p \geq 1$, for every $f \in \mathcal{F}$, for every $\beta > 0$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,*

$$\inf_{y\notin B_{f(\varepsilon)}} |y|^2\left(\log\left(\frac{|y|}{\varepsilon}\right)\right)^{1-p}\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^p\right] \geq (1-\beta)\,p!\left(\frac{\overline{\varphi}}{2\pi^2}\right)^p\varepsilon^{4p}. \tag{76}$$

We prove the claim by induction on p. Let us first consider the case $p = 1$. We fix $f \in \mathcal{F}$. Using (72), for $\varepsilon > 0$ and $y \in \mathbb{R}^4$ such that $|y| > f(\varepsilon)$, we have

$$\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle\right] = \int_{B_\varepsilon} dz\varphi_\varepsilon(z)G(y,z) \geq \varepsilon^4\overline{\varphi}\inf_{z\in B_\varepsilon} G(y,z). \tag{77}$$

Since $\lim_{\varepsilon\to 0}(f(\varepsilon)/\varepsilon) = \infty$, we see that

$$\inf_{y\notin B_{f(\varepsilon)}}\left(|y|^2\inf\big\{G(y,z), z\in B_\varepsilon\big\}\right) \underset{\varepsilon\to 0}{\to} \frac{1}{2\pi^2}.$$

We thus deduce from (77) that for ε small enough,

$$\inf_{y\notin B_{f(\varepsilon)}} |y|^2\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle\right] \geq \frac{1-\beta}{2\pi^2}\,\overline{\varphi}\,\varepsilon^4,$$

which gives our claim for $p = 1$.

Let $p \geq 2$ and suppose that the claim holds up to order $p-1$. Fix $f \in \mathcal{F}$ and $\beta \in (0,1)$. Let $\beta' \in (0,1)$ be such that $(1-\beta')^4 = 1-\beta$, and let $C > 0$ be such that $(1+C^{-1})^{-2} = 1-\beta'$. Introduce the function $\hat{f}$ defined by

$$\hat{f}(\varepsilon) = \varepsilon \log\left(\frac{f(\varepsilon)}{\varepsilon}\right). \tag{78}$$

Clearly, $\hat{f} \in \mathcal{F}$. Furthermore, we have

$$\lim_{\varepsilon \to 0} \frac{\log(\hat{f}(\varepsilon)/\varepsilon)}{\log(f(\varepsilon)/\varepsilon)} = 0. \tag{79}$$

Using (73), we obtain, for any $y \notin B_{f(\varepsilon)}$

$$\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^p\right] \geq \sum_{j=1}^{p-1} \binom{p}{j} \int_{B_{|y|/C} \setminus B_{\hat{f}(\varepsilon)}} dz\, G(y,z)\, \mathbb{N}_z\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^j\right] \mathbb{N}_z\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^{p-j}\right].$$

Using the induction hypothesis, we get, provided ε is small enough,

$$\begin{aligned}
&\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^p\right] \\
&\quad \geq (1-\beta')^2 p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p (p-1)\varepsilon^{4p} \int_{B_{|y|/C} \setminus B_{\hat{f}(\varepsilon)}} dz G(y,z) \frac{1}{|z|^4}\left(\log\frac{|z|}{\varepsilon}\right)^{p-2}.
\end{aligned}$$

From the definition of C, for any $z \in B_{|y|/C}$, we have $G(y,z) \geq (1-\beta')G(0,y)$. Using the fact that the area of the unit sphere S^3 is $2\pi^2$, we obtain

$$\begin{aligned}
(p-1) &\int_{B_{|y|/C} \setminus B_{\hat{f}(\varepsilon)}} dz G(y,z) \frac{1}{|z|^4}\left(\log\frac{|z|}{\varepsilon}\right)^{p-2} \\
&\geq 2\pi^2 (1-\beta')\, G(0,y)(p-1) \int_{\hat{f}(\varepsilon)}^{|y|/C} \frac{dr}{r}\left(\log\frac{r}{\varepsilon}\right)^{p-2} \\
&= (1-\beta')|y|^{-2}\left(\left(\log\frac{|y|}{C\varepsilon}\right)^{p-1} - \left(\log\frac{\hat{f}(\varepsilon)}{\varepsilon}\right)^{p-1}\right).
\end{aligned}$$

Moreover, using the property $f \in \mathcal{F}$ and (79), we see that, if ε is sufficiently small, for any $y \notin B_{f(\varepsilon)}$

$$\left(\log\frac{|y|}{C\varepsilon}\right)^{p-1} - \left(\log\frac{\hat{f}(\varepsilon)}{\varepsilon}\right)^{p-1} \geq (1-\beta')\left(\log\frac{|y|}{\varepsilon}\right)^{p-1}.$$

From the preceding bounds, we get that, if ε is sufficiently small,

$$\inf_{y \notin B_{f(\varepsilon)}} |y|^2 \left(\log\frac{|y|}{\varepsilon}\right)^{1-p} \mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^p\right] \geq (1-\beta')^4 p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p \varepsilon^{4p},$$

which is our claim at order p.

5.2.2. *Upper bound.* Without loss of generality we assume that $\varphi \le 1$. We need to get upper bounds on $\mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right]$ for y belonging to different subsets of $\mathbb{R}^4$.

We will prove that, for every $p \ge 1$, for every $f \in \mathcal{F}$ and every $\beta \in (0,1)$ the following bounds hold for $\varepsilon > 0$ sufficiently small :

$$(\mathfrak{H}_p^1) \qquad \sup_{|y|\le 4\varepsilon} \mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right] \le p!\,\varepsilon^{4p-2}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-1},$$

$$(\mathfrak{H}_p^2) \qquad \sup_{4\varepsilon\le|y|\le f(\varepsilon)} |y|^2\mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right] \le p!\,\varepsilon^{4p}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-1},$$

$$(\mathfrak{H}_p^3) \qquad \sup_{|y|\ge f(\varepsilon)} |y|^2\left(\log\frac{|y|}{\varepsilon}\right)^{1-p}\mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right] \le \left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p p!\,\varepsilon^{4p}.$$

Only $(\mathfrak{H}_p^3)$ is needed in our proof of Lemma IV.3. However, we will proceed by induction on p to get $(\mathfrak{H}_p^3)$, and we will use $(\mathfrak{H}_p^1)$ and $(\mathfrak{H}_p^2)$ in our induction argument. The bounds $(\mathfrak{H}_p^1)$ and $(\mathfrak{H}_p^2)$ are not sharp, but they will be sufficient for our purposes. Notice that $(\overline{\varphi}+\beta)/(2\pi^2) < 1/3$ because $\varphi \le 1$ and $\beta < 1$.

We first note that when $p = 1$ the bounds $(\mathfrak{H}_1^1)$, $(\mathfrak{H}_1^2)$ and $(\mathfrak{H}_1^3)$ are easy consequences of (72). Let $p \ge 2$ and assume that $(\mathfrak{H}_k^1)$, $(\mathfrak{H}_k^2)$ and $(\mathfrak{H}_k^3)$ hold for every $1 \le k \le p-1$, for any choice of β and f. Let us fix $f \in \mathcal{F}$ and $\beta_0 \in (0,1)$. In our induction argument we will use $(\mathfrak{H}_k^3)$, for $1 \le k \le p-1$, with $\beta \in (0,1)$ chosen small enough so that

$$(1+\beta)^2(\overline{\varphi}+\beta)^p < (\overline{\varphi}+\beta_0)^p.$$

For every $j \in \{1,\dots,p-1\}$, every $y \in \mathbb{R}^4$ and every Borel subset A of $\mathbb{R}^4$, we set

$$I^\varepsilon_{p,j}(A,y) := \frac{1}{j!(p-j)!}\int_A dz G(y,z)\mathbb{N}_z\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^j\right]\mathbb{N}_z\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^{p-j}\right].$$

We also set $I^\varepsilon_{p,j}(y) = I^\varepsilon_{p,j}(\mathbb{R}^4,y)$.

We first verify $(\mathfrak{H}_p^1)$, and so we assume that $|y| \le 4\varepsilon$. We fix $j \in \{1,\dots,p-1\}$ and we split the the integral in $I^\varepsilon_{p,j}(y)$ into three parts corresponding to the sets

$$A_1^{(1)} = B_{8\varepsilon},\ A_2^{(1)} = B_{f(\varepsilon)} \setminus B_{8\varepsilon},\ A_3^{(1)} = \mathbb{R}^4 \setminus B_{f(\varepsilon)}.$$

From (73), we have

$$\mathbb{N}_y\left[\langle \mathcal{Z},\varphi_\varepsilon\rangle^p\right] = p!\sum_{j=1}^{p-1}\left(I^\varepsilon_{p,j}(A_1^{(1)},y) + I^\varepsilon_{p,j}(A_2^{(1)},y) + I^\varepsilon_{p,j}(A_3^{(1)},y)\right). \tag{80}$$

If ε is small enough, we deduce from the bounds $(\mathfrak{H}_k^1)$ and $(\mathfrak{H}_k^2)$, with $1 \le k \le p-1$, that

$$I^\varepsilon_{p,j}(A_1^{(1)},y) \le \varepsilon^{4p-4}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{p-2}\int_{B_{8\varepsilon}}\frac{dz}{2\pi^2|z-y|^2}.$$

It follows that

$$\lim_{\varepsilon\to 0}\left(\sup_{|y|\le 4\varepsilon}\varepsilon^{2-4p}\left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_1^{(1)},y)\right) = 0.$$

If $z \in A_2^{(1)} \cup A_3^{(1)}$, we have $G(y,z) \leq 4G(0,z)$. Using $(\mathfrak{H}_k^2)$ with $1 \leq k \leq p-1$, we obtain, if ε is small enough,

$$I_{p,j}^{\varepsilon}(A_2^{(1)}, y) \quad \leq \quad 4\,\varepsilon^{4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \int_{8\varepsilon}^{f(\varepsilon)} \frac{dr}{r^3},$$

and thus

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \leq 4\varepsilon} \varepsilon^{2-4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_2^{(1)}, y)\right) = 0.$$

Using finally $(\mathfrak{H}_k^3)$ with $1 \leq k \leq p-1$, we obtain, for ε sufficiently small,

$$I_{p,j}^{\varepsilon}(A_3^{(1)}, y) \leq 4\,\varepsilon^{4p} \int_{f(\varepsilon)}^{\infty} \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2}.$$

Since

$$\int_{f(\varepsilon)}^{\infty} \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2} \underset{\varepsilon \to 0}{\sim} \frac{1}{2 f(\varepsilon)^2} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2}$$

we get

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \leq 4\varepsilon} \varepsilon^{2-4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_3^{(1)}, y)\right) = 0.$$

Combining the estimates we obtained for $I_{p,j}^{\varepsilon}(A_1^{(1)}, y), I_{p,j}^{\varepsilon}(A_2^{(1)}, y)$ and $I_{p,j}^{\varepsilon}(A_3^{(1)}, y)$, we arrive at

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \leq 4\varepsilon} \varepsilon^{2-4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(y)\right) = 0.$$

From (80), we obtain that $(\mathfrak{H}_p^1)$ holds.

We now turn to the proof of $(\mathfrak{H}_p^2)$, and so we assume that $4\varepsilon \leq |y| \leq f(\varepsilon)$. Again we fix $j \in \{1, \ldots, p-1\}$. We split the integral in $I_{p,j}^{\varepsilon}(y)$ into five parts corresponding to the sets

– $A_1^{(2)} = B_{2\varepsilon}$,
– $A_2^{(2)} = B(y, |y|/2)$,
– $A_3^{(2)} = B_{\hat{f}(\varepsilon)} \setminus (B_{2\varepsilon} \cup B(y, |y|/2))$,
– $A_4^{(2)} = B_{2f(\varepsilon)} \setminus \left(B_{\hat{f}(\varepsilon)} \cup B(y, |y|/2)\right)$,
– $A_5^{(2)} = \mathbb{R}^4 \setminus B_{2f(\varepsilon)}$,

where $\hat{f}(\varepsilon) = \varepsilon \log(f(\varepsilon)/\varepsilon)$ as in (78). We have thus

$$\mathbb{N}_y \left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^p\right] = p! \sum_{j=1}^{p-1} \sum_{i=1}^{5} I_{p,j}^{\varepsilon}(A_i^{(2)}, y). \tag{81}$$

Notice first that if $z \in A_1^{(2)}$, we have $|z| \leq |y|/2$ so that $G(z-y) \leq 4G(y)$. Using $(\mathfrak{H}_k^1)$ with $1 \leq k \leq p-1$, we obtain, provided ε is small enough

$$I_{p,j}^{\varepsilon}(A_1^{(2)}, y) \leq \varepsilon^{4p-4} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \frac{2}{\pi^2 |y|^2} \int_{\{|z| \leq 2\varepsilon\}} dz,$$

so that

$$\lim_{\varepsilon \to 0} \left(\sup_{4\varepsilon < |y| < f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_1^{(2)}, y)\right) = 0.$$

If $z \in A_2^{(2)}$, using the bound $|z|^{-2} \le 4|y|^{-2}$, we deduce from $(\mathfrak{H}_k^1)$ and $(\mathfrak{H}_k^2)$ for $1 \le k \le p-1$ that for sufficiently small ε,

$$I^\varepsilon_{p,j}(A_2^{(2)}, y) \le \varepsilon^{4p} \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \frac{16^3}{|y|^4} \int_{B(y,|y|/2)} G(y,z)dz.$$

It follows that

$$\lim_{\varepsilon \to 0} \left(\sup_{4\varepsilon < |y| < f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_2^{(2)}, y) \right) = 0.$$

If $z \in A_3^{(2)}$, $G(y,z) \le 4G(0,y)$. Since $\hat{f} \in \mathcal{F}$, we can use $(\mathfrak{H}_k^1)$ and $(\mathfrak{H}_k^2)$ with $1 \le k \le p-1$ to get that, for ε small enough,

$$I^\varepsilon_{p,j}(A_3^{(2)}, y) \le \varepsilon^{4p} \left(\log \frac{\hat{f}(\varepsilon)}{\varepsilon}\right)^{p-2} \frac{4 \times 16^2}{|y|^2} \int_\varepsilon^{\hat{f}(\varepsilon)} r^{-1} dr.$$

It then follows from (79) that

$$\lim_{\varepsilon \to 0} \left(\sup_{4\varepsilon < |y| < f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_3^{(2)}, y) \right) = 0.$$

If $z \in A_4^{(3)}$, we still have $G(y,z) \le 4G(0,y)$. Again, $\hat{f} \in \mathcal{F}$, and we can use $(\mathfrak{H}_k^3)$ with $1 \le k \le p-1$, recalling that $(\overline{\varphi} + \beta)/(2\pi^2) < 1/3$, to obtain for ε small

$$\begin{aligned} I^\varepsilon_{p,j}(A_4^{(2)}, y) &\le 3^{-p} \varepsilon^{4p} \frac{4}{|y|^2} \int_{\hat{f}(\varepsilon)}^{2f(\varepsilon)} \frac{dr}{r} \left(\log \frac{r}{\varepsilon}\right)^{p-2} \\ &= \frac{4 \times 3^{-p}}{p-1} \varepsilon^{4p} \frac{1}{|y|^2} \left(\log \frac{2f(\varepsilon)}{\hat{f}(\varepsilon)}\right)^{p-1}. \end{aligned}$$

It follows that

$$\limsup_{\varepsilon \to 0} \left(\sup_{4\varepsilon < |y| < f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_4^{(2)}, y) \right) \le \frac{4 \times 3^{-p}}{p-1} < \frac{1}{p-1}.$$

Finally, if $|z| \ge 2f(\varepsilon)$, we have $G(y,z) \le 4(2\pi^2)^{-1}|z|^{-2}$ and using again $(\mathfrak{H}_k^3)$ with $1 \le k \le p-1$, we get for ε sufficiently small,

$$I^\varepsilon_{p,j}(A_5^{(2)}, y) \le 4\, \varepsilon^{4p} \int_{2f(\varepsilon)}^\infty \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2},$$

and as before in the estimate for $I^\varepsilon_{p,j}(A_3^{(1)}, y)$, it follows that

$$\lim_{\varepsilon \to 0} \left(\sup_{4\varepsilon < |y| < f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_5^{(2)}, y) \right) = 0.$$

We get $(\mathfrak{H}_p^2)$ by combining the preceding estimates on $I^\varepsilon_{p,j}(A_i^{(2)}, y)$ for $1 \le i \le 5$, and using (81).

We now prove $(\mathfrak{H}_p^3)$, and we thus assume that $|y| \ge f(\varepsilon)$. Let $C' > 1$ be such that $1 - (C')^{-1} = (1+\beta)^{-1}$. For $\varepsilon > 0$ sufficiently small, we can split the integral in $I^\varepsilon_{p,j}(y)$ into five parts corresponding to the sets

– $A_1^{(3)} = B_{4\varepsilon}$,
– $A_2^{(3)} = B_{\hat{f}(\varepsilon)} \setminus B_{4\varepsilon}$,

– $A_3^{(3)} = B_{|y|/C'} \setminus B_{\hat{f}(\varepsilon)}$,
– $A_4^{(3)} = B_{2|y|} \setminus B_{|y|/C'}$,
– $A_5^{(3)} = \mathbb{R}^4 \setminus B_{2|y|}$.

We have then

$$\mathbb{N}_y\left[\langle \mathcal{Z}, \varphi_\varepsilon\rangle^p\right] = p! \sum_{j=1}^{p-1} \sum_{i=1}^{5} I^\varepsilon_{p,j}(A_i^{(3)}, y). \tag{82}$$

Using $(\mathfrak{H}_k^1)$ with $1 \leq k \leq p-1$, we get for ε small that

$$I^\varepsilon_{p,j}(A_1^{(3)}, y) \leq \varepsilon^{4p-4}\left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \frac{4}{|y|^2}\int_0^{4\varepsilon} r^3 dr,$$

so that

$$\lim_{\varepsilon\to 0}\left(\sup_{|y|\geq f(\varepsilon)} \varepsilon^{-4p}|y|^2 \left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_1^{(3)}, y)\right) = 0.$$

Then, using $(\mathfrak{H}_k^2)$ with $1 \leq k \leq p-1$, we obtain for ε small enough,

$$I^\varepsilon_{p,j}(A_2^{(3)}, y) \leq \varepsilon^{4p}\left(\log \frac{f(\varepsilon)}{\varepsilon}\right)^{p-2} \frac{4}{|y|^2}\int_{4\varepsilon}^{\hat{f}(\varepsilon)} r^{-1} dr.$$

Thus, using (79), we have

$$\lim_{\varepsilon\to 0}\left(\sup_{|y|\geq f(\varepsilon)} \varepsilon^{-4p}|y|^2 \left(\log\frac{f(\varepsilon)}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_2^{(3)}, y)\right) = 0.$$

If $z \in A_3^{(3)}$, we have $G(y,z) \leq (1-(C')^{-1})^{-2} G(0,y) = (1+\beta)^2 G(0,y)$. Since $\hat{f} \in \mathcal{F}$, we can use $(\mathfrak{H}_k^3)$ with $1 \leq k \leq p-1$ to get for ε sufficiently small,

$$\begin{aligned} I^\varepsilon_{p,j}(A_3^{(3)}, y) &\leq (1+\beta)^2 \left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p \varepsilon^{4p} \frac{1}{|y|^2} \int_{\hat{f}(\varepsilon)}^{|y|/C'} \frac{dr}{r}\left(\log\frac{r}{\varepsilon}\right)^{p-2} \\ &= \frac{(1+\beta)^2}{p-1}\left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p \varepsilon^{4p}\frac{1}{|y|^2}\left(\log\frac{|y|}{C'\hat{f}(\varepsilon)}\right)^{p-1}, \end{aligned}$$

Recalling (79), we obtain that

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\left(\sup_{|y|\geq f(\varepsilon)} \varepsilon^{-4p}|y|^2\left(\log\frac{|y|}{\varepsilon}\right)^{1-p} I^\varepsilon_{p,j}(A_3^{(3)}, y)\right) \\ &\quad\leq \frac{(1+\beta)^2}{p-1}\left(\frac{\overline{\varphi}+\beta}{2\pi^2}\right)^p < \frac{1}{p-1}\ \left(\frac{\overline{\varphi}+\beta_0}{2\pi^2}\right)^p, \end{aligned}$$

using our choice of β. If $z \in A_4^{(3)}$, we have $|z|^{-2} \leq C'^2|y|^{-2}$. Since $\hat{f} \in \mathcal{F}$, we obtain from $(\mathfrak{H}_k^3)$ with $1 \leq k \leq p-1$ that for sufficiently small ε,

$$\begin{aligned}
I_{p,j}^{\varepsilon}(A_4^{(3)}, y) &\leq \varepsilon^{4p} \frac{C'^4}{|y|^4} \int_{\{|y|/C' \leq |z| \leq 2|y|\}} dz G(y,z) \left(\log \frac{|z|}{\varepsilon}\right)^{p-2} \\
&\leq \varepsilon^{4p} \frac{C'^4}{|y|^4} \int_{\{|z| \leq 3|y|\}} \frac{dz}{2\pi^2|z|^2} \left(\log_+ \left(\frac{|z+y|}{\varepsilon}\right)\right)^{p-2} \\
&\leq \varepsilon^{4p} \frac{9C'^4}{2|y|^2} \left(\log \frac{4|y|}{\varepsilon}\right)^{p-2}.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \geq f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{|y|}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_4^{(3)}, y) \right) = 0.$$

Finally, using $(\mathfrak{H}_k^3)$ with $1 \leq k \leq p-1$, we get for ε small,

$$\begin{aligned}
I_{p,j}^{\varepsilon}(A_5^{(3)}, y) &\leq 4\,\varepsilon^{4p} \int_{2|y|}^{\infty} \frac{dr}{r^3} \left(\log \frac{r}{\varepsilon}\right)^{p-2} \\
&\leq K\,\varepsilon^{4p} \frac{1}{|y|^2} \left(\log \frac{2|y|}{\varepsilon}\right)^{p-2},
\end{aligned}$$

for some constant K depending only on p. Thus,

$$\lim_{\varepsilon \to 0} \left(\sup_{|y| \geq f(\varepsilon)} \varepsilon^{-4p} |y|^2 \left(\log \frac{|y|}{\varepsilon}\right)^{1-p} I_{p,j}^{\varepsilon}(A_5^{(3)}, y) \right) = 0.$$

Combining our estimates on $I_{p,j}^{\varepsilon}(A_i^{(3)}, y)$ for $1 \leq i \leq 5$ and then summing over j using (82), we get that $(\mathfrak{H}_p^3)$ holds for the given f and β_0. This completes the proof of the bounds $(\mathfrak{H}_p^1)$, $(\mathfrak{H}_p^2)$ and $(\mathfrak{H}_p^3)$, for every $p \geq 1$.

It now follows from (76) and $(\mathfrak{H}_p^3)$ that, for every $p \geq 1$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-4p} \left(\log \frac{|x|}{\varepsilon}\right)^{1-p} \mathbb{N}_x\left[\langle \mathcal{Z}, \varphi_\varepsilon \rangle^p\right] = p! \left(\frac{\overline{\varphi}}{2\pi^2}\right)^p |x|^{-2},$$

uniformly when x varies over a compact subset of $\mathbb{R}^4 \setminus \{0\}$. This completes the proof of Lemma IV.3. $\square$

CHAPITRE V

Hitting probability of a distant point for the voter model

Abstract. The goal of this work is to find the asymptotics of the probability for the voter model started from a single 1 to hit a distant point. In dimensions $d = 2$ or 3, we obtain the precise asymptotic behaviour of this probability. We use the scaling limit of the voter model started from a single 1 at the origin in terms of super-Brownian motion under its excursion measure. This invariance principle was stated by Bramson, Cox and Le Gall, as a consequence of a theorem of Cox, Durrett and Perkins. Less precise estimates are derived in dimension $d \geq 4$.

1. Introduction, Notation and Statement of result

The voter model is one of the most classical interacting particle systems. This model is of great interest because it exhibits a range of interesting phenomena and also because it is dual to a system of coalescing random walks. The voter model was first introduced in [**CS 73**], [**HL 75**], and some of its basic properties were investigated by Liggett [**Li 85**], Sawyer [**Sa 79**], Arratia [**A 81**], Bramson and Griffeath [**BG 80**].

More recently, Cox, Durrett and Perkins [**CDP 00**] showed an important invariance principle, establishing that, after a suitable renormalization, voter models in dimension $d \geq 2$ converge to super-Brownian motion. Super-Brownian motion is a continuous measure-valued process which arises as the weak limit of branching particle systems (see Watanabe [**W 68**]). It was discussed by Dawson [**D 75**], and studied extensively in the nineties (see in particular [**D 93**], [**Pe 99**], [**LG 99**]). In the recent years, it was shown that super-Brownian motion also appears in scaling limits of a wide range of lattice systems such as lattice trees, contact processes or oriented percolation. The main idea of this work is to exploit known properties of super-Brownian motion to get asymptotic results for the voter model.

Let us now describe the voter model and state our main result. Let $d \geq 2$. At each site of the integer lattice $\mathbb{Z}^d$ there is a voter holding an opinion. We will study here a two-type model, where there are only two possible opinions, say 0 or 1. At rate 1 exponential times, the voter at $x \in \mathbb{Z}^d$ chooses a neighbor y according to a given jump kernel p and adopts the opinion of y. The voting times and neighbor selections are supposed independent. The jump kernel $p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ will be supposed symmetric, translation invariant, irreducible, centered, isotropic, and having exponential moments :

- $p(x, y) = p(0, y - x), \qquad p(x, y) = p(y, x), \qquad p(0, 0) = 0,$
- $\sum_{y \in \mathbb{Z}^d} y p(0, y) = 0,$
- $\sum_{y \in \mathbb{Z}^d} p(0, y) y^i y^j = \sigma^2 \delta_{ij}$ for some $0 < \sigma^2 < \infty,$

– there exists a constant $C > 0$ such that $\sum_{y \in \mathbb{Z}^d} p(0, y) \exp(C|y|) < \infty$.
If $t \geq 0$, we denote by ξ_t the set of sites where voters hold opinion 1 at time t ; $(\xi_t)_{t \geq 0}$ is the two-type voter model. If $A \subset \mathbb{Z}^d$, we write P_A for the probability measure under which $\xi_0 = A$. Throughout this paper, we will consider the particular case when $\xi_0 = \{0\}$. In this case, $(\xi_t^0)_{t \geq 0}$ will denote the two-type voter model started from a single opinion 1 at the origin, and for simplicity, we will write P for $P_{\{0\}}$.

It is often convenient to work with the associated measure-valued processes

$$X_t := \sum_{y \in \xi_t} \delta_y, \quad X_t^0 := \sum_{y \in \xi_t^0} \delta_y.$$

For $\alpha > 0$ we define the conditional probability

$$P_\alpha^*(.) := P(.|\xi_\alpha^0 \neq \emptyset).$$

We are interested in estimating the probability that a voter located at a distance of order c from the origin ever holds opinion 1. If $x \in \mathbb{R}^d$, we denote by $[x]_c$ the point in $c^{-1}\mathbb{Z}^d$ closest to x. If there is more than one such point, we choose the point closest to the origin. Our goal is to find the asymptotic order as $c \to \infty$ of

$$P(\exists t \geq 0 : c[x]_c \in \xi_t^0).$$

We introduce the notation $T_{c[x]_c} = \inf\{t \geq 0 : c[x]_c \in \xi_t^0\}$ so that the previous quantity can also be written $P(T_{c[x]_c} < \infty)$. Set $\beta_2 = 2\pi$, and for $d \geq 3$, let β_d be the probability that a rate 1 continuous time random walk with jump kernel p started from the origin never returns to it.

Theorem V.1. *Let $x \in \mathbb{R}^d \setminus 0$ be fixed. Let us define*

$$\phi_d(c) = \begin{cases} \frac{c^2}{2\ln(c)} & \textit{if } d = 2, \\ c^2 & \textit{if } d = 3, \\ c^{d-2} & \textit{if } d \geq 5. \end{cases}$$

Then, if $d = 2$ or $d = 3$,

$$\lim_{c \to \infty} \phi_d(c) P(T_{c[x]_c} < \infty) = \frac{2\sigma^2}{\beta_d} \left(2 - \frac{d}{2}\right) |x|^{-2}.$$

If $d \geq 5$, there exist positive constants a_d, b_d depending on x such that

$$a_d \leq \liminf_{c \to \infty} \phi_d(c) P(T_{c[x]_c} < \infty) \leq \limsup_{c \to \infty} \phi_d(c) P(T_{c[x]_c} < \infty) \leq b_d.$$

In dimension 4 we obtain less precise results. We will prove the existence of a positive constant a_4 and we conjecture the existence of a positive b_4 such that a statement similar to the one in $d \geq 5$ holds for $d = 4$ with the function $\phi_4(c) := c^2 \ln(c)$. The upper bound in dimension 4 seems more difficult than the corresponding results in other dimensions. Adapting the proof of the upper bound for $2 \leq d \leq 3$ to the case $d = 4$ only gives $\limsup_{c \to \infty} c^2 P(T_{c[x]} < \infty) = 0$.

Theorem V.1 immediately extends to the multitype voter model $\overline{\xi}_t$, which is described as follows. We assume that the initial opinions are all distinct. The dynamics of the multitype voter model are the same as those of the two-type voter model. In this multitype setting, Theorem V.1 gives the asymptotics of the probability that the voter at x ever adopts the initial opinion of y, as $|x - y|$ tends to infinity.

In dimensions 2 and 3, we will let $T > 0$ and argue under the measure $P_{c^2T}^*$. Motivated by the results of [**CDP 00**], Bramson, Cox and Le Gall [**BCLG 01**] proved that for $T > 0$, the voter model ξ^0 under $P^*_{c^2T}$ converges as $c \to \infty$ modulo a suitable rescaling to a nondegenerate limit that can be expressed in terms of the excursion measure $\mathbb{N}_0$ of super-Brownian motion (see Theorem V.2 below). This invariance principle of [**BCLG 01**] will be our main tool in the proof of Theorem V.1 for small dimensions. We will also need properties of super-Brownian motion under its excursion measure $\mathbb{N}_0$. The Brownian snake approach of Le Gall [**LG 99**] gives a good understanding of the measure $\mathbb{N}_0$, and will be used to prove an intermediate result.

As mentioned earlier, the voter model and coalescing random walks are dual processes. In a coalescing random walk system, particles are assumed to execute rate 1 random walks with jump kernel p. Particles move independently until they meet, then coalesce and move together afterwards. The duality property also serves as a major tool for our results.

In Section 2.1, we introduce super-Brownian motion and its excursion measure $\mathbb{N}_0$. Scaling limits of the voter model (invariance principles) are discussed in Section 2.2. The duality property is explained in Section 2.3, and preliminary results on rate 1 random walks and system of coalescing random walks are discussed in Section 2.4 and 2.5.

We establish the asymptotic upper bounds on $P(T_{c[x]} < \infty)$ in Section 3. This requires interesting intermediate results. Lemma V.4 expresses that the probability for the voter model under P^*_α to escape $B(0, A)$ before time 2α decays exponentially with A. Lemma V.3 informally expresses that for any fixed $\epsilon > 0$, then, $\cup_{t \geq \epsilon\alpha} \xi^0_t$ does not contain any "isolated" point, with arbitrarily high probability under P^*_α, when α is taken large enough.

We prove the asymptotic lower bounds in Section 4. Sections 4.1 is devoted to the case $d \geq 4$, and Sections 4.2 and 4.3 to the case $d = 2$ or 3. Finally, we prove the results of Sections 2.4 and 2.5 in Section 5.

2. Further notation and preliminary results

Let f and g be two functions from $\mathbb{R}$ into $(0, \infty)$. We will write $f(x) = o(g(x))$ as $x \to \infty$, respectively $f(x) \sim g(x)$ as $x \to \infty$ whenever $\lim_{x\to\infty} f(x)(g(x))^{-1}$ is equal to 0, respectively 1.

For $x \in \mathbb{R}^d, r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^d$ centered at x with radius r, and $B(x, r)^c$ its complement.

Finally, for real numbers $x \leq y$, the set $\{n \in \mathbb{Z} : x \leq n \leq y\}$ of integers between x and y will be denoted by $[\![x, y]\!]$; also, the integer part of $x : \max\{n \in \mathbb{Z} : n \leq x\}$ will be denoted by $\lfloor x \rfloor$, while $\lfloor x \rfloor + 1 = \min\{n \in \mathbb{Z} : n > x\}$ will be denoted by $\lceil x \rceil$.

2.1. Super-Brownian motion. Let $M_F(\mathbb{R}^d)$ be the space of all finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. For $\mu \in M_F(\mathbb{R}^d)$, f a function on $\mathbb{R}^d$, the notation $\mu(f)$ will stand for $\int_{\mathbb{R}^d} f(x)\mu(dx)$ whenever this integral is well-defined. We let $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ be the space of continuous paths from $\mathbb{R}_+$ into $M_F(\mathbb{R}^d)$, and we let $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$ be the Skorohod space of cadlag functions from $\mathbb{R}^d$ into $M_F(\mathbb{R}^d)$. We denote by $(Y_t, t \geq 0)$ the canonical process on either $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ or $D(\mathbb{R}_+, M_F(\mathbb{R}^d))$.

The law of super-Brownian motion with branching rate γ and diffusion coefficient σ^2, starting from $\mu \in M_F(\mathbb{R}^d)$, is the probability measure $\mathbf{Q}^{\gamma;\sigma^2}_\mu$ on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$

that solves the following well-posed martingale problem (see [**Pe 99**], Theorem II.5.1) :

(MP) For any $\phi \in C_b^\infty(\mathbb{R}^d)$,

$$Y_t(\phi) \;=\; \mu(\phi) + M_t(\phi) + \frac{1}{2}\int_0^t Y_s(\sigma^2\Delta\phi)ds,$$

where $M_t(\phi)$ is a $\mathbf{Q}_\mu^{\gamma,\sigma^2}$ continuous square integrable martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$< M(\phi) >_t = \int_0^t Y_s(\gamma\phi^2)ds.$$

One can show (see for example Section II.7 of [**Pe 99**]) that there exists a family $\{R_t^{\gamma,\sigma^2}(y,.), y \in \mathbb{R}^d, t > 0\}$ of finite measures on $M_F(\mathbb{R}^d)$, called the canonical measures of super-Brownian motion, which assign zero mass to the 0 measure and are such that the following holds. The law of Y_t under $\mathbf{Q}_\mu^{\gamma,\sigma^2}$ is the same as the law of $\sum_{i\in I} Y_t^i$, where $\sum_{i\in I}\delta_{Y_t^i}$ is a Poisson measure with intensity $\int_{M_F(\mathbb{R}^d)} R_t^{\gamma,\sigma^2}(y,.)\mu(dy)$. It follows that for any Borel subset $\mathcal{Y}$ of $M_F(\mathbb{R}^d)$ with $0 \notin \mathcal{Y}$,

$$\lim_{\epsilon\to 0}\epsilon^{-1}\mathbf{Q}_{\epsilon\delta_y}^{\gamma,\sigma^2}(Y_t \in \mathcal{Y}) = R_t^{\gamma,\sigma^2}(y,\mathcal{Y}). \tag{83}$$

It is also well-known (see [**Pe 99**], Theorem II.7.2) that for any $y \in \mathbb{R}^d$,

$$R_t^{\gamma,\sigma^2}(y, M_F(\mathbb{R}^d)) = \frac{2}{\gamma t}. \tag{84}$$

From [**Pe 99**], Theorem II.7.3 (see also formula (3.10) in [**BCLG 01**]), for each $y \in \mathbb{R}^d$ there is a σ-finite measure $\mathbb{N}_y$ on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ called the excursion measure of super-Brownian motion with branching rate γ and diffusion coefficient σ^2 such that the following holds. For any $\alpha > 0$ fixed, then for any bounded continuous function F on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$, such that $F(\omega) = 0$ for any ω with $\omega(t) = 0$ for all $t \geq \alpha$,

$$\lim_{\epsilon\to 0}\epsilon^{-1}\mathbf{Q}_{\epsilon\delta_y}^{\gamma,\sigma^2}(F((Y_t, t\geq 0))) = \mathbb{N}_y(F). \tag{85}$$

The convergence (83) is a particular case of (85). Thus, for any Borel subset $\mathcal{Y}$ of $M_F(\mathbb{R}^d)$ with $0 \notin \mathcal{Y}$, we have

$$\mathbb{N}_y(Y_t \in \mathcal{Y}) = R_t^{\gamma,\sigma^2}(y,\mathcal{Y}).$$

Also, for any $T > 0, y \in \mathbb{R}^d$, we get from (84)

$$\mathbb{N}_y(Y_T \neq 0) = \frac{2}{\gamma T}, \tag{86}$$

and we can define the probability measure $\mathbb{N}_y^{(T)} := \mathbb{N}_y(.|Y_T \neq 0)$.

A better understanding of the measures $\mathbb{N}_y$ is given by the Brownian snake approach of Le Gall (see [**LG 99**], and Section 4.4 below). The Brownian snake approach corresponds to $\gamma = 4$, but scaling properties of super-Brownian motion can then be used to deal with a general value of γ.

Finally, we will use the following result about hitting probabilities of a single point. Let $\mathcal{R}_t$ denote the topological support of the measure Y_t, and $\mathcal{R} = \bigcup_{t>0} \mathcal{R}_t$. It follows from [**LG 99**], Section 6.1 that

$$\mathbb{N}_0(x \in \mathcal{R}) = \frac{4\sigma^2}{\gamma}\left(2 - \frac{d}{2}\right)^+ |x|^{-2}. \tag{87}$$

In particular, in the case $d \geq 4$, $\mathbb{N}_0(x \in \mathcal{R}) = 0$, which explains why our results are less precise. Also, as (87) suggests, the case of dimension 4 is critical, and thus harder.

Since $\mathbb{N}_0(Y_T = 0 \big| x \in \mathcal{R}) \to 0$ as T goes to 0, we deduce from (86) and (87) that

$$\mathbb{N}_0^{(T)}(x \in \mathcal{R}) \underset{T \to 0}{\sim} 2\sigma^2 T \left(2 - \frac{d}{2}\right)^+ |x|^{-2}. \tag{88}$$

2.2. Extinction probability, invariance principle. Set $p_t := P(\xi_t^0 \neq \emptyset)$. The asymptotic rate at which p_t converges to 0 was found in [**BG 80**]. As $t \to \infty$,

$$p_t \sim \begin{cases} \log(t)/(\beta_2 t) \text{ if } d = 2 \\ 1/(\beta_d t) \text{ if } d \geq 3, \end{cases} \tag{89}$$

where $\beta_d, d \geq 2$ was defined before Theorem V.1. Hence, for any $d \geq 2$ there exist a positive κ depending only on d such that for any $1/4 < t' \leq t$,

$$\frac{p_{t'}}{p_t} \leq \kappa \frac{t}{t'} \tag{90}$$

If $|C|$ denote the cardinality of a finite set C, Bramson and Griffeath ([**BG 80**]) established that that the law of $p_t|\xi_t^0|$ under P_t^* converges as $t \to \infty$ to an exponential distribution with parameter 1.

Bramson and Griffeath [**BG 80**] also conjectured that ξ_t^0 would obey a certain asymptotic shape theorem. Such a result was derived in 2001 by Bramson, Cox and Le Gall [**BCLG 01**] using the invariance principle relating the voter model and super-Brownian motion, which was proved by Cox, Durrett and Perkins in [**CDP 00**]. We rescale the voter model as follows. For $N > 0$, the lattice is now $S_N := \mathbb{Z}^d/\sqrt{N}$. Individuals change opinion at rate N instead of 1, and the jump kernel becomes $p_N : S_N \times S_N \to \mathbb{R}_+$ such that $p_N(x,y) = p(\sqrt{N}x, \sqrt{N}y)$. We denote by $(\xi_t^{N,0})_{t\geq 0}$ the corresponding process ($\xi_t^{N,0}$ represents the set of sites having opinion 1 at time t). If we let

$$m_N := \frac{N}{\log(N)} \text{ if } d = 2, \qquad m_N := N \text{ if } d \geq 3,$$

we can define the corresponding measure-valued processes :

$$X_t^{N,0} := \frac{1}{m_N} \sum_{y \in \xi_t^{N,0}} \delta_y.$$

Similarly, when at time 0, opinion 1 is started from a given set ξ_0, we can define for $N > 0$ a rescaled voter model ξ_t^N and its associated measure valued process X_t^N. Theorem 1.2 of [**CDP 00**] states that whenever X_0^N converges to a non-degenerate measure $X_0 \in M_F(\mathbb{R}^d)$, then $(X_t^N)_{t\geq 0}$ converges to a super-Brownian motion on $\mathbb{R}^d$ with branching rate $2\beta_d$ and diffusion coefficient σ^2, started from X_0.

Theorem 2 below states the convergence in law of the process $(X_t^{N,0})_{t\geq 0}$ under the conditional distribution $P(.\big|X_\alpha^{N,0}\neq 0)$ towards super-Brownian motion under $\mathbb{N}_0^{(\alpha)}$. This result, which is taken from [**BCLG 01**] (Theorem 4) will be a key ingredient of the proof of Theorem V.1 in dimensions 2 and 3.

THEOREM V.2. *Assume $d\geq 2$, and let $\mathbb{N}_0$ be the excursion measure of super-Brownian motion on $\mathbb{R}^d$ with branching rate $2\beta_d$ and diffusion coefficient σ^2. Let $\alpha>0$, and let F be a bounded continuous function on $D(\mathbb{R}_+,M_F(\mathbb{R}^d))$. Then*

$$\lim_{N\to\infty} E\left[F\left((X_t^{N,0})_{t\geq 0}\right)\big|X_\alpha^{N,0}\neq 0\right]=\mathbb{N}_0^{(\alpha)}[F]. \tag{91}$$

Let us now turn to the relation between the voter model and coalescing random walks

2.3. Dual process to the voter model. Let us introduce some further notation, in order to describe the dual process to the voter model. The times at which the voter at x adopts the opinion of the voter at y are the jump times of a standard Poisson process with rate $p(x,y)$. We denote by $\Lambda(x,y)$ this set of times. Then, $\{\Lambda(x,y),x,y\in\mathbb{Z}^d\}$ forms a family of independent Poisson point processes on $[0,\infty)$. We will use the graphical representation of the voter model ($\mathbb{Z}^d$ will be represented by the horizontal axis while time will be represented by the vertical axis).

For $x,y\in\mathbb{Z}^d$ we draw an arrow from y to x at each time $s\in\Lambda(x,y)$. For $s<t$ we say there is a path up from (y,s) to (x,t) or equivalently a path down from (x,t) to (y,s) and we will write

$$(y,s)\nearrow(x,t)\Leftrightarrow(x,t)\searrow(y,s)$$

if there exist times $s=s_0<s_1<...<s_n\leq s_{n+1}=t$ and sites $y=x_0,x_1,...,x_n=x$ such that
– for $1\leq i\leq n$ there is an arrow pointing from x_{i-1} towards x_i at time s_i,
– for $0\leq i\leq n$, there is no arrow pointing towards x_i in the time interval (s_i,s_{i+1}).
Clearly for every $x\in\mathbb{Z}^d$ and every choice of $0\leq s\leq t$, there is a unique $y\in\mathbb{Z}^d$ such that $(y,s)\nearrow(x,t)$. In such a case, the opinion of (x,t) is "descended" from that at (y,s). We will say that x at time t is a "descendant" of y at time s, or equivalently that y at time s is an "ancestor" of x at time t.

We are now in a position to describe the dual process to the voter model. For $t>0$ and $x\in\mathbb{Z}^d$ we define $(Z_s^{x,t})_{0\leq s\leq t}$ by setting $Z_0^{x,t}=x$ and for $0<s\leq t$, $Z_s^{x,t}=y$ if and only if $(x,t)\searrow(y,t-s)$. Clearly, $(Z_s^{x,t})_{0\leq s\leq t}$ is a rate 1 random walk with jump kernel p starting from x. Moreover, for $x\in\mathbb{Z}^d$, $y\in\mathbb{Z}^d$, the two walks $(Z_s^{x,t})_{0\leq s\leq t},(Z_s^{y,t})_{0\leq s\leq t}$ start respectively from x and y, move independently until they meet, and move together afterwards. That is, $(Z_s^{x,t})_{0\leq s\leq t,x\in\mathbb{Z}^d}$ forms a coalescing random walk system with jump kernel p. Furthermore

$$\xi_t^0=\{y\in\mathbb{Z}^d:Z_t^{y,t}=0\}. \tag{92}$$

For $t\geq 0$, we denote by $\hat{\xi}_s^{y,t}$ the set of descendants at time $t+s$ of y at time t, that is

$$\hat{\xi}_s^{y,t}:=\{z\in\mathbb{Z}^d:(y,t)\nearrow(z,t+s)\}=\{z\in\mathbb{Z}^d:Z_s^{z,t+s}=y\}$$

Notice that $(\hat{\xi}_s^{y,t})_{s\geq 0}$ has the same law as $(\xi_s^0+y)_{s\geq 0}$. For $u\leq t$ we will denote by Ω_u^t the set of points having opinion 1 at time u and having descendants at time t, that is

$$\Omega_u^t := \{y \in \xi_u^0 : \hat{\xi}_{t-u}^{y,u} \neq \emptyset\} = \xi_u^0 \cap \{Z_{t-u}^{z,t}, z \in \mathbb{Z}^d\}.$$

The coalescing random walk perspective, combined with the Bramson and Griffeath results and Theorem V.2, gives us a heuristic explanation of our main result Theorem V.1. If $c[x]_c$ has opinion 1 at time t, then $Z_t^{c[x]_c,t} = 0$ so that from well-known properties of random walks, t should be of order c^2. The probability for the voter model to survive a time of order c^2 is of order p_{c^2}, and conditionally on that event, the rescaled voter model converges to super-Brownian motion under its excursion measure. Then, formula (87) is exactly what we need to conclude in the case $2 \leq d \leq 3$. Also, informally (and not rigourously), one should expect that for $d \geq 4$, the probability of hitting $c[x]_c$ should be of order $p_{c^2} \times \mathbb{N}_0(Y \text{ hits } B(x, 1/c)) \approx A_d(x) \times \phi_d(c)^{-1}$, where $A_d(x)$ is a constant depending only on d and $|x|$ (see [**DIP 89**]).

In order to use the dual process, we will need a few well-known facts about random walks, and then some estimates for coalescing random walks.

2.4. Random walks with jump kernel p. We denote by $(Z_t, t \geq 0)$ a continuous-time random walk on $\mathbb{Z}^d$ with jump kernel p and exponential holding times with parameter 1. For $x \in \mathbb{Z}^d$, Z starts from x under the probability measure $\mathbb{P}_x$. For $x, y \in \mathbb{Z}^d, t \geq 0$ we let

$$q_t(x, y) = q_t(y - x) := \mathbb{P}_x(Z_t = y)$$

be the transition kernel of our random walk. For $x, y \in \mathbb{R}^d$ and $t > 0$ let

$$p_t(x, y) = p_t(x - y) := (2\pi\sigma^2 t)^{-d/2} \exp\left(-\frac{|x - y|^2}{2\sigma^2 t}\right)$$

be the transition density of d-dimensional Brownian motion. We denote by P_t the associated semigroup. For $d \geq 3$ and $x \in \mathbb{R}^d \setminus 0$, we also denote by $G(x)$ the Green function associated with p :

$$G(x) = \int_0^\infty p_s(x) ds = c_d |x|^{2-d}.$$

The asymptotic behaviour of $q_t(y)$ as $t \to \infty$ is given by standard local limit theorems (see [**Sp 76**], and [**La 91**] for an equivalent statement for discrete random walks).

THEOREM V.3. *If q and p are defined as above,*

$$\lim_{t \to \infty} \sup_{y \in \mathbb{Z}^d} \left| t^{d/2} q_t(y) - p_1(y t^{-1/2}) \right| = 0.$$

We will also need an upper bound on the transition kernel q that is valid for any $t \geq 1/2$:

LEMMA V.1. *There exist two positive constants a_1, a_2 such that for every $t \geq 1/2$, $y \in \mathbb{Z}^d$,*

$$q_t(y) \leq \frac{a_1}{t^{d/2}} \exp\left(-\frac{a_2 |y|}{\sqrt{t}}\right).$$

For the reader's convenience, we provide a short proof of Lemma V.1 in Section 5. For $t > 0$ and $y \in \mathbb{R}^d$ let us define

$$f_t(y) := \frac{a_1}{t^{d/2}} \exp\left(-\frac{a_2|y|}{\sqrt{t}}\right).$$

We also set for $t > 0$ and $y \in \mathbb{R}^d$

$$\tilde{a}_1 := 2a_1;\ \tilde{a}_2 := a_2/4; \qquad \tilde{f}_t(y) := \frac{\tilde{a}_1}{t^{d/2}} \exp\left(-\frac{\tilde{a}_2|y|}{\sqrt{t}}\right);$$
$$\hat{a}_1 := 4a_1;\ \hat{a}_2 := a_2/8; \qquad \hat{f}_t(y) := \frac{\hat{a}_1}{t^{d/2}} \exp\left(-\frac{\hat{a}_2|y|}{\sqrt{t}}\right),$$

so that $f_t(x) \leq \tilde{f}_t(x) \leq \hat{f}_t(x)$. We need to control integrals of these functions. Note that, for $x \neq 0$, the supremum of the function $t \to \hat{f}_t(x)$ is reached at $t_0 = \frac{|x|^2\hat{a}_2^2}{d^2}$. Let us introduce for $r > 0$

$$\psi_2(r) = 2\ln(r \vee e),$$
$$\psi_d(r) = r^{d-2} \text{ if } d \geq 3.$$

We then observe that for $T > 0$, there exists a constant L_0 depending only on d and T such that for any $x \in \mathbb{R}^d \setminus 0$,

$$\int_0^T f_t(x)dt \leq \int_0^T \tilde{f}_t(x)dt \leq \int_0^T \hat{f}_t(x)dt \leq L_0\psi_d\left(|x|^{-1}\right). \tag{93}$$

Furthermore, whenever $|x| \geq \frac{d\sqrt{T}}{\hat{a}_2}$ we have

$$\int_0^T \hat{f}_t(x)dt \leq \hat{a}_1 T^{1-d/2} \exp(-\frac{\hat{a}_2|x|}{\sqrt{T}}). \tag{94}$$

Finally, when $d = 2$, the integral $\int_0^T f_t(x)dt$ diverges when $T \to \infty$, but, when $d \geq 3$, there exist a constant K_0 depending only on d such that

$$\int_0^\infty \hat{f}_t(x)dt \leq K_0\psi_d\left(|x|^{-1}\right) = K_0|x|^{2-d}. \tag{95}$$

We also need an exponential bound on the probability for a random walk with jump kernel p to escape $B(0, A\sqrt{t})$ before time t. As a consequence of Lemma V.1 and Doob's maximal inequality applied to a suitable exponential martingale of the random walk, there exist positive constants κ_1, κ_2 such that for any $t \geq 1/2$, for any $A > 0$,

$$\mathbb{P}_0(\sup_{s\in[0,t]} |Z_s| \geq A\sqrt{t}) \leq \kappa_1 \exp(-\kappa_2 A). \tag{96}$$

We may and will assume that the constant a_2 in Lemma V.1 is such that $\kappa_2 \geq 4a_2$.

We then deduce easy consequences of Theorem V.3 and Lemma V.1. From Theorem V.3, we obtain, for $x \neq 0$ and $s > 0$,

$$c^d q_{c^2s}(c[x]_c) = s^{-d/2}(c^2s)^{d/2}q_{c^2s}(c[x]_c) \underset{c\to\infty}{\longrightarrow} s^{-d/2}p_1\left(\frac{x}{\sqrt{s}}\right) = p_s(x). \tag{97}$$

On the other hand, using (96), we get

$$c^d \int_0^{c^{-2}} q_{c^2s}(c[x]_c)ds \leq \kappa_1 c^{d-2} \exp\left(-\kappa_2 c[x]_c\right) \underset{c\to\infty}{\longrightarrow} 0,$$

whereas, from Lemma V.1, for any $s \geq c^{-2}$ we have

$$c^d q_{c^2 s}(c[x]_c) \leq f_s([x]_c). \tag{98}$$

We can use (97) and dominated convergence to deduce that for $x \neq 0$ and $T > 0$ we have

$$c^d \int_0^T q_{c^2 s}(c[x]_c) ds \underset{c\to\infty}{\longrightarrow} \int_0^T p_s(x) ds. \tag{99}$$

By a similar argument, we obtain, for any $T > 0$, $y \in \mathbb{Z}^d$,

$$\int_0^T q_{c^2 s}(y) ds \leq c^{-2}\kappa_1 \exp(-\kappa_2 |y|) + c^{-d} \int_{c^{-2}}^T f_s(y/c) ds.$$

Using (93), (94), it is then easy to establish that there exist constants L_0, L_0', depending only on T and d, such that for any $c \geq 1$, we have

$$\int_0^T q_{c^2 s}(y) ds \leq \begin{cases} L_0 c^{-d} \psi_d(c) & \text{if } y = 0, \\ L_0 c^{-d} \psi_d\left(c|y|^{-1}\right) & \text{if } y \in \mathbb{Z}^d \setminus 0, \\ L_0 c^{-d} \exp\left(-L_0' \frac{|y|}{c}\right) & \text{if } y \in \mathbb{Z}^d, |y| > c. \end{cases} \tag{100}$$

We now discuss some preliminary results on coalescing random walks.

2.5. Preliminary results on coalescing random walks. Consider two independent copies Z^1, Z^2 of the random walk Z with transition kernel q, starting respectively at points $y_1, y_2 \in \mathbb{Z}^d$ under the probability measure $\mathbb{P}_{y_1,y_2}$. The time at which Z^1 and Z^2 first meet is the stopping time $T_1 = \inf\{t \geq 0 : Z_t^1 = Z_t^2\}$. We will need the following result. The first bound below holds for $d \geq 3$ (recall that when $d \geq 3$, $\psi_d(r) = r^{d-2}$), the second one holds for $d = 2$ (recall $\psi_2(r) = 2\ln(r \vee e)$).

LEMMA V.2. *Let $d \geq 2$ and $T > 0$. There exists a positive constant L_1 depending only on T and d such that for any $x \in \mathbb{R}^d \setminus 0$, for any $c \geq 1 \vee |x|^{-2}$ and for any $y \in \mathbb{Z}^d \setminus 0$,*

$$\begin{cases} \textit{if } d \geq 3, & c^d \psi_d(|y|) \quad \int_0^T dt \mathbb{P}_{0,y}\left[T_1 \leq c^2 t, Z_{c^2 t}^1 = c[x]_c\right] \leq L_1 \ \psi_d(|x|^{-1}), \\ \textit{if } d = 2, & c^2 \frac{\psi_2(|y|)}{\psi_2(c/|y|)} \int_0^T dt \mathbb{P}_{0,y}\left[T_1 \leq c^2 t, Z_{c^2 t}^1 = c[x]_c\right] \leq L_1 \psi_2(|x|^{-1}). \end{cases}$$

We postpone the proof of this result to Section 5.

3. Upper bound

In the case $d \geq 5$, the upper bound of Theorem V.1 follows from the next proposition.

PROPOSITION V.1. *Let $d \geq 5, x \in \mathbb{R}^d$. For c large enough*

$$P(T_{c[x]_c} < \infty) \leq 2ec^{2-d} G(x).$$

In the case $d \leq 3$, we will argue under $P^*_{c^2 T}$ and use Theorem V.2 to establish the following sharp asymptotic upper bound. This bound also holds when $d \geq 4$ but is not sharp in that case.

PROPOSITION V.2. *Let $d \geq 2$, $T > 0, x \in \mathbb{R}^d \setminus \{0\}$,*

$$\limsup_{c\to\infty} P^*_{c^2 T}(T_{c[x]_c} < \infty) \leq \mathbb{N}_0^{(T)} \ (x \in \mathcal{R}). \tag{101}$$

In the cases $d=2$ or $d=3$, we will see in Section 3.3 that Proposition V.2 implies the asymptotic upper bound in Theorem V.1. Notice that the right-hand side of (101) is 0 if $d \geq 4$. We begin with the proof of Proposition V.1, which only requires very simple arguments.

3.1. The case $d \geq 5$**.** Fix $x \in \mathbb{R}^d \setminus \{0\}$. Proving Proposition V.1 reduces to establishing the following two results :

$$E\left[\int_0^\infty ds \mathbf{1}_{\{c[x]_c \in \xi_s^0\}}\right] \underset{c\to\infty}{\sim} c^{2-d} \int_0^\infty ds p_s(x), \tag{102}$$

$$P\left(T_{c[x]_c} < \infty\right) \leq eE\left[\int_0^\infty ds \mathbf{1}_{\{c[x]_c \in \xi_s^0\}}\right]. \tag{103}$$

Let us fix $T>0$, and observe that

$$\begin{aligned} E\left[\int_0^{c^2T} ds \mathbf{1}_{\{c[x]_c \in \xi_s^0\}}\right] &= c^2 E\left[\int_0^T ds \mathbf{1}_{\{c[x]_c \in \xi_{c^2 s}^0\}}\right] \\ &= c^2 \int_0^T ds P\left(Z_{c^2 s}^{c[x]_c, c^2 s} = 0\right) = c^2 \int_0^T ds q_{c^2 s}(c[x]_c) \\ &\underset{c\to\infty}{\sim} c^{2-d} \int_0^T p_s(x) ds, \end{aligned} \tag{104}$$

where the asymptotics at the last line come from (99). Furthermore, using (98), we have similarly

$$\begin{aligned} E\left[\int_{c^2T}^\infty \mathbf{1}_{\{c[x]_c \in \xi_s^0\}} ds\right] &= c^2 \int_T^\infty q_{c^2 s}(c[x]_c) ds \\ &\leq c^{2-d} \int_T^\infty f_s([x]_c) ds, \end{aligned}$$

and since $d \geq 3$, $\int_T^\infty f_s([x]_c)ds$ goes to 0 as $c \to \infty$. Thus from (104), we obtain (102). Let us now prove (103).

When $T_{c[x]_c} < \infty$, denote by N the numbers of arrows pointing towards $c[x]_c$ in the time interval $(T_{c[x]_c}, T_{c[x]_c}+1]$. Under $P\left(.\middle|T_{c[x]_c} < \infty\right)$, N is a Poisson variable with parameter 1. It follows that

$$P(T_{c[x]_c} < \infty) = eP(T_{c[x]_c} < \infty, N=0).$$

Furthermore, on the event $\{N=0\}$ we have $c[x]_c \in \xi_s^0$ for every $s \in [T_{c[x]_c}, T_{c[x]_c}+1]$. Hence,

$$E\left[\int_0^\infty ds \mathbf{1}_{\{c[x]_c \in \xi_s^0\}}\right] \geq P(T_{c[x]_c} < \infty, N=0).$$

This completes the proof of (103), and of Proposition V.1. □

3.2. Proof of Proposition V.2. Let $d \geq 2$ and fix $T>0$, $x \in \mathbb{R}^d \setminus 0$, and $\eta \in (0, |x|/2)$. Recall the notation m_N from Section 2.2. We have for any $\delta > 0$, $\varepsilon > 0$:

$$\begin{aligned} P_{c^2T}^*(T_{c[x]_c} < \infty) \leq\ & P_{c^2T}^*\left[\int_0^\infty ds \mathbf{1}_{\{X_s^0(\overline{B}(cx,\eta c)) \geq \delta m_{c^2}\}} \geq \varepsilon c^2\right] \\ & + P_{c^2T}^*\left[\int_0^\infty ds \mathbf{1}_{\{X_s^0(\overline{B}(cx,\eta c)) \geq \delta m_{c^2}\}} < \varepsilon c^2, T_{c[x]_c} < \infty\right]. \end{aligned} \tag{105}$$

Intuitively, when c tends to infinity, the second term of the sum above should remain small when ε and δ are small enough, while the first term, using the invariance principle, should be bounded by a corresponding rescaled quantity under $\mathbb{N}_0^{(T)}$. Let us be more precise. Using rescaling, the first term of the sum in the right-hand side of (105) is equal to

$$P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right].$$

It is easy to see that for any $A>0$, the set

$$\left\{\omega\in D(\mathbb{R}_+,M_F(\mathbb{R}^d)):\int_0^A ds\mathbf{1}_{\{\omega_s(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right\}$$

is closed for the Skorohod J_1 topology. Then, Theorem V.2 implies that

$$\begin{aligned}\limsup_{c\to\infty}P^*_{c^2T}\left[\int_0^A ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]&\leq\mathbb{N}_0^{(T)}\left[\int_0^A ds\mathbf{1}_{\{Y_s(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]\\&\leq\mathbb{N}_0^{(T)}\left[Y\text{ hits }\overline{B}(x,\eta)\right].\end{aligned}$$

Furthermore, we have, for $A\geq T$,

$$P^*_{c^2T}\left[\int_A^\infty ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]\leq P^*_{c^2T}(X_A^{c^2,0}\neq0)=\frac{p_{c^2A}}{p_{c^2T}},$$

which goes to 0 as $A\to\infty$. Hence, we obtain for every $\delta>0,\epsilon>0$,

$$\limsup_{c\to\infty}P^*_{c^2T}\left[\int_0^\infty ds\mathbf{1}_{\{X_s^{c^2,0}(\overline{B}(x,\eta))\geq\delta\}}\geq\varepsilon\right]\leq\mathbb{N}_0^{(T)}\left[Y\text{ hits }\overline{B}(x,\eta)\right].\tag{106}$$

To control the second term of the sum in the right-hand side of (105), we will use the following argument. When c is large and point $c[x]_c$ is hit by opinion 1, then with arbitrarily high probability, a sufficient number (of order m_{c^2}) of its neighbors (at distance less than ηc) should also be hit by opinion 1 during a certain time interval (with length of order c^2).

We will prove a somewhat more general result, which will be valid uniformly over all points in ξ_t^0, with the restriction that t should be at least of order c^2.

Lemma V.3. *Let $T>0,\rho>0$, $\eta>0$ be fixed. We can find $\varepsilon_0>0$ so that for any $\varepsilon\in(0,\varepsilon_0]$, there exists $\delta>0$ such that for c sufficiently large,*

$$P^*_{c^2T}\left(\exists t\geq4\varepsilon c^2\ \exists x\in\xi_t^0:\inf_{s\in[t-3\varepsilon c^2,t-2\varepsilon c^2]}|\xi_s^0\cap\overline{B}(x,\eta c)|<\delta m_{c^2}\right)\leq\rho.\tag{107}$$

We will also need a useful exponential bound on the probability for the voter model to escape a ball of radius $A\sqrt{\alpha}$ before time 2α :

Lemma V.4. *There exists constants $K_1>0,K_2>0$ such that for any $\alpha>1$, for any $A>0$,*

$$P^*_\alpha\left(\sup_{t\leq2\alpha}\sup_{x\in\xi_t^0}|x|>A\sqrt{\alpha}\right)\leq K_1\exp(-K_2A).\tag{108}$$

Let us postpone the proofs of Lemma V.3 and Lemma V.4, and finish the proof of Proposition 2. Recall $x, T, \eta \in (0, |x|/2)$ have been fixed. Notice that, when c is large enough, $\overline{B}(c[x]_c, \eta c/2) \subset \overline{B}(cx, \eta c)$. Thus,

$$\begin{aligned} &P^*_{c^2T}\left[\int_0^\infty ds \mathbf{1}_{\{X^0_s(\overline{B}(cx,\eta c))\geq \delta m_{c^2}\}} < \varepsilon c^2, 4\varepsilon c^2 \leq T_{c[x]_c} < \infty\right] \\ &\leq P^*_{c^2T}\left[\int_0^\infty ds \mathbf{1}_{\{X^0_s(\overline{B}(c[x]_c,\eta c/2))\geq \delta m_{c^2}\}} < \varepsilon c^2, 4\varepsilon c^2 \leq T_{c[x]_c} < \infty\right]. \end{aligned}$$

Hence, using Lemma V.3, for any $\rho > 0$, we can choose $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists $\delta > 0$ such that for c large enough,

$$P^*_{c^2T}\left[\int_0^\infty ds \mathbf{1}_{\{X^0_s(\overline{B}(c[x]_c,\eta c/2))\geq \delta m_{c^2}\}} < \varepsilon c^2, 4\varepsilon c^2 \leq T_{c[x]_c} < \infty\right] \leq \rho. \tag{109}$$

Furthermore, provided $2\varepsilon \leq T$, we have

$$P^*_{c^2T}\left[T_{c[x]_c} < 4\varepsilon c^2\right] \leq \frac{p_{2\varepsilon c^2}}{p_{c^2T}} P^*_{2\varepsilon c^2}\left[T_{c[x]_c} < 4\varepsilon c^2\right].$$

If c is sufficiently large, we can thus use (90) and the fact that $c|[x]_c| \geq c|x|/\sqrt{2}$, then Lemma V.4 with $\alpha = 2\varepsilon c^2$ and $A = \frac{|x|}{2\sqrt{\varepsilon}}$ to get

$$\begin{aligned} P^*_{c^2T}\left[T_{c[x]_c} < 4\varepsilon c^2\right] &\leq \kappa \frac{T}{2\varepsilon} P^*_{2\varepsilon c^2}\left(\sup_{t\leq 4\varepsilon c^2} \sup_{y\in\xi^0_t} |y| > \frac{c|x|}{\sqrt{2}}\right) \\ &\leq K_1 \kappa \frac{T}{2\varepsilon} \exp\left(-\frac{K_2|x|}{2\sqrt{\varepsilon}}\right). \end{aligned}$$

Combining (105), (106), (109) and the last inequality now yields

$$\limsup_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty) \leq \mathbb{N}^{(T)}_0\left[Y \text{ hits } \overline{B}(x,\eta)\right] + \rho + K_1\kappa\frac{T}{2\varepsilon}\exp\left(-\frac{K_2|x|}{2\sqrt{\varepsilon}}\right),$$

for any $\rho > 0$ and $\varepsilon \in (0, \varepsilon_0(\eta, \rho)]$. By letting ε and then ρ go to 0, we get

$$\limsup_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty) \leq \mathbb{N}^{(T)}_0\left[Y \text{ hits } \overline{B}(x,\eta)\right]. \tag{110}$$

Our reasonning is valid for any $\eta \in (0, |x|/2)$. Thus, letting η go to 0 in (110) finishes the proof of Proposition V.2. □

It remains to prove Lemma V.4 and Lemma V.3. We start with the proof of Lemma V.4, since it will appear to be a key tool in the proof of Lemma V.3.

3.2.1. *Proof of Lemma V.4.* Let us first outline the proof and summarize the intermediate results. We need to discretize the time scale. Introduce the integer

$$N := \min\{n \in \mathbb{N} : \alpha 2^{-n} < 1\} = \left\lfloor \frac{\ln(\alpha)}{\ln(2)} \right\rfloor,$$

e and the time intervals

$$B_n := [(n-1)2^{-N-1}\alpha, n2^{-N-1}\alpha], \quad n \in [\![1, 2^{N+2}]\!].$$

Let us introduce the set of points having, for some odd $n \in [\![1, 2^{N+2}]\!]$, opinion 1 at a time belonging to B_n, and descendants at time $(n+1)2^{-N-1}\alpha$:

$$\Xi_{N+1} := \bigcup_{\substack{n=1\\ n \text{ odd}}}^{2^{N+1}} \bigcup_{u\in B_n} \Omega_u^{(n+1)2^{-N-1}\alpha}.$$

Informally, our interest in this set Ξ_{N+1} comes from the fact that if $x \in \bigcup_{t\leq 2\alpha} \xi_t^0$, a "close" ancestor of x belongs to Ξ_{N+1}, and hence, Ξ_{N+1} should not be too far from $\bigcup_{t\leq 2\alpha} \xi_t^0$. More precisely, for $t<1$, set $u_t = 0$, and for $t \in [1, 2\alpha]$, let us choose

$$u_t \in \left[t - \frac{\alpha}{2^N}, t - \frac{\alpha}{2^{N+1}}\right] \bigcap \bigcup_{\substack{n=1 \\ n \text{ odd}}}^{2^{N+1}} B_n.$$

We have $t - u_t \leq \alpha 2^{-N} < 1$, and, if $x \in \xi_t^0$ for some $t \in [0, 2\alpha]$, the ancestor of x at time u_t indeed belongs to Ξ_{N+1}.

We will show that, under P_{α^*}, Ξ_{N+1} intersects $B\left(0, \frac{A}{2}\sqrt{\alpha}\right)$ with a probability which decays exponentially with A.

Lemma V.5. *There exist positive constants K_3, K_4 such that for any $A > 0$, for any $\alpha > 1$,*

$$P_\alpha^* \left(\Xi_{N+1} \not\subset B(0, \frac{A}{2}\sqrt{\alpha})\right) \leq K_3 \exp(-K_4 A). \tag{111}$$

Then, we will argue that the probability under P_α^* for $\bigcup_{t\leq 2\alpha} \xi_t^0$ to escape $B(0, A\sqrt{\alpha})$ and simultaneously to have $\Xi_{N+1} \subset B(0, \frac{A}{2}\sqrt{\alpha})$ also decays exponentially with A. This is seen below as a consequence of the following result.

Lemma V.6. *There exist positive constants K_5, K_6 such that for any $A > 0$,*

$$P\left(\exists t \in [0,1]\ \exists x \in \mathbb{Z}^d \setminus B(0,A)\ \exists y \in B(0, \frac{A}{2}) : (t,x) \searrow (0,y)\right) \leq K_5 \exp(-K_6 A).$$

Let us postpone the proofs of Lemmas V.5 and V.6 and show how Lemma V.4 is deduced from these two results. Introduce the event

$$\mathcal{A} := \left\{\exists t \leq 2\alpha\ \exists x \in \xi_t^0 : |x| > A\sqrt{\alpha}\right\} \bigcap \left\{\Xi_{N+1} \subset B(0, \frac{A}{2}\sqrt{\alpha})\right\}.$$

Clearly, $\mathcal{A} = \bigcup_{n=1}^{2^{N+2}} \mathcal{A}_n$, where

$$\mathcal{A}_n := \left\{\exists t \in B_n\ \exists x \in \xi_t^0\ :\ |x| > A\sqrt{\alpha}\right\} \bigcap \left\{\Xi_{N+1} \subset B(0, \frac{A}{2}\sqrt{\alpha})\right\}.$$

As we noticed earlier, when $x \in \xi_t^0$, the ancestor of x at time u_t belongs to Ξ_{N+1}. Hence, using the Markov property at time u_t, we get, for every $n \in [\![1, 2^{N+2}]\!]$,

$$P(\mathcal{A}_n) \leq P\left(\exists s \in \left[0, \frac{\alpha}{2^N}\right]\ \exists x \in \mathbb{Z}^d \setminus B(0, A\sqrt{\alpha})\ \exists y \in B(0, \frac{A}{2}\sqrt{\alpha})\ :\ (s,x) \searrow (0,y)\right),$$

where we used that $t - u_t \leq \alpha 2^{-N}$ Since $\alpha 2^{-N} < 1$, we can use Lemma V.6 to obtain

$$P(\mathcal{A}) \leq 2^{N+2} K_5 \exp(-K_6 A\sqrt{\alpha}).$$

From the definition of N, $2^{N+2} \leq 8\alpha$, hence, there exists positive constants K_1', K_2' such that $P_\alpha^*(\mathcal{A}) \leq K_1' \exp(-K_2' A)$. This fact and Lemma V.5 imply Lemma V.4. □

It now remains to prove Lemmas V.5 and V.6. We first establish Lemma V.6.
Proof of Lemma V.6 : Fix $x \in \mathbb{Z}^d \setminus B(0, A)$. There is a Poisson number n_x with parameter 1 of arrows pointing towards x during the time interval $[0,1]$. Denote by $1 \geq T_1 > T_2 > ... > T_{n_x} \geq 0$ the times at which these arrows occur and by

$z_1, z_2, ..., z_{n_x}$ the respective origins of these arrows. We also set $T_i = 0$ when $i > n_x$. For $t \in [0,1]$ and $y \in B(0, A/2)$, a path up $(0,y) \nearrow (x,t)$ has to "follow" one of the n_x arrows pointing towards x in the time interval $[0,1]$, say the ith one at time T_i, in this case we then have $(z_i, T_i) \searrow (y, 0)$.

For $t \in [0,1]$, let us define $\mathcal{G}_t$ the σ-field which is generated by the random sets $(\Lambda(x,y) \cap [1-t, 1])$ for all $x, y \in \mathbb{Z}^d$.

The times $1 - T_i, i \in \mathbb{N}$ are stopping times for the filtration $(\mathcal{G}_t)_{t\in[0,1]}$, and conditionally on $\{n_x = k\}$, the points $z_i, 1 \le i \le k$ are located independently according to $p(x, .)$. In particular, using the exponential moments assumption on p, there exist positive κ_1', κ_2' such that for any $1 \le i \le k$,

$$P\left(z_i \in B\left(0, \frac{3|x|}{4}\right) \middle| n_x = k\right) \le \kappa_1' \exp(-\kappa_2'|x|). \tag{112}$$

For $i \in [\![1, k]\!]$, let us define $(\tilde{Z}_s^{x,T_i})_{0\le s\le T_i}$ as follows

– for $0 < s \le T_i$, $\tilde{Z}_s^{x,T_i} = Z_s^{x,T_i}$

– $\tilde{Z}_0^{x,T_i} = z_i$.

For $t > 0$, conditionally on $\{T_i = t\}$, $(\tilde{Z}_s^{x,T_i})_{0\le s\le t}$ is a rate 1 random walk with jump kernel p started from z_i, and is thus distributed as $(Z_s)_{0\le s\le t}$ under $\mathbb{P}_{z_i}$. Furthermore, using (96), we have for any $z \in \mathbb{Z}^d \setminus B\left(0, \frac{3|x|}{4}\right)$

$$\mathbb{P}_z\left(\exists s \in [0,1] : Z_s \in B\left(0, \frac{|x|}{2}\right)\right) \le \kappa_1 \exp\left(-\kappa_2 \frac{|x|}{4}\right). \tag{113}$$

Combining (112) and (113), we see that there exist positive constants K_5', K_6' such that for any $x \in \mathbb{Z}^d \setminus B(0, A)$, for any $k \in \mathbb{N}$

$$P\left(\exists i \in [\![1, k]\!] : Z_{T_i}^{x,T_i} \in B\left(0, \frac{A}{2}\right) \;\middle|\; n_x = k\right) \le K_5' k \exp\left(-K_6'|x|\right).$$

We now get

$$\begin{aligned} &P\left(\exists t \in [0,1]\ \exists x \in \mathbb{Z}^d \setminus B(0,A)\ \exists y \in B\left(0, \frac{A}{2}\right)\ :\ (t,x) \searrow (0,y)\right) \\ &\le \sum_{x\in\mathbb{Z}^d\setminus B(0,A)} \sum_{k=1}^{\infty} P\left(n_x = k\ ;\ \exists i \in [\![1,k]\!]\ :\ Z_{T_i}^{x,T_i} \in B(0, \frac{A}{2})\right) \\ &\le \sum_{x\in\mathbb{Z}^d\setminus B(0,A)} \frac{1}{e} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} K_5' \exp\left(-K_6'|x|\right). \end{aligned}$$

Lemma V.6 follows. □

To prove Lemma V.5, we need the following key result.

Lemma V.7. *Let $t \ge 0, s > r > 0$ and $A > A' \ge 0$.*

$$P\left(\bigcup_{u\in[0,t]} \Omega_u^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]} \Omega_u^{t+s} \not\subseteq B(0,A)\right) \le p_{s-r}\kappa_1 \exp\left(-\kappa_2 \frac{A-A'}{\sqrt{r}}\right).$$

Proof of Lemma V.7 : The event $\left\{\Omega_t^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]} \Omega_u^{t+s} \not\subseteq B(0,A)\right\}$ considered in Lemma V.7 is contained in the event that there exists a point $z \in \xi_{t+r}^0$ having descendants at time $t+s$, such that the ancestor of z at time t belongs to $B(0,A')$, and moreover, z has an ancestor in $B(0,A)^c$ at a time belonging to $[t,t+r]$. More precisely, using duality over the time interval $[0,t+r]$, and then decomposing over all possible values of the point z,

$$\begin{aligned}
& P\left(\Omega_t^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]} \Omega_u^{t+s} \not\subseteq B(0,A)\right) \\
& \leq P\left(\exists z \in \xi^0_{t+r} : \hat{\xi}^{z,t+r}_{s-r} \neq \emptyset, \sup_{u\in[0,r]} |Z_u^{z,t+r}| > A, |Z_r^{z,t+r}| \leq A', Z^{z,t+r}_{t+r} = 0\right) \\
(114) \quad & \leq \sum_{z\in\mathbb{Z}^d} P\left(\hat{\xi}^{z,t+r}_{s-r} \neq \emptyset, \sup_{u\in[0,r]} |Z_u^{z,t+r}| > A, |Z_r^{z,t+r}| \leq A', Z^{z,t+r}_{t+r} = 0\right).
\end{aligned}$$

Using the Markov property at time $(t+r)$, we obtain that the quantity in the right-hand side of (114) is equal to

$$\begin{aligned}
& p_{s-r} \sum_{z\in\mathbb{Z}^d} \mathbb{P}_z\left(\sup_{u\in[0,r]} |Z_u| > A, |Z_r| \leq A', Z_{t+r} = 0\right) \\
& = p_{s-r} \sum_{z\in\mathbb{Z}^d} \mathbb{P}_0\left(|Z_r| \leq A', \sup_{u\in[t,t+r]} |Z_u| > A, Z_{t+r} = z\right) \\
(115) \quad & \leq p_{s-r}\mathbb{P}_0\left(|Z_r| \leq A', \sup_{u\in[t,t+r]} |Z_u| > A\right),
\end{aligned}$$

where, at the second line above, we used a time-reversal argument together with the symmetry assumption we made on the jump kernel p. From (114), (115) and the Markov property for the random walk at time t, we now obtain

$$\begin{aligned}
& P\left(\Omega_t^{t+s} \subset B(0,A'), \bigcup_{u\in[t,t+r]} \Omega_u^{t+s} \not\subseteq B(0,A)\right) \\
& \leq p_{s-r}\mathbb{P}_0(\sup_{u\in[0,r]} |Z_u| > A - A'),
\end{aligned}$$

and we conclude using (96). $\square$

<u>Proof of Lemma V.5</u> : Let us first note that we only need to establish the existence of positive K_3', K_4' such that (111) holds for any $A \geq 1$ and $\alpha > 1$. Indeed, Lemma V.5 will follow from taking $K_3 = K_3' \vee \exp(K_4'), K_4 := K_4'$. For $p \in [\![0, N+1]\!]$, let us introduce the sets

$$\Xi_p := \bigcup_{\substack{n=1 \\ n \text{ odd}}}^{2^{p+1}} \bigcup_{u\in[(n-1)2^{-p}\alpha, n2^{-p}\alpha]} \Omega_u^{(n+1)2^{-p}\alpha}.$$

For convenience, we also set $\Xi_{-1} := \emptyset$. In the case $p = N+1$, this is of course consistent with our definition of Ξ_{N+1}. For $p \geq 0$, let $A_p := \frac{A}{14}\sum_{i=0}^p 2^{-i/4}$ and set $A_{-1} = 0$. so that for any $k \in \mathbb{N}$, $A_k \leq \frac{A}{2}$. For $p \in [\![0, N+1]\!]$, let

$$\mathcal{E}_p := \left\{\Xi_{p-1} \subset B(0, A_{p-1}\sqrt{\alpha}), \Xi_p \not\subseteq B(0, A_p\sqrt{\alpha})\right\}.$$

Note that $\mathcal{E}_p$ is a subset of

$$\mathcal{F}_p := \left\{ \exists n \in [\![1, 2^{p+1}]\!], n \text{ odd } : \Omega_{(n-1)2^{-p}\alpha}^{(n+1)2^{-p}\alpha} \subset B(0, A_{p-1}\sqrt{\alpha}), \bigcup_{u \in (n-1)2^{-p}\alpha, n2^{-p}\alpha} \Omega_u^{(n+1)2^{-p}\alpha} \not\subseteq B(0, A_p\sqrt{\alpha}) \right\}.$$

Hence,

$$P_\alpha^*\left(\Xi_{N+1} \not\subseteq B(0, \frac{A}{2}\sqrt{\alpha})\right) \leq \sum_{p=0}^{N+1} P_\alpha^*\left(\mathcal{E}_p\right) \leq \sum_{p=0}^{N+1} P_\alpha^*\left(\mathcal{F}_p\right) \tag{116}$$

From Lemma V.7, we obtain

$$P_\alpha^*\left(\mathcal{F}_p\right) \leq 2^p p_\alpha^{-1} p_{2^{-p}\alpha} \exp\left(-\kappa_2 2^{p/2}(A_p - A_{p-1})\right) \leq \kappa 2^{2p} \exp\left(-\kappa_2 2^{p/4}\frac{A}{14}\right),$$

where we used (90) and our definition of the numbers $A_p, p \geq -1$. From (116) and the last inequality, elementary arguments then give Lemma V.5. □

This completes the proof of Lemma V.4. To finish the one of Proposition V.2, it remains to establish Lemma V.3.

3.2.2. *Proof of Lemma V.3.* Fix $T > 0, \rho > 0, \eta > 0$. Recall K_1, K_2 are the constants appearing in the statement of Lemma V.4. We can choose $\varepsilon_0 \in (0,1)$ so that for any $\varepsilon \in (0, \varepsilon_0]$,

$$K_1 \exp\left(-K_2 \frac{\eta}{4\sqrt{\varepsilon}}\right) \leq \frac{\rho^2\varepsilon^2}{8T^2}.$$

Let us now fix $\varepsilon \in (0, \varepsilon_0]$. We can then choose $\delta > 0$ small enough so that

$$\mathbb{N}_0^{(4\varepsilon)}\left[\inf_{t \in [\varepsilon, 3\varepsilon]} Y_t(1) \leq \delta\right] \leq \frac{\rho^2\varepsilon^2}{8T^2}.$$

The reasons for our choices of ϵ_0 and δ will become clear in the following.

We first need to reduce the problem to a finite time interval. Notice that $P_{c^2T}^*(\xi_{c^2\frac{4T}{\rho}} \neq \emptyset) \leq p_{c^2T}^{-1} p_{c^2\frac{4T}{\rho}}$ which, using (89), is bounded by $\rho/2$ for c large enough. Thus, to establish Lemma V.3 we only need to prove that provided c is sufficiently large,

$$P_{c^2T}^*\left(\exists t \in [4\varepsilon c^2, \frac{4T}{\rho}c^2]\ \exists x \in \xi_t^0 : \inf_{s \in [t-3\varepsilon c^2, t-2\varepsilon c^2]} |\xi_s^0 \cap \overline{B}(x, \eta c)| < \delta m_{c^2}\right) \leq \frac{\rho}{2}, \tag{117}$$

Set $M := \left\lceil \frac{4T}{\rho\varepsilon} \right\rceil$. Let us discretize the time scale via introducing the levels $L_k := k\varepsilon c^2, k \in [\![0, M-4]\!]$.

We are going to establish, using Lemma V.4, that with arbitrarily high probability, when c is large enough, each point holding opinion 1 at such a level L_k and having descendants at time $L_k + 4\varepsilon c^2$ is close (at a distance less than $\eta c/2$) to all its descendants during the time interval $[L_k, L_k + 5\varepsilon c^2]$. Then, using Theorem V.2, we will prove that such a point has more than δm_{c^2} descendants in the time interval $[L_k + \varepsilon c^2, L_k + 3\varepsilon c^2]$.

Let us be more precise. We shall prove that if c is large enough,

$$P_{c^2T}^*\left(\exists k \in [\![0, M-4]\!]\ \ \exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \bigcup_{s \in [0, 5\varepsilon c^2]} \hat{\xi}_s^{y, k\varepsilon c^2} \not\subset \overline{B}(y, \eta c/2)\right) \leq \frac{\rho}{4}, \tag{118}$$

$$P^*_{c^2T}\left(\exists k \in [\![0, M-4]\!] \quad \exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\hat{\xi}_s^{y,k\varepsilon c^2}| < \delta m_{c^2}\right) \le \frac{\rho}{4}. \tag{119}$$

Let us postpone the proof of these two results and show how (117) follows from (118) and (119). Consider $t \in [4\varepsilon c^2, M\varepsilon c^2]$ and $x \in \xi_t^0$. Introduce

$$k_t := \sup\{n \in \mathbb{N} : n\varepsilon c^2 \le t - 4\varepsilon c^2\}; \quad z_x := Z^{x,t}_{t-k_t\varepsilon c^2},$$

so that $z_x \in \Omega^t_{k_t\varepsilon c^2} \subset \Omega^{(k_t+4)\varepsilon c^2}_{k_t\varepsilon c^2}$ (indeed z_x is the ancestor of $x \in \xi_t^0$ at time $k_t\varepsilon c^2$). Thus,

- using (118), with probability at least $1-\rho/4$, for any $x \in \xi_t^0$, $t \in [4\varepsilon c^2, M\varepsilon c^2]$, all descendants of z_x until time $(k_t+5)\varepsilon c^2$ belong to $\overline{B}(x, \eta c)$.
- using (119), with probability at least $1-\rho/4$, for any $x \in \xi_t^0$, $t \in [4\varepsilon c^2, M\varepsilon c^2]$, there are more than δm_c^2 descendants of z_x at every time $s \in [(k_t+1)\varepsilon c^2, (k_t+3)\varepsilon c^2]$.

Since $[t-3\varepsilon c^2, t-2\varepsilon c^2] \subset [(k_t+1)\varepsilon c^2, (k_t+3)\varepsilon c^2]$, we now deduce from the above that with probability at least $1-\rho/2$, for any $x \in \xi_t^0$, z_x has at least δm_c^2 descendants in $\overline{B}(x, \eta c)$ at every time $s \in [t-3\varepsilon c^2, t-2\varepsilon c^2]$. Assertion (117) follows.

Let us now prove (118). Let us consider $k \in [\![0, M-4]\!]$, and c large enough so that $4\varepsilon c^2 > 1$. Using the Markov property at time $k\varepsilon c^2$ and the fact that $(\hat{\xi}_s^{y,k\varepsilon c^2})_{s\ge 0}$ has the same law as $(y+\xi_s^0)_{s\ge 0}$ we get :

$$\begin{aligned}
&P\left(\exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \bigcup_{s\in[0,5\varepsilon c^2]} \hat{\xi}_s^{y,k\varepsilon c^2} \not\subset \overline{B}(y, \eta c/2)\right) \\
&\le \sum_{y\in\mathbb{Z}^d} P(y \in \xi^0_{k\varepsilon c^2}) P\left(\xi^0_{4\varepsilon c^2} \ne \emptyset, \bigcup_{s\in[0,5\varepsilon c^2]} \xi_s^0 \not\subset \overline{B}(0, \eta c/2)\right) \\
&= \sum_{y\in\mathbb{Z}^d} q_{k\varepsilon c^2}(y) p_{4\varepsilon c^2} P^*_{4\varepsilon c^2}\left(\sup_{s\le 5\varepsilon c^2}\sup_{x\in\xi_s^0} |x| > \frac{\eta c}{2}\right). \\
&\le p_{4\varepsilon c^2} K_1 \exp\left(-K_2 \frac{\eta}{4\sqrt{\varepsilon}}\right), \qquad (120)
\end{aligned}$$

where at the last line we used Lemma V.4 with $\alpha = 4\varepsilon c^2 > 1$ and $A = \eta(4\sqrt{\varepsilon})^{-1}$, and the fact that $\sum_{y\in\mathbb{Z}^d} q_{k\varepsilon c^2}(y) = 1$ from the symmetry assumption on p. Since $M - 3 \le 4T(\varepsilon\rho)^{-1}$, we deduce from the above that

$$\begin{aligned}
&P^*_{c^2T}\left(\exists k \in [\![0, M-4]\!] \ \exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \bigcup_{s\in[0,5\varepsilon c^2]} \hat{\xi}_s^{y,k\varepsilon c^2} \not\subset \overline{B}(y, \eta c/2)\right) \\
&\le p^{-1}_{c^2T} \frac{4T}{\varepsilon\rho} p_{4\varepsilon c^2} K_1 \exp\left(-K_2 \frac{\eta}{4\sqrt{\varepsilon}}\right). \qquad (121)
\end{aligned}$$

We now get (118) from (121), (89), and our choice of ε_0, provided c is taken sufficiently large.

Let us now prove (119). Fix $k \in [\![0, M-4]\!]$. Using the same arguments as in the proof of (118), we obtain

$$\begin{aligned} & P\left(\exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\hat{\xi}_s^{y,k\varepsilon c^2}| < \delta m_{c^2}\right) \\ & \leq \sum_{y\in\mathbb{Z}^d} q_{k\varepsilon c^2}(y) P\left(\inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\hat{\xi}_s^{0,k\varepsilon c^2}| < \delta m_{c^2}, \hat{\xi}_{4\varepsilon c^2}^{0,k\varepsilon c^2} \neq \emptyset\right) \\ & = p_{4\varepsilon c^2} P_{4\varepsilon c^2}^* \left(\inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\xi_s^0| < \delta m_{c^2}\right). \end{aligned} \tag{122}$$

Furthermore, by rescaling, when c is large enough so that $(m_{c^2})^{-1} m_{\varepsilon c^2} \leq 1$ (recall $\varepsilon < 1$ from our choice of ε_0), we get

$$\begin{aligned} P_{4\varepsilon c^2}^*\left(\inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\xi_s^0| < \delta m_{\varepsilon c^2}\right) & = P_{4\varepsilon c^2}^*\left(\inf_{s\in[\varepsilon, 3\varepsilon]} X_s^{c^2,0}(1) < \delta \frac{m_{\varepsilon c^2}}{m_{c^2}}\right) \\ & \leq P_{4\varepsilon c^2}^*\left(\inf_{s\in[\varepsilon, 3\varepsilon]} X_s^{c^2,0}(1) \leq \delta\right). \end{aligned} \tag{123}$$

Since the set $\{\omega \in D(\mathbb{R}_+, M_F(\mathbb{R}^d)) : \inf_{s\in[\varepsilon,3\varepsilon]} \omega_s(1) \leq \delta\}$ is closed for the Skorohod J_1 topology, Theorem V.2 implies

$$\limsup_{c\to\infty} P_{4\varepsilon c^2}^*\left(\inf_{s\in[\varepsilon,3\varepsilon]} X_s^{c^2,0}(1) \leq \delta\right) \leq \mathbb{N}_0^{(4\varepsilon)}\left[\inf_{t\in[\varepsilon,3\varepsilon]} Y_t(1) \leq \delta\right] \leq \frac{\varepsilon^2\rho^2}{8T^2},$$

by our choice of δ. Assertions (122), (123), and the above now imply

$$\begin{aligned} & \limsup_{c\to\infty} P_{c^2T}^*\left(\exists y \in \Omega_{k\varepsilon c^2}^{(k+4)\varepsilon c^2} : \inf_{s\in[\varepsilon c^2, 3\varepsilon c^2]} |\hat{\xi}_s^{y,k\varepsilon c^2}| < \delta m_{c^2}\right) \\ & \leq \limsup_{c\to\infty} p_{c^2T}^{-1} p_{4\varepsilon c^2} \frac{\varepsilon^2\rho^2}{8T^2} = \frac{\varepsilon\rho^2}{32T}, \end{aligned}$$

where we used (89) at the last line. Hence, using the fact that $M - 3 \leq 4T(\rho\varepsilon)^{-1}$, we get (119), provided c is sufficiently large. This ends the proof of Lemma V.3. □

We have thus finished the proof of the asymptotic upper bound on $P_{c^2T}^*(T_{c[x]_c} < \infty)$ (Proposition V.2). However, to complete the proof of the asymptotic upper bound for $d = 2$ or 3 in Theorem 1, we need to establish a corresponding result under the measure P. Let us briefly explain how Lemma V.4 allows us to do so.

3.3. Back to non-conditioned results. First, we shall prove a result corresponding to Lemma V.4 without conditioning upon survival.

Claim 1. - There exists a positive K_0 such that for any $\alpha > 1$, for any $A \geq 1$,

$$P\left(\sup_{t\leq 2\alpha} \sup_{x\in\xi_t^0} |x| > A\sqrt{\alpha}\right) \leq K_0 p_\alpha \exp(-K_2 A). \tag{124}$$

Proof of Claim 1 : For any $i \in [\![0, N-1]\!]$ we have

$$\begin{aligned}
&P\left(\sup_{t\le 2^{1-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)\\
&=p_{2^{-i}\alpha}P^*_{2^{-i}\alpha}\left(\sup_{t\le 2^{1-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)+P\left(\sup_{t\le 2^{-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha},\xi^0_{2^{-i}\alpha}=\emptyset\right)\\
&\le p_{2^{-i}\alpha}K_1\exp\left(-K_2A2^{i/2}\right)+P\left(\sup_{t\le 2^{-i}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha},\xi^0_{2^{-i}\alpha}=\emptyset\right),
\end{aligned}$$

where we used Lemma V.4 at the last line. It easily follows that

$$\begin{aligned}
&P\left(\sup_{t\le 2\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)\\
&\le\sum_{i=0}^{N-1}p_{2^{-i}\alpha}K_1\exp\left(-K_2A2^{i/2}\right)+P\left(\sup_{t\le 2^{1-N}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right).
\end{aligned}\tag{125}$$

Furthermore, by an easy application of Lemma V.6,

$$P\left(\sup_{t\le 2^{1-N}\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)\le 2K_5\exp\left(-K_6A\sqrt{\alpha}/2\right).$$

Thus, from (125) and (90), we obtain

$$P\left(\sup_{t\le 2\alpha}\sup_{x\in\xi^0_t}|x|>A\sqrt{\alpha}\right)\le\kappa K_1\sum_{i=0}^{N-1}2^ip_\alpha\exp\left(-K_2A2^{i/2}\right)+2K_5\exp(-K_6A\sqrt{\alpha}/2),$$

and Claim 1 follows. $\square$

Let us now finish the proof of the upper bound in Theorem V.1 in dimensions 2 and 3. As before, $x\in\mathbb{R}^d\setminus 0$ is fixed. Simply observe that, for every $T>0$,

$$P(T_{c[x]_c}<\infty)=P(\xi^0_{c^2T}=\emptyset,T_{c[x]_c}<\infty)+p_{c^2T}P^*_{c^2T}(T_{c[x]_c}<\infty).\tag{126}$$

On the one hand

$$\begin{aligned}
\phi_d(c)P(\xi^0_{c^2T}=\emptyset,T_{c[x]_c}<\infty)\quad&\le\quad\phi_d(c)P\left(\sup_{t\le c^2T}\sup_{y\in\xi^0_t}|y|\ge c|[x]_c|\right)\\
&\le\quad\phi_d(c)K_0p_{c^2T}\exp\left(-K_2\frac{|[x]_c|}{\sqrt{T}}\right),
\end{aligned}$$

where we used (124) at the last line. We can now use (89) to obtain

$$\limsup_{c\to\infty}\phi_d(c)P(\xi^0_{c^2T}=\emptyset,T_{c[x]_c}<\infty)\le\frac{K_0}{\beta_dT}\exp\left(-K_2\frac{|x|}{2\sqrt{T}}\right),$$

which goes to 0 as $T\to 0$.

On the other hand, using Proposition V.2 and (89), we get, for every $T>0$,

$$\limsup_{c\to\infty}\phi_d(c)p_{c^2T}P^*_{c^2T}(T_{c[x]_c}<\infty)\le\frac{1}{T\beta_d}\mathbb{N}^{(T)}_0(x\in\mathcal{R}),$$

and by (88), the right-hand side converges, as $T\to 0$, to

$$\frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}.$$

From (126) and the preceeding observations we get

$$\limsup_{c\to\infty} \phi_d(c)P(T_{c[x]_c} < \infty) \le \frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}, \tag{127}$$

which completes the proof of the upper bound in Theorem 1, in the case $d = 2$ or 3. We already noticed that the case $d \ge 5$ follows from Proposition 1. Finally, note that in the case $d = 4$, Proposition V.2 and a similar proof imply

$$\limsup_{c\to\infty} c^2 P(T_{c[x]_c} < \infty) = 0,$$

as we already mentioned in the introduction.

4. Lower bound.

In this section we finish the proof of Theorem 1 by establishing the required asymptotic lower bounds on $P(T_{c[x]_c} < \infty)$. We will also prove a similar result in dimension 4.

PROPOSITION V.3. *Fix $x \in \mathbb{R}^d$, $x \neq 0$.*

ROUGH LOWER BOUND : *Let $d \ge 4$. There exists a positive constant a_d depending on $|x|$ and d such that*

$$\liminf_{c\to\infty} \phi_d(c)P(\exists t \ge 0 : c[x]_c \in \xi_t^0) \ge a_d,$$

where we recall that for $d \ge 5$, $\phi_d(c) = c^{d-2}$, and $\phi_4(c) = c^2\ln(c)$.

SHARP LOWER BOUND : *Let $d = 2$ or 3. Recall $\phi_2(c) = c^2(\ln(c))^{-1}$ and $\phi_3(c) = c^2$. Then*

$$\liminf_{c\to\infty} \phi_d(c)P(\exists t \ge 0 : c[x]_c \in \xi_t^0) \ge \frac{2\sigma^2}{\beta_d}\left(2-\frac{d}{2}\right)|x|^{-2}.$$

4.1. Proof of the rough lower bound, $d \ge 4$. For $T > 0$, and $x \in \mathbb{R}^d \setminus 0$, let us introduce the random variable

$$U_T := \int_0^T \mathbf{1}_{\{c[x]_c \in \xi^0_{c^2 s}\}} ds,$$

so that $c^2 U_T$ is the occupation time of opinion 1 for the voter at $c[x]_c$ in the time interval $[0, c^2T]$.

We clearly have for any $T > 0$, $P(\exists t \ge 0 : c[x]_c \in \xi_t^0) \ge P(U_T > 0)$. Using the Cauchy-Schwarz inequality, we thus obtain

$$P(\exists t \ge 0 : c[x]_c \in \xi_t^0) \ge \frac{(E[U_T])^2}{E[(U_T)^2]}. \tag{128}$$

Hence, proving the lower bound reduces to establishing the following two estimates

$$c^d E[U_T] \underset{c\to\infty}{\longrightarrow} \int_0^T ds p_s(x), \tag{129}$$

$$\limsup_{c\to\infty} c^{2d}\phi_d(c)^{-1} E[(U_T)^2] \le K, \tag{130}$$

where K is a constant depending only on $|x|, d$ and T.

The first moment of U_T is

$$E[U_T] = \int_0^T dsP\left(Z_{c^2s}^{c[x]_c,c^2s} = 0\right) = \int_0^T dsq_{c^2s}(c[x]_c),$$

so that (129) is a consequence of (99). Let us now estimate the second moment of U_T. We have

$$\frac{1}{2}E[(U_T)^2] = \int_0^T dt \int_t^T drP\left[c[x]_c \in \xi^0_{c^2t}, c[x]_c \in \xi^0_{c^2r}\right].$$

Let us fix r and t with $0 < t < r \leq T$. Using duality over the time interval $[0, c^2r]$ and setting $s := r - t$, we see that $P\left[c[x]_c \in \xi^0_{c^2t}, c[x]_c \in \xi^0_{c^2r}\right]$ is the probability for two coalescing random walks starting at point $c[x]_c$ respectively at times 0 and c^2s, to be both located at point 0 at time c^2r. Recall the notation of Section 2.5. Using the symmetry properties of p and the Markov property for the first walk at time c^2s we get

$$\frac{1}{2}E[(U_T)^2] = \int_0^T dt \int_0^{T-t} ds \sum_{y\in\mathbb{Z}^d} q_{c^2s}(y)\mathbb{P}_{0,y}\left[T_1 \leq c^2t, Z^1_{c^2t} = c[x]_c\right]. \tag{131}$$

With a slight abuse of notation, in the remaining part of the section we use K to denote a positive constant that only depends on T and $|x|$ and may change from line to line. We suppose that $c \geq 1 \vee |x|^{-2}$ in order to use Lemma V.2. Let us set

$$I(y) := \int_0^T dt \left(\int_0^{T-t} ds\ q_{c^2s}(y)\right)\mathbb{P}_{0,y}\left[T_1 \leq c^2t, Z^1_{c^2t} = c[x]_c\right].$$

Note that $I(y)$ also depends on d, T and x, although this does not appear in our notation. From (131), we have $E[(U_T)^2] = 2\sum_{y\in\mathbb{Z}^d} I(y)$. Hence, we need to bound $I(y)$ over different regions of $\mathbb{Z}^d$, in order to control $c^{2d}(\phi_d(c))^{-1}E[(U_T)^2]$. Recall from Section 2.4 that $\psi_d(c) = c^{d-2}$ for $d \geq 4$. Using (100) twice in the case $y = 0$, and using (100) together with Lemma V.2 in the case $y \neq 0$, we get
– for $y = 0$,

$$I(0) \leq Kc^{-2d}\psi_d(c),$$

– for $y \in \mathbb{Z}^d, y \neq 0$,

$$I(y) \leq Kc^{-2d}\psi_d(c/|y|)\psi_d(|y|)^{-1},$$

– for $y \in \mathbb{Z}^d, |y| > c^2$,

$$I(y) \leq Kc^{-2d}\exp(-L'_0\sqrt{|y|})\psi_d(|y|)^{-1}.$$

Since $\phi_d(c) \geq \psi_d(c)$ when $d \geq 4$, we then obtain

$$c^{2d}(\phi_d(c))^{-1}I(0) \leq K. \tag{132}$$

Furthermore, if $y \in \mathbb{Z}^d \setminus 0, d \geq 4$, we have $\psi_d(c/|y|) = \psi_d(c)(\psi_d(|y|))^{-1}$. Hence,

$$\begin{aligned} c^{2d}(\phi_d(c))^{-1}\sum_{y\in\mathbb{Z}^d,0<|y|\leq c^2} I(y) &\leq K(\phi_d(c))^{-1}\psi_d(c)\sum_{y\in\mathbb{Z}^d,0<|y|\leq c^2}|y|^{4-2d} \\ &\leq K(\phi_d(c))^{-1}\psi_d(c)\sum_{k=1}^{c^2}k^{d-1}k^{4-2d} \leq K, \end{aligned} \tag{133}$$

Finally,

$$c^{2d}(\phi_d(c))^{-1}\sum_{y\in\mathbb{Z}^d,|y|>c^2} I(y) \le K(\phi_d(c))^{-1}\sum_{|y|>c^2}|y|^{-d+2}\exp(-L_0'\sqrt{|y|}) \underset{c\to\infty}{\longrightarrow} 0. \tag{134}$$

Since $E[(U_T)^2] = 2\sum_{y\in\mathbb{Z}^d} I(y)$, the desired result (130) follows from (132), (133) and (134). This finishes the proof of the lower bound for $d \ge 4$. $\square$

A similar proof in the case $d = 2$ or 3 would give us a rough lower bound, but we need to get sharper estimates.

4.2. Outline of the proof of the sharp lower bound, $d = 2$ or 3.

Fix $x \in \mathbb{R}^d \setminus \{0\}$. For any $T > 0$, $c > 0$, we have

$$\phi_d(c)P(T_{c[x]_c} < \infty) \ge \phi_d(c)p_{c^2T}P^*_{c^2T}(T_{c[x]_c} < \infty).$$

We deduce from (89) that for any $T > 0$, $\lim_{c\to\infty}\phi_d(c)p_{c^2T} = (\beta_d T)^{-1}$ so that

$$\liminf_{c\to\infty}\phi_d(c)P(T_{c[x]_c} < \infty) \ge (\beta_d T)^{-1}\liminf_{c\to\infty}P^*_{c^2T}(T_{c[x]_c} < \infty). \tag{135}$$

Claim 2. - For any $\rho > 0$, if $T > 0$ is sufficiently small,

$$\liminf_{c\to\infty}P^*_{c^2T}(T_{c[x]_c} < \infty) \ge (1-\rho)\mathbb{N}_0^{(T)}(x \in \mathcal{R}).$$

The desired lower bound follows from (135), the above claim and (88) by letting T go to 0. Let us now outline the proof of Claim 2.

For $z \in \mathbb{R}^d$, $\varepsilon' > \varepsilon > 0$, let

$$\mathcal{C}(z,\varepsilon,\varepsilon') = \left\{y \in \mathbb{R}^d : \varepsilon < |y - z| < \varepsilon'\right\}.$$

We also set for $\varepsilon \in (0, 1/e)$, $r \in (0,1)$,

$$h(\varepsilon) := \varepsilon^2\ln(\ln(\varepsilon^{-1})), \qquad g_d(r) = \begin{cases} 2^{-\left(\frac{\ln(r)}{\ln(2)}\right)^4} & \text{if } d = 2,\\ r^{16} & \text{if } d = 3.\end{cases}$$

For $\alpha > 0, T > 0$ and $\varepsilon_0 \in (0,1)$ we consider the events

$$\mathcal{E}^{(c)}_{\varepsilon_0} = \left\{\exists s \ge 0\ \exists\varepsilon \in (g_d(\varepsilon_0),\varepsilon_0) : X^0_s\left(\mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) \ge \alpha h(\varepsilon)\phi_d(c)\right\},$$
$$\mathcal{F}^{(c)}_{\varepsilon_0} = \left\{\exists s \ge 0\ \exists\varepsilon \in (g_d(\varepsilon_0),\varepsilon_0) : X^0_s\left(\mathcal{C}\left(cx, \frac{c\varepsilon}{2}, c\varepsilon\right)\right) > \alpha h(\varepsilon)\phi_d(c)\right\}.$$

For c large enough, we have $\mathcal{F}^{(c)}_{\varepsilon_0} \subset \mathcal{E}^{(c)}_{\varepsilon_0}$, hence

$$P^*_{c^2T}(T_{c[x]_c} < \infty) \ge P^*_{c^2T}\left(\mathcal{F}^{(c)}_{\varepsilon_0}\right) \times P^*_{c^2T}\left(T_{c[x]_c} < \infty \middle| \mathcal{E}^{(c)}_{\varepsilon_0}\right). \tag{136}$$

The idea of the proof of Claim 2 is the following. Rescaling and using Theorem V.2, we will show that for c large, the first term of the product in (136), namely $P^*_{c^2T}(\mathcal{F}^{(c)}_{\varepsilon_0})$, is bounded below by a corresponding rescaled quantity under $\mathbb{N}_0^{(T)}$. For α small enough, this quantity will then be bounded from below by a quantity arbitrarily close to $\mathbb{N}_0^{(T)}(Y \text{ hits } \{x\})$, that is $\mathbb{N}_0^{(T)}(x \in \mathcal{R})$ (see Lemma V.8 and assertion (139) below). To finish the proof of Claim 2 we shall then establish that if we take T, ε_0 small enough, the second term of the product in (136), $P^*_{c^2T}\left(T_{c[x]_c} < \infty \middle| \mathcal{E}^{(c)}_{\varepsilon_0}\right)$, is, for c large, arbitrarily close to 1 (see Lemma V.9 below).

Let us reformulate the preceeding discussion in more precise terms. Using rescaling we have

(137)
$$P^*_{c^2T}\left(\mathcal{F}^{(c)}_{\varepsilon_0}\right) = P^*_{c^2T}\left(\exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : X^{c^2,0}_s\left(\mathcal{C}(x, \frac{\varepsilon}{2}, \varepsilon)\right) > \alpha h(\varepsilon)\right).$$

It is easy to see that the set

$$\left\{\omega \in D(\mathbb{R}_+, M_F(\mathbb{R}^d)) : \exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0),\ \omega_s\left(\mathcal{C}(x, \frac{\varepsilon}{2}, \varepsilon)\right) > \alpha h(\varepsilon)\right\}$$

is open for the Skorohod J_1 topology. Theorem V.2 thus implies that

(138)
$$\liminf_{c\to\infty} P^*_{c^2T}\left(\mathcal{F}^{(c)}_{\varepsilon_0}\right) \geq \mathbb{N}^{(T)}_0(\exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : Y_s(C(x, \frac{\varepsilon}{2}, \varepsilon)) > \alpha h(\varepsilon)).$$

Lemma V.8. *Let $d = 2$ or 3. We can choose $\alpha > 0$ so that, for any $\delta > 0$, there exists $\varepsilon_1 \in \left(0, 1 \wedge \frac{|x|}{2}\right)$ such that for any $\varepsilon_0 \in (0, \varepsilon_1)$,*

$$\mathbb{N}_0\left[\exists s \geq 0\ \exists\ \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : Y_s\left(\mathcal{C}\left(x, \frac{\varepsilon}{2}, \varepsilon\right)\right) > \alpha h(\varepsilon)\ \middle|\ x \in \mathcal{R}\right] \geq 1 - \delta.$$

In the following, we fix α as in Lemma V.8. The following lemma estimates the second term of the product in the right-hand side of (136).

Lemma V.9. *For any fixed $\gamma > 0$, there exists $\varepsilon_2 \in (0, 1)$ such that for any $\varepsilon_0 \in (0, \varepsilon_2)$, we have*

(a) $$\liminf_{c\to\infty} P\left(T_{c[x]_c} < \infty \mid \mathcal{E}^{(c)}_{\varepsilon_0}\right) \geq 1 - \gamma,$$

(b) $$\liminf_{T\to 0}\left(\liminf_{c\to\infty} P^*_{c^2T}\left(T_{c[x]_c} < \infty \middle| \mathcal{E}^{(c)}_{\varepsilon_0}\right)\right) \geq 1 - \gamma.$$

Let us now fix $\delta \in (0, 1)$, $\gamma > 0$ and let ε_1 and ε_2 be as in Lemma V.8 and V.9 respectively. Since

$$\mathbb{N}_0\left(\left\{\sup\{|y| : y \in \mathcal{R}\} > \frac{|x|}{2}\right\} \cap \left\{Y_T = 0\right\}\right) \underset{T\to 0}{\longrightarrow} 0,$$

we deduce from Lemma V.8 that for any $\varepsilon_0 \in (0, \varepsilon_1)$, for $T > 0$ sufficiently small,

(139)
$$\mathbb{N}^{(T)}_0\left[\exists s \geq 0\ \exists \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : Y_s\left(C\left(x, \frac{\varepsilon}{2}, \varepsilon\right)\right) > \alpha h(\varepsilon)\right] \geq (1 - 2\delta)\mathbb{N}^{(T)}_0(x \in \mathcal{R}).$$

From (138) and (139), we have for T sufficiently small

$$\liminf_{c\to\infty} P^*_{c^2T}(\mathcal{F}^{(c)}_{\varepsilon_0}) \geq (1 - 2\delta)\mathbb{N}^{(T)}_0(x \in \mathcal{R}). \tag{140}$$

Now use (136) and Lemma V.9 (b) to get for T small,

$$\liminf_{c\to\infty} P^*_{c^2T}(T_{c[x]_c} < \infty) \geq (1 - 2\delta)(1 - 2\gamma)\mathbb{N}^{(T)}_0(x \in \mathcal{R}),$$

which gives Claim 2, hence Proposition V.3.

To complete our proof of the lower bound Proposition V.3, we still need to establish Lemma V.8 and Lemma V.9. Establishing that part (b) of Lemma V.9 follows from part (a) requires a result which is a consequence of Lemma V.8. However, we first give the proof of Lemma V.9 (Section 4.3 below), because it is more closely related to our results. We then provide a proof of Lemma V.8 in Section 4.4. In these two sections, we will assume for simplicity that $\sigma = 1$. Adapting the proofs to a general σ is easy.

4.3. Proof of Lemma V.9. We assume in this section that Lemma V.8 has been proved, and in particular that (140) holds. Let us first explain how to derive part (b) from part (a). We have

$$
\begin{aligned}
&P^*_{c^2T}\left(T_{c[x]_c} < \infty \,\middle|\, \mathcal{E}^{(c)}_{\varepsilon_0}\right) \\
&= \frac{P\left(\{T_{c[x]_c} < \infty\} \cap \mathcal{E}^{(c)}_{\varepsilon_0}\right)}{P\left(\mathcal{E}^{(c)}_{\varepsilon_0} \cap \{\xi^0_{c^2T} \neq \emptyset\}\right)} - \frac{P\left(\{T_{c[x]_c} < \infty\} \cap \mathcal{E}^{(c)}_{\varepsilon_0} \cap \{\xi_{c^2T} = \emptyset\}\right)}{P\left(\mathcal{E}^{(c)}_{\varepsilon_0} \cap \{\xi^0_{c^2T} \neq \emptyset\}\right)}
\end{aligned}
$$

$$
\geq P\left(T_{c[x]_c} < \infty \,\middle|\, \mathcal{E}^{(c)}_{\varepsilon_0}\right) - \frac{P\left(\{T_{c[x]_c} < \infty\} \cap \{\xi_{c^2T} = \emptyset\}\right)}{p_{c^2T} P^*_{c^2T}(\mathcal{E}^{(c)}_{\varepsilon_0})}. \tag{141}
$$

Take $\delta = 1/2$ in Lemma V.8, and choose ε_1 so that the conclusion of this lemma holds. From the fact that $\mathcal{F}^{(c)}_{\varepsilon_0} \subset \mathcal{E}^{(c)}_{\varepsilon_0}$, and then from (140), we get that for $\varepsilon_0 \in (0, \varepsilon_1)$, for $T > 0$ small,

$$
\liminf_{c\to\infty} T^{-1} P^*_{c^2T}(\mathcal{E}^{(c)}_{\varepsilon_0}) \geq \frac{1}{2T}\mathbb{N}^{(T)}_0(x \in \mathcal{R}) \geq \frac{1}{2\beta_d}\left(2 - \frac{d}{2}\right)|x|^{-2}, \tag{142}
$$

using (88). Since $|[x]_c| > |x|/2$ for c large enough, we have

$$
P\left(\{T_{c[x]_c} < \infty\} \cap \{\xi^0_{c^2T} = \emptyset\}\right) \leq P\left(\sup_{t \leq c^2T}\sup_{y\in\xi^0_t}|y| > c|x|/2\right).
$$

For T small enough so that $\frac{|x|}{\sqrt{2T}} \geq 1$, and c large enough so that $\frac{c^2T}{2} > 1$, we can use (124) to deduce that

$$
P\left(\{T_{c[x]_c} < \infty\} \cap \{\xi_{c^2T} = \emptyset\}\right) \leq p_{c^2T/2} K_0 \exp\left(-K_2\frac{|x|}{\sqrt{2T}}\right).
$$

Combining (142) and this last inequality, we obtain that for any $\varepsilon_0 \in (0, \varepsilon_1)$

$$
\limsup_{T\to 0}\limsup_{c\to\infty}\frac{P\left(\{T_{c[x]_c} < \infty\} \cap \{\xi_{c^2T} = \emptyset\}\right)}{p_{c^2T}P^*_{c^2T}\left(\mathcal{E}^{(c)}_{\varepsilon_0}\right)} = 0.
$$

It is now clear from (141) that part (b) of Lemma V.9 follows from part (a). $\square$

Proving part (a) of Lemma V.9 requires the following intermediate result. Recall that for $A \subset \mathbb{Z}^d$ the voter model ξ starts from $\xi_0 = A$ under P_A.

Lemma V.10. *For any $\gamma > 0$, there exists $M > 0$ and $U > 0$ such that for every $u \geq U$ and every subset A of $\mathbb{Z}^d \cap \mathcal{C}(0, \frac{u}{4}, 2u)$ with $|A| \geq M\phi_d(u)$, one has*

$$
P_A(\exists t \geq 0 : 0 \in \xi_t) \geq 1 - \gamma.
$$

Let us postpone the proof of Lemma V.10 and proceed to the proof of Lemma V.9 (a).

Let us fix $\gamma > 0$. Let us choose $M > 0$ and $U > 0$ such that the conclusion of Lemma V.10 holds. We can then choose $\varepsilon_2 > 0$ small enough so that

$$
\frac{\alpha}{2}\ln\left(\ln\left(\frac{1}{\varepsilon_2}\right)\right) \geq M.
$$

Let us fix $\varepsilon_0 \in (0, \varepsilon_2)$. Let $c > 0$ be large enough so that

$$
cg_d(\varepsilon_0) \geq U, \qquad \text{and} \qquad 2\ln(cg_d(\varepsilon_0)) \geq \ln(c).
$$

We then set

$$T_{\varepsilon_0}^{(c)} := \inf\left\{t \geq 0 : X_t^0\left(\mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) \geq \alpha h(\varepsilon)\phi_d(c) \text{ for some } \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0)\right\}.$$

Clearly $T_{\varepsilon_0}^{(c)}$ is a stopping time of the filtration generated by the voter model, and $\mathcal{E}_{\varepsilon_0}^{(c)} = \{T_{\varepsilon_0}^{(c)} < \infty\}$.

From the definition of $T_{\varepsilon_0}^{(c)}$, on the event $\mathcal{E}_{\varepsilon_0}^{(c)}$ we can choose a random $\varepsilon \in (g_d(\varepsilon_0), \varepsilon_0)$ such that

$$X^0_{T_{\varepsilon_0}^{(c)}}\left(\mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) \geq \alpha h(\varepsilon)\phi_d(c).$$

On the event $\mathcal{E}_{\varepsilon_0}^{(c)}$, we can consider the set

$$\mathcal{A}_\varepsilon^{(c)} := \left(\xi^0_{T_{\varepsilon_0}^{(c)}} \cap \mathcal{C}\left(c[x]_c, \frac{c\varepsilon}{4}, 2c\varepsilon\right)\right) - c[x]_c,$$

where, for $A \subset \mathbb{Z}^d$, $z \in \mathbb{Z}^d$, $A - z = \{y \in \mathbb{Z}^d : y + z \in A\}$. The random set $\mathcal{A}_\varepsilon^{(c)}$ is a subset of $\mathbb{Z}^d \cap \mathcal{C}(0, \frac{c\varepsilon}{4}, 2c\varepsilon)$, and has cardinality $|\mathcal{A}_\varepsilon^{(c)}| = \lceil \alpha h(\varepsilon)\phi_d(c)\rceil$.

Let us argue on $\mathcal{E}_{\varepsilon_0}^{(c)}$ and set $u(\varepsilon, c) = c\varepsilon$. Note that $u(\varepsilon, c) \geq c g_d(\varepsilon_0) \geq U$. When $d = 3$, we have

$$|\mathcal{A}_\varepsilon^{(c)}| \geq \alpha h(\varepsilon) c^2 \geq \alpha \ln\left(\ln\left(\frac{1}{\varepsilon_2}\right)\right)\varepsilon^2 c^2 \geq M\phi_d(u(\varepsilon, c)).$$

When $d = 2$, recalling that $2\ln(cg_d(\varepsilon_0)) \geq \ln(c)$, we also have

$$|\mathcal{A}_\varepsilon^{(c)}| \geq \alpha h(\varepsilon)\frac{c^2}{\ln(c)} \geq \alpha \ln\left(\ln\left(\frac{1}{\varepsilon_2}\right)\right)\frac{c^2\varepsilon^2}{2\ln(c\varepsilon)} \geq M\phi_d(u(\varepsilon, c)).$$

From Lemma V.10, we deduce that, on the event $\mathcal{E}_{\varepsilon_0}^{(c)}$,

$$P_{\mathcal{A}_\varepsilon^{(c)}}(\exists t \geq 0 : 0 \in \xi_t) \geq 1 - \gamma.$$

Using the strong Markov property for ξ^0 at time $T_{\varepsilon_0}^{(c)}$, then the fact that $\mathcal{A}_\varepsilon^{(c)} + c[x]_c \subset \xi^0_{T_{\varepsilon_0}^{(c)}}$, we obtain

$$\begin{aligned} P\left(\mathcal{E}_{\varepsilon_0}^{(c)} \cap \{T_{c[x]_c} < \infty\}\right) &\geq E\left(\mathbf{1}_{\{T_{\varepsilon_0}^{(c)} < \infty\}} P_{\xi^0_{T_{\varepsilon_0}^{(c)}}}(\exists t \geq 0 : c[x]_c \in \xi_t)\right) \\ &\geq E\left(\mathbf{1}_{\{T_{\varepsilon_0}^{(c)} < \infty\}} P_{\mathcal{A}_\varepsilon^{(c)}}(\exists t \geq 0 : 0 \in \xi_t)\right) \\ &\geq (1-\gamma)P(\mathcal{E}_{\varepsilon_0}^{(c)}), \end{aligned}$$

which gives part (a) of Lemma V.9. □

Let us now fix $\gamma > 0$ and establish Lemma V.10. First, notice that the function $A \to P_A(\exists t \geq 0 : 0 \in \xi_t)$ is increasing. It thus suffices to find $M > 0$ and $U > 0$ such that for $u \geq U$,

$$\inf\left\{P_A(\exists t \geq 0 : 0 \in \xi_t) : A \subset \mathbb{Z}^d \cap \mathcal{C}\left(0, \frac{u}{4}, 2u\right), |A| = \lceil M\phi_d(u)\rceil\right\} \geq 1 - \gamma. \tag{143}$$

For $M > 0, u > 0$ let us introduce

$$\mathfrak{A}_u^{(M)} := \left\{A \subset \mathbb{Z}^d \cap \mathcal{C}\left(0, \frac{u}{4}, 2u\right) \ : \ |A| = \lceil M\phi_d(u)\rceil\right\}.$$

We then use a similar method as for establishing the rough lower bound. Let us set $V_T = \int_0^T \mathbf{1}_{\{0\in\xi_{u^2t}\}}dt$. As in Section 4.1, we use the Cauchy-Schwarz inequality to get for any $u > 0$, $A \in \mathfrak{A}_u^{(M)}$

$$P_A(\exists t \geq 0 : 0 \in \xi_t) \geq \frac{(E_A[V_T])^2}{E_A[(V_T)^2]}. \tag{144}$$

We will verify that for any fixed $M > 0$, there exists a constant $U(M)$ such that if $u \geq U$, then for any $A \in \mathfrak{A}_u^{(M)}$, we have

$$E_A[V_T] \geq \frac{M}{2}(\psi_d(u))^{-1}\int_{T/2}^{T}\frac{1}{(2\pi s)^{d/2}}\exp\left(-\frac{2}{s}\right)ds, \tag{145}$$

$$E_A[(V_T)^2] \leq (E_A[V_T])^2 + L'M\,(\psi_d(u))^{-2}, \tag{146}$$

where L' is a constant depending only on d and T. Let us postpone the proof of these two assertions and finish the proof of Lemma V.10. We can choose $M > 0$ sufficiently large so that

$$L'M \leq \gamma\frac{M^2}{4}\left(\int_{T/2}^{T}ds\frac{1}{(2\pi s)^{d/2}}\exp\left(-\frac{2}{s}\right)\right)^2.$$

From (145) and (146), we then deduce that for $u \geq U(M)$, for any $A \in \mathfrak{A}_u^{(M)}$, we have

$$E_A[(V_T)^2] \leq (1+\gamma)\,(E_A[V_T])^2\,.$$

Lemma V.10 now follows from (144).

<u>Proof of (145)</u> : Let us now fix $M > 0$ and establish that (145) is valid for u sufficiently large, and for any $A \in \mathfrak{A}_u^{(M)}$. Note that for any $A \in \mathfrak{A}_u^{(M)}$, $u^{-1}A$ is a subset of $u^{-1}\mathbb{Z}^d \cap \mathcal{C}(0, 1/4, 2)$ and has cardinality $\lceil M\phi_d(u)\rceil$. For any $u > 0$, $A \in \mathfrak{A}_u^{(M)}$, duality gives

(147)

$$E_A[V_T] = \int_0^T \mathbb{P}_0(Z_{u^2t} \in A)dt = \sum_{y\in u^{-1}A}\int_0^T q_{u^2t}(uy)dt \geq \sum_{y\in u^{-1}A}\int_{T/2}^{T} q_{u^2t}(uy)dt.$$

It is easy to deduce from Theorem V.3 that uniformly in $t \in [T/2, T]$,

$$\lim_{u\to\infty}\sup_{y\in u^{-1}\mathbb{Z}^d}|u^d q_{u^2t}(uy) - p_t(y)| = 0.$$

Thus, if u is sufficiently large, for any $y \in u^{-1}\mathbb{Z}^d$ with $|y| \leq 2$,

$$u^d q_{u^2t}(uy) \geq \frac{1}{2}p_t(y) \geq \frac{1}{2}(2\pi t)^{-d/2}\exp\left(-\frac{2}{t}\right).$$

We deduce from the above and (147) that for u large enough, and for any $A \in \mathfrak{A}_u^{(M)}$, we have

$$E_A[V_T] \geq \frac{1}{2}u^{-d}|A|\int_{T/2}^{T}(2\pi s)^{-d/2}\exp\left(-\frac{2}{s}\right)dt.$$

(145) now follows from the fact that $u^{-d}\phi_d(u) = (\psi_d(u))^{-1}$.

Proof of (146) : Let us now estimate the second moment of V_T and prove (146). Using the same arguments as in the proof of the rough estimate, we obtain for any $u > 0$, $A \in \mathfrak{A}_u^{(M)}$,

$$E_A[(V_T)^2] = 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')\mathbb{E}_{0,z'}[Z^1_{u^2t} \in A, Z^2_{u^2t} \in A],$$

so that

$$E_A[(V_T)^2] = 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')(H_{1,u}(t,z') + H_{2,u}(t,z')), \tag{148}$$

where

$$\begin{aligned} H_{1,u}(t,z') &:= \sum_{y\in A} \mathbb{P}_{0,z'}[T_1 \le u^2 t, Z^1_{u^2t} = y], \\ H_{2,u}(t,z') &:= \sum_{y\in A}\sum_{y'\in A} \mathbb{P}_{0,z'}[T_1 > u^2 t, Z^1_{u^2t} = y, Z^2_{u^2t} = y']. \end{aligned}$$

Since two coalescing walks behave independently before they meet, we can bound $\mathbb{P}_{0,z'}[T_1 > u^2 t, Z^1_{u^2t} = y, Z^2_{u^2t} = y']$ by $q_{u^2t}(y)q_{u^2t}(z'-y')$ so we obtain

$$\begin{aligned} & 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')H_{2,u}(t,z') \\ \le\ & 2\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')\sum_{y\in A}\sum_{y'\in A} q_{u^2t}(y)q_{u^2t}(z'-y') \\ =\ & 2\int_0^T dt \int_0^{T-t} ds \sum_{y\in A}\sum_{y'\in A} q_{u^2t}(y)q_{u^2(t+s)}(y') \\ =\ & \left(\int_0^T dt \sum_{y\in A} q_{u^2t}(y)\right)^2 = (E_A[V_T])^2 \end{aligned} \tag{149}$$

With a slight abuse of notation, in the remaining part of the section we use L to denote a constant depending only on d and T and which may change from line to line.

Using (100), we obtain

$$\begin{aligned} &\int_0^T dt \int_0^{T-t} ds \sum_{z'\in\mathbb{Z}^d} q_{u^2 s}(z')H_{1,u}(t,z') \\ &\le L_0 u^{-d}\Bigg[L_0\psi_d(u)u^{-d}\sum_{y\in A}\psi_d(|y|^{-1}u) + \sum_{z'\in\mathbb{Z}^d, 0<|z'|\le u} \psi_d\left(\frac{u}{|z'|}\right)\int_0^T H_{1,u}(t,z') \\ &\qquad + \sum_{z'\in\mathbb{Z}^d, |z'|>u} \exp\left(-L_0'\frac{|z'|}{u}\right)\int_0^T H_{1,u}(t,z')\Bigg]. \end{aligned} \tag{150}$$

Note that we used (100) a second time to bound $\int_0^T dt H_1(t,0)$ and get the first term in the sum above. Then, Lemma V.2 is exactly what we need to bound $\int_0^T H_{1,u}(t,z')$ when $z' \neq 0$. However, the cases $d = 2$ and $d = 3$ are slightly different.

When $d = 3$, for any $u > 0$, for any $A \in \mathfrak{A}_u^{(M)}$, and any $z' \neq 0$, we obtain from Lemma V.2 that

$$\int_0^T dt H_{1,u}(t, z') \quad \leq \quad L_1 u^{-3} \left(\psi_3(|z'| \vee 1)\right)^{-1} \sum_{y \in u^{-1}A} \psi_3\left(\frac{1}{|y|}\right) \tag{151}$$

From the fact that $\min_{y \in u^{-1}A} |y| > 1/4$ and $|A| = \lceil M\phi_3(u) \rceil = \lceil Mu^2 \rceil$, we obtain that

$$\sum_{y \in u^{-1}A} \psi_3\left(\frac{1}{|y|}\right) \leq 4\lceil Mu^2 \rceil.$$

Hence, from (150) and (151), we deduce that, when $d = 3$, for u sufficiently large, and for any $A \in \mathfrak{A}_u^{(M)}$,

$$\begin{aligned} & \int_0^T dt \int_0^{T-t} ds \sum_{z' \in \mathbb{Z}^d} q_{u^2 s}(z') H_{1,u}(t, z') \\ \leq \quad & L u^{-6} \lceil Mu^2 \rceil \left[u + \sum_{z' \in \mathbb{Z}^3, 0 < |z'| \leq u} |z'|^{-2} + \sum_{z' \in \mathbb{Z}^3, |z'| \geq u} |z'|^{-1} \exp\left(-L_0' \frac{|z'|}{u}\right) \right] \\ \leq \quad & L u^{-6} \lceil M \phi_d(u) \rceil u^2, \end{aligned}$$

where we used that $\sum_{z' \in \mathbb{Z}^3, |z'| \geq u} |z'|^{-1} \exp\left(-L_0' \frac{|z'|}{u}\right) \leq L \int_u^\infty \rho \exp(-L_0' \rho / u) d\rho$.

Similarly, when $d = 2$, for any $u > 1$, for any $A \in \mathfrak{A}_u^{(M)}$, and any $z' \neq 0$, we get from Lemma V.2 that

$$\begin{aligned} \int_0^T dt H_{1,u}(t, z') \quad & \leq \quad L_1 u^{-2} \frac{\psi_2(u/|z'|)}{\psi_2(|z'|)} \sum_{y \in u^{-1}A} \ln(|y|^{-1}) \\ & \leq \quad \ln(4) L_1 u^{-2} \lceil M \frac{u^2}{\ln(u)} \rceil \frac{\psi_2(u/|z'|)}{\psi_2(|z'|)}, \end{aligned} \tag{152}$$

Furthermore, we have $\min_{y \in u^{-1}A} |y| > 1/4$ and $|A| = \lceil M\phi_2(u) \rceil = \lceil Mu^2/(2\ln(u)) \rceil$, so that

$$\sum_{y \in u^{-1}A} \psi_2\left(\frac{1}{|y|}\right) \leq 4\lceil Mu^2 / 2(\ln(u)) \rceil.$$

Hence, from (150) and (152), we deduce that for any u sufficiently large, for any $A \in \mathfrak{A}_u^{(M)}$,

$$\begin{aligned} & \int_0^T dt \int_0^{T-t} ds \sum_{z' \in \mathbb{Z}^d} q_{u^2 s}(z') H_{1,u}(t,z') \\ \leq \; & Lu^{-4} \lceil M \frac{u^2}{2\ln(u)} \rceil \Bigg[\ln(u) + \sum_{z' \in \mathbb{Z}^3, 0<|z'|\leq u} \ln\left(\frac{u}{|z'|} \vee e\right)^2 \times \frac{1}{\ln(|z'| \vee e)} \\ & \qquad + \sum_{z' \in \mathbb{Z}^3, |z'| \geq u} \frac{1}{\ln(|z'|)} \exp\left(-L_0' \frac{|z'|}{u}\right) \Bigg] \\ \leq \; & Lu^{-4} \lceil M \frac{u^2}{2\ln(u)} \rceil \left[\ln(u) + \int_{\sqrt{2}}^u \rho \frac{(\ln(u/\rho))^2}{\ln(\rho)} d\rho + \int_u^\infty \frac{\rho}{\ln(\rho)} \exp(-L_0' \rho/u) d\rho \right] \\ \leq \; & Lu^{-2} (\ln(u))^{-1} \lceil M \frac{u^2}{2\ln(u)} \rceil. \end{aligned}$$

In both $d = 2$ and $d = 3$, we have thus obtained that for any u sufficiently large, for any $A \in \mathfrak{A}_u^{(M)}$,

$$\int_0^T dt \int_0^{T-t} ds \sum_{z' \in \mathbb{Z}^d} q_{u^2 s}(z') H_{1,u}(t,z') \leq L' M (\psi_d(u))^{-2},$$

where L' is a constant depending on d and T. From (148), (149) and the above, we deduce (146). As explained earlier, this completes the proof of Lemma V.10. $\square$

4.4. Proof of Lemma V.8. The proof of Lemma V.8 is somewhat lengthy. It is inspired by the first part of [**LGP 95**], where an upper bound for the Hausdorff measure of the support of two-dimensional super-Brownian motion is established. In particular, we use the Brownian snake as a main tool. The Brownian snake gives an alternative construction of super-Brownian motion under its excursion measure. Moreover, this object introduces time dynamics in the analysis of super-Brownian motion which prove to be critical for our arguments to work. We briefly introduce the Brownian snake and related notation in paragraph 4.4.1, then discuss the link between Brownian snake and super-Brownian motion.

For convenience, we work in this section with super-Brownian motion with branching rate 4 and diffusion coefficient 1 under its excursion measure. Simple scaling arguments then give the general case.

We only give a detailed proof of Lemma V.8 in the three-dimensional case (paragraph 4.4.4), after having summarized the basic idea (paragraph 4.4.2), and presented three intermediate lemmas (paragraph 4.4.3). Using the results of the first part of [**LGP 95**], the case $d = 2$ easily adapts. In fact, we even establish a stronger result in the plane (see Lemma V.14 below), which we discuss in paragraph 4.4.5.

4.4.1. *Brownian snake.* For a precise definition of the Brownian snake, we refer to [**LG 99**], Chapter IV. Let $\mathcal{W}$ be the set of continuous finite paths from $\mathbb{R}_+$ into $\mathbb{R}^d$. For $w \in \mathcal{W}$, we denote by ζ_w the lifetime of w, and by $\hat{w}$ the terminal point of the path w, that is $w(\zeta_w)$. The trivial path $\overline{y}$ is the path with initial point $y \in \mathbb{R}^d$ and lifetime 0. The space $\mathcal{W}$ is Polish when equipped with the distance

$$d(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{r \geq 0} |w(r \wedge \zeta_w) - w'(r \wedge \zeta_{w'})|.$$

We then consider $\Omega = \mathcal{C}(\mathbb{R}_+, \mathcal{W})$, the space of continuous paths from $\mathbb{R}_+$ into $\mathcal{W}$ with the topology of uniform convergence on compact sets, and $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ-field on Ω. The canonical process on this space is denoted $(W_s, s \geq 0)$, and we define for $s \geq 0$, $\zeta_s := \zeta(W_s)$, and $\sigma(\zeta) := \inf\{s > 0 : \zeta_s = 0\}$. We also let $(\mathcal{F}_t)_{t\geq 0}$ be the canonical filtration on Ω.

For $w \in \mathcal{W}$, we let Π_w be the law on $(\Omega, \mathcal{F})$ of the Brownian snake starting from the path w. Under Π_w, $(W_s, s \geq 0)$ is a $\mathcal{W}$-valued diffusion and $(\zeta_s, s \geq 0)$ is a one-dimensional reflecting Brownian motion. Informally, when ζ_s "increases", the path W_s grows like a d-dimensional Brownian motion, whereas it is erased when ζ_s "decreases" (see [**LG 99**], Chapter IV for more precisions).

For $y \in \mathbb{R}^d$, the measure $\mathbb{N}_y$ is the excursion measure of W away from the trivial path $\overline{y}$. We abuse the notation by using the same notation $\mathbb{N}_y$ for the excursion measure of the Brownian snake away from $\overline{y}$ and for the excursion measure of super-Brownian motion (cf Section 2.1). This abuse will be justified below when we construct the excursion measure of super-Brownian motion from the Brownian snake under $\mathbb{N}_y$. Under $\mathbb{N}_y$, the law of ζ is the Itô measure of positive Brownian excursions and $\sigma(\zeta)$ is the length of this excursion.

Denote by Π^*_w the law under Π_w of $(W_{s\wedge\sigma_{\zeta_s}}, s \geq 0)$, that is the law of the Brownian snake stopped when its lifetime process hits 0. The strong Markov property of W under $\mathbb{N}_y$ can be expressed in the following way. Let θ_t denotes the usual shift operator on Ω. If T is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time such that $T > 0$ $\mathbb{N}_y$-a.e., then, for any nonnegative $\mathcal{F}_T$-measurable F, for any nonnegative $\mathcal{F}$-measurable G,

$$\mathbb{N}_y\left(\mathbf{1}_{\{T<\infty\}}F \times G \circ \theta_T\right) = \mathbb{N}_y\left(\mathbf{1}_{\{T<\infty\}}F \times \Pi^*_{W_T}(G)\right). \tag{153}$$

The link between Brownian snake and super-Brownian motion can be expressed as follows. Let L^t_s denote the local time of ζ at time s and level t. Since the law of $(\zeta_s, s \geq 0)$ under $\mathbb{N}_y$ is the Itô measure of positive Brownian excursions, $(L^t_s, s \geq 0)$ is, for any $t \geq 0$, well-defined, increasing and continuous, $\mathbb{N}_y$-a.s. We denote by $d_sL^t_s$ the measure associated with the function $u \to L^t_u$ and we let $(Y_t(W), t \geq 0)$ be the measure-valued process defined by the formula

$$Y_t(W)(.) = \int_0^{\sigma(\zeta)} d_sL^t_s\mathbf{1}_{\{\hat{W}_s\in .\}}.$$

Then, the law of $(Y_t(W), t \geq 0)$ under $\mathbb{N}_y$ is the excursion measure of super-Brownian motion with branching rate 4 and diffusion coefficient 1[1].

4.4.2. *Informal sketch of the proof of Lemma V.8.* Using a symmetry argument, we can interchange the roles of 0 and x, and we will thus work under the probability measure $\mathbb{N}_x(.|0 \in \mathcal{R}) = \mathbb{N}_x(.|T_0 < \infty)$, where $T_0 = \inf\{t \geq 0 : \hat{W}_t = 0\}$. It is possible to precise the law of $(|W_{T_0}|)_{t\leq\zeta_{T_0}}$ under $\mathbb{N}_x(.|0 \in \mathcal{R})$ (see Lemma V.11 below).

[1]Moreover, if we let $\mu \in M_F(\mathbb{R}^d)$ and $\sum_{i\in I}\delta_{y_i,W_i}$ be a Poisson measure with intensity $\mu(dy)\mathbb{N}_y(dW)$, then a super-Brownian motion $(\overline{Y}_t, t \geq 0)$ starting from μ can be obtained by setting

$$\overline{Y}_t = \sum_{i\in I} Y_t(W_i).$$

For $j \in \mathbb{N}$, let us introduce $r_j = \exp(-j^2)$. To $n_1 \in \mathbb{N}$ we associate $\varepsilon_1 := r_{2^{n_1}}$, and for $\varepsilon_0 > 0$, we set $n_0 := \min\{p \in \mathbb{N}^* : r_{2^p} \leq \varepsilon_0\}$. Note that we have

$$r_j \in (g_d(\varepsilon_0), \varepsilon_0) \quad \forall j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!].$$

Hence, to prove Lemma V.8 it suffices to establish the following claim.

Claim 3. - One can choose $\alpha > 0$ such that, for any $\delta > 0$, there exists $n_1 \in \mathbb{N}$ such that for any $n_0 \geq n_1$, one has

$$\mathbb{N}_x \left(\forall j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!] : Y_{\zeta_{T_0}}(C(0, r_j/2, r_j)) \leq \alpha h(r_j) \;\middle|\; T_0 < \infty \right) \leq \delta.$$

The idea of the proof of Claim 3 is the following. For given $w \in \mathcal{W}$, $n_0 \in \mathbb{N}$ and $j \in [\![2^{n_0}, 2^{n_0+1} - 1]\!]$, we will express further the contribution $Z_w(r_j)$ to $Y_{\zeta_w}(\mathcal{C}(0, r_j/2, r_j))$ of particules which split off the path w in the time interval $[\zeta_w - r_j^2 \ln(1/r_j), \zeta_w - r_j^2]$ (see (159) below). We will observe that the contributions $(Z_w(r_j))_{j \geq 0}$ are independent. Using estimates on these contributions (see Lemma V.13 below), this independence will lead us to a bound on the probability that for any $j \in [\![2^n, 2^{n+1} - 1]\!]$, $Z_w(r_j)$ remains smaller than $\alpha h(r_j)$ (see (165) below).

For a well-choosen $\alpha > 0$, we will deduce from this bound and the knowledge of the law of the path $|W_{T_0}|$ the existence of integers N_0, N, and of a family of sets of "good paths" $(\mathbb{W}_n, n \geq N_0)$ such that, with a probability arbitrarily close to 1 when N is large enough,

– $(|W_{T_0}|)_{t \leq \zeta_{T_0}}$ belongs to $\mathbb{W}_n$ for any $n \geq N$.

– for any $w \in \mathbb{W}_n$, $n \geq N$, there exists $j \in [\![2^n, 2^{n+1} - 1]\!]$ such that $Z(r_j) > \alpha h(r_j)$.

The desired claim will follow (see assertions (160), (163) and (164) below).

We now present three intermediate lemmas.

4.4.3. *Preliminary results.* Let us start by investigating the law of the path $|W_{T_0}|$ under $\mathbb{N}_x[.|0 \in \mathcal{R}]$.

Lemma V.11. *Under* $\mathbb{N}_x[.|0 \in \mathcal{R}]$, $|W_{T_0}(t)|_{t \in [0, \zeta_{T_0}]}$ *has the law of a Bessel process with index* $-5/2$ *started from* $|x|$ *and stopped when it first hits* 0.

Proof of Lemma V.11. Introduce a d-dimensional Brownian motion $(B_t)_{t \geq 0}$ and the function $u(y) := \mathbb{N}_y(T_0 < \infty) = (2 - d/2)\,|y|^{-2}$. Let us denote by Q the law of the solution of the stochastic differential equation

$$dY_t = dB_t + \frac{\nabla u}{u}(Y_t), Y_0 = x,$$

stopped when it hits 0.

We know from [**De 99**], Proposition 1.4, that for any nonnegative continuous function F on $\mathcal{W}$,

$$\mathbb{N}_x \left[\mathbf{1}_{\{T_0 < \infty\}} F(W_{T_0}) \right] = \int \pi_{0,x}(dW) F(W), \tag{154}$$

where $\pi_{0,x} := \left(2 - \frac{d}{2}\right) |x|^{-2} Q$.

For $t \in [0, \zeta_{T_0}]$ let us set $R_t := |W_{T_0}(t)|$. From (154) we deduce that under the probability measure $\mathbb{N}_x[.|T_0 < \infty]$, $(R_t)_{0 \leq t \leq \zeta_{T_0}}$ solves the stochastic differential equation

$$dR_t = d\beta_t - \frac{2}{R_t} dt,$$

where $(\beta_t)_{t \geq 0}$ is a linear Brownian motion. Thus, $(|W_{T_0}|(t))_{0 \leq t \leq \zeta_{T_0}}$ has under $\mathbb{N}_x(.|\{T_0 < \infty\})$ the law of a Bessel process with index $-5/2$ started from $|x|$ and stopped when it first hits 0. We have completed the proof of Lemma V.11. □

For $x \in \mathbb{R}^d$, $p \in \mathbb{N}^*$, $\varepsilon > 0$ and $t > 0$, we will need a lower bound on

$$\psi(t, x, \varepsilon, p) := \mathbb{N}_x \left[Y_t \left(\mathcal{C}(0, \varepsilon/2, \varepsilon) \right)^p \right].$$

Recall P_t denotes the semigroup of d-dimensional Brownian motion. We know (see [**LGP 95**], Proposition 3.2) that we have

$$\psi(t, x, \varepsilon, 1) = P_t \mathbf{1}_{\mathcal{C}(0,\varepsilon/2,\varepsilon)}(x),$$

and the following recursion relation for $p \geq 2$

$$(155) \quad \psi(t, x, \varepsilon, p) = 2 \sum_{j=1}^{p-1} \binom{p}{j} \int_0^t P_{t-s} \left(\psi(s, ., \varepsilon, j) \psi(s, ., \varepsilon, p - j) \right)(x) ds.$$

Fix $c_1 > 1$ and let $c_2 = 1 - \frac{1}{2c_1}$. Note that $1/2 < c_2 < 1$. First observe that there exists a positive c_3 such that

$$(156) \quad \psi(t, x, \varepsilon, 1) = P_t \mathbf{1}_{\mathcal{C}(0,\varepsilon/2,\varepsilon)}(x) \geq c_3 \varepsilon^3 \exp\left(-\frac{|x|^2}{2t} \right) t^{-3/2} \mathbf{1}_{\{t \geq \varepsilon^2\}}.$$

Lemma V.12. *For $d = 3$, there exists a positive constant c_4 so that for any $p \in \mathbb{N}^*$, $t > 0$, $x \in \mathbb{R}^3$ and $\varepsilon \geq 0$,*

$$(\mathcal{H}_p) \qquad \psi(t, x, \varepsilon, p) \geq c_4^p p! \frac{\varepsilon^{2p+1}}{t^{3/2}} \exp\left(-\frac{c_1 |x|^2}{t} \right) \mathbf{1}_{\{t \geq c_2^{-2} \varepsilon^2\}}.$$

Corollary 3.3 of [**LGP 95**] is the corresponding result for the two-dimensional case.
Proof of Lemma V.12. Note that there exists a constant $C \geq 1$ such that for any $p \geq 2$,

$$\sum_{j=1}^{p-1} (j(p-j))^{3/2} \geq \frac{1}{C} p^4.$$

Let us set $c_8 := c_3 C^{-1} (\pi c_1)^{-3/2}$. We first verify that for any $p \in \mathbb{N}^*$,

$$(\tilde{\mathcal{H}}_p) \qquad \psi(t, x, \varepsilon, p) \geq \pi^{3/2} C c_8^p p! \varepsilon^{3p} p_{t/2pc_1}(x) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{t}} \right)^{p-1} \mathbf{1}_{\{t \geq \varepsilon^2\}}.$$

We use induction on p to establish $(\tilde{\mathcal{H}}_p)$. If $p = 1$, using (156) and our definition of c_8, we obtain

$$\psi(t, x, \varepsilon, 1) \geq (\pi c_1)^{3/2} C c_8 \varepsilon^3 t^{-3/2} \exp\left(-\frac{|x|^2}{2t} \right) \mathbf{1}_{\{t \geq \varepsilon^2\}}.$$

Since $c_1 > 1$, $(\tilde{\mathcal{H}}_1)$ follows.

Let $p \geq 2$ and assume that the result holds for all $p' < p$. Using (155) and the induction assumption we get for $t \geq \varepsilon^2$

$$\psi(t, x, \varepsilon, p) \geq 2\pi^3 C^2 c_8^p p! \varepsilon^{3p} \times \sum_{j=1}^{p-1} \int_{\varepsilon^2}^t ds \int_{\mathbb{R}^3} dy \; p_{t-s}(x - y) p_{s/2jc_1}(y) p_{s/2(p-j)c_1}(y) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}} \right)^{p-2}.$$

For any $j \in [\![1, p-1]\!]$, $s > 0$ and $y \in \mathbb{R}^3$ we have

$$p_{s/2jc_1}(y) p_{s/2(p-j)c_1}(y) = \left(\frac{c_1 j(p-j)}{\pi p s} \right)^{3/2} p_{s/2pc_1}(y)$$

From the last two displays, the choice of C and the fact that $p_{t-s} * p_{s/2pc_1} = p_{t-s+(s/2pc_1)}$ we obtain

$$(157)\quad \psi(t,x,\varepsilon,p) \geq 2\pi^{3/2} C c_8^p p! \varepsilon^{3p} p^{5/2} c_1^{3/2} \int_{\varepsilon^2}^t \frac{ds}{s^{3/2}} p_{t-s+s/2pc_1}(x) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2}.$$

Since $c_1 > 1$ the function $s \to t - s + s/2pc_1$ is decreasing, so that for any $s \in [\varepsilon^2, t]$, we have $t \geq t - s + \frac{s}{2pc_1} \geq \frac{t}{2pc_1}$. Thus,

$$(2pc_1)^{3/2} p_{t-s+\frac{s}{2pc_1}}(x) \geq p_{\frac{t}{2pc_1}}(x).$$

It follows that

$$\begin{aligned}\psi(t,x,\varepsilon,p) &\geq 2^{-1/2}\pi^{3/2} C c_8^p p.p! \varepsilon^{3p} p_{\frac{t}{2pc_1}}(x) \int_{\varepsilon^2}^t \frac{ds}{s^{3/2}} \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2} \\ &= 2^{1/2}\pi^{3/2} C c_8^p p! \varepsilon^{3p} p_{\frac{t}{2pc_1}}(x) \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{t}}\right)^{p-1},\end{aligned}$$

which finishes the proof of $(\tilde{\mathcal{H}}_p)$. We have established $(\tilde{\mathcal{H}}_p)$ for any $p \in \mathbb{N}^*$. Note in particular that (157) holds for any $p \in \mathbb{N}^*$.

Let us now complete the proof of Lemma V.12. We set $c_4 := (1 - c_2^{1/2})c_8$. The case $p = 1$ follows from (156) since $c_3 \geq c_4$.

Let us now suppose $p \geq 2$ and $t \geq c_2^{-2}\varepsilon^2$. Since $c_2 = 1 - \frac{1}{2c_1} \leq \frac{2c_1 - 1}{2c_1 - 1/p}$, we get, for any $s \leq c_2 t$, $t - s + s/2pc_1 \geq t/2c_1$, so that

$$p_{t-s+\frac{s}{2pc_1}}(x) \geq (2\pi t)^{-3/2} \exp\left(-\frac{c_1|x|^2}{t}\right).$$

Hence, it follows from (157) that for any $p \geq 2$,

$$\begin{aligned}\psi(t,x,\varepsilon,p) &\geq 2^{-1/2} C c_8^p p^{5/2} p! \varepsilon^{3p} t^{-3/2} \exp\left(-\frac{c_1|x|^2}{t}\right) \int_{\varepsilon^2}^{c_2 t} \frac{ds}{s^{3/2}} \left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{s}}\right)^{p-2} \\ &\geq 2^{1/2} C c_8^p p^{3/2} p! \varepsilon^{3p} t^{-3/2} \exp\left(-\frac{c_1|x|^2}{t}\right)\left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{c_2 t}}\right)^{p-1}.\end{aligned}$$

Since $t \geq c_2^{-2}\varepsilon^2$, we have $\left(\frac{1}{\varepsilon} - \frac{1}{\sqrt{c_2 t}}\right)^{p-1} \geq \varepsilon^{-p+1}(1 - c_2)^{p-1}$. Moreover, $C \geq 1$, hence, $(\mathcal{H}_p)$ follows for $p \geq 2$, which completes the proof of Lemma V.12. □

As explained briefly in paragraph 4.4.2, we will need to estimate, for a fixed $w \in \mathcal{W}$, the contribution under Π_w to $Y_{\zeta_w}(\mathcal{C}(0, \varepsilon/2, \varepsilon))$ of particules which split off the path w shortly before ζ_w. For a given $t > 0$, Lemma V.5 of [**LG 99**] allows one to decompose Y_t under Π_w^* as the sum of independent contributions corresponding to the decomposition of the path ζ into its excursions above its minimum-to-date. Let us state this more precisely.

Fix $w \in \mathcal{W}$ with $\zeta_w > 0$. Under Π_w^* we can construct a Poisson point measure Λ on $[0, \zeta_w] \times \Omega$ with intensity $2dt\mathbb{N}_{w(t)}(dW)$ such that

$$(158)\qquad Y_{\zeta_w} = \int_{[0,\zeta_w]\times\Omega} Y_{\zeta_w - t}(W)\Lambda(dt, dW), \qquad \Pi_w^* - \text{a.s.}$$

Hence, when $w \in \mathcal{W}$ is such that $\zeta_w \geq \varepsilon^2 \ln(1/\varepsilon)$, the contribution to $Y_{\zeta_w}(\mathcal{C}(0, \varepsilon/2, \varepsilon))$ of particules which split off the path w in the time interval $[\zeta_w - \varepsilon^2 \ln(1/\varepsilon), \zeta_w - \varepsilon^2]$

can be written

$$\varepsilon^2 Z_w(\varepsilon) := \int_{\zeta_w-\varepsilon^2\ln(1/\varepsilon)}^{\zeta_w-\varepsilon^2}\int_\Omega Y_{\zeta_w-t}(\mathcal{C}(0,\varepsilon/2,\varepsilon))\Lambda(dt,dW). \tag{159}$$

We then estimate, for $\varepsilon>0, w\in\mathcal{W}$, the moments of $Z_w(\varepsilon)$ under Π^*_w.

LEMMA V.13. *For $w\in\mathcal{W}$ and $\varepsilon>0$ such that $\zeta_w\geq\varepsilon\ln(1/\varepsilon)$, let us set*

$$I_w(\varepsilon) := \varepsilon\int_{\varepsilon^2}^{\varepsilon^2\ln(1/\varepsilon)}\exp\left(-c_1\frac{|w(\zeta_w-s)|^2}{s}\right)s^{-3/2}\mathbf{1}_{\{s\geq\varepsilon^2c_2^{-2}\}}ds.$$

There exist positive constants c_5,c_6,c_7 such that for any $w\in\mathcal{W}$, $\varepsilon_2\in(0,1/e)$ such that $\zeta_w\geq\varepsilon_2^2\ln(1/\varepsilon_2)$, the following holds.

(a): *For any $\varepsilon\in(0,\varepsilon_2)$ and for any $p\in\mathbb{N}$*

$$c_5^pp^p\geq\Pi^*_w(Z_w(\varepsilon)^p)\geq c_6^pp^pI_w(\varepsilon).$$

(b): *For any $A>0$, let $p=\lceil 2A/c_6\rceil$. Then, for any $\varepsilon\in(0,\varepsilon_2)$*

$$\Pi^*_w(Z_w(\varepsilon)\geq A)\geq\exp(-c_7A)\left((2^pI_w(\varepsilon)-1)^+\right)^2.$$

The lower bounds on $\Pi^*_w(Z_w(\varepsilon)^p), p\in\mathbb{N}$ in Lemma V.13 (a) are a direct consequence of Lemma V.12. Furthermore, the proof of the upper bound in Lemma V.13 (a) easily adapts from the one of Lemma 3.4 in [**LGP 95**]. Then, part (b) of Lemma V.13 is deduced from part (a) in the exact same manner as, in [**LGP 95**], Lemma 3.5 is deduced from Lemma 3.4. We leave details to the reader. □

4.4.4. Let us now complete the proof of Lemma V.8 by establishing Claim 3. We let $\alpha=1/(4c_7)$ and fix $\delta>0$. Note that T_0 is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq0}$. Using the strong Markov property (153) at time T_0, we have, for $n_0>0$,

$$\begin{aligned}(160)\quad &\mathbb{N}_x\left[\exists j\in[\![2^{n_0},2^{n_0+1}-1]\!]:Y_{\zeta_{T_0}}(C(0,r_j/2,r_j))>\alpha h(r_j),\ T_0<\infty\right]\\ &\geq\ \mathbb{N}_x\left[\Pi^*_{W_{T_0}}\left(\exists j\in[\![2^{n_0},2^{n_0+1}-1]\!]:Y_r(C(0,r_j/2,r_j))>\alpha h(r_j)\right)_{r=\zeta_{T_0}},T_0<\infty\right].\end{aligned}$$

Notice that (160) is an inequality and not an equality, because we used that

$$Y_r\left(\mathcal{C}(0,r_j/2,r_j)\right)=\int_0^{\sigma(\zeta)}d_sL^r_s\mathbf{1}_{\mathcal{C}(0,r_j/2,r_j)}(\hat{W}_s)\geq\int_{T_0}^{\sigma(\zeta)}d_sL^r_s\mathbf{1}_{\mathcal{C}(0,r_j/2,r_j)}(\hat{W}_s),$$

on the event $\{T_0<\infty\}$.

Introduce the sequence $u_j:=r^2_{2^j}\ln(1/r_{2^j})$, and choose n_2 large enough so that

$$r_{2^{n_2}}\leq e^{-4}\qquad\text{and}\qquad\mathbb{N}_x\left[\zeta_{T_0}\geq s_{n_2}\middle|T_0<\infty\right]\leq\delta/4.$$

Let us set

$$\mathfrak{W}:=\{w\in\mathcal{W}\ :\ w(0)=x,\hat{w}=0,\zeta_w\geq s_{n_2}\}.$$

Our choice of n_2 ensures that

$$\mathbb{N}_x[W_{T_0}\in\mathfrak{W}|T_0<\infty]\geq1-\delta/4. \tag{161}$$

Since $c_2>1/2$, it also guarantees that for any $w\in\mathfrak{W}$, $j\geq2^{n_2}$, $2r_j^2c_2^{-2}\leq\zeta_w$. We can then define, for $B>0$, $j\geq2^{n_2}$,

$$\mathscr{W}_{B,j}:=\left\{w\in\mathfrak{W}\ :\ \sup_{s\leq2r_j^2c_2^{-2}}|w(\zeta(w)-s)|>Br_jc_2^{-1}\right\}.$$

For $w \in \mathfrak{W}$, $n \geq n_2$, we then introduce

$$F_{n,B}(w) := 2^{-n} \sum_{j=2^n}^{2^{n+1}-1} \mathbf{1}_{\{w \in \mathscr{W}_{B,j}\}},$$

and for $n \geq n_2$, we finally let

$$\mathbb{W}_n := \{w \in \mathfrak{W} \ : \ \forall p \geq n \quad F_{p,B}(w) < 1/2\}.$$

From (161), it follows that

$$\mathbb{N}_x\left[W_{T_0} \notin \mathbb{W}_n \middle| T_0 < \infty\right] \leq \frac{\delta}{4} + \sum_{p \geq n} \mathbb{N}_x\left[W_{T_0} \in \mathfrak{W}, F_{p,B}(W_{T_0}) \geq 1/2 \middle| T_0 < \infty\right].$$

Using Lemma V.11 and following the arguments of the proof of Lemma 1 in [**LG 94**], one can easily establish that there exist constants $B > 0, K > 0$ such that

$$\mathbb{N}_x\left[W_{T_0} \in \mathfrak{W}, F_{p,B}(W_{T_0}) \geq 1/2 | T_0 < \infty\right] \leq K e^{-n}.$$

Hence, there exists $n_3 \geq n_2$ large enough so that for any $n \geq n_3$,

$$\mathbb{N}_x\left[W_{T_0} \notin \mathbb{W}_n \middle| T_0 < \infty\right] \leq \delta/2, \tag{162}$$

which yields

$$\begin{aligned} &\mathbb{N}_x\left[\Pi^*_{W_{T_0}}\left(\forall j \in [\![2^n, 2^{n+1}-1]\!] : Y_r(C(0, r_j/2, r_j)) < \alpha h(r_j)\right)_{r=\zeta_{T_0}} \middle| \ T_0 < \infty\right] \\ (163) \leq\ & \delta/2 + \sup_{w \in \mathbb{W}_n} \left\{\Pi^*_w\left[\exists j \in [\![2^n, 2^{n+1}-1]\!] : Y_{\zeta_w}(C(0, r_j/2, r_j)) < \alpha h(r_j)\right]\right\}. \end{aligned}$$

Let $n \geq n_3$ and $w \in \mathbb{W}_n$. Since $r_j^2 \geq r_{j+1}^2 \ln(1/r_{j+1})$ for $2^n \leq j \leq 2^{n+1}-1$, the independence properties of Poisson measures imply that for any $n \in \mathbb{N}$, the variables $Z_w(r_j), 2^n \leq j \leq 2^{n+1}-1$ are independent under Π^*_w. Using (158) and the definition of $Z_w(\varepsilon)$, we then get

$$\begin{aligned} (164)\ \ &\Pi^*_w\left[\forall j \in [\![2^n, 2^{n+1}-1]\!] : Y_{\zeta_{T_0}}(C(0, r_j/2, r_j)) \geq \alpha h(r_j)\right] \\ \leq\ \ &\Pi^*_w\left[Z_w(r_j) \leq \alpha\theta(r_j)\ \forall j \in [\![2^n, 2^{n+1}-1]\!]\right] = \prod_{j=2^{n_0}}^{2^{n_0+1}-1} \Pi^*_w\Big(Z_w(r_j) \leq \alpha\theta(r_j)\Big), \end{aligned}$$

where we set, for $r > 0$, $\theta(r) := r^{-2}h(r)$. Furthermore, Lemma V.13 (b) leads to

$$\begin{aligned} \prod_{j=2^{n_0}}^{2^{n_0+1}-1} \Pi^*_w\Big(Z_w(r_j) \leq \alpha\theta(r_j)\Big) &\leq \prod_{j=2^{n_0}}^{2^{n_0+1}-1} \left(1 - e^{-c_7\alpha\theta(r_j)}\left(\left(2^{p_j} I_w(r_j) - 1\right)^+\right)^2\right) \\ &\leq \exp\left\{-\sum_{j=2^n}^{2^{n+1}-1} j^{-2c_7\alpha}\left(\left(2^{p_j} I_w(r_j) - 1\right)^+\right)^2\right\}. \end{aligned} \tag{165}$$

We then note that, if $j \geq 2^{n_2}$ and $w \in \mathfrak{W} \setminus \mathscr{W}_{B,j}$, an easy computation provides

$$I_w(r_j) \geq c_2 \exp\left(-c_1 B^2\right) =: K(B).$$

Hence, from our choice of α, there exists $n_4 \geq n_3$ so that for any $w \in \mathbb{W}_n$, $n \geq n_4$, one has

$$\sum_{j=2^n}^{2^{n+1}-1} j^{-2c_7\alpha} \left((2^{p_j} I_w(r_j) - 1)^+ \right)^2 \geq 2^{n+1/2}.$$

We finally choose $n_0 > n_1 \geq n_4 \geq n_3 \geq n_2$ large enough so that $\exp(-2^{n+1/2}) \leq \delta/2$, and combine (160), (163), (164) and (165) with the above inequality to obtain Claim 3. As explained in paragraph 4.4.2, Lemma V.8 follows. □

4.4.5. *The case $d = 2$.* We know from [**Pe 99**], Section III.3 that for any $t > 0$, h is for $d = 3$ the correct Hausdorff measure function of $\mathcal{R}_t$. On the other hand, when $d = 2$, the correct Hausdorff measure function of $\mathcal{R}_t$, is, as it is proven in [**LGP 95**], the function

$$h_2(\varepsilon) = \varepsilon^2 \ln(\varepsilon^{-1}) \ln(\ln(\ln(\varepsilon^{-1}))).$$

Not surprinsingly, when $d = 2$, one can in fact establish a stronger result than Lemma V.8.

LEMMA V.14. *We can choose $\alpha > 0$ so that, for any $\delta > 0$, there exists $\varepsilon_1 \in \left(0, 1 \wedge \frac{|x|}{2}\right)$ such that for any $\varepsilon_0 \in (0, \varepsilon_1)$,*

$$\mathbb{N}_0 \left[\exists s \geq 0 \ \exists \ \varepsilon \in (g_d(\varepsilon_0), \varepsilon_0) : Y_s \left(\mathcal{C}\left(x, \frac{\varepsilon}{2}, \varepsilon\right)\right) > \alpha h_2(\varepsilon) \ \middle| \ x \in \mathcal{R} \right] \geq 1 - \delta.$$

Lemma V.14 clearly implies the two-dimensional case of Lemma V.8. The proof is similar to that of Lemma V.8 in the three-dimensional case. Let us only point out the main differences, and leave details to the reader.

Obviously, one should work with h_2 instead of h, g_2 instead of g and the function θ_2 such that $\theta_2(r) := \ln \ln \ln(1/r)$ instead of θ. Moreover, the sequence r_j is to be replaced with $r_j^{(2)} = 2^{-2^j}$, so that $r^{(2)}_{2^{n+1}-1} \geq g_2(r^{(2)}_{2^n})$. We already noted that Lemma V.12 for the three-dimensional case corresponds to Corollary 3.3 of [**LGP 95**] in the plane. In particular, note that c_1, c_2, c_4 should be replaced with $c_1^{(2)} > 1/2, c_2 = (4c_1^{(2)} - 2)/(4c_1^{(2)} - 1), c_4^{(2)} = c_3^{(2)}(2c_1^{(2)})^{-1}$. We also already remarked that Lemma V.13 (a) and (b) are to be respectively related with Lemma 3.4, respectively Lemma 3.5 of [**LGP 95**]. Lemma V.13 remains valid in the plane when one replaces $Z_w(\varepsilon)$ with

$$Z_w^{(2)}(\varepsilon) := \left(\varepsilon^2 \ln(\varepsilon^{-1})\right)^{-2} \int_{\zeta_w - \varepsilon^2 \ln(1/\varepsilon)}^{\zeta_w - \varepsilon^2} Y_{\zeta_w - t}(\mathcal{C}(0, \varepsilon/2, \varepsilon)) \Lambda(dt, dW),$$

then $I_w(\varepsilon)$ with

$$I_w^{(2)}(\varepsilon, p) := p \left(\log(\varepsilon^{-1})\right)^{-p} \times \int_{\varepsilon^2}^{\varepsilon^2 \ln(1/\varepsilon)} \exp\left(-c_1^{(2)} \frac{|w(\zeta_w - s)|^2}{s}\right) \left(\log^+\left(s (c_2^{(2)})^p \varepsilon^{-2}\right)\right)^{p-1} s^{-1} ds.$$

Is is then straightforward to check that, once all these changes have been made, the exact same proof as in paragraph 4.4.4 leads to assertions similar to (162), (160), (161), (163), (164), (165) and (3).

5. Results on coalescing random walks

In this section, we prove Lemmas V.1 and V.2, which we used in the proof of Theorem V.1. First, let us introduce further notation. We write $\mathbb{P}_x^{(2)}$ for a probability measure under which $(Z_t, t \geq 0)$ is a continuous time random walk with rate 2 (instead of rate 1 for $\mathbb{P}_x$) and jump kernel p, starting from x. Let us also denote

$$H_t(x) := \mathbb{P}_x(Z \text{ hits } 0 \text{ before } t); \qquad G_t(x) := \mathbb{E}_x\left[\int_0^t \mathbf{1}_{\{Z_s=0\}} ds\right];$$

and $H_t^{(2)}(x), G_t^{(2)}(x)$ the corresponding quantities under $\mathbb{P}_x^{(2)}$. It is well-known that when $d = 2$,

$$G^{(2)}_{c^2 t}(0) \underset{c\to\infty}{\sim} G^{(2)}_{c^2\varepsilon}(0) \underset{c\to\infty}{\sim} \frac{1}{2\pi}\ln(c). \tag{166}$$

When $d = 3$, from the definition of k_d, we have

$$G^{(2)}_{c^2 t}(0) \underset{c\to\infty}{\longrightarrow} k_d^{-1}. \tag{167}$$

In dimension $d = 2$, an easy adaptation of [**La 91**], Theorem 1.6.1, to the continuous time setting, ensures the existence of

$$a(x) := \lim_{t\to\infty}[G_t(0) - G_t(x)], \quad a^{(2)}(x) := \lim_{t\to\infty}[G_t^{(2)}(0) - G_t^{(2)}(x)].$$

An easy consequence of the proof of Theorem 1.6.2 in [**La 91**] is the existence of a constants L, L' depending only on p such that

$$G_t(0) - G_t(x) \leq La(x), \quad G_t^{(2)}(0) - G_t^{(2)}(x) \leq La^{(2)}(x). \tag{168}$$

Furthermore, from Theorem 1.6.2 of [**La 91**], both $a(x)$ and $a^{(2)}(x)$ are $O(\ln|x|)$.

5.1. Proof of Lemma V.1. This proof is due to [**LG u**]. In the following, we denote by $C_i, i \geq 0$ positive constants depending only on d. Let us denote by $(S_n, n \in \mathbb{N})$ a discrete time random walk with jump kernel p, starting from x under the probability measure $\mathbb{Q}_x$. By combining the well-known bound
$\mathbb{Q}_x[S_n = y] \leq Kn^{-d/2}$ and the martingale inequality of Ledoux and Talagrand ([**LT 91**], Lemma 1.5), we get for any $n \geq 1, y \in \mathbb{Z}^d$,

$$\mathbb{Q}_x[S_n = y] \leq \frac{C_1}{n^{d/2}} \exp\left(-\frac{C_2|y-x|^2}{n}\right).$$

Let $(N_t, t \geq 0)$ be a standard Poisson process. Then

$$\begin{aligned} \mathbb{Q}_0[Z_t = y] &= \sum_{n=0}^{\infty} P[N_t = n]\mathbb{Q}_0[S_n = y] \\ &\leq \exp(-t)\mathbf{1}_{\{y=0\}} + \sum_{n=1}^{\infty} P[N_t = n]\frac{C_1}{n^{d/2}}\exp\left(-\frac{C_2|y|^2}{n}\right). \end{aligned} \tag{169}$$

We also have for any $t \geq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{d/2}} P[N_t = n] \leq C_3 t^{-d/2}.$$

It follows from the above that

$$\sum_{n=1}^{\lfloor 2t\rfloor} P[N_t = n]\frac{C_1}{n^{d/2}}\exp\left(-\frac{C_2|y|^2}{n}\right) \le \frac{C_1C_3}{t^{d/2}}\exp\left(-\frac{C_2|y|^2}{2t}\right). \tag{170}$$

For values of n greater then $2t$, a simple large deviation estimate gives for every $t > 0, m \ge 1$,

$$P[N_t \ge 2^m t] \le C_4 \exp(-C_5 2^m t).$$

Hence,

$$\begin{aligned} & \sum_{n>2t} P[N_t = n]n^{-d/2}\exp\left(-\frac{C_2|y|^2}{n}\right) \\ \le \quad & \sum_{m=1}^{\infty} C_4 \exp(-C_5 2^m t)(2^m t)^{-d/2}\exp\left(-\frac{C_2|y|^2}{2^{m+1}t}\right) \\ = \quad & C_4 t^{-d/2}\sum_{m=1}^{\infty} 2^{-md/2}\exp\left(-C_5 2^m t - \frac{C_2|y|^2}{2^{m+1}t}\right). \end{aligned}$$

Setting $C_6 = \sqrt{2C_2C_5}$ we now obtain from the above

$$\sum_{n>2t} P[N_t = n]n^{-d/2}\exp\left(-\frac{C_2|y|^2}{n}\right) \le C_7 t^{-d/2}\exp(-C_6|y|).$$

Combining (169), (170) and the above now gives

$$q_t(y) \le \exp(-t)\mathbf{1}_{\{y=0\}} + \frac{C_1C_3}{t^{d/2}}\exp\left(-\frac{C_2|y|^2}{2t}\right) + \frac{C_7}{t^{d/2}}\exp(-C_6|y|),$$

which clearly implies Lemma V.1. $\square$

5.2. Proof of Lemma V.2. In the following, we use K, K' to denote positive constants depending only on d, T and which may change from line to line. We consider only the case when c and x are such that $cx \in \mathbb{Z}^d \setminus 0$. The general case immediately follows.

We are first going to rule out small values of t. We deal with the integral over the interval $[0, c^{-2}]$. Note that

$$\mathbb{P}_{0,y}\left[T_1 \le c^2 t, Z^1_{c^2t} = cx\right] \le \min\left\{q_{c^2t}(0, cx), q_{c^2t}(y, cx)\right\}.$$

Considering separately the cases $|y| \le 2c|x|$ and $|y| > 2c|x|$ and using (96), we easily obtain

$$\begin{aligned} & c^d\psi_d(|y|)\int_0^{c^{-2}} dt\mathbb{P}_{0,y}\left[T_1 \le c^2 t, Z^1_{c^2t} = cx\right] \\ \le \quad & \kappa_1 c^{d-2}\exp(-\kappa_2 c|x|/2) \times \psi_d(|y|)\exp(-\kappa_2(|y|/4)) \\ \le \quad & K|x|^{2-d} \le K\psi_d(|x|^{-1}). \end{aligned}$$

Let us now deal with small values of T_1. Ina similar way as in the prvious computation, one gets

$$\begin{aligned} & c^d\psi_d(|y|)\int_0^{c^{-2}} dt\mathbb{P}_{0,y}\left[T_1 \le 1, Z^1_{c^2t} = cx\right] \\ \le \quad & \kappa_1 c^d\exp(-\kappa_2 c|x|/2) \times \psi_d(|y|)\exp(-\kappa_2(|y|/4\sqrt{2})) \\ \le \quad & K|x|^{2-d} \le K\psi_d(|x|^{-1}). \end{aligned}$$

Note that to obtain the last line above, we used the assumption $c \geq |x|^{-2}$.

Let us then deal with large values of T_1. For $t \geq c^{-2}$, we have

$$\begin{aligned} & c^d \psi_d(|y|) \mathbb{P}_{0,y}\left[c^2 t - 1 \leq T_1 \leq c^2 t, Z^1_{c^2 t} = cx\right] \\ \leq \;& e c^d \psi_d(|y|) \mathbb{P}_{0,y}\left[c^2 t - 1 \leq T_1 \leq c^2 t, Z^1_{c^2 t} = Z^1_{T_1} = cx\right] \\ \leq \;& e c^d \psi_d(|y|) \mathbb{P}_{0,y}\left[Z^1_{c^2 t} = Z^2_{c^2 t} = cx\right] \\ \leq \;& e c^{-d} \psi_d(|y|) f_t(x) f_t(x - y/c), \end{aligned}$$

where we used Lemma V.1 at the last line above. Hence,

$$\begin{aligned} & c^d \psi_d(|y|) \int_{c^{-2}}^{T} dt \mathbb{P}_{0,y}\left[c^2 t - 1 \leq T_1 \leq c^2 t, Z^1_{c^2 t} = cx\right] \\ \leq \;& K c^{-d} \psi_d(|y|) \left(|x|^{2-2d} \wedge \left(\frac{|y|}{c}\right)^{2-2d} \right) \leq K |x|^{2-d} \leq K \psi_d(|x|^{-1}), \end{aligned}$$

using (93) then the assumption $c \geq |x|^{-2}$ at the last line above.

We can now suppose $1 \leq c^2 t - 1$, that is $t \geq 2c^{-2}$, and restrict our attention to estimating

$$\int_{2c^{-2}}^{T} dt \mathbb{P}_{0,y}\left[1 \leq T_1 \leq c^2 t - 1, Z^1_{c^2 t} = cx\right].$$

Using the Markov property at time T_1, then Lemma V.1, we obtain

$$\begin{aligned} & \int_{2c^{-2}}^{T} dt\ \mathbb{P}_{0,y}\left[1 \leq T_1 \leq c^2 t - 1, Z^1_{c^2 t} = cx\right] \\ = & \int_{2c^{-2}}^{T} dt\ \mathbb{E}_{0,y}\left[\mathbf{1}_{\{1 \leq T_1 \leq c^2 t - 1\}} \sum_{z \in \mathbb{Z}^d} \mathbf{1}_{\{Z^1_{T_1} = z\}} q_{c^2 t - T_1}(z - cx)\right] \\ \leq & K c^{-d} \int_{2c^{-2}}^{T} dt \sum_{z \in \mathbb{Z}^d} \mathbb{E}_{0,y}\left[\mathbf{1}_{\{1 \leq T_1 \leq c^2 t - 1, Z^1_{T_1} = z\}} f_{t - \frac{T_1}{c^2}}\left(\frac{z}{c} - x\right)\right]. \end{aligned} \tag{171}$$

In order to bound the above quantity, we need the following intermediate result.

Lemma V.15. *Let $d \geq 2$. There exists a positive constant L_2 depending only on d such that for any $c \geq 1$, $z \in \mathbb{Z}^d$, $t \geq 2c^{-2}$, $y \in \mathbb{Z}^d \setminus 0$ and every measurable function $\phi : \mathbb{R}_+ \times \mathbb{Z}^d \to \mathbb{R}_+$,*

$$\begin{aligned} & |y|^d \psi_d(|y|) \mathbb{E}_{0,y}\left[\mathbf{1}_{\{1 \leq T_1 \leq c^2 t - 1, Z^1_{T_1} = z\}} \phi\left(\frac{T_1}{|y|^2}, Z^1_{T_1}\right)\right] \\ & \leq L_2 \int_{|y|^{-2}}^{\frac{tc^2}{|y|^2} - \frac{1}{2|y|^2}} du\ \Phi(u, z)\, \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z - y}{|y|}\right), \end{aligned}$$

where $\Phi(u, z) = \sup_{\left(u - \frac{1}{2|y|^2}\right)^+ \leq r \leq u} \phi(r, z)$, and $\tilde{f}_u$ was defined in Section 2.4.

Proof of lemma V.15. In this proof, we use L to denote a constant depending only on d and which may change from line to line. Obviously,

$$\phi(r, z) \leq 2|y|^2 \int_{r}^{r + \frac{1}{2|y|^2}} \Phi(u, z) du.$$

It follows that

$$
\begin{aligned}
&|y|^d\psi_d(|y|)\mathbb{E}_{0,y}\left[\mathbf{1}_{\{1\le T_1\le c^2t-1, Z^1_{T_1}=z\}}\phi\left(\frac{T_1}{|y|^2}, Z^1_{T_1}\right)\right]\\
&\le 2|y|^{d+2}\psi_d(|y|)\mathbb{E}_{0,y}\Bigg[\mathbf{1}_{\{1\le T_1\le c^2t-1, Z^1_{T_1}=z\}}\\
&\qquad\qquad\qquad\times\int_{|y|^{-2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\Phi(u,z)\mathbf{1}_{\{\frac{T_1}{|y|^2}\le u\le \frac{T_1}{|y|^2}+\frac{1}{2|y|^2}\}}\Bigg]\\
&= 2|y|^{d+2}\psi_d(|y|)\int_{|y|^{-2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\Phi(u,z)\\
&\qquad\qquad\qquad\times\mathbb{P}_{0,y}\left[1\le T_1\le u|y|^2\le T_1+\frac{1}{2}\le c^2t-\frac{1}{2}, Z^1_{T_1}=z\right].
\end{aligned}
\tag{172}
$$

where we use the Fubini theorem at the last line. Hence, proving Lemma V.15 reduces to establishing that, if $u>|y|^{-2}$,

$$
\begin{aligned}
&2|y|^{d+2}\psi_d(|y|)\mathbb{P}_{0,y}\left[1\le T_1\le u|y|^2\le T_1+\frac{1}{2}\le c^2t-\frac{1}{2}, Z^1_{T_1}=z\right]\\
&\le L_2\tilde f_u\left(\frac{z}{|y|}\right)\tilde f_u\left(\frac{z-y}{|y|}\right).
\end{aligned}
\tag{173}
$$

The case $d\ge 3$ is simple, noticing that

$$
\mathbb{P}_{0,y}\left[T_1\le |y|^2u\le T_1+\frac{1}{2}, Z^1_{T_1}=z\right]\le e q_{|y|^2u}(z)q_{|y|^2u}(y-z),
$$

and using Lemma 1 to conclude.

Let us now give the proof of (173) in the case $d=2$. Using the same argument as in the case $d\ge 3$ only gives

$$
\begin{aligned}
&|y|^4\ln(|y|\vee e)\mathbb{P}_{0,y}\left[1\le T_1\le |y|^2u\le T_1+\frac{1}{2}, Z^1_{T_1}=z\right]\\
&\le e\ln(|y|\vee e)f_u\left(\frac{z}{|y|}\right)f_u\left(\frac{z-y}{|y|}\right).
\end{aligned}
$$

However, in the particular cases when $|y|\le A$ for some fixed constant $A\ge 1$, or when $|y|^{-2}\le u\le |y|^{-1}$, we have

$$
\ln(|y|\vee e)\exp\left(-\frac{a_2|z|}{2|y|\sqrt{u}}\right)\exp\left(-\frac{a_2|z-y|}{2|y|\sqrt{u}}\right)\le L.
$$

This easily leads to (173) in these particular cases. Thus, it only remains to establish (173) for $|y|\ge A:=6^8$ and $u\ge |y|^{-1}$.

We have

$$
\begin{aligned}
&\mathbb{P}_{0,y}\left[1\le T_1\le |y|^2u\le T_1+\frac{1}{2}, Z^1_{T_1}=z\right]\\
&\le e\mathbb{P}_{0,y}\left[1\le T_1\le |y|^2u\le T_1+\frac{1}{2}, Z^1_{|y|^2u}=Z^2_{|y|^2u}=z\right]\\
&\le e\mathbb{P}_{z,z}\left[Z^1_s\ne Z^2_s\ \forall s\in[1,|y|^2u], Z^1_{|y|^2u}=0, Z^2_{|y|^2u}=y\right],
\end{aligned}
\tag{174}
$$

where we used a time-reversal argument in the last line.

Let us give the informal sketch of the proof of (173). We are going to use (174) and argue under $\mathbb{P}_{z,z}$. With high probability, both $Z^1_{|y|^{3/2}(u\wedge 1)}$ and $Z^2_{|y|^{3/2}(u\wedge 1)}$ should remain close to z (see the definition of $B_{z,y}$ and (175) below). Thus, whenever $Z^1_{|y|^{3/2}(u\wedge 1)}$ or $Z^2_{|y|^{3/2}(u\wedge 1)}$ happen to be too far from z, we should be able to conclude easily (see (176) below). On the other hand, when both $Z^1_{|y|^{3/2}(u\wedge 1)}, Z^2_{|y|^{3/2}(u\wedge 1)}$ are close to z, we will use Lemma V.1 to get that the probability for two walks started respectively at $Z^1_{|y|^{3/2}(u\wedge 1)}$ and at $Z^2_{|y|^{3/2}(u\wedge 1)}$ to be respectively at 0 and at y at time $|y|^2u - |y|^{3/2}(u\wedge 1)$ is bounded above by $|y|^{-4}\tilde{f}_u(z/|y|)\tilde{f}_u((z-y)/|y|)$ (see (178) below). We will conclude by proving that for $u \geq |y|^{-1}$, the probability for (Z^1, Z^2) to avoid each other in the time interval $[1, |y|^{3/2}(u\wedge 1)]$ should be of order $\ln(|y|\vee e)^{-1}$ (see (179) below).

Let us be more precise. For the sake of a lighter notation, we set

$$B_{z,y} := B\left(z, |y|^{7/8}\left(\frac{|z|}{3|y|}\vee 1\right)\right), \qquad t(u,y) := |y|^2u - |y|^{3/2}(u\wedge 1).$$

We first deal with the case when either $Z^1_{|y|^{3/2}(u\wedge 1)}$ or $Z^2_{|y|^{3/2}(u\wedge 1)}$ escapes $B_{z,y}$. As a consequence of (96),

$$\begin{aligned}
&\ln(|y|)\mathbb{P}_{z,z}\Big[Z^1_{|y|^{3/2}(u\wedge 1)} \notin B_{z,y} \text{ or } Z^2_{|y|^{3/2}(u\wedge 1)} \notin B_{z,y}\Big]\\
&\leq 2\kappa_1 \ln(|y|)\exp\left(-\kappa_2\frac{|y|^{1/8}}{\sqrt{u\wedge 1}}\left(\frac{|z|}{3|y|}\vee 1\right)\right)\\
&\leq L\exp\left(-\kappa_2\frac{|y|^{1/8}}{2\sqrt{u\wedge 1}}\left(\frac{|z|}{3|y|}\vee 1\right)\right),
\end{aligned} \tag{175}$$

where at the last line, we used the bound $\ln(|y|)\exp\left(-\kappa_2|y|^{1/8}/2\right) \leq L$. By studying separately the cases $|z| \geq 3|y|$, $|z| \leq 3|y|$, and using the fact that $\kappa_2/4 \geq a_2$, we get

$$\exp\left(-\kappa_2\frac{|y|^{1/8}}{2\sqrt{u\wedge 1}}\left(\frac{|z|}{3|y|}\vee 1\right)\right) \leq \exp\left(-a_2\frac{|z|}{4|y|\sqrt{u}}\right)\exp\left(-a_2\frac{|z-y|}{4|y|\sqrt{u}}\right),$$

Furthermore, since $t(u,y) \geq |y|^2u/2$, we get from Lemma V.1 that

$$\sup\{|y|^2 q_{t(u,y)}(x,x'), x \in \mathbb{Z}^d, x' \in \mathbb{Z}^d\} \leq Lu^{-1}$$

Hence, using the Markov property for the walks Z^1, Z^2 at time $|y|^{3/2}(u\wedge 1)$, we deduce that

$$\begin{aligned}
&\ln(|y|)|y|^4\mathbb{P}_{z,z}\Big[Z^1_{|y|^{3/2}(u\wedge 1)} \notin B_{z,y} \text{ or } Z^2_{|y|^{3/2}(u\wedge 1)} \notin B_{z,y}, Z^1_{|y|^2u} = 0, Z^2_{|y|^2u} = y\Big]\\
&\leq L\tilde{f}_u\left(\frac{z}{|y|}\right)\tilde{f}_u\left(\frac{z-y}{|y|}\right).
\end{aligned} \tag{176}$$

To complete the proof of (173), it remains to deal with the case when both $Z^1_{|y|^{3/2}(u\wedge 1)}$ and $Z^2_{|y|^{3/2}(u\wedge 1)}$ belong to $B_{z,y}$. Using again the Markov property for the walks Z^1, Z^2 at time $|y|^{3/2}(u \wedge 1)$, we obtain

$$\begin{aligned} &\mathbb{P}_{z,z}\Big[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^{3/2}(u\wedge 1)],\ Z^1_{|y|^2 u} = 0,\ Z^2_{|y|^2 u} = y, \\ &\qquad\qquad Z^1_{|y|^{3/2}(u\wedge 1)} \in B_{z,y},\ Z^2_{|y|^{3/2}(u\wedge 1)} \in B_{z,y}\Big] \\ (177)\ &\leq \mathbb{P}_{z,z}\Big[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^{3/2}(u\wedge 1)]\Big] \sup_{(x_1,x_2)\in B^2_{z,y}} q_{t(u,y)}(x_1) q_{t(u,y)}(x_2 - y). \end{aligned}$$

We now establish that

$$\sup_{(x_1,x_2)\in B^2_{z,y}} |y|^4 q_{t(u,y)}(x_1) q_{t(u,y)}(x_2 - y) \leq \tilde{f}_u\left(\frac{z}{|y|}\right) \tilde{f}_u\left(\frac{z-y}{|y|}\right); \tag{178}$$

Lemma V.1 implies that

$$\sup_{(x_1,x_2)\in B_{z,y}} q_{t(u,y)}(x_1) q_{t(u,y)}(x_2 - y) \leq 4|y|^{-4} \sup_{(x_1,x_2)\in B^2_{z,y}} f_u\left(\frac{x_1}{|y|}\right) f_u\left(\frac{x_2-y}{|y|}\right).$$

The bound (178) in the case $|z| \geq 3|y|$ easily follows from the fact that for x_1, x_2 both in $B_{z,y}$, we have $|x_1| \geq 2|z|/3$, $|x_2 - y| \geq |z|/3 \geq |z-y|/4$.

Let us now suppose $|z| \leq 3|y|$, and recall that we assumed $|y| \geq 6^8$. Consequently the balls $B(0, 3|y|^{7/8})$ and $B(y, 3|y|^{7/8})$ are disjoint. Now, if $z \in B(0, 2|y|^{7/8})$ we easily see that for any $x_2 \in B(z, |y|^{7/8})$, we have $|x_2 - y| \geq |z-y|/2 \geq |z|/2$. It follows that

$$\exp\left(-\frac{a_2|x_2-y|}{|y|\sqrt{u}}\right) \leq \exp\left(-\frac{a_2|z-y|}{4|y|\sqrt{u}}\right)\exp\left(-\frac{a_2|z|}{4|y|\sqrt{u}}\right).$$

Together with the preceeding bound this gives (178) in the case $z \in B(0, 2|y|^{7/8})$. We can use a similar argument to conclude in the case $z \in B(y, 2|y|^{7/8})$. Finally, if z is not in any of these balls, for any $x_1, x_2 \in B(z, |y|^{7/8})$ we have $|x_1| \geq |z|/2$, $|x_2 - y| \geq |z-y|/2$, and (178) easily follows.

Finally, we establish that

$$\ln(|y| \vee e)\mathbb{P}_{z,z}\Big[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^{3/2}(u\wedge 1)]\Big] \leq L. \tag{179}$$

First note that under $\mathbb{P}_{z,z}$, $Z^1 - Z^2$ has law $\mathbb{P}^{(2)}_0$. Recall that the notation $a^{(2)}(x)$, $G^{(2)}_x$, $H^{(2)}_t(x)$ have been introduced at the beginning of the section. From the simple bound $H^{(2)}_t(x) \geq G^{(2)}_t(0)^{-1} G^{(2)}_t(x)$, then (168), we get

$$1 - H^{(2)}_{|y|}(x) \leq \frac{G^{(2)}_{|y|}(0) - G^{(2)}_{|y|}(x)}{G^{(2)}_{|y|}(0)} \leq \frac{La^{(2)}(x)}{\ln(|y|)}. \tag{180}$$

From the Markov property for $Z^1 - Z^2$ at time 1, we have

$$\begin{aligned}
&\mathbb{P}_{z,z}\left[Z^1_s \neq Z^2_s \ \forall s \in [1, |y|^{3/2}(u\wedge 1)]\right] \\
=\ & \mathbb{E}_{z,z}\left[\sum_{x\in\mathbb{Z}^d} \mathbf{1}_{\{Z^1_1 - Z^2_1 = x\}}\mathbf{1}_{\{Z^1_s - Z^2_s \neq 0\ \forall s\in[1,|y|^{3/2}(u\wedge 1)]\}}\right] \\
=\ & \sum_{x\in\mathbb{Z}^d} q_2(x)\left(1-\mathbb{P}^{(2)}_x\left(Z \text{ hits } 0 \text{ before time } |y|^{3/2}(u\wedge 1)\right)\right) \\
\leq\ & \sum_{x\neq 0} q_2(x)\frac{La^{(2)}(x)}{\ln\left(|y|^{3/2}(u\wedge 1)\right)}
\end{aligned}$$

where we used (180) at the last line. Since $u \geq |y|^{-1}$, we have $\ln\left(|y|^{3/2}(u\wedge 1)\right) \geq \ln(|y|)/2$. Hence, using Lemma V.1 and the fact that $a^{(2)}(x) = O(\ln(|x|))$, we get (179).

As explained earlier on, (173) is a consequence of assertions (174), (176), (177), (178) and (179), and Lemma V.15 follows. $\square$

Let us now complete the proof of Lemma V.2. We will apply Lemma V.15 to bound the right-hand side of (171). Fix $t \in [2c^{-2}, T]$. Let us consider the nonnegative functions $\Phi_c(u,z) = f_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c} - x\right)$. For $u \in [\frac{1}{|y|^2}, \frac{2tc^2-1}{2|y|^2}]$ we have

$$\Phi_c(u,z) := \sup_{u-(2|y|^2)^{-1}\leq r\leq u} \phi_c(r,z) \leq \tilde{f}_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c} - x\right).$$

Thus, from (171) and Lemma V.15, it follows that

$$\begin{aligned}
(181)\qquad & c^d\psi_d(|y|)\int_{2c^{-2}}^T dt\mathbb{P}_{0,y}\left[1\leq T_1\leq c^2t-1, Z^1_{c^2t} = cx\right] \\
&\leq K|y|^{-d}\int_{2c^{-2}}^T dt\sum_{z\in\mathbb{Z}^d}\int_{|y|^{-2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\tilde{f}_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c}-x\right)\tilde{f}_u\left(\frac{z}{|y|}\right)\tilde{f}_u\left(\frac{z-y}{|y|}\right) \\
&\leq K|y|^{-d}\int_{2c^{-2}}^T dt\sum_{z\in\mathbb{Z}^d}\int_{|y|^{-2}}^{\frac{tc^2}{2|y|^2}} du\hat{f}_t\left(\frac{z}{c}-x\right)\tilde{f}_u\left(\frac{z}{|y|}\right)\tilde{f}_u\left(\frac{z-y}{|y|}\right) \\
&+K|y|^{-d}\int_{2c^{-2}}^T dt\sum_{z\in\mathbb{Z}^d}\int_{\frac{tc^2}{2|y|^2}}^{\frac{tc^2}{|y|^2}-\frac{1}{2|y|^2}} du\tilde{f}_{t-\frac{u|y|^2}{c^2}}\left(\frac{z}{c}-x\right)\hat{f}_{\frac{tc^2}{|y|^2}}\left(\frac{z}{|y|}\right)\hat{f}_{\frac{tc^2}{|y|^2}}\left(\frac{z-y}{|y|}\right).
\end{aligned}$$

For convenience, let us define, for $z \in \mathbb{Z}^d$, $y \in \mathbb{Z}^d\backslash 0$, $x \in c^{-1}\mathbb{Z}^d\backslash 0$ and $c \geq |x|^{-1}\vee 1$,

$$\begin{aligned}
F_1(c,x,y,z) &:= \int_{2c^{-2}}^T dt\int_{|y|^{-2}}^{\frac{tc^2}{2|y|^2}} du\hat{f}_t\left(\frac{z}{c}-x\right)\tilde{f}_u\left(\frac{z}{|y|}\right)\tilde{f}_u\left(\frac{z-y}{|y|}\right) \\
F_2(c,x,y,z) &:= \int_{2|y|^{-2}}^{\frac{Tc^2}{|y|^2}} dt'\int_{\frac{1}{2c^2}}^{\frac{t}{2}} du'\tilde{f}_{u'}\left(\frac{z}{c}-x\right)\hat{f}_{t'}\left(\frac{z}{|y|}\right)\hat{f}_{t'}\left(\frac{z-y}{|y|}\right).
\end{aligned}$$

so that (181) can be rewritten

$$c^d\psi_d(|y|)\int_{2c^{-2}}^{T} dt\mathbb{P}_{0,y}\left[1\leq T_1\leq c^2t-1, Z^1_{c^2t}=cx\right] \leq K|y|^{-d}\sum_{z\in\mathbb{Z}^d}F_1(c,x,y,z)+K|y|^{-d}\sum_{z\in\mathbb{Z}^d}F_2(c,x,y,z). \tag{182}$$

Thus, completing the proof of Lemma V.2, in the case $d=2$, reduces to verify the bounds

$$|y|^{-2}\ln\left(\frac{c}{|y|}\vee e\right)^{-1}\sum_{z\in\mathbb{Z}^2}F_1(c,x,y,z)\leq K\ln(|x|^{-1}\vee e), \tag{183}$$

$$|y|^{-2}\ln\left(\frac{c}{|y|}\vee e\right)^{-1}\sum_{z\in\mathbb{Z}^2}F_2(c,x,y,z)\leq K\ln(|x|^{-1}\vee e). \tag{184}$$

Similarly, in the case $d=3$, in order to complete the proof of Lemma V.2, we need to establish that

$$|y|^{-3}\sum_{z\in\mathbb{Z}^3}F_1(c,x,y,z)\leq K(|x|^{-1}\vee 1), \tag{185}$$

$$|y|^{-3}\sum_{z\in\mathbb{Z}^3}F_2(c,x,y,z)\leq K(|x|^{-1}\vee 1). \tag{186}$$

We first deal with the first term of the sum in the right-hand side of (182).
<u>Proof of (183), (185)</u> : We have

$$\int_{|y|^{-2}}^{\frac{tc^2}{2|y|^2}} du\tilde{f}_u\left(\frac{z}{|y|}\right)\tilde{f}_u\left(\frac{z-y}{|y|}\right)\leq K\left(\frac{|z|}{|y|}\vee 1\right)^{2-2d}. \tag{187}$$

Then, from (93) and (94), we easily get

$$\int_{2c^{-2}}^{T} dt\hat{f}_t(\frac{z}{c}-x)\leq\begin{cases}K\psi_d(c) & \text{if } z=cx,\\ K\psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right) & \text{if } z\neq cx,\\ K\exp\left(-K'\left|\frac{z}{c}-x\right|\right) & \text{if } \left|\frac{z}{c}-x\right|\geq\sqrt{T}.\end{cases} \tag{188}$$

We then split $\mathbb{Z}^d$ into the following subsets
$D_0:=\{cx\},\ D_1:=\left(\mathbb{Z}^d\cap B(cx,c|x|/2)\right)\setminus D_0,\ D_2:=\left(\mathbb{Z}^d\cap B(0,|y|\wedge 2c^2T)\right)\setminus D_1,$
$D_3:=\left(\mathbb{Z}^d\cap B(0,|y|\vee 2c^2T)\right)\setminus(D_2\cup D_1),\ D_4:=\mathbb{Z}^d\setminus(D_1\cup D_3).$

We now combine the displays (187), (188), in order to obtain bounds on $F_1(c,x,y,z)$ over the regions D_i, $0\leq i\leq 4$. We also use that, for $z\in D_1$, $|z|\geq Kc|x|$, while, for $z\notin D_1$, $\psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right)\leq K\psi_d(|x|^{-1})$. We have

$$F_1(c,x,y,z)\leq K\times\begin{cases}\psi_d(c)\left(\frac{c|x|}{|y|}\vee 1\right)^{2-2d} & \text{if } z=cx,\\ \psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right)\left(\frac{c|x|}{|y|}\vee 1\right)^{2-2d} & \text{if } z\in D_1,\\ \psi_d(|x|^{-1}) & \text{if } z\in D_2,\\ \psi_d(|x|^{-1})\left(\frac{|z|}{|y|}\vee 1\right)^{2-2d} & \text{if } z\in D_3\cup D_4,\\ \exp\left(-K'\frac{|z|}{c}\right)|z|^{2-2d}|y|^{2d-2} & \text{if } z\in D_4.\end{cases} \tag{189}$$

Then, observe that

$$
\begin{aligned}
&\sum_{z\in D_1} \psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right) \le Kc^{d-2}(c|x|)^2,\\
&|D_2| \le K(|y|\wedge c^2)^d,\\
&|D_3| \le K(|y|\vee c^2)^d,\\
&\text{if } d=3, \quad \sum_{z\in D_3\cup D_4} |z|^{-4} \le K(|y|\wedge c^2)^{-1}\\
&\text{if } d=2, \quad \begin{cases} \sum_{z\in D_3} |z|^{-2} \le K\ln\left(\frac{c}{|y|}\vee e\right),\\ \sum_{z\in D_4} |z|^{-2}\exp\left(-K'\frac{|z|}{c}\right) \le K. \end{cases}
\end{aligned}
\tag{190}
$$

Combining the bounds (189) and (190), and doing some elementary computations then leads to (183), (185).

<u>Proof of (184), (186)</u> : From (93) and (94), we obtain

$$
\int_{\frac{1}{2c^2}}^{\frac{t}{2}} du' \tilde{f}_{u'}\left(\frac{z}{c}-x\right) \le \begin{cases} K\psi_d(c) & \text{if } z=cx,\\ K\psi_d\left(\left|\frac{z}{c}-x\right|^{-1}\right) & \text{if } z\ne cx,\\ K\exp\left(-K'\left|\frac{z}{c}-x\right|\right) & \text{if } \left|\frac{z}{c}-x\right| \ge \sqrt{T}. \end{cases}
\tag{191}
$$

Also, from (95),

$$
\int_{2|y|^{-2}}^{\frac{Tc^2}{|y|^2}} dt' \hat{f}_{t'}\left(\frac{z}{|y|}\right)\hat{f}_{t'}\left(\frac{z-y}{|y|}\right) \le K\left(\frac{|z|}{|y|}\vee 1\right)^{2-2d}.
\tag{192}
$$

Thus, the bounds in (189) remain true when replacing F_1 with F_2, and (184), (186) follow. This completes the proof of Lemma V.2. $\square$

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