A RANDOM WALK ON $\mathbb{Z}$ WITH DRIFT DRIVEN BY ITS OCCUPATION TIME AT ZERO

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Abstract. We consider a nearest neighbor random walk on the one-dimensional integer lattice with drift towards the origin determined by an asymptotically vanishing function of the number of visits to zero. We show the existence of distinct regimes according to the rate of decay of the drift. In particular, when the rate is sufficiently slow, the position of the random walk, properly normalized, converges to a symmetric exponential law. In this regime, in contrast to the classical case, the range of the walk scales differently from its position.

1. Introduction

We consider a self-interacting random walk $X := (X_n)_{n\geq 0}$ on $\mathbb{Z}$ whose drift is a function of the number of times it has already visited the origin. The random variable X_n represents the position of the walker at time $n \in \mathbb{Z}_+$. We assume that $|X_{n+1} - X_n| = 1$ for all $n \geq 0$, that is X is a nearest neighbor model. Let η_0 be a positive integer and, for $n \geq 1$, let

$$\eta_n = \eta_0 + \#\{i \in (0, n] : X_i = 0\}. \tag{1}$$

Thus, $\eta_n - \eta_0$ describes the number of visits of the walker to the origin by time n. Let $\varepsilon := (\varepsilon_n)_{n\geq 1}$ be a sequence taking values in $[0, 1)$. For $x \in \mathbb{Z}$ and $l \in \mathbb{N}$, let $P^{\varepsilon}_{(x,l)}$ denote a measure on the nearest neighbor random walk paths defined as follows:

$$P^{\varepsilon}_{(x,l)}(X_0 = x, \eta_0 = l) = 1$$

$$P^{\varepsilon}_{(x,l)}(X_{n+1} = j | X_n = i, \eta_n = m) = \begin{cases} \frac{1}{2} & \text{if } i = 0 \text{ and } |j| = 1 \\ \frac{1}{2}\big(1 - \operatorname{sign}(i)\varepsilon_m\big) & \text{if } i \neq 0 \text{ and } j - i = 1. \\ \frac{1}{2}\big(1 + \operatorname{sign}(i)\varepsilon_m\big) & \text{if } i \neq 0 \text{ and } j - i = -1. \end{cases} \tag{2}$$

Here $\operatorname{sign}(x)$ is $-1, 0$, or 1 according to whether x is a negative, zero, or positive respectively. The corresponding expectation is denoted by $E^{\varepsilon}_{(x,l)}$.

To simplify the notation, we usually denote $P^{\varepsilon}_{(0,1)}$ by P and $E^{\varepsilon}_{(0,1)}$ by E. If $\varepsilon_n = 0$ for all $n \geq 1$, we denote P by $\mathbb{P}$, E by $\mathbb{E}$, and refer to X as the *simple random walk on $\mathbb{Z}$*.

We note that, unless ε is a constant sequence, X is not a Markov chain. However, the pairs $(X_n, \eta_n)_{n\geq 0}$ form a time-homogeneous Markov chain.

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Let $d_n = -\text{sign}(X_n)\varepsilon_{\eta_n}$, and let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ denote the σ-algebra generated by the random walk paths up to time n. Then

$$E\big(X_{n+1} - X_n | \mathcal{F}_n\big) = d_n. \tag{3}$$

That is d_n is the local drift of the random walk at time n. Note that the drift is always toward the origin.

The aim of this paper is to prove limit theorems for the model described above in the case when $\lim_{n\to\infty} \varepsilon_n = 0$. If the convergence is fast enough, the asymptotic behavior of X is similar to that of the simple random walk. In Theorem 2.1 we show that the functional central limit theorem holds when $n\varepsilon_n \to 0$ and that P and $\mathbb{P}$ are mutually absolutely continuous if and only if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. We refer to this regime as supercritical. On the other hand, when ε_n converges to 0 slowly, the process exhibits a different limiting behavior. This case is treated in Theorems 2.5–2.7. In particular, we show that when ε is a regularly varying sequence converging to 0 and satisfying $n\varepsilon_n \to \infty$, the position of the walk X_n, properly normalized, converges in distribution to a symmetric exponential random variable. In this case, in contrast to the simple random walk, the range of the walk up to time n scales differently from X_n. We call this regime subcritical. The critical regime, which essentially corresponds to sequences satisfying $c_1 \leq n\varepsilon_n \leq c_2$ for some $0 < c_1 \leq c_2 < \infty$, is subject of future work.

The above definition of the random walk was inspired by a branching tree model arising in [1] in the study of the invasion percolation cluster (denoted IPC) on a regular tree. The scaling limit of this branching tree is further studied in [2] and is related to the critical regime of our model.

It is well-known that there is a one-to-one correspondence between discrete random trees and certain random walk paths (cf. [19]). More precisely, a rooted ordered tree θ is a graph which is formally described in the following way. Vertices of θ belong to $\bigcup_{n\geq 0} \mathbb{N}^n$. By convention, $\mathbb{N}^0 = \emptyset$ is always a vertex of θ which is called the root. For a vertex $v \in \theta$, we let $k_v = k_v(\theta)$ be the number of children of v and whenever $k_v = k \in \mathbb{N}$, these children are denoted $v1, \ldots, vk$. In particular, the ith child of the root is simply i, and if $vi \in \theta$ then $\forall 1 \leq j < i$, $vj \in \theta$ as well. Edges of θ are the pairs of vertices (v, vi), with $v, vi \in \theta$ for some $i \in \mathbb{N}$. Define $\#\theta$ to be the total number of vertices in θ, possibly infinite.

Let $\big(v^i, 0 \leq i < \#\theta\big)$ be the vertices of θ listed in lexicographic order, so that $v^0 = \emptyset$. The *Lukaciewicz path* of θ is the piecewise constant function $\big(V_t^\theta, t \in [0, \#\theta]\big)$ defined as follows: For $t \in [0, \#\theta]$, and $n := [t]$,

$$V_t^\theta = V_n^\theta := \sum_{i=0}^{n-1} \big(k_{v^i} - 1\big),$$

It is straightforward to check that the Lukaciewicz path (or in fact, its integer values) uniquely determines – and hence represents – any finite tree θ. When the tree is infinite, note that one can only recover from the path the part of the tree which lies left of its first infinite branch. One may

The interest in such a coding comes from the fact that the Lukaciewicz path of a Galton-Watson tree simply is a part of a certain random walk path (cf Corollary 1.6 of [19]), which for instance, makes it easy to discuss scaling limits of sequences of Galton-Watson trees, or also Galton-Watson trees conditioned to be large (see [19]).

Moreover, this representation helped finding the scaling limit of the IPC on a regular tree (cf [2]). Indeed it was shown in [1] that IPC on a regular tree consists of a uniformly distributed single infinite rising branch (backbone) from which emerge subcritical percolation clusters. Furthermore, the parameters of these subcritical percolation clusters depends on the height at which they branch off the backbone, and when moving up the backbone, these parameters tend to being critical. Note that a (sub)critical percolation cluster on a regular tree is a (sub)critical Galton-Watson tree, and the corresponding Lukaciewicz path is above the origin (except for its terminal value) and is the path of a random walk drifted downwards.

On the other hand, the absolute value of our random walk is the Lukaciewicz path of a discrete random binary tree with an infinite rightmost branch, a backbone, from which emerge off-backbone tree. Every off-backbone tree has a single vertex at its first generation, from which emerges a subcritical Galton-Watson tree, the branching law in this off-backbone tree depends on the height at which it branches off the backbone. Again in this case, the further up one goes, the closer these trees are to being critical.

Thus, the two models are related, however, they exhibit several differences. Most importantly, our assumption that the random walk is simple corresponds to the fact that every vertex in the Galton-Watson trees branching above the backbone has either 0 or 2 offsprings. In order to recover the case of invasion percolation, one would need to consider a walk that can not only move up or down, but also stay put. Note however that the reason for our restriction to simple random walks was mainly computational, and one would expect a very similar behviour for a more general random walk. The other main difference is that in the case of IPC, the successive drifts are random, whereas in our study we choose the sequence of drifts to be deterministic. On the other hand, for the IPC, a typical realization of the sequence of drifts will be constant for long stretches of time, which simplifies the study in this case (see [2]).

Another related class of random processes are oscillating random walks, namely time-homogeneous Markov chains in $\mathbb{R}^d$ with transition function which depends on the position of the chain with respect to a fixed hyperplane, cf. [18, 9].

We remark that the model can be interpreted as describing a gambler (Sisyphus) who learns from his experience and adopts a new strategy whenever a ruin event occurs. This paper intends to be a first step towards a more general study of random walks in $\mathbb{Z}^d$ for which the transition probabilities are updated each time the walk visits a certain set. Another possible extension would be to consider a random environment version of the random walk X.

The paper is organized as follows. The main results are collected in Section 2. Some general facts about random walks and regular varying sequence are recalled in Section 3. The proofs are contained in Section 4 (supercritical case) and Section 5 (subcritical case).

2. Statement of main results

This section presents the main results of this paper. It is divided into two parts. The first is devoted to the supercritical case while the second one covers the results for the subcritical regime. Throughout the paper we assume that the drift sequence ε is fixed and consider the random walk X under the measure P defined above.

2.1. **Supercritical Regime.**

Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. For a sequence of random variables $Z := (Z_n)_{n\geq 0}$ and each $n \geq 0$, let $\mathcal{I}_n^Z \in C(\mathbb{R}_+, \mathbb{R})$ denote the following linear interpolation of $Z_{[nt]}$:

$$\mathcal{I}_n^Z(t) = \frac{1}{\sqrt{n}}\big(([nt]+1-nt)Z_{[nt]} + (nt-[nt])Z_{[nt]+1}\big). \tag{4}$$

Here and henceforth $[x]$ denotes the integer part of a real number x.

We say that Z satisfies the invariance principle, if the sequence of processes $\big(\mathcal{I}_n^Z(t)\big)_{t\in\mathbb{R}_+}$ converges weakly in $C(\mathbb{R}_+,\mathbb{R})$, as $n\to\infty$, to the standard Brownian motion. We have:

Theorem 2.1.

(i) *Assume that* $\lim_{n\to\infty} n\varepsilon_n = 0$. *Then* X *satisfies the invariance principle.*
(ii) *The distribution of* X *under* P *is either equivalent or orthogonal to the law* $\mathbb{P}$ *of the simple random walk, according to whether* $\sum_{n=1}^\infty \varepsilon_n$ *is finite or not.*

For the sake of comparison with the subcritical regime, we now state some consequences of this result. Let

$$M_n := \max_{i\le n} X_i, \quad \mathcal{M}_n := \max_{i\le n} |X_i|. \tag{5}$$

We have:

Corollary 2.2.

(i) *Assume that* $\lim_{n\to\infty} n\varepsilon_n = 0$. *Then* $M_n/\sqrt{n}$ *(respectively* $\mathcal{M}_n/\sqrt{n}$*) converge in distribution, as* $n\to\infty$, *to* $\sup_{0\le t\le 1} B_t$ *(respectively to* $\sup_{0\le t\le 1}|B_t|$*), where* B_t *is the standard Brownian motion.*
(ii) *Assume that* $\sum_{n=1}^\infty \varepsilon_n < \infty$. *Then* $\limsup_{n\to\infty} \dfrac{X_n}{\sqrt{2n\log\log n}} = 1$, P-a.s.

This corollary extends to our model the limit theorem for the maxima and the law of the iterated logarithm of the simple random walk.

2.2. **Subcritical Regime.** First, we recall the definition of regularly varying sequences (see for example [7] or Section 1.9 of [6]).

Definition 2.3. *Let* $r := (r_n)_{n\ge1}$ *be a sequence of positive reals. We say that* r *is* regularly varying *with index* $\rho\in\mathbb{R}$, *if* $r_n = n^\rho \ell_n$, *where* $\ell := (\ell_n)_{n\ge1}$ *is such that for any* $\lambda > 0$, $\lim_{n\to\infty} \ell_{[\lambda n]}/\ell_n = 1$.
The set of regularly varying sequences with index ρ *is denoted by* $RV(\rho)$. *If* $r\in RV(0)$, *we say that* r *is* slowly varying.

In this section we make the following assumption:

Assumption 2.4 (subcritical regime).
Assume that $\varepsilon\in RV(-\alpha)$ *for some* $\alpha\in[0,1]$. *Moreover,*

- *if* $\alpha=0$, *assume in addition that* $\lim_{n\to\infty}\varepsilon_n = 0$;
- *if* $\alpha=1$, *assume in addition that* $\lim_{n\to\infty} n\varepsilon_n/\log n = \infty$.

To state our results for this regime, we need to introduce some additional notations. We say that two sequences of real numbers $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ are asymptotically equivalent and write $x_n\sim y_n$ if $\lim_{n\to\infty} x_n/y_n = 1$. Let

$$T_0 = 0 \quad\text{and}\quad T_{n+1} = \inf\{k > T_n : X_k = 0\},\ n\ge 0. \tag{6}$$

That is, T_n is the time of the n-th return to 0. Let

$$a_n = n + \sum_{i=1}^{n} \frac{1}{\varepsilon_i}, \qquad c_n = \min\{i \in \mathbb{N} : a_i \geq n\}, \quad \text{and} \quad b_n = \frac{1}{\varepsilon_{c_n}}. \tag{7}$$

Lemma 3.1 below shows that $a_n = E(T_n)$. The sequence $(c_n)_{n\geq 1}$ is an inverse of $(a_n)_{n\geq 1}$, and, by a renewal theorem of Smith [21], $c_n \sim E(\eta_n)$. Therefore, b_n can be understood as a typical lifetime of the last excursion from the origin completed before time n. The sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, and $(c_n)_{n\geq 1}$ are regularly varying, and their asymptotic behavior, as $n \to \infty$, can be deduced from the standard results collected in Theorem 3.4 (see Corollary 3.5). For the distinguished case $\varepsilon_n = n^{-\alpha}$ with $\alpha \in (0,1)$, we have $a_n \sim (1+\alpha)^{-1} n^{1+\alpha}$, $c_n \sim (1+\alpha)^{\frac{1}{1+\alpha}} n^{\frac{1}{1+\alpha}}$, and hence $b_n \sim (1+\alpha)^{\frac{\alpha}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}}$.

We have:

Theorem 2.5. *Let Assumption 2.4 hold. Then, as $n \to \infty$, X_n/b_n converges in distribution to a random variable with density $e^{-2|x|}$, $x \in (-\infty, \infty)$.*

Due to the symmetry of the law of X, the theorem is equivalent to the statement that $|X_n|/b_n$ converges in distribution to a rate-2 exponential random variable. The proof of Theorem 2.5 is based on a comparison of the distribution of X_n to a stationary distribution of an oscillating random walk with constant drift ε_{c_n} toward the origin.

We proceed with a more precise description of X, from which Theorem 2.5 can be in fact derived in an alternative way (see Remark 5.6 below). Interestingly, the method we use to establish these more precise results could possibly be adapted to the non nearest neighbor case, provided one could show in this more general setting that the number of visits to the origin is well-localized around its typical value. In this more general case, the method evoked in Remark 5.6 would also remain valid.

Let $\mathfrak{N}^{(c)}$ denote Ito's excursion measure associated with the excursions of the Brownian motion with drift $c < 0$ above its infimum process, and let ζ denote the lifetime of an excursion above the infimum (see Section 3.3 below for details). Let

$$V_n = \max\{i \leq n : X_i = 0\}, S_n := \inf\{i \geq V_n : X_i = 0\}$$

We have:

Theorem 2.6. *Let Assumption 2.4 hold. Then:*

(i) $\lim_{n\to\infty} b_{2n} P(X_{2n} = 0) = 2$.

(ii) *For $t > 0$,* $\lim\limits_{n\to\infty} b_{2n}^2 P\big(V_{2n} = 2n - 2[tb_{2n}^2]\big) = 2\mathfrak{N}^{(-1)}(\zeta > 2t)$.
In particular, $\lim_{n\to\infty} P\big((2n - V_{2n})/b_{2n}^2 \leq x\big) = \int_0^{2x} \mathfrak{N}^{(-1)}(\zeta > t)dt$ *for all $x > 0$.*

(iii) *For $n \in \mathbb{N}$, let $Z_n = \big(Z_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $Z_n\big(k \cdot b_{2n}^{-2}\big) = |X_{(V_{2n}+k)\wedge S_n}| \cdot b_{2n}^{-1}$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere. Then, as $n \to \infty$, the process Z_n converges weakly in $C(\mathbb{R}_+, \mathbb{R})$ to a non-negative process with the law* $\displaystyle\int_0^\infty \mathfrak{N}^{(-1)}(\,\cdot\,, \zeta > t)dt$.

Part (i) states that, similarly to the classical renewal theory (cf. [14, 15]), the probability to find the random walk at the origin at time $2n$ is asymptotically reciprocal to the expected duration of the of the last excursion away from the origin completed before that time. Part (ii) provides limit results on the law of the last visit time to the origin before a given time. It turns out that under Assumption 2.4, b_{2n}^2 is of smaller order that n (see Lemma 3.5 below). In particular, in contrast to the classical arc-sine law (cf. [12, p. 196]), $V_{2n}/2n$ converges in probability to 1. Finally, part (iii) is a limit theorem for the law of excursion away from 0 straddling time $2n$.

The next theorem concerns the asymptotic behavior of the maxima of X. Let

$$h_n := \frac{1}{2} b_n \log(c_n/b_n) = \frac{\log(\varepsilon_{c_n} c_n)}{2\varepsilon_{c_n}}. \tag{8}$$

Note that by Assumption 2.4, $\varepsilon_{c_n} c_n \to \infty$ as $n \to \infty$. Moreover, Corollary 3.5-(v) below shows that

$$\lim_{n\to\infty} \frac{\log(\varepsilon_{c_n} c_n)}{\log n} = \frac{1-\alpha}{1+\alpha}.$$

When $\varepsilon_n = n^{-\alpha}$ with $\alpha \in (0,1)$, we have $h_n \sim \frac{1}{2}(1-\alpha)(1+\alpha)^{\frac{-1+\alpha}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}} \log n$ as $n \to \infty$.

Recall the random variables M_n defined in (5). We prove in Section 5:

Theorem 2.7. *Let Assumption 2.4 hold. Then*

$$\lim_{n\to\infty} \frac{1}{\log(\varepsilon_{c_n} c_n)} \log\bigl(-\log P(M_n \le x h_n)\bigr) = 1 - x, \qquad x \in (0,\infty)\backslash\{1\}.$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log(\varepsilon_{c_n} c_n)} \log P\bigl(M_n > x h_n\bigr) = 1 - x, \qquad x > 1.$$

The above limits remain true when M_n is replaced with $\mathcal{M}_n$.

Corollary 2.8. *Let Assumption 2.4 hold. Then*

$$\limsup_{n\to\infty} X_n/h_n = \lim_{n\to\infty} M_n/h_n = \lim_{n\to\infty} \mathcal{M}_n/h_n = 1,$$

where the limits hold P-a.s. when $\alpha < 1$ and in probability when $\alpha = 1$.

We remark that under Assumption 2.4, $\lim_{n\to\infty} h_n/b_n = \infty$, and hence $\lim_{n\to\infty} X_n/M_n = 0$ in probability. In particular, Theorem 2.5 cannot be extended to a functional CLT for a piecewise-linear interpolation of X_n/b_n in $C(\mathbb{R}_+, \mathbb{R})$.

3. Preliminaries

The goal of this section is threefold. First, in a series of lemmas we state in Section 3.1 some general facts about the measure P^ε in the case when ε is a constant sequence. Second, in Section 3.2, we recall some useful properties of regularly varying sequences (see Theorem 3.4), and then apply this theorem (see Corollary 3.5) to draw conclusions regarding a_n, b_n, and c_n defined in (7). Finally, in Section 3.3 we deal with the asymptotic behavior of a sequence of random walks with a negative drift conditioned to stay positive. Lemma 3.6 is the key to the proof of the last two parts of Theorem 2.6.

3.1. **Random walks with a negative drift and oscillating random walks.** For a real $\delta \in [0,1)$, let (δ) denote the constant sequence $\delta, \delta, \ldots$ To simplify the notations we write $P_j^{(\delta)}$ for $P_{(j,1)}^{(\delta)}$, $P^{(\delta)}$ for $P_{(0,1)}^{(\delta)}$, and let $E_j^{(\delta)}$ and $E^{(\delta)}$ denote the respective expectation operators. We remark that $P^{(0)} = \mathbb{P}$ while $P^{(\delta)}$ with $\delta \in (0,1)$ correspond to so-called oscillating random walks (cf. [18, 9]). If μ is a probability distribution on $\mathbb{Z}$, we write $P_\mu^{(\delta)}$ for the probability measure $\sum_{j\in\mathbb{Z}} \mu(j) P_j^{(\delta)}$ and let $E_\mu^{(\delta)}$ denote the corresponding expectation.

Recall T_n from (6) and set

$$\tau_n = T_n - T_{n-1}, \quad n \geq 1, \tag{9}$$

with the convention that $\infty - \infty = \infty$. That is, τ_n is the duration of the n-th excursion away from 0. In the following lemma we recall a well-known explicit expression for the moment generating function of τ_n (see for instance [14, p. 273] or [12, p. 276]). The moments of τ_n can be computed as appropriate derivatives of the generating function.

Lemma 3.1. *Let $\delta \in [0,1)$. Then*

$$E^{(\delta)}\big(s^{\tau_1}\big) = \frac{1-\sqrt{1-(1-\delta^2)s^2}}{1-\delta} \text{ for } 0 < s < \frac{1}{\sqrt{1-\delta^2}}.$$

In particular,

$E\big(s^{\tau_n}\big) = \dfrac{1-\sqrt{1-(1-\varepsilon_n^2)s^2}}{1-\varepsilon_n}$ *for* $0 < s < (1-\varepsilon_n^2)^{-1/2}$.
$E(\tau_n) = 1 + \varepsilon_n^{-1}$.
$E(\tau_n^2) = 1 + \varepsilon_n^{-1} + \varepsilon_n^{-2} + \varepsilon_n^{-3}$.
$E(\tau_n^3) = 1 + \varepsilon_n^{-1} + 3\varepsilon_n^{-4} + 3\varepsilon_n^{-5}$.

For our proofs in Sections 4 and 5, we need the following monotonicity result.

Lemma 3.2. *Let $\varepsilon^1 := (\varepsilon_n^1)_{n\geq 1}$ and $\varepsilon^2 := (\varepsilon_n^2)_{n\geq 1}$ be two sequences such that $\varepsilon_n^j \in (0,1)$ for $j = 1,2$ and $n \in \mathbb{N}$, and $\sup_{n\geq 1} \varepsilon_n^2 \leq \inf_{n\geq 1} \varepsilon_n^1$. Further, let $x_1, x_2 \in \mathbb{Z}_+$ be such that $x_2 - x_1 \in 2\mathbb{Z}_+$. Then there exist two processes $Y^j := (Y_n^j)_{n\geq 0}$, $j = 1,2$, defined on the same probability space, such that*

(i) *For $j = 1,2$, Y^j has the same distribution as X under $P_{x_j}^{\varepsilon^j}$.*
(ii) *$|Y_n^1| \leq |Y_n^2|$ for all $n \geq 0$.*

Proof. Let $(U_n)_{n\geq 1}$ be an IID sequence of uniform random variables on $[0,1]$. For $j = 1,2$, set $Y_0^1 = x_1$, $Y_0^2 = x_2$, $\eta_0^j = 1$, and let

$$Y_{n+1}^j = Y_n^j + 2\mathbf{I}_{\left\{U_n \geq \frac{1}{2}\left(1+\text{sign}(Y_n^j)\varepsilon^j_{\eta_n^j}\right)\right\}} - 1 \quad \text{and} \quad \eta_{n+1}^j = \eta_n^j + \mathbf{I}_{\{Y_{n+1}^j = 0\}}.$$

Clearly, $(Y_n^j)_{n\geq 0}$ has the same distribution as X under $P_{x_j}^{\varepsilon^j}$. Moreover, using induction, it is not hard to check that for all $n \geq 0$, $|Y_{n+1}^2| - |Y_n^2| \geq |Y_{n+1}^1| - |Y_n^1|$, unless $Y_n^1 = 0$. But, since $Y_n^2 - Y_n^1$ is an even integer, $|Y_{n+1}^1| = 1 \leq |Y_{n+1}^2|$ also in the latter case. □

In the next lemma, to avoid dealing with a periodic Markov chain, we focus on the process $(X_{2n})_{n\geq 0}$ rather than on $X = (X_n)_{n\geq 0}$ itself. It is well-known (see [9] for a closely related general result) that the law of the Markov chain X_{2n} under $P^{(\delta)}$ converges to its unique stationary distribution μ_δ. The latter is given by

$$\mu_\delta(0) = \frac{2\delta}{1+\delta}, \quad \mu_\delta(2i) = \frac{2\delta(1-\delta)}{(1+\delta)^3}\Big(\frac{1-\delta}{1+\delta}\Big)^{2(|i|-1)}, \ i \in \mathbb{Z}\setminus\{0\}. \tag{10}$$

Let $T = \inf\{n \geq 0 : X_n = 0\}$. A standard coupling construction for countable stationary Markov chains (see for instance [12, p. 315]) implies that

$$\sup_{A\subset 2\mathbb{Z}_+} |P^{(\delta)}(X_{2n} \in A) - \mu_\delta(A)| \leq P^{(\delta)}_{\mu_\delta}(T > 2n). \tag{11}$$

Estimating the righthand side of (11) we get:

Lemma 3.3. *For all* $\delta \in (0,1)$ *and* $n \geq 1$,

$$\sup_{A\subset 2\mathbb{Z}_+} |P^{(\delta)}(X_{2n} \in A) - \mu_\delta(A)| \leq 2(1+\delta^2)^{-n}.$$

Proof of Lemma 3.3. By Chebyshev's inequality, for every $\lambda > 0$,

$$P_{\mu_\delta}(T > 2n) \leq e^{-2\lambda n} E^{(\delta)}_{\mu_\delta}(e^{\lambda T}). \tag{12}$$

By Lemma 3.1, for $j \in \mathbb{Z}$,

$$E^{(\delta)}_j(e^{\lambda T}) = \big[E^{(\delta)}_1(e^{\lambda T})\big]^{|j|} = \Big[\frac{1-\sqrt{1-(1-\delta^2)e^{2\lambda}}}{(1-\delta)e^\lambda}\Big]^{|j|},\ e^{2\lambda}(1-\delta^2) < 1. \tag{13}$$

Note that the extra term e^λ (comparing to the statement of Lemma 3.1) in the denominator corresponds to the difference between the definition of τ_1, the time of the first *return* to 0, and T, the time of the first *visit* to 0.

Choose $\lambda > 0$ such that $e^{2\lambda} = 1+\delta^2$. Clearly, $e^{2\lambda}(1-\delta^2) = (1-\delta^4) < 1$. Therefore,

$$\begin{aligned} P_{\mu_\delta}(T > 2n) &\underset{(12)}{\leq} (1+\delta^2)^{-n} \sum_{j\in\mathbb{Z}} \mu_\delta(2j) E^{(\delta)}_{2j}(e^{\lambda T}) \\ &\underset{(10),(13)}{=} \frac{1}{(1+\delta^2)^n}\frac{2\delta}{1+\delta}\Big[1 + \frac{2(1-\delta)}{(1+\delta)^2}\sum_{j=1}^\infty \Big(\frac{1-\delta}{1+\delta}\Big)^{2(j-1)}\Big(\frac{1-\sqrt{1-(1-\delta^2)e^{2\lambda}}}{(1-\delta)e^\lambda}\Big)^{2j}\Big] \\ &= \frac{1}{(1+\delta^2)^n}\frac{2}{1+\delta} \leq 2(1+\delta^2)^{-n}, \end{aligned}$$

completing the proof. □

3.2. **Regularly varying sequences.** We next recall some fundamental properties of regularly varying sequences that are required for our proofs in the subcritical regime.

Theorem 3.4. [6], [7] *Let* $r := (r_n)_{n\geq 1} \in RV(\rho)$ *for some* $\rho \in \mathbb{R}$.

(i) *Suppose that* $\rho > -1$. *Then* $\lim_{n\to\infty} \frac{1}{nr_n}\sum_{m=1}^n r_m = \frac{1}{1+\rho}$.
(ii) *Suppose that* $\rho \geq 0$. *Let* $(j_n)_{n\geq 1}$ *be a sequence of integers such that* $\lim_{n\to\infty} j_n/n = \gamma$ *for some* $\gamma \in (0,1]$. *Then* $\max_{j_n\leq i\leq n} r_i \sim r_n$ *and* $\min_{j_n\leq i\leq n} r_i \sim \gamma^\rho r_n$ *as* $n \to \infty$.
(iii) *Suppose that* $\rho > 0$. *Let* $r^{\mathrm{inv}} := (r^{\mathrm{inv}}_n)_{n\geq 1}$, *where* $r^{\mathrm{inv}}_n = \min\{i \geq 1 : r_i \geq n\}$. *Then* $r^{\mathrm{inv}}_n \in RV(1/\rho)$ *and* $r^{\mathrm{inv}}_{[r_n]} \sim r_{[r^{\mathrm{inv}}_n]} \sim n$ *as* $n \to \infty$.
(iv) *Suppose that* $\rho = 0$. *Then* $\lim_{n\to\infty} \frac{\log r_n}{\log n} = 0$.

Corollary 3.5. *Let Assumption 2.4 hold and recall* $a = (a_n)_{n\in\mathbb{N}}$, $b = (b_n)_{n\in\mathbb{N}}$, *and* $c = (c_n)_{n\in\mathbb{N}}$ *introduced in* (7). *We have*

(i) $a_n \sim (1+\alpha)^{-1} n\varepsilon_n^{-1}$ *as* $n \to \infty$. *In particular,* $a \in RV(1+\alpha)$.
(ii) $c \in RV(1/(1+\alpha))$.
(iii) $b_n = \varepsilon_{c_n}^{-1} \sim (1+\alpha)n/c_n$ *as* $n \to \infty$. *In particular,* $b \in RV(\alpha/(1+\alpha))$.

(iv) $\lim_{n\to\infty} \frac{n}{b_n^2 \log b_n} = \lim_{n\to\infty} \frac{c_n^2}{n \log b_n} = \infty.$

(v) $\lim_{n\to\infty} \frac{\log(b_n/c_n)}{\log n} = \frac{1-\alpha}{1+\alpha}.$

Part (i) of the corollary follows from Theorem 3.4-(i). Once this is established, part (ii) follows from Theorem 3.4-(iii). Next, claims (i) and (iii) of Theorem 3.4 imply that

$$c_n \varepsilon_{c_n}^{-1} \sim (1+\alpha) a_{c_n} \sim (1+\alpha) n, \qquad \text{as } n \to \infty,$$

which proves (iii). To see that (iv) holds true observe that part (iii) along with Assumption 2.4 imply:

$$\frac{n\varepsilon_{c_n}^2}{\log(\varepsilon_{c_n}^{-1})} \sim \frac{1}{1+\alpha} \cdot \frac{c_n \varepsilon_{c_n}}{\log(\varepsilon_{c_n}^{-1})} \to \infty \text{ as } n \to \infty.$$

Finally, (v) follows from (ii) and (iii) combined with Theorem 3.4-(iv).

3.3. **Random walks conditioned to stay positive.** The aim of this section is to prove Lemma 3.6 below. We start by recalling some features of the excursion measure of negatively drifted Brownian motion above its infimum (cf Chapter VI.8 in [20], in particular Lemma VI.55.1).

Let $(Z_t)_{t\geq 0}$ be the canonical process on $C(\mathbb{R}_+,\mathbb{R})$, namely $Z_t(\omega) = \omega(t)$ for $\omega \in C(\mathbb{R}_+,\mathbb{R})$, and, for $c \leq 0$, let $\mathbf{W}^{(c)}$ be the law on $C(\mathbb{R}_+,\mathbb{R})$ which makes $Z_t - ct$ into the standard Brownian motion. For $t \in \mathbb{R}_+$, let $Y_t = Z_t - \inf\{Z_s : s \leq t\}$, $\zeta = \inf\{t > 0 : Y_t = 0\}$, and define $\widetilde{Y}_t = Y_{t\wedge\zeta}$. Then $\widetilde{Y} = \big(\widetilde{Y}_t\big)_{t\geq 0}$ is a time-homogeneous continuous Markov process "killed at zero" with taboo transition density function

$$\mathcal{P}_t^{(c)}(x,y) := \frac{\mathbf{W}^{(c)}\big(\widetilde{Y}_t \in dy, \zeta > t \big| \widetilde{Y}_0 = x\big)}{dy} = \\ = \frac{1}{\sqrt{2\pi t}} e^{c(y-x)-c^2t/2}\big[e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t}\big], \qquad x,y > 0, t > 0.$$

In words, $\widetilde{Y}$ is an excursion of the Brownian motion with drift $c \leq 0$ above its infimum process and ζ is its lifetime.

For $\omega \in C(\mathbb{R}_+,\mathbb{R})$ let $\zeta(\omega) = \inf\{t > 0 : \omega(t) = 0\}$, and let

$$U = \{f \in C(\mathbb{R}_+,\mathbb{R}) : \omega(0) = 0 \text{ and } \omega(t) = 0 \text{ for } t > \zeta(f)\}$$

be the space of excursions from zero. By Ito's theorem, under $\mathbf{W}^{(c)}$, the excursions of the process $Y = (Y_t)_{t\geq 0}$ away from zero form a Poisson point process on $(0,+\infty) \times U$ with intensity $dt \times \mathfrak{N}^{(c)}$. The finite-dimensional distributions of $\mathfrak{N}^{(c)}$ can be expressed as follows. Let

$$\mathcal{R}_t^{(c)}(y) := \frac{2y}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(y-ct)^2}{2t}\Big), \qquad y > 0, t > 0.$$

Then, for $0 < t_1 < \ldots < t_m$ and $x_1,\ldots,x_m > 0$,

$$\mathfrak{N}^{(c)}\{f(t_k) \in dx_k : 1 \leq k \leq m\} = \mathcal{R}_{t_1}^{(c)}(x_1)dx_1 \prod_{k=2}^{m} \mathcal{P}_{t_{k-1},t_k}^{(c)}(x_{k-1},x_k)dx_k. \tag{14}$$

The law $\mathcal{R}_t^{(c)}(y)dy$ is called the entrance law associated with $\mathfrak{N}^{(c)}$. Note that

$$\mathfrak{N}^{(c)}(\zeta > t) = \int_0^\infty \mathcal{R}_t^{(c)}(y)dy. \tag{15}$$

In particular, $\mathfrak{N}^{(0)}(\zeta > t) = \sqrt{\frac{2}{\pi t}}$ whereas for $c < 0$, $\mathfrak{N}^{(c)}(\zeta > t) = |c| \cdot \mathfrak{N}^{(-1)}(\zeta > tc^2)$. More generally, (14) implies that for any constant $c < 0$,

$$\mathfrak{N}^{(c)}\big(\big(|c| \cdot f(t/c^2)\big)_{t\in\mathbb{R}_+} \in \cdot\,\big) = |c| \cdot \mathfrak{N}^{(-1)}\big(\big(f(t)\big)_{t\in\mathbb{R}_+} \in \cdot\,\big). \tag{16}$$

For $m > 0$, let $C[0,m] := \{f : [0,m] \to \mathbb{R}, f \text{ continuous }\}$, equipped with the topology of uniform convergence. Let $\pi_m : C(\mathbb{R}_+,\mathbb{R}) \to C[0,m]$ be the canonical projection defined by $\pi_m\omega(t) = \omega(t)$ for $t \in [0,m]$. Let $\mathfrak{N}^{(c)}(\,\cdot\,|\zeta > t) := \frac{\mathfrak{N}^{(c)}(\,\cdot\,;\zeta > t)}{\mathfrak{N}^{(c)}(\zeta > t)}$. A non-homogeneous in time Markov process W_+ on $C[0,1]$ with the law

$$\mathfrak{M}^{(0)}(A) := P(W_+ \in A\,) = \mathfrak{N}^{(0)}(\pi_1^{-1}A|\zeta > 1), \quad A \text{ is a Borel subset of } C[0,1],$$

is called Brownian meander (see for instance [4, 13] and references therein for further background). The meander is a weak limit of zero-mean random walks conditioned to stay positive (see [8, 16] and [10]). Its finite-dimensional distributions were first computed in [3], and it is not hard to check that these are consistent with our definition of the meander. The Brownian meander can also be understood as a Brownian motion in $C[0,1]$ conditioned to stay positive up to time 1, defined rigorously with the help of an appropriate h-transform.

Analogously, for $c < 0$, we call a non-homogeneous in time Markov process $W_+^{(c)}$ on $C[0,1]$ with the law

$$\mathfrak{M}^{(c)}(A) := P(W_+^{(c)} \in A\,) = \mathfrak{N}^{(c)}(\pi_1^{-1}A|\zeta > 1), \quad A \text{ is a Borel subset of } C[0,1],$$

a drifted Brownian meander with drift c.

It is well-known a sequence of random walks with well-chosen asymptotically vanishing drifts converges in distribution to drifted Brownian motion (see for instance Theorem II.3.2 in [17]). Part (ii) of the following lemma asserts that such walks, when conditioned to stay positive up to the scaling time, also converge to a non-degenerate limit, which, not surprisingly, is the drifted Brownian meander. Part (iii) is then a direct consequence of this fact. Recall the notation $P^{(\delta)}$ was introduced in the first paragraph of the section and corresponds to a constant sequence $\delta, \delta, ...$ Define

$$\Lambda_n = \{X_1 > 0, \ldots, X_n > 0\}. \tag{17}$$

Lemma 3.6. *Let $(j_n)_{n\in\mathbb{N}}$ be a sequence of positive reals and $(m_n)_{n\in\mathbb{N}}$ be sequence of positive integers such that $\lim_{n\to\infty} j_n = \infty$, $\lim_{n\to\infty} j_n/j_{n+1} = 1$, and $\lim_{n\to\infty} \varepsilon_{m_n} j_n = \gamma \in (0,\infty)$.*

Then,

(i) $\lim_{n\to\infty} j_n P^{(\varepsilon_{m_n})}(\Lambda_{[j_n^2]}) = \frac{1}{2}\mathfrak{N}^{(-\gamma)}(\zeta > 1)$.

(ii) *For $n \in \mathbb{N}$, let $Y_n = \big(Y_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $Y_n\big(j_n^{-2}k\big) = j_n^{-1}X_k$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere.*
Then the process $\pi_1 Y_n$ under $P^{(\varepsilon_{m_n})}(\,\cdot\,|\Lambda_{[j_n^2]})$ converges weakly in $C[0,1]$ to a drifted Brownian meander with drift $-\gamma$.

(iii) *For $n \in \mathbb{N}$, let $\widetilde{Y}_n = \big(\widetilde{Y}_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $\widetilde{Y}_n\big(j_n^{-2}k\big) = j_n^{-1}X_{k\wedge T_1}$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere.*

Then the process $\widetilde{Y}_n$ under $P^{(\varepsilon_{m_n})}(\,\cdot\,|\Lambda_{[j_n^2]})$ converges weakly in $C(\mathbb{R}_+,\mathbb{R})$ to a process with law $\mathfrak{N}^{(-\gamma)}(\,\cdot\,|\zeta>1)$.

Proof. Since $P^{(\varepsilon)}(\Lambda_j)$ is a non-increasing function of j and $j_n/j_{n+1}\sim 1$ as $n\to\infty$, we can assume without loss of generality that $[j_n^2]\in 2\mathbb{Z}_+$.

The proof of the lemma is based on the fact that, as we already mentioned, the result is known for a symmetric random walk, and that we can explicitly compare the law of a nearest-neighbor drifted walk and the distribution $\mathbb{P}$ of the simple random walk. Set $\varepsilon_{m_n}=\delta_n$ and $J_n=\{y\in\mathbb{R}: yj_n\in\mathbb{N}\}$. Counting the number of steps to the right and to the left, we obtain for any $m\in\mathbb{N}$, $0<t_1<\ldots<t_m\le 1$ and $y_1,\ldots,y_m\in\mathbb{R}$, $y\in J_n$,

$$
\begin{aligned}
&P^{(\delta_n)}\big(Y_n(t_k)=y_k,\ k=1,\ldots,m; X_{[j_n^2]}=2yj_n\big)=\\
&\quad=\mathbb{P}\big(Y_n(t_k)=y_k,\ k=1,\ldots,m; X_{[j_n^2]}=2yj_n\big)\frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^{yj_n},
\end{aligned}
\tag{18}
$$

where the extra factor $(1-\delta_n)^{-1}$ is due to the fact that the transition kernels of the random walk under $P^{(\delta_n)}$ and $\mathbb{P}$ coincide at the origin. In particular,

$$
P^{(\delta_n)}(\Lambda_{[j_n^2]})=\sum_{y\in J_n}\mathbb{P}\big(\Lambda_{[j_n^2]},X_{[j_n^2]}=2yj_n\big)\frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\Big(\frac{1-\delta_n^2}{1+\delta_n}\Big)^{yj_n}.
$$

(i) Using the identity $\mathbb{P}(\Lambda_{[j_n^2]})=\frac{1}{2}\mathbb{P}(X_{[j_n^2]}=0)$ (see for instance [12, p. 198]), we obtain:

$$
P^{(\delta_n)}(\Lambda_{[j_n^2]}) \;=\; \frac{j_n}{2}\int_0^\infty du\mathbb{P}(X_{[j_n^2]}=0)\mathbb{P}\big(X_{[j_n^2]}=2[j_nu]\big|\Lambda_{[j_n^2]}\big)\frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^{[j_nu]}.
$$

The local limit theorem for the simple random walk (see for instance [12, p. 199]) implies that

$$
\lim_{n\to\infty} j_n\mathbb{P}(X_{[j_n^2]}=0)=2\lim_{n\to\infty} j_n\mathbb{P}(\Lambda_{[j_n^2]})=\sqrt{\frac{2}{\pi}}.
\tag{19}
$$

Furthermore (see for instance [16]), the sequence of probability measures (ν_n) defined on Borel sets $A\subset\mathbb{R}_+$ by

$$
\nu_n(A):=j_n\int_A du\mathbb{P}\big(X_{[j_n^2]}=2[j_nu]\big|\Lambda_{[j_n^2]}\big)
$$

converges weakly to the Rayleigh distribution on $\mathbb{R}_+$ with the density $ue^{-\frac{u^2}{2}}du$. Using the dominated convergence theorem, we conclude that

$$
\lim_{n\to\infty} j_nP^{(\delta_n)}(\Lambda_{[j_n^2]})=\int_0^\infty du\ \frac{u}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}-u\gamma-\frac{\gamma^2}{2}\Big),
$$

which proves Lemma 3.6-(i) in view of (15).

(ii) First we will prove the convergence of finite-dimensional distributions. It follows from (18) that for any $m\in\mathbb{N}$, positive reals $0<t_1<\cdots<t_m\le 1$, and Borel sets $A_k\subset\mathbb{R}_+$,

$k = 1, \ldots, m$,

$$P^{(\delta_n)}\Big(Y_n(t_k) \in A_k,\ k = 1, \ldots, m \big| \Lambda_{[j_n^2]}\Big) = \tag{20}$$

$$\frac{\sum_{y \in J_n} \mathbb{P}\Big(Y_n(t_k) \in A_k,\ k = 1, \ldots, m; X_{[j_n^2]} = 2yj_n \big| \Lambda_{[j_n^2]}\Big) \mathbb{P}\big(\Lambda_{[j_n^2]}\big) \frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n} \Big(\frac{1-\delta_n}{1+\delta_n}\Big)^{yj_n}}{P^{(\delta_n)}\big(\Lambda_{[j_n^2]}\big)}.$$

Therefore, by the central limit theorem for random walks conditioned to stay positive (see [8, 16]) combined with the first part of the lemma and (19),

$$\begin{aligned}
&\lim_{n\to\infty} P^{(\delta_n)}\Big(Y_n(t_k) \in A_k,\ k = 1, \ldots, m \big| \Lambda_{[j_n^2]}\Big) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{\mathfrak{N}^{-\gamma}(\zeta > 1)} \int_0^\infty du\ \mathfrak{M}^{(0)}(Y_{t_k} \in A_k,\ k = 1, \ldots, m; Y_1 \in du) \exp\Big(-u\gamma - \frac{\gamma^2}{2}\Big) \\
&= \frac{1}{\mathfrak{N}^{(-\gamma)}(\zeta > 1)} \int_0^\infty du\ \mathfrak{N}^{(0)}(Y_{t_k} \in A_k,\ k = 1, \ldots, m; Y_1 \in du) \exp\Big(-u\gamma - \frac{\gamma^2}{2}\Big) \\
&= \mathfrak{M}^{(-\gamma)}(Y_{t_k} \in A_k,\ k = 1, \ldots, m).
\end{aligned}$$

Next, tightness of the family of discrete distributions follows from the corresponding result for the simple random walk available in Section 3 of [16], along with (20). This completes the proof of Lemma 3.6-(ii).

(iii) We use the second part of the lemma, along with the fact that the process $(Y_n(t))_{t\ge1}$ converges weakly in $C(\mathbb{R}_+, \mathbb{R})$ to a Brownian motion with drift $-\gamma$ (see for instance [17, Theorem II.3.2]). The claim then follows immediately from the Markov property (applied at time $t = 1$) under $\mathfrak{N}^{(-\gamma)}(\,\cdot\,|\zeta > 1)$ (cf. [20, Section VI.48]). □

4. Supercritical Regime

This section is devoted to the proof of Theorem 2.1 and is correspondingly divided into two parts. The proof of the invariance principle for X_n given in Section 4.1 uses a decomposition representing X_n as a sum of a martingale and a drift term. It is then shown that the drift term is asymptotically small compared to the martingale, and that the martingale satisfies the invariance principle. The criterion for the equivalence of P and $\mathbb{P}$ is proved in Section 4.2 by a reduction to a similar question for the law of the sequence of independent variables τ_n defined in (9).

4.1. **Invariance principle for X_n.** The first part of the following proposition states that T_n/n^2 converges in distribution, as $n \to \infty$, to the hitting time of level 1 of the standard Brownian motion, a non-degenerate stable random variable of index 1/2. The second part is required to evaluate both the variance of the martingale term as well as the magnitude of the drift in decomposition (25) below.

Proposition 4.1. *Assume that* $\lim_{n\to\infty} n\varepsilon_n = 0$. *Then*

(i) *For* $\lambda \ge 0$, $\lim_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) = e^{-\sqrt{2\lambda}}$.

(ii) $\frac{1}{n}\sum_{i=1}^n \varepsilon_i \tau_i$ *converges to zero in probability as* $n \to \infty$.

Proof.
(i) It is well-known (see for instance [12, p. 394]) that

$$\lim_{n\to\infty} \mathbb{E}\big(e^{-\lambda T_n/n^2}\big) = \lim_{n\to\infty} \big(\mathbb{E}\big(e^{-\lambda \tau_1/n^2}\big)\big)^n = e^{-\sqrt{2\lambda}},\ \lambda \geq 0.$$

By Lemma 3.2-(i), $E\big(e^{-\lambda T_n/n^2}\big) \geq \mathbb{E}\big(e^{-\lambda T_n/n^2}\big)$. Hence $\liminf_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) \geq e^{-\sqrt{2\lambda}}$. It remains to show that $\limsup_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) \leq e^{-\sqrt{2\lambda}}$.

Let $\delta \in (0,1)$. Clearly,

$$E\big(e^{-\lambda T_n/n^2}\big) \leq \prod_{k=[\delta n]}^{n} E\big(e^{-\lambda \tau_k/n^2}\big). \tag{21}$$

Thanks to Assumption 2.4, we can take n large enough so that $k\varepsilon_k \leq \delta^2/2$ for all $k \geq [\delta n]$. Then, for $k \geq [\delta n]$,

$$\varepsilon_k \leq \frac{\delta^2}{2k} \leq \frac{\delta^2}{2[\delta n]} < \frac{\delta}{n}. \tag{22}$$

Using Lemma 3.2 to estimate the product in the righthand side of (21), we get

$$E\big(e^{-\lambda T_n/n^2}\big) \leq \Big(E^{(\delta/n)}\big(e^{-\lambda \tau_1/n^2}\big)\Big)^{(1-\delta)n}. \tag{23}$$

Next, we observe that, using Lemma 3.1,

$$\begin{aligned} E^{(\delta/n)}\big(e^{-\lambda\tau_1/n^2}\big) &= \frac{1-\sqrt{1-(1-\delta^2/n^2)e^{-2\lambda/n^2}}}{1-\delta/n} \leq \frac{1-\sqrt{1-e^{-(\delta^2+2\lambda)/n^2}}}{1-\delta/n} \\ &= \mathbb{E}\big(e^{-(\delta^2/2+\lambda)\tau_1/n^2}\big)(1-\delta/n)^{-1}. \end{aligned} \tag{24}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) &\underset{(23),\,(24)}{\leq} \limsup_{n\to\infty} \big(\mathbb{E}\big(e^{-(\delta^2/2+\lambda)\tau_1/n^2}\big)\big)^{[(1-\delta)n]}(1-\delta/n)^{-(1-\delta)n} \\ &\underset{(21)}{=} e^{-(1-\delta)\sqrt{\delta^2+2\lambda}}e^{\delta(1-\delta)}. \end{aligned}$$

Letting $\delta \to 0$ completes the proof of Proposition 4.1-(i).

(ii) Fix $\delta \in (0,1)$ and let $S_1 = \frac{1}{n}\sum_{k=1}^{[\delta n]-1} \varepsilon_k \tau_k$, $S_2 = \frac{1}{n}\sum_{k=[\delta n]}^{n} \varepsilon_k \tau_k$. As before, we assume that n is large enough, so that (22) holds true for all $k \geq [\delta n]$. In particular, $S_2 \leq \delta T_n/n^2$. Next,

$$P(S_1 + S_2 \geq 2\sqrt{\delta}) \leq P(S_1 \geq \sqrt{\delta}) + P(S_2 \geq \sqrt{\delta}) \leq \delta^{-1/2}E(S_1) + P(T_n/n^2 \geq \delta^{-1/2}).$$

By Lemma 3.1, $E(S_1) \leq \frac{1}{n}\sum_{k=1}^{[\delta n]}(1+\varepsilon_k) \leq 2\delta$. Therefore,

$$P(S_1 + S_2 \geq 2\sqrt{\delta}) \leq 2\sqrt{\delta} + P(T_n/n^2 \geq \delta^{-1/2}).$$

By part (i), the second term goes to 0 as $n \to \infty$. Letting δ go to 0 finishes the proof. □

We are now in position to give the proof of the first part of Theorem 2.1.

Proof of Theorem 2.1-(i). Recall $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $d_n = \mathrm{sign}(X_n)\varepsilon_{\eta_n}$, and identity (3). Let:

$$H_n = X_n - \overline{D}_n \quad \text{with} \quad \overline{D}_n := \sum_{k=0}^{n-1} d_k. \tag{25}$$

It follows from (3) that $H := (H_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale.

Let $S_n = \sum_{k=1}^{\eta_n} \varepsilon_k \tau_k$. We next prove the following estimate:

$$\lim_{n\to\infty} S_n/\sqrt{n} = 0, \text{ in probability.} \tag{26}$$

Let $\delta > 0$ and $m > 0$. Then,

$$\{S_n > \delta\sqrt{n}\} \subseteq \{\eta_n \geq [\sqrt{mn}]\} \cup \{ \sum_{k=1}^{[\sqrt{mn}]} \varepsilon_k \tau_k > \delta\sqrt{n}\}.$$

Hence, by Proposition 4.1-(ii), $\limsup_{n\to\infty} P(S_n \geq \delta\sqrt{n}) \leq \limsup_{n\to\infty} P(\eta_n \geq [\sqrt{mn}])$. However, $\{\eta_n \geq [\sqrt{mn}]\} = \{T_{[\sqrt{mn}]} \leq n\}$. Therefore,

$$\limsup_{n\to\infty} P(S_n \geq \delta\sqrt{n}) \leq \limsup_{k\to\infty} P(T_k/k^2 \leq 2/m).$$

By letting $m \to \infty$, and since δ is arbitrary, (26) follows Proposition 4.1-(i).

We next apply the martingale central limit theorem [12, pp. 412] to show that H satisfies the invariance principle. Let

$$V_n = \sum_{k=1}^{n} E\big((H_{k+1} - H_k)^2 \big| \mathcal{F}_k\big) = \sum_{k=1}^{n} E\big((X_{k+1} - X_k - d_k)^2 \big| \mathcal{F}_k\big),$$

Due to the fact that H has bounded increments, it is enough to verify that $\lim_{n\to\infty} V_n/n = 1$ in probability. Note that by (3)

$$V_n = \sum_{k=1}^{n} \big(1 - 2d_k^2 + d_k^2\big) = n - \sum_{k=1}^{n} d_k^2,$$

and $\sum_{k=1}^{n} d_k^2 \leq \sum_{k=1}^{n} |d_k| \leq S_n$. It follows from (26) that $\lim_{n\to\infty} \sum_{k=1}^{n} d_k^2/n = 0$ in probability, and, consequently, the invariance principle holds for H.

In order to complete the proof, by [5, Theorem 2.1, p.11], it suffices to show that for all $m > 0$ and any continuous function $\varphi : C[0, m] \to \mathbb{R}$, we have $\lim_{n\to\infty} E\big(\varphi(\mathcal{I}_n^X)\big) = \lim_{n\to\infty} E\big(\varphi(\mathcal{I}_n^{H,m})\big)$, where $\mathcal{I}_n^{H,m}(t)$ coincides with $\mathcal{I}_n^H(t)$ on $[0, m]$. Note that the limit in the righthand side exists due to the invariance principle for H. Since φ is bounded, uniformly continuous, this will follow once we prove that

$$K_n := \max_{t\in[0,m]} \big|\mathcal{I}_n^X(t) - \mathcal{I}_n^{H,m}(t)\big| \underset{n\to\infty}{\to} 0, \quad \text{in } P\text{-probability.}$$

By its definition in (4), $\mathcal{I}_n^X(t)$ (resp. $\mathcal{I}_n^{H,m}(t)$) is a convex combination of $X_{[nt]}$ and $X_{[nt]+1}$ (resp. $H_{[nt]}$ and $H_{[nt]+1}$). Since $|X_{[nt]} - X_{[nt]+1}| = 1$ and $|H_{[nt]} - H_{[nt]+1}| \leq 2$, it follows that

$$K_n \leq \max_{t\in[0,m]} \frac{|X_{[nt]} - H_{[nt]}| + 3}{\sqrt{n}} \leq \max_{t\in[0,m]} \frac{S_{[nt]} + 3}{\sqrt{n}} \leq \frac{S_{nm} + 3}{\sqrt{n}} \underset{n\to\infty}{\to} 0 \text{ in } P\text{-probability,}$$

where the limit in the righthand side is due to (26). □

4.2. Criterion for the equivalence of P and $\mathbb{P}$.

Proof of Theorem 2.1-(ii). Recall $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ and let $\mathcal{G}_n = \mathcal{F}_{T_n}$, the σ-algebra generated by the paths of X up to time T_n. Let $\mathcal{F} = \sigma(\cup_{n\ge 0}\mathcal{F}_n)$ and $\mathcal{G} = \sigma(\cup_{n\ge 0}\mathcal{G}_n)$.

Under both P and $\mathbb{P}$, $\lim_{n\to\infty} T_n = \infty$ with probability one and hence $\mathcal{G} = \mathcal{F}$ up to null-measure sets. Therefore, the measures P and $\mathbb{P}$ are equivalent if $P|_{\mathcal{G}}$ and $\mathbb{P}|_{\mathcal{G}}$, their restrictions to $\mathcal{G}$, are equivalent.

Let $\gamma := (\gamma_0, \gamma_1, \dots)$ be a random walk path starting from the origin. That is, $\gamma_0 = 0$ and $|\gamma_{n+1} - \gamma_n| = 1$ for all n. Let $T_0(\gamma) = 0$ and, for $n \ge 1$,

$$T_n(\gamma) = \min\{i > T_{n-1}(\gamma) : X_i = 0\} \quad \text{and} \quad \tau_n(\gamma) = T_n(\gamma) - T_{n-1}(\gamma). \tag{27}$$

Counting the number of the steps to the left and to the right during each excursion of the random walk from zero, we obtain

$$\begin{aligned} P(X_k = \gamma_k,\ \forall k \le T_n) &= \prod_{k=1}^{n} \frac{1}{2}\Big(\frac{1}{2}(1+\varepsilon_k)\Big)^{\tau_k(\gamma)/2}\Big(\frac{1}{2}(1-\varepsilon_k)\Big)^{\tau_k(\gamma)/2-1} \\ &= 2^{-T_n(\gamma)} \prod_{k=1}^{n} \frac{(1-\varepsilon_k^2)^{\tau_k(\gamma)/2}}{1-\varepsilon_k}, \end{aligned} \tag{28}$$

where the difference between the powers in the righthand side of the first line is due to the fact that from 0, the probability of going either to the right or to the left is $\frac{1}{2}$. On the other hand, $\mathbb{P}(X_k = \gamma_k,\ \forall k \le T_n) = 2^{-T_n(\gamma)}$.

For $n \ge 1$, let

$$F_n(\gamma) := \frac{P(X_k = \gamma_k,\ \forall k \le n)}{\mathbb{P}(X_k = \gamma_k,\ \forall k \le n)} = \prod_{k=1}^{n} \frac{(1-\varepsilon_k^2)^{\tau_k(\gamma)/2}}{1-\varepsilon_k}, \tag{29}$$

and set $F_\infty = \limsup_{n\to\infty} F_n$. Note that $F_n \in \mathcal{G}_n$ and hence $F_\infty \in \mathcal{G}$. By [12, Theorem 3.3, p. 242],

$$P|_{\mathcal{G}} \sim \mathbb{P}|_{\mathcal{G}} \text{ if and only if } F_\infty < \infty,\ P|_{\mathcal{G}}\text{-almost surely;}$$
$$P|_{\mathcal{G}} \perp \mathbb{P}|_{\mathcal{G}} \text{ if and only if } F_\infty = \infty,\ P|_{\mathcal{G}}\text{-almost surely.}$$

Identity (29) with $n = 1$ shows that distribution of τ_k under P is absolutely continuous with respect to its distribution under $\mathbb{P}$, and the corresponding Radon-Nikodym derivative is $(1-\varepsilon_k)^{-1}(1-\varepsilon_k^2)^{\tau_k/2}$. Since $(\tau_k)_{k\ge 1}$ is a sequence of independent random variables under both measures, Kakutani's dichotomy theorem (see [12, p. 244]) implies that

$$F_\infty < \infty \text{ or } = \infty,\ P|_{\mathcal{G}} - \text{a.s., according to whether } \lim_{n\to\infty} \mathbb{E}\big(\sqrt{F_n}\big) > 0 \text{ or } = 0.$$

We have:

$$\mathbb{E}\big(\sqrt{F_n}\big) = \prod_{k=1}^{n} \mathbb{E}\Big(\frac{(1-\varepsilon_k^2)^{\tau_k/4}}{\sqrt{1-\varepsilon_k}}\Big) \underset{\text{Lemma 3.1}}{=} \prod_{k=1}^{n} \frac{1-\sqrt{1-(1-\varepsilon_k^2)^{1/2}}}{\sqrt{1-\varepsilon_k}}.$$

Choose any $\delta \in (0, \sqrt{1/2} - 1/2)$. Since $\lim_{k\to\infty} \varepsilon_k = 0$, we have for all k large enough,

$$1 - \varepsilon_k\sqrt{1/2+\delta} \le 1 - \sqrt{1-(1-\varepsilon_k^2)^{1/2}} \le 1 - \varepsilon_k\sqrt{1/2},$$

and

$$1 - (1/2+\delta)\varepsilon_k \le \sqrt{1-\varepsilon_k} \le 1 - \varepsilon_k/2.$$

In particular, $\lim_{n\to\infty} \mathbb{E}\big(\sqrt{F_n}\big) > 0$ if and only if $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. □

5. Subcritical Regime

The goal of this section is to prove the results presented in Section 2.2. In Section 5.1 we obtain auxiliary limit theorems and large deviations estimates for η_n, the occupation time at the origin. We first prove corresponding results for T_n, and then use the correspondence between $(T_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$. Section 5.2 contains the proof of the limit theorem for X_n stated in Theorem 2.5. In Section 5.3 we prove the more refined result given by Theorem 2.6. Finally, Theorem 2.7 and Corollary 2.8, describing the asymptotic behavior of the range of the random walk, are proved in Section 5.4.

5.1. **Limit theorems and large deviations estimates for T_n and η_n.** Let $N(0,\sigma^2)$ denote a normal random variable with zero mean and variance σ^2. We write $X_n \Rightarrow Y$ when a sequence of random variables $(X_n)_{n\geq 1}$ converges to random variable Y in distribution. Let

$$g_n := \sqrt{E(T_n^2) - (E(T_n))^2} = \Big[\sum_{i=1}^{n} (\varepsilon_i^{-3} - \varepsilon_i^{-1})\Big]^{1/2}, \tag{30}$$

where the first equality is the definition of g_n while the second one follows from Lemma 3.1.

First, we prove the following limit theorem for the sequence $(T_n)_{n\geq 1}$.

Proposition 5.1. *Let Assumption 2.4 hold. Then*

$$\frac{T_n - a_n}{g_n} \Rightarrow N(0,1), \ \textit{as } n \to \infty.$$

In particular, $\lim_{n\to\infty} T_n/a_n = 1$*, where the convergence is in probability.*

Next, we derive from this proposition the following limit theorem for $(\eta_n)_{n\geq 1}$.

Proposition 5.2. *Let Assumption 2.4 hold. Then*

$$\frac{\eta_n - c_n}{\sqrt{n}} \Rightarrow N\Big(0, \frac{1+\alpha}{1+3\alpha}\Big).$$

In particular, $\lim_{n\to\infty} \eta_n/c_n = 1$*, where the convergence is in probability.*

Finally, we complement the above limit results by the following large deviation estimates.

Proposition 5.3. *Let Assumption 2.4 hold. Then, for $x > 0$,*

$$\lim_{n\to\infty} \frac{1}{n\varepsilon_n} \log P\Big(\Big|\frac{T_n}{a_n} - 1\Big| > x\Big) < 0.$$

Corollary 5.4. *Let Assumption 2.4 hold. Then, for $x > 0$,*

$$\lim_{n\to\infty} \frac{b_n^2}{n} \log P\Big(\Big|\frac{\eta_n}{c_n} - 1\Big| > x\Big) < 0.$$

In both the corollaries above, the existence of the limit is a part of the claim.

Corollary 5.5. *Let Assumption 2.4 hold. Then, there exists a sequence $(\theta_n)_{n\geq 1}$ such that $\theta_n \in (0,1)$ for all n, $\lim_{n\to\infty} \theta_n = 0$ and*

$$\lim_{n\to\infty} \exp\Big(\frac{n}{b_n^2 \log n}\Big) \cdot P\Big(\Big|\frac{\eta_n}{c_n} - 1\Big| > \theta_n\Big) = \lim_{n\to\infty} b_n^2 P\Big(\Big|\frac{\eta_n}{c_n} - 1\Big| > \theta_n\Big) = 0.$$

We remark that the estimates stated in Corollary 5.5 are not optimal and, furthermore, the second is actually implied by the first one. However, the statement in the form given above is particularly convenient for reference in the sequel.

Corollary 5.4 is deduced from Proposition 5.3 using a routine argument similar to the derivation of Proposition 5.2 from Proposition 5.1, and thus its proof will be omitted. In turn, Corollary 5.5 is an immediate consequence of Corollary 5.4 and Corollary 3.5-(iv). Indeed, these two results combined together imply that

$$\lim_{n\to\infty} \exp\Big(\frac{n}{b_n^2 \log n}\Big) \cdot P\Big(\Big|\frac{\eta_n}{c_n} - 1\Big| > x\Big) = \lim_{n\to\infty} b_n^2 P(|\eta_n/c_n - 1| > x) = 0$$

for all $x > 0$. Let $n_0 = 1$, for $p \in \mathbb{N}$ let n_p be the smallest integer greater than n_{p-1} such that $\exp\big(\frac{n}{b_n^2 \log n}\big) \cdot P\big(\big|\frac{\eta_n}{c_n} - 1\big| > 1/p\big) < 1/p$ for all $n \geq n_p$, and set $\theta_n = 1/p$ for $n = n_p, \ldots, n_{p+1} - 1$.

Proof of Proposition 5.1. Let $S_n = (T_n - a_n)/g_n$. By Lemma 3.1, $E(S_n) = 0$ and $E(S_n^2) =$. By Lyapunov's version of the CLT for the partial sums of independent random variables, [12, p. 121], $S_n \Rightarrow N(0,1)$ if

$$\lim_{n\to\infty} \frac{1}{g_n^3} \sum_{m=1}^{n} E\big(|\tau_m - 1 - E(\tau_m)|^3\big) = 0.$$

By Lemma 3.1, and using the fact that $\varepsilon_m \in (0,1)$,

$$E\big(|\tau_m - 1 - \varepsilon_m^{-1}|^3\big) \leq 4E\big((\tau_m - 1)^3 + \varepsilon_m^{-3}\big) \leq 4(8\varepsilon_m^{-5} + \varepsilon_m^{-3}) \leq 36\varepsilon_m^{-5}.$$

Next, by Theorem 3.4-(i), as $n \to \infty$, $\sum_{m=1}^n \varepsilon_m^{-5} \sim (1+5\alpha)^{-1} n \varepsilon_n^{-5}$ and

$$g_n^2 \sim (1+3\alpha)^{-1} n \varepsilon_n^{-3}. \tag{31}$$

Therefore,

$$\frac{1}{g_n^3} \sum_{m=1}^{n} \varepsilon_m^{-5} \sim \frac{(1+5\alpha)^{-1} n \varepsilon_n^{-5}}{(1+3\alpha)^{-3/2} n^{3/2} \varepsilon_n^{-9/2}} = \frac{(1+3\alpha)^{3/2}}{(1+5\alpha)} \frac{1}{\sqrt{n\varepsilon_n}} \to 0, \quad \text{as } n \to \infty,$$

where we use Assumption 2.4 to obtain the last limit. This completes the proof of the weak convergence of $(T_n - a_n)/g_n$.

The convergence of T_n/a_n in probability will follow, provided that $\lim_{n\to\infty} a_n/g_n = \infty$. Using again Theorem 3.4-(i), and then Assumption 2.4, we obtain, as $n \to \infty$,

$$\frac{a_n}{g_n} \sim \frac{(1+\alpha)^{-1} n \varepsilon_n^{-1}}{(1+3\alpha)^{-1/2} n^{1/2} \varepsilon_n^{-3/2}} \sim \frac{(1+3\alpha)^{1/2}}{1+\alpha} \sqrt{n\varepsilon_n} \to \infty, \quad \text{as } n \to \infty.$$

The proof of the proposition is completed. □

Proof of Proposition 5.2. First, we observe that the second statement of the proposition follows from the first one and the fact that $\lim_{n\to\infty} c_n/\sqrt{n} = \infty$ (cf. Corollary 3.5-(iv)).

We next prove the central limit theorem for η_n. As in Proposition 5.1, let g_m denote the variance of T_m and let $\widetilde{T}_m = (T_m - a_m)/g_m$. Fix $x \in \mathbb{R}$. By Corollary 3.5-(iv), $x\sqrt{n} + c_n \sim c_n$

as $n \to \infty$, and hence

$$P\Big(\frac{\eta_n - c_n}{\sqrt{n}} \le x\Big) = P(\eta_n \le c_n + x\sqrt{n}) = P(T_{[c_n + x\sqrt{n}]+1} > n)$$

$$= 1 - P\Big(\widetilde{T}_{[c_n+x\sqrt{n}]+1} \le \frac{n - a_{[c_n+x\sqrt{n}]+1}}{g_{[c_n+x\sqrt{n}]+1}}\Big). \tag{32}$$

By (31), as $n \to \infty$, $g_{[c_n+x\sqrt{n}]+1} \sim (1+3\alpha)^{-1/2}(c_n + x\sqrt{n})^{1/2}\varepsilon^{-3/2}_{[c_n+x\sqrt{n}]} \sim \sqrt{\frac{c_n\varepsilon_{c_n}^{-3}}{1+3\alpha}}$, and hence

$$\frac{a_{[c_n+x\sqrt{n}]+1} - n}{g_{[c_n+x\sqrt{n}]+1}} \sim \frac{\sum_{i=c_n}^{[c_n+x\sqrt{n}]+1} \varepsilon_i^{-1}}{\sqrt{c_n\varepsilon_{c_n}^{-3}/(1+3\alpha)}} \underset{\text{Theorem } 3.4-(ii)}{\sim} \frac{x\sqrt{n}\cdot\varepsilon_{c_n}^{-1}\sqrt{1+3\alpha}}{\sqrt{c_n\varepsilon_{c_n}^{-3}}}.$$

The rightmost expression above tends to $x\sqrt{\frac{(1+3\alpha)}{1+\alpha}}$, as $n \to \infty$. Therefore,

$$\lim_{m\to\infty} P\Big(\widetilde{T}_m \le x\sqrt{\frac{1+3\alpha}{1+\alpha}}\Big) \underset{\text{Proposition } 5.1}{=} \lim_{m\to\infty} 1 - P\Big(\widetilde{T}_m \le -x\sqrt{\frac{1+3\alpha}{1+\alpha}}\Big)$$

$$\underset{(32)}{=} \lim_{n\to\infty} P\Big(\frac{\eta_n - c_n}{\sqrt{n}} \le x\Big),$$

completing the proof of Proposition 5.2. □

Proof of Proposition 5.3. Let $\rho_n = \min_{k\le n}\varepsilon_k$. Theorem 3.4-(ii) implies that $\rho_n \sim \varepsilon_n$ as $n \to \infty$. Let $\lambda \in (-\infty, \frac{1}{2})$ and define $\Lambda(\lambda) = \int_0^1 \big(x^{-\alpha} - \sqrt{x^{-2\alpha} - 2\lambda}\big)dx$. We shall prove that

$$\lim_{n\to\infty} \frac{1}{n\varepsilon_n} \log E\big(e^{\lambda\rho_n^2 T_n}\big) = \Lambda(\lambda). \tag{33}$$

Once this result is established, we will deduce the proposition by applying standard Chebyshev's bounds for the tail probabilities of T_n.

To prove (33) we first observe that, by Lemma 3.1,

$$\frac{1}{n\varepsilon_n} \log E\big(e^{\lambda\rho_n^2 T_n}\big) = \frac{1}{n\varepsilon_n}\sum_{i=1}^n \log\Big(1 + \frac{\varepsilon_i - \sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big). \tag{34}$$

Fix $\delta \in (0,1)$. We next show that, when n is large enough, the contribution of the first $[\delta n]$ summands on the righthand side of (34) is bounded by a continuous function of δ which vanishes at 0. We have

$$\Big|\frac{1}{n\varepsilon_n}\sum_{i=1}^{[\delta n]} \log\Big(1 + \frac{\varepsilon_i - \sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big)\Big| \le \frac{1}{n\varepsilon_n}\sum_{i=1}^{[\delta n]} \frac{(1+\varepsilon_i)\big|e^{2\rho_n^2\lambda} - 1\big|}{\varepsilon_i + \sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}$$

$$\le \frac{2}{n\varepsilon_n}\sum_{i=1}^{[\delta n]} \frac{\big|e^{2\rho_n^2\lambda} - 1\big|}{\varepsilon_i} \le \frac{2a_{[\delta n]}}{n\varepsilon_n}\big|e^{2\rho_n^2\lambda} - 1\big|.$$

Since $(a_n)_{n\ge1} \in \mathrm{RV}(1+\alpha)$, Theorem 3.4 implies that, as $n \to \infty$, $a_{[\delta n]} \sim \delta^{1+\alpha}a_n \sim \frac{\delta^{1+\alpha}}{1+\alpha}\varepsilon_n^{-1}$. Therefore,

$$\frac{2a_{[\delta n]}}{n\varepsilon_n}\big|e^{2\rho_n^2\lambda} - 1\big| \underset{n\to\infty}{\sim} \frac{2\delta^{1+\alpha}\varepsilon_n^{-1}}{(1+\alpha)n\varepsilon_n}2\varepsilon_n^2\lambda = \frac{4\lambda\delta^{1+\alpha}}{1+\alpha} \underset{\delta\to0}{\longrightarrow} 0\ .$$

Hence,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\Big|\frac{1}{n\varepsilon_n}\sum_{i=1}^{[\delta n]}\log\Big(1+\frac{\varepsilon_i-\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big)\Big| = 0.$$

Next, using elementary estimates on remainders of Taylor's series, we obtain

$$\lim_{n\to\infty}\frac{1}{n\varepsilon_n}\log E\big(e^{\lambda\rho_n^2 T_n}\big) = \lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n\varepsilon_n}\sum_{i=[\delta n]}^{n}\log\Big(1+\frac{\varepsilon_i-\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big)$$

$$= \lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n\varepsilon_n}\sum_{i=[\delta n]}^{n}\frac{(1+\varepsilon_i)(e^{2\rho_n^2\lambda}-1)}{\varepsilon_i+\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}} = \lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=[\delta n]}^{n}\frac{2\lambda\rho_n}{\varepsilon_i+\sqrt{\varepsilon_i^2-2\rho_n^2\lambda}}$$

$$= \lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=[\delta n]}^{n}\frac{2\lambda}{\varepsilon_i/\rho_n+\sqrt{(\varepsilon_i/\rho_n)^2-2\lambda}} = \int_0^1\frac{2\lambda}{x^{-\alpha}+\sqrt{x^{-2\alpha}-2\lambda}}\,dx = \Lambda(\lambda).$$

This completes the proof of (33).

We note that $\lim_{\lambda\to-\infty}\Lambda(\lambda) = -\infty$. In addition,

$$\Lambda'(\lambda) = \int_0^1\big(x^{-2\alpha}-2\lambda\big)^{-1/2}\,dx.$$

This function is strictly increasing and hence Λ is strictly convex. Note also that $\Lambda'(0) = \frac{1}{1+\alpha}$, $\lim_{\lambda\to-\infty}\Lambda'(\lambda) = 0$, and $\lim_{\lambda\to\frac{1}{2}}\Lambda'(\lambda) = \infty$.

For $z > 0$, let $J_z(\lambda) = \Lambda(\lambda) - \lambda z/(1+\alpha)$. This function is convex and $J_z(0) = 0$. Since $J_z'(\lambda) = \Lambda'(\lambda) - z/(1+\alpha)$, the minimum of J_z is uniquely attained at some $\lambda^* \in (-\infty, \frac{1}{2})$, and $J_z(\lambda^*) < 0$ for $z \neq 1$. In addition, if $z > 1$, $\lambda^* > 0$ and if $z < 1$, $\lambda^* < 0$.
By Theorem 3.4-(i), as $n\to\infty$,

$$a_n\rho_n^2 \sim \frac{n\varepsilon_n^{-1}\varepsilon_n^2}{1+\alpha} = \frac{n\varepsilon_n}{1+\alpha}.$$

It follows that if $\lambda\in(0,\frac{1}{2})$, then for $x > 0$, as $n\to\infty$,

$$\frac{1}{n\varepsilon_n}\log P\big(T_n/a_n \geq 1+x\big) \leq \frac{1}{n\varepsilon_n}\big[\log E\big(e^{\lambda\rho_n^2 T_n}\big) - \lambda a_n\rho_n^2(1+x)\big] \sim J_{1+x}(\lambda).$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{n\varepsilon_n}\log P\big(T_n/a_n \geq 1+x\big) \leq \min_{0<\lambda<\frac{1}{2}} J_{1+x}(\lambda) < 0.$$

If $\lambda < 0$, then for $x\in(0,1)$, as $n\to\infty$,

$$\frac{1}{n\varepsilon_n}\log P\big(T_n/a_n \leq 1-x\big) \leq \frac{1}{n\varepsilon_n}\big[\log E\big(e^{\lambda\rho_n^2 T_n}\big) - \lambda a_n\rho_n^2(1-x)\big] \sim J_{1-x}(\lambda).$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{n\varepsilon_n}\log P\big(T_n/a_n \leq 1-x\big) \leq \min_{\lambda<0} J_{1-x}(\lambda) < 0.$$

Moreover, since $\lim_{\lambda\to\frac{1}{2}}\Lambda'(\lambda) = \infty$, the log-generating function $\Lambda(\lambda)$ is steep in the terminology of [11]. Therefore, by the Gärtner–Ellis theorem (cf. p. 44 in [11], see also Remark (a) following the theorem), the above upper limits are in fact the limits. The proof of Proposition 5.3 is completed. □

5.2. **Proof of Theorem 2.5.** Since the law of X is symmetric about 0, the theorem is equivalent to the claim that $\lim_{n\to\infty} P(X_n > xb_n) = e^{-2x}/2$ for all $x > 0$. Furthermore, since $\lim_{n\to\infty} b_n = \infty$ and $b_n \sim b_{n+1}$, it suffices to show that

$$\lim_{n\to\infty} P(X_{2n} > xb_{2n}) = e^{-2x}/2, \quad x > 0.$$

The idea of the proof is the following. In this subcritical regime, we have seen in the beginning of the section that the number of visits to the origin by time $2n$ is very-well localized around its typical value c_{2n} (cf Proposition 5.2, Corollaries 5.4 and 5.5). From properties of regular varying sequences, this will imply that the drift at time $2n$ is also very-well localized around its typical value $\varepsilon_{c_{2n}}$ (see assertions (35) and (41) below). Then, by Lemma 3.2, we are able to compare our walk with oscillating walks with a drift close to $\varepsilon_{c_{2n}}$ (see (36) and (44) below), for which we know the stationary distribution. In particular, Lemma 3.3 allows us to show that the distribution of X_n is close to that stationary distribution. Let us now turn to the precise argument.

Fix $x > 0$. We begin with an upper bound for $P(X_{2n} > xb_{2n})$. Recall the definition of (θ_n) from Corollary 5.5. For $n \geq 1$, let

$$\Gamma_n = \{X_n > xb_n,\ \eta_n \leq (1+\theta_n)c_n\}.$$

We have

$$P(X_{2n} > xb_{2n}) \leq P(\Gamma_{2n}) + P(\eta_{2n} > (1+\theta_{2n})c_{2n}).$$

We proceed with an estimate of the righthand side. By Theorem 3.4-(ii), as $n \to \infty$,

$$\xi_n := \min_{i\leq(1+\theta_n)c_n} \varepsilon_i \sim \varepsilon_{(1+\theta_n)c_n} \sim \varepsilon_{c_n}. \tag{35}$$

For $n \geq 1$ consider the sequence $\alpha_n = (\alpha_{n,k})_{k\geq 1}$ defined as follows: $\alpha_{n,k} = \varepsilon_k$ for $k \leq (1+\theta_n)c_n$ and $\alpha_{n,k} = \xi_n$ for $k > (1+\theta_n)c_n$. Since on event Γ_n we have $\eta_n \leq (1+\theta_n)c_n$, it follows that $P(\Gamma_{2n}) = P^{\alpha_{2n}}(\Gamma_{2n}) \leq P^{\alpha_{2n}}(X_{2n} > xb_{2n})$.

Recall the notation $P^{(\delta)}$ introduced in the second paragraph of Section 3 (this notation is distinct from P^{δ} and emphasizes that the sequence (δ) is constant). Since $\xi_n = \min_{k\geq 1}\alpha_{n,k}$, Lemma 3.2 implies:

$$P^{\alpha_{2n}}(X_{2n} > xb_{2n}) \leq P^{(\xi_{2n})}(X_{2n} > xb_{2n}) \leq P^{(\xi_{2n})}_{\mu_{\xi_{2n}}}(X_{2n} > xb_{2n}) = \mu_{\xi_{2n}}\big((xb_{2n},\infty)\big). \tag{36}$$

Therefore

$$P(X_{2n} > xb_{2n}) \leq \mu_{\xi_{2n}}\big((xb_{2n},\infty)\big) + P(\eta_{2n} > (1+\theta_{2n})c_{2n}). \tag{37}$$

The second term on the righthand side of (37) converges to 0, as $n \to \infty$, due to Proposition 5.2. Furthermore, (10) and (35) yield that, as $n \to \infty$,

$$\mu_{\xi_{2n}}\big((xb_{2n},\infty)\big) \sim 2\xi_{2n} \sum_{j=[xb_{2n}/2]}^{\infty} \Big(\frac{1-\xi_{2n}}{1+\xi_{2n}}\Big)^{2(j-1)} \sim \frac{1}{2}\Big(\frac{1-\xi_{2n}}{1+\xi_{2n}}\Big)^{xb_{2n}} \underset{\rho\to 0}{\longrightarrow} \frac{1}{2}e^{-2x}. \tag{38}$$

Using (37), we conclude that

$$\limsup_{n\to\infty} P(X_{2n} > xb_{2n}) \leq e^{-2x}/2.$$

We now turn to a lower bound on $P(X_{2n} > xb_{2n})$. It follows from Corollary 3.5-(iv) that there exists a sequence $(\kappa_n)_{n\geq 1}$ taking values in $2\mathbb{Z}_+$ and satisfying

$$\lim_{n\to\infty} \kappa_n/n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\kappa_n \varepsilon_{c_n}^2}{\log(\varepsilon_{c_n}^{-1})} = \infty. \tag{39}$$

Note that the second limit in (39) ensures that $\lim_{n\to\infty} \kappa_n = \infty$. Let

$$\Upsilon_n = \{m \in \mathbb{N} : |m - c_n| \leq \theta_n c_n\}. \tag{40}$$

By Theorem 3.4-(ii), we have, as $n \to \infty$,

$$\beta_n := \max_{m\in\Upsilon_n} \varepsilon_m \sim \varepsilon_{c_n}. \tag{41}$$

Since the function $z \to z^2/\log(z^{-1})$ is increasing on $(0,1)$, the second limit in (39) along with (41) imply that $\lim_{n\to\infty} \frac{\kappa_n\beta_n^2}{\log(\beta_n^{-1})} = \lim_{n\to\infty} \frac{\kappa_n\beta_{n-\kappa_n}^2}{\log(\beta_{n-\kappa_n}^{-1})} = \infty$. Therefore,

$$\lim_{n\to\infty} (1+\beta_n^2)^{-\kappa_n}\beta_n^{-s} = \lim_{n\to\infty} (1+\beta_{n-\kappa_n}^2)^{-\kappa_n}\beta_{n-\kappa_n}^{-s} = 0 \text{ for all } s \in \mathbb{R}. \tag{42}$$

We have

$$\begin{aligned}
P(X_{2n} > xb_{2n}) &= \frac{1}{2}P(|X_{2n}| > xb_{2n}) \geq \frac{1}{2}P(|X_{2n}| > xb_{2n}, \eta_{2n-\kappa_{2n}} \in \Upsilon_{2n-\kappa_{2n}}) \\
&= \frac{1}{2}\sum_{m\in\Upsilon_{2n-\kappa_{2n}}} P(|X_{2n}| > xb_{2n}, \eta_{2n-\kappa_{2n}} = m) \\
&= \frac{1}{2}\sum_{m\in\Upsilon_{2n-\kappa_{2n}}}\sum_{j\in 2\mathbb{Z}} E\big(\mathbf{I}_{\{\eta_{2n-\kappa_{2n}}=m, X_{2n-\kappa_{2n}}=j\}} P_{(j,m)}(|X_{\kappa_{2n}}| > xb_{2n})\big).
\end{aligned} \tag{43}$$

For $j \in 2\mathbb{Z}$ and $m \in \Upsilon_{2n-\kappa_{2n}}$, Lemma 3.2 implies that

$$P_{(j,m)}(|X_{\kappa_{2n}}| > xb_{2n}) \geq P_j^{(\beta_{2n-\kappa_{2n}})}(|X_{\kappa_{2n}}| > xb_{2n}) \geq P^{(\beta_{2n-\kappa_{2n}})}(|X|_{\kappa_{2n}} > xb_{2n}). \tag{44}$$

Plugging this inequality into the righthand side of (43), we obtain

$$P(X_{2n} > xb_{2n}) \geq P(\eta_{2n-\kappa_{2n}} \in \Upsilon_{2n-\kappa_{2n}}) P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}} > xb_{2n}). \tag{45}$$

The first term on the righthand side of (45) converges to 1, as $n \to \infty$, by Corollary 5.5. Moreover, by Lemma 3.3,

$$P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}} > xb_{2n}) \geq \mu_{\beta_{2n-\kappa_{2n}}}\big((xb_{2n},\infty)\big) - 2(1+\beta_{2n-\kappa_{2n}}^2)^{-\kappa_{2n}}.$$

The second term on the righthand side converges to 0 due to (42). Therefore, (38) and (41) imply that

$$\liminf_{n\to\infty} P(X_{2n} > xb_{2n}) \geq e^{-2x}/2,$$

which completes the proof of Theorem 2.5. □

5.3. **Proof of Theorem 2.6.**

Proof of Theorem 2.6-(i). As in the previous paragraph, this proof once again relies on Lemma 3.3 and Corollary 5.4. We adopt notation from the proof of Theorem 2.5 above.

It follows from (37) that

$$P(X_{2n}=0)\geq \mu_{\xi_{2n}}(0)-P(\eta_{2n}\geq (1+\theta_{2n})c_{2n}).$$

Therefore,

$$b_{2n}P(X_{2n}=0)\underset{(10)}{\geq}\frac{2}{1+\xi_{2n}}\frac{\xi_{2n}}{\varepsilon_{c_{2n}}}-b_{2n}P(\eta_{2n}\geq (1+\theta_{2n})c_{2n}).$$

The second term on the righthand side converges to 0 due to Corollary 5.4 while he first term converges to 2 due to (35). Hence,

$$\liminf_{n\to\infty} b_{2n}P(X_{2n}=0)\geq 2.$$

The upper bound is obtained in a similar way. Recall Υ_n was defined in (40). By (45),

$$\begin{aligned}
P(X_{2n}=0) &\leq 1-P(\eta_{2n-\kappa_{2n}}\in \Upsilon_{2n-\kappa_{2n}})P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}}\geq 2)\\
&\leq 1-(1-P(\eta_{2n-\kappa_{2n}}\notin \Upsilon_{2n-\kappa_{2n}})P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}}\geq 2)\\
&= P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}}=0)+P(\eta_{2n-\kappa_{2n}}\notin \Upsilon_{2n-\kappa_{2n}})\\
&\leq \mu_{\beta_{2n-\kappa_{2n}}}(0)+2(1+\beta^2_{2n-\kappa_{2n}})^{-\kappa_{2n}}+P(\eta_{2n-\kappa_{2n}}\notin \Upsilon_{2n-\kappa_{2n}}),
\end{aligned}$$

where in the last step we used Lemma 3.3. Therefore,

$$\begin{aligned}
b_{2n}P(X_{2n}=0) \underset{(10)}{\leq} & \frac{2}{1+\beta_{2n-\kappa_{2n}}}\frac{\beta_{2n-\kappa_{2n}}}{\varepsilon_{c_{2n}}}+2(1+\beta^2_{2n-\kappa_{2n}})^{-\kappa_{2n}}\beta^{-1}_{2n-\kappa_{2n}}\frac{\beta_{2n-\kappa_{2n}}}{\varepsilon_{c_{2n}}}\\
&+b_{2n}P(\eta_{2n-\kappa_{2n}}\notin \Upsilon_{2n-\kappa_{2n}}).
\end{aligned}$$

The third term on the righthand side converges to 0 due to Corollary 5.4. The second term on the righthand side converges to 0 by (41) and (42). Finally, the first term on the righthand side converges to 2 by (41). Hence,

$$\limsup_{n\to\infty} b_{2n}P(X_{2n}=0)\leq 2.$$

This completes the proof of the first part of Theorem 2.6. □

Proof of Theorem 2.6-(ii). Recall Υ_n from (40) and Λ_n from (17). By Corollary 5.5, and using the Markov property, we obtain for $t>0$,

$$\begin{aligned}
&\liminf_{n\to\infty} b^2_{2n}P(V_{2n}=2n-2[tb^2_{2n}])\\
&\quad=\liminf_{n\to\infty} b^2_{2n}\sum_{m\in\Upsilon_{2n-2[tb^2_{2n}]}}P(V_{2n}=2n-2[tb^2_{2n}],\eta_{2n-2[tb^2_{2n}]}=m)\\
&\quad=\liminf_{n\to\infty} b^2_{2n}\sum_{m\in\Upsilon_{2n-2[tb^2_{2n}]}}P(X_{2n-2[tb^2_{2n}]}=0,\eta_{2n-2[tb^2_{2n}]}=m)\cdot 2P^{(\varepsilon_m)}(\Lambda_{2[tb^2_{2n}]}),
\end{aligned}\tag{46}$$

The factor 2 in the last line comes from the fact that we also want count excursions to the negative half-line and a symmetry argument.

Recall (41). By Lemma 3.2,

$$\liminf_{n\to\infty} b_{2n}^2 P(V_{2n} = 2n - 2[tb_{2n}^2])$$
$$\geq 2 \liminf_{n\to\infty} b_{2n}^2 P\big(X_{2n-2[tb_{2n}^2]} = 0, \eta_{2n-2[tb_{2n}^2]} \in \Upsilon_{2n-2[tb_{2n}^2]}\big) P^{(\beta_{2n-2[tb_{2n}^2]})}(\Lambda_{2[tb_{2n}^2]}).$$

Using again Corollary 5.5, and taking in account that $\lim_{n\to\infty} b_{2n}/b_{2n-2[tb_{2n}^2]} = 1$, we get

$$\liminf_{n\to\infty} b_{2n}^2 P(V_{2n} = 2n - 2[tb_{2n}^2]) \geq 2 \liminf_{n\to\infty} b_{2n}^2 P(X_{2n-2[tb_{2n}^2]} = 0) P^{(\beta_{2n-2[tb_{2n}^2]})}(\Lambda_{2[tb_{2n}^2]}).$$

Using Lemma 3.6-(i) and Theorem 2.6-(i), we conclude that

$$\begin{aligned}\liminf_{n\to\infty} b_{2n}^2 P(V_{2n} = 2n - 2[tb_{2n}^2]) &\geq 2 \cdot \frac{1}{\sqrt{2t}} \int_0^\infty du \, \frac{2u}{\sqrt{2\pi}} \exp\Big(-\frac{u^2}{2} - u\sqrt{2t} - t\Big) \\ &= \sqrt{2}\mathfrak{N}^{(-\sqrt{2})}(\zeta > t).\end{aligned}$$

A very similar argument shows that

$$\limsup_{n\to\infty} b_{2n}^2 P(V_{2n} = 2n - 2[tb_{2n}^2]) \leq \sqrt{2}\mathfrak{N}^{(-\sqrt{2})}(\zeta > t),$$

from which Theorem 2.6-(ii) follows in view of (16).

Proof of Theorem 2.6-(iii). Fix a bounded continuous function $F : C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$, a constant $t > 0$. Let Z_n be the process defined in the statement of the theorem and let $\big(\widetilde{Z}_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $\widetilde{Z}_n\big(k/b_{2n}^2\big) = |X_{k\wedge S_n}|/b_{2n}$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere.

For n large enough, so that the quantities below are well defined, the Markov property implies that

$$E\big(F(Z_n)|V_{2n} = 2n - 2[tb_{2n}^2], \eta_{2n} = m\big) = E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big).$$

Let

$$H_n(t) := b_{2n}^2 \sum_{m\in\Upsilon_{2n}} E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) P\big(V_{2n} = 2n - 2[tb_{2n}^2], \eta_{2n} = m\big). \tag{47}$$

Since F is bounded, Corollary 5.5 implies that

$$\lim_{n\to\infty} \big|b_{2n}^2 E\big(F(Z_n); V_{2n} = 2n - 2[tb_{2n}^2]\big) - H_n(t)\big| = 0. \tag{48}$$

Recall Υ_n was defined in (40). For $m \in \mathbb{N}$, let $\underline{m}_n \in \Upsilon_{2n}$ and $\overline{m}_n \in \Upsilon_{2n}$ be such that

$$E^{(\varepsilon_{\underline{m}_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = \min_{m\in\Upsilon_{2m}} E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big)$$

and

$$E^{(\varepsilon_{\overline{m}_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = \max_{m\in\Upsilon_{2m}} E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big).$$

Since $\lim_{n\to\infty} \varepsilon_{\underline{m}_n} b_{2n} = \lim_{n\to\infty} \varepsilon_{\overline{m}_n} b_{2n} = 1$, Lemma 3.6 implies that there exists

$$\lim_{n\to\infty} E^{(\varepsilon_{\underline{m}_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = \lim_{n\to\infty} E^{(\varepsilon_{\overline{m}_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = E\big(F(\overline{Y})\big) \tag{49}$$

where $\overline{Y} = (\overline{Y}(s))_{s\in\mathbb{R}_+}$ is a non-negative process in $C(\mathbb{R}_+, \mathbb{R})$ such that $\frac{1}{\sqrt{2t}}\big(\overline{Y}(2ts)\big)_{s\in\mathbb{R}_+}$ is distributed according to the law $\mathfrak{N}^{(-\sqrt{2t})}(\,\cdot\,|\zeta > 1)$, and the underlying probability space is enlarged, if needed, to include this process. The scaling property (16) implies that $\overline{Y}$ is distributed according to the law $\mathfrak{N}^{(-1)}(\,\cdot\,|\zeta > 2t)$.

In virtue of Theorem 2.6-(ii) and Corollary 5.5, the claim of Theorem 2.6-(iii) follows from the above convergence and (47). □

Remark 5.6. *Theorem 2.6-(ii) along with* (49) *yield*

$$\lim_{n\to\infty} P(|X_{2n}| > xb_{2n}) = \int_0^\infty dt\ \mathfrak{N}^{(-1)}(X_t > x, \zeta > t) = \int_x^\infty dy \int_0^\infty dt \frac{2y}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(y-ct)^2}{2t}\Big).$$

It is not hard to check that the right-hand side above is $\exp(-2x)$ *in agreement with Theorem 2.5.*

5.4. **Proof of Theorem 2.7 and Corollary 2.8.**

Proof of Theorem 2.7. For $i \geq 1$ let $S_i = \max_{T_{i-1}\leq k<T_i} |X_k|$. For $x > 0$ let $x_n = xh_n$, where $h_n$is defined in the statement of the theorem. Recall $(\theta_n)_{n\geq 1}$ from Corollary 5.5 and Υ_n from (40).

Fix any $x \in (0,\infty)\backslash\{1\}$, $\lambda \in (0,1)$, and assume that $n \in \mathbb{N}$ below is large enough, so that $1-\theta_n > \lambda$. Then, on one hand,

$$\begin{aligned} P\big(\mathcal{M}_n \leq x_n\big) &= P\big(\mathcal{M}_n \leq x_n, \eta_n < c_n(1-\theta_n)\big) + P\big(\mathcal{M}_n \leq x_n, \eta_n \geq c_n(1-\theta_n)\big) \\ &\leq P(\eta_n \not\in \Upsilon_n) + \prod_{i=[\lambda c_n]}^{[c_n(1-\theta_n)]} P\big(S_i \leq x_n\big), \end{aligned} \tag{50}$$

and on the other hand,

$$\begin{aligned} P\big(\mathcal{M}_n \leq x_n\big) &\geq P\big(\mathcal{M}_n \leq x_n, \eta_n \leq c_n(1+\theta_n)\big) \\ &\geq -P(\eta_n \not\in \Upsilon_n) + \prod_{i=1}^{[c_n(1+\theta_n)]} P\big(S_i \leq x_n\big). \end{aligned} \tag{51}$$

Observe now that

$$\lim_{n\to\infty} \frac{c_n\varepsilon_{c_n}}{n\varepsilon_{c_n}^2} \underset{\text{Theorem } 3.4-(iii)}{=} \lim_{n\to\infty} \frac{c_n}{a_{c_n}\varepsilon_{c_n}} \underset{\text{Theorem } 3.4-(i)}{=} (1+\alpha) < \infty,$$

and hence, by Corollary 5.5, for all $z > 0$,

$$\lim_{n\to\infty} \frac{P(\eta_n \not\in \Upsilon_n)}{e^{-(c_n\varepsilon_{c_n})^z}} = 0. \tag{52}$$

Next, by the well-known formula for the ruin probability (see for instance [12, p. 274]),

$$P(S_i \leq x_n) = 1 - \frac{\rho_i}{(1+\rho_i)^{x_n} - 1}, \tag{53}$$

where $\rho_i = \dfrac{2\varepsilon_i}{1-\varepsilon_i}$. For $n \in \mathbb{N}$ let

$$\chi_n := \min_{1\leq i\leq(1+\theta_n)c_n} \rho_i \sim 2\varepsilon_{c_n} \quad \text{and} \quad \beta_{n,\lambda} := \max_{\lambda c_n\leq i\leq(1+\theta_n)c_n} \rho_i \sim 2\lambda^{-\alpha}\varepsilon_{c_n}, \tag{54}$$

where we use Theorem 3.4-(iii) to state the equivalence relations. Since the righthand side in (53) is an increasing function of ρ_i, we obtain:

$$\log \prod_{i=1}^{[(1+\theta_n)c_n]} P(S_i \le x_n) \ \ge [(1+\theta_n)c_n] \log\Big(1-\tfrac{\chi_n}{(1+\chi_n)^{x_n}-1}\Big) \underset{n\to\infty}{\sim} -\frac{c_n\chi_n}{(1+\chi_n)^{x_n}}.$$

We next estimate the rightmost expression above. Using (54) and the definition of h_n given in the statement of Theorem 2.7, we have, as $n \to \infty$,

$$\frac{1}{\log(\varepsilon_{c_n} c_n)} \cdot \log \frac{c_n\chi_n}{(1+\chi_n)^{x_n}} \sim 1 - \frac{2x\varepsilon_{c_n} h_n}{\log(\varepsilon_{c_n} c_n)} \sim 1 - x,$$

Similarly, as $n \to \infty$,

$$\log \prod_{i=[\lambda c_n]}^{[(1-\theta_n)c_n]} P(S_i \le x_n) \ \le [(1-\theta_n-\lambda)c_n] \log\left(1 - \frac{\beta_{n,\lambda}}{(1+\beta_{n,\lambda})^{x_n}-1}\right) \sim -\frac{(1-\lambda)c_n\beta_{n,\lambda}}{(1+\beta_{n,\lambda})^{x_n}},$$

and

$$\frac{1}{\log(\varepsilon_{c_n} c_n)} \cdot \log \frac{(1-\lambda)c_n\beta_{n,\lambda}}{(1+\beta_{n,\lambda})^{x_n}} \sim 1 - \frac{2x\lambda^{-\alpha}\varepsilon_{c_n} h_n}{\log(\varepsilon_{c_n} c_n)} \sim 1 - x\lambda^{-\alpha},$$

Since $\lambda \in (0,1)$ is arbitrary, we conclude from (50), (51), and (52) that

$$\lim_{n\to\infty} \frac{1}{\log(c_n\varepsilon_{c_n})} \log\big(-\log P(\mathcal{M}_n \le x_n)\big) = 1 - x.$$

Note that if $x > 1$, this is equivalent to $\lim_{n\to\infty} \frac{1}{\log(c_n\varepsilon_{c_n})} \log\big(P(\mathcal{M}_n > x_n)\big) = 1 - x$.

To complete the proof of Theorem 2.7, observe that

$$P\big(\max_{T_{i-1}\le k<T_i} X_k \le x_n\big) = \frac{1}{2} + \frac{1}{2}P\big(S_i \le x_n\big) = 1 - \frac{1}{2}\frac{\rho_i}{(1+\rho_i)^{x_n}-1}.$$

Therefore, replacing $\mathcal{M}_n$ with M_n and S_i with $\max_{T_{i-1}\le k<T_i} X_k$ in (50) and (51), the proof given above for $\mathcal{M}_n$ goes through verbatim for M_n. □

Proof of Corollary 2.8. Theorem 2.7 implies $\lim_{n\to\infty} M_n/h_n = 1$ in probability. Furthermore, by Corollary 3.5-(iii), $\varepsilon_{c_n} c_n \in \mathrm{RV}((1-\alpha)/(1+\alpha))$. Therefore, if $\alpha < 1$, Theorem 2.7 implies that for any $x > 0$ there exists a constant $z = z(x) > 0$ such that

$$P\big(|M_n - h_n| > xh_n\big) \le n^{-z}$$

for all n sufficiently large.

Once this point is reached, the rest of the proof is standard (see for instance [12, Section 1.7]). Fix $\gamma > 1$ and let $m_n = [\gamma^n]$. Using the Borel-Cantelli lemma, we obtain that

$$P\big(|M_{m_n} - h_{m_n}| > xh_{m_n} \text{ i.o.}\big) = 0, \qquad x > 0.$$

Therefore $\lim_{n\to\infty} M_{m_n}/h_{m_n} = 1$, a.s. Moreover, if $m_n \le k < m_{n+1}$,

$$\frac{M_{m_n}}{h_{m_n}}\frac{h_{m_n}}{h_k} \le \frac{M_k}{h_k} \le \frac{M_{m_{n+1}}}{h_{m_{n+1}}}\frac{h_{m_{n+1}}}{h_k}.$$

Since $\lim_{n\to\infty} m_{m+1}/m_m = \gamma$ and $(h_n)_{n\ge1} \in \mathrm{RV}(\alpha/(1+\alpha))$, Theorem 3.4-(ii) implies that

$$\gamma^{-\frac{\alpha}{1+\alpha}} \le \liminf_{k\to\infty} \frac{M_k}{h_k} \le \limsup_{k\to\infty} \frac{M_k}{h_k} \le \gamma^{\frac{\alpha}{1+\alpha}}, \qquad P-\text{a.s.}$$

Since $\gamma > 1$ is arbitrary, it follows that $\lim_{k\to\infty} M_k/h_k = 1$, P-a.s. Furthermore, if $(k_n)_{n\geq 1}$ is a random sequence such that $X_{k_n} = M_{k_n}$, we have:

$$\limsup_{n\to\infty} \frac{X_n}{h_n} \geq \limsup_{n\to\infty} \frac{X_{k_n}}{h_{k_n}} = \lim_{n\to\infty} \frac{M_{k_n}}{h_{k_n}} = 1,$$

where the limits hold P-a.s. when $\alpha < 1$ and in probability when $\alpha = 1$. Since $X_n \leq M_n$, this finishes the proof of the corollary. □

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