Reminders

In what follows we consider X a Markov chain with kernel P with state space E.
Recall x is recurrent
iff $\mathbb{P}_x(T_x^+ < \infty) = 1$
iff $\mathcal{V}_x = +\infty$ a.s. under $\mathbb{P}_x$
iff $\sum_{n\geq 0} P^n(x,x) = +\infty$.
Recall further that recurrence is a class property, that is is $x \leftrightarrow y$ they are either both transient or recurrent.
Finally recall that λ is invariant iff $\lambda P = \lambda$.

Invariant measures of an irreducible, recurrent chain

Theorem

Assume X irreducible, and x is recurrent. Introduce

$$\nu_x(y) := \mathbb{E}_x\left[\sum_{k=0}^{T_x^+-1} \mathbb{1}_{\{X_k=y\}}\right], y \in E. \quad \textit{Then}$$

- ν_x *is an invariant measure for the chain.*
- ν_x *is the unique invariant measure of X attributing mass 1 to x. The chain admits a unique invariant probability iff $\mathbb{E}_x[T_x^+] < \infty$.*

When $\mathbb{E}_x[T_x^+] < \infty$ we say x is *positive recurrent* and the invariant probability is $\pi = \nu_x / \mathbb{E}_x[T_x^+]$. Because of the above theorem positive recurrence remains a class property.
When $\mathbb{E}_x[T_x^+] = +\infty$ we say x is *null recurrent*, and there exists no invariant probability for the chain.

Invariant measures of more general chains :remarks

Note that the irreducibility assumption is easy to relax : the chain restricted to the recurrent class of x is irreducible, thus the theorem implies there is a unique invariant measure charging the states in the class of x which attributes mass 1 to x, and it is ν_x. If x is positive recurrent, there is a unique (extreme) invariant probability charging only states in the class of x. If x is null recurrent, there is no invariant probability charging states in the class of x.

When there are several positive recurrent classes, then any invariant probability is a convex combination of these extreme invariant probabilities.

On the other hand, if there is only one positive recurrent class, then again there exists a unique invariant probability, $\pi = \nu_x / \mathbb{E}_x[T_x^+]$.

Proof of theorem, invariance of ν_x

By definition of T_x^+, there is exactly one visit to x between 0 and $T_x^+ - 1$, thus $\nu_x(x) = 1$. For any $n \in \mathbb{N}$, $\nu_x(y)P^n(y,x) \leq \nu_x(x)$ therefore $\nu_x(y) < \infty$ and so ν_x is indeed a nonnegative measure. Now

$$
\begin{aligned}
\nu_x P(y) &= \sum_{z \in E} \nu_x(z) P(z,y) \\
&= \sum_{z \in E} \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \mathbb{1}_{\{X_k = z\}} \right] P(z,y) \\
&= \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \sum_{z \in E} \mathbb{1}_{\{X_k = z\}} P(z,y) \right]
\end{aligned}
$$

where the interchange of summation comes from Fubini-Tonelli

Proof of theorem, invariance of ν_x

Hence

$$
\begin{aligned}
\nu_x P(y) &= \sum_{z \in E} \nu_x(z) P(z,y) \\
&= \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \sum_{z \in E} \mathbb{1}_{\{X_k = z\}} P(X_k, y) \right] \\
&= \mathbb{E}_x \left[\underbrace{\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \sum_{z \in E} \mathbb{1}_{\{X_k = z\}}}_{\mathcal{F}_k-\text{measurable}} \mathbb{E}\left[\mathbb{1}_{\{X_{k+1} = y\}} \mid \mathcal{F}_k \right] \right],
\end{aligned}
$$

where $\mathbb{E}\left[\mathbb{1}_{\{X_{k+1}=y\}} \mid \mathcal{F}_k\right] = P(X_k, y)$ comes from the Markov property at time k, and the above-mentioned measurability follows from the fact that T_x^+ is a stopping time.

Proof of theorem, invariance of ν_x

Thus

$$
\begin{aligned}
\nu_x P(y) &= \sum_{z \in E} \nu_x(z) P(z,y) \\
&= \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \sum_{z \in E} \mathbb{1}_{\{X_k = z\}} P(X_k, y) \right] \\
&= \mathbb{E}_x \left[\underbrace{\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \sum_{z \in E} \mathbb{1}_{\{X_k = z\}}}_{\mathcal{F}_k-\text{measurable}} \mathbb{E}\left[\mathbb{1}_{\{X_{k+1} = y\}} \mid \mathcal{F}_k \right] \right] \\
&= \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \mathbb{1}_{\{X_{k+1} = y\}} \right]
\end{aligned}
$$

where we used $\sum_{z \in E} \mathbb{1}_{\{X_k = z\}} = 1$ and the definition of conditional expectation.

Proof of theorem, invariance of ν_x

Finally

$$\begin{aligned} \nu_x P(y) &= \sum_{z \in E} \nu_x(z) P(z, y) \\ &= \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{\{k < T_x^+\}} \mathbb{1}_{\{X_{k+1} = y\}} \right] \\ &= \mathbb{E}_x \left[\sum_{k'=1}^{T_x^+} \mathbb{1}_{\{X_{k'} = y\}} \right] = \nu_x(y) \end{aligned}$$

where we set $k' = k + 1$ for obtaining the before-to-last equality. To obtain the last we used that at both times 0 and T_x^+, the chain is at x so that whatever $y \in E$, $\sum_{k=0}^{T_x^+ - 1} \mathbb{1}_{\{X_k = y\}} = \sum_{k'=1}^{T_x^+} \mathbb{1}_{\{X_{k'} = y\}}$.

Proof of theorem

We have established that ν_x indeed is an invariant measure attributing mass 1 to state x. It is also obvious that $\nu_x(E) = \sum_{y \in E} \nu_x(y) = \mathbb{E}_x[T_x^+]$.
Before turning to proving unicity of such invariant measure, we establish

Lemma

Assume μ is an invariant, non-zero measure of X. Then for any $y \in E$, $\mu(y) > 0$.

Since μ is non-zero there must exist $x \in E$ such that $\mu(x) > 0$. For any $n \in \mathbb{N}$, $\mu(y) = \mu P^n(y) \geq \mu(x) P^n(x, y)$, so the claim follows from the irreducibility assumption.

Proof of theorem, unicity

It remains to prove unicity of an invariant measure which assigns mass 1 to x. Let λ be such a measure. Since λ is invariant and $\lambda(x) = 1$, we have

$$\begin{aligned}
\lambda(y) &= \lambda P(y) \\
&= P(x,y) + \sum_{z_1 \neq x} \lambda(z_1) P(z_1, y) \\
&= P(x,y) + \sum_{z_1 \neq x} \lambda P(z_1) P(z_1, y) \\
&= P(x,y) + \sum_{z_1 \neq x} P(x, z_1) P(z_1, y) + \sum_{z_1, z_2 \neq x} \lambda(z_2) P(z_2, z_1) P(z_1, y)
\end{aligned}$$

Proof of theorem, unicity

Iterating

$$\begin{aligned}
\lambda(y) &= \lambda P(y) \\
&= P(x,y) + \sum_{z_1 \neq x} P(x,z_1)P(z_1,y) + \sum_{z_1,z_2 \neq x} \lambda(z_2)P(z_2,z_1)P(z_1,y) \\
&= P(x,y) + \sum_{z_1 \neq x} P(x,z_1)P(z_1,y) + \cdots + \\
&\quad + \sum_{z_1,\ldots,z_{n-1} \neq x} P(x,z_{n-1})P(z_{n-1},z_{n-2})\ldots P(z_2,z_1)P(z_1,y) \\
&\quad + \sum_{z_1,\ldots,z_n \neq x} \lambda(z_n)P(z_n,z_{n-1})\ldots P(z_2,z_1)P(z_1,y)
\end{aligned}$$

Proof of theorem, unicity

The last term in the sum being nonnegative, we have for any n,

$$\begin{aligned}\lambda(y) &\geq P(x,y) + \sum_{z_1 \neq x} P(x,z_1)P(z_1,y) + \cdots + \\ &\quad + \sum_{z_1,\ldots,z_{n-1} \neq x} P(x,z_{n-1})P(z_{n-1},z_{n-2})\ldots P(z_2,z_1)P(z_1,y) \\ &\geq \mathbb{P}_x(X_1 = y) + \mathbb{P}_x(T_x^+ \geq 2, X_2 = y) + \cdots + \\ &\quad \mathbb{P}_x(T_x^+ \geq n, X_n = y)\end{aligned}$$

which, by MCT, converges to $\nu_x(y)$ when $n \to \infty$.

Proof of theorem, unicity

We have established that if λ is invariant with $\lambda(x) = 1$ then $\lambda \geq \nu_x$.
This implies $\mu = \lambda - \nu_x$ is also a (nonnegative) invariant measure.
Since $\mu(x) = 0$, by the lemma it has to be the zero measure, and we conclude that $\lambda = \nu_x$, as required.

Transient chain : no unicity

The recurrence assumption is crucial in the theorem.
Simple, asymmetric random walk on $\mathbb{Z}$ has in fact infinitely many invariant measures charging the origin with mass one. Indeed for any $\alpha \in [0,1]$, the measure λ_α such that

$$\lambda(x) = \alpha + (1-\alpha)\left(\frac{p}{1-p}\right)^x, \; x \in \mathbb{Z}$$

is invariant, and such that $\lambda(0) = 1$.

Transient chain : no invariant probability

Theorem

Assume X is irreducible, transient. Then X possesses no invariant probability.

By contradiction assume there exists π a probability invariant for X. Pick $x_0 \in E$ such that $\pi(x_0) > 0$. There must exists a finite $A \subset E$ such that $\pi(A) \geq 1 - \pi(x_0)/3$. We claim that for any $x, y \in E$, $P^n(x, y) \to 0$ as $n \to \infty$. Indeed transience implies $\sum_{n \geq 0} P^n(x, y) \leq \sum_{n \geq 0} P^n(y, y) < \infty$. Hence, since A is finite, there exists n_0 such that for any $x \in A$, $P^{n_0}(x, x_0) \leq \pi(x_0)/3$. Now

$$\pi(x_0) = \pi P^{n_0}(x_0) \leq \pi(A^c) + \sum_{x \in A} \pi(x) P^n(x, x_0) \leq 2\pi(x_0)/3,$$

a contradiction.