Reminder

Assume X is Markov (λ, P), and $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration. Recall that by the Markov property at time n, the conditional distribution of X_{n+1} given $\mathcal{F}_n$ is $P(X_n, \cdot)$. Thus, for $f : E \to \mathbb{R}$ bounded or satisfying some intergrability conditions, we have

$$\begin{aligned}\mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] &= \sum_{y\in E} P(X_n, y) f(y) \\ &= Pf(X_n)\end{aligned}$$

Martingale problem for a Markov chain

Theorem (2.1)

TFAAE :

(i) *X is Markov with kernel P.*

(ii) *For any $f : E \to \mathbb{R}$ bounded, if*

$$M_n^f := f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P - Id) f(X_k),$$

then $(M_n^f)_{n \geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Of course (ii) above can be generalized to any unbounded function f such that M_n^f remains integrable.
Note that the converse part in the theorem implies that knowing the distribution of X_0 and that all processes M^f are martingales is enough to fully determine the law of X.

Martingale problem for a Markov chain

If X is Markov with kernel P, $f : E \to \mathbb{R}$ bounded, it is clear that for all n, M_n^f is $\mathcal{F}_n$-measurable, integrable, and

$$\begin{aligned} &\mathbb{E}[M_{n+1}^f \mid \mathcal{F}_n] \\ = \; &\mathbb{E}[f(X_{n+1} \mid \mathcal{F}_n] - f(X_0) - \sum_{k=0}^{n}(P - Id)f(X_k) \end{aligned}$$

using that $f(X_0) + \sum_{k=0}^{n-1}(P - Id)f(X_k)$ is $\mathcal{F}_n$-measurable. By the above computation $\mathbb{E}[f(X_{n+1} \mid \mathcal{F}_n] = Pf(X_n)$, whence

$$\mathbb{E}[M_{n+1}^f \mid \mathcal{F}_n] = Pf(X_n) - f(X_0) - \sum_{k=0}^{n}(P - Id)f(X_k)$$

The nth term in the sum if $Pf(X_n) - f(X_n)$, yielding the desired result.

Martingale problem for a Markov chain

Assume now that for any $f : E \to \mathbb{R}$ bounded, M^f is a $(\mathcal{F}_n)$-martingale. With $f = \mathbb{1}_{\{y\}}$ we find that

$$\mathbb{P}[X_{n+1} = y \mid \mathcal{F}_n] = P(X_n, y),$$

in particular

$$\mathbb{P}(X_{n+1} = y \mid X_0 = x_0, ..., X_n = x_n) = P(x_n, y),$$

hence X is indeed Markov with kernel P.

Conditions for $f(n, X_n)$ to be a martingale

Theorem (2.2)

Assume $f : \mathbb{N} \times E \to \mathbb{R}$ is such that $f(n, X_n)$ is integrable for any $n \in \mathbb{N}$, and satisfies $Pf(n+1, x) = f(n, x)$ for any $n \in \mathbb{N}, x \in E$. Then $(f(n, X_n))_{n \in \mathbb{N}}$ is a $(\mathcal{F}_n)$-martingale.

Proof : $\mathbb{E}[f(n+1, X_{n+1}) \mid \mathcal{F}_n] = Pf(n+1, X_n) = f(n, X_n)$

Corollary (2.2.1)

Assume that f_λ is an eigenfunction of P associated with eigenvalue $\lambda \in \mathbb{C}^*$, i.e. $Pf_\lambda = \lambda f_\lambda$, and that $f_\lambda(X_n)$ is integrable. Then $\left(\frac{f_\lambda(X_n)}{\lambda^n}\right)_{n \in \mathbb{N}}$ is a $(\mathcal{F}_n)$-martingale.

In particular, if f is *P-harmonic*, that is $Pf = f$, then $(f(X_n))_{n \geq 0}$ is a $(\mathcal{F}_n)$-martingale.

We have $Pf(x) = pf(x+1) + (1-p)f(x-1)$.
In particular $PId(x) = x + (2p-1)$.
It follows from Theorem (2.1) (or rather its generalization to unbounded functions) that $(S_n - n(2p-1))_{n\in\mathbb{N}}$ is a $(\mathcal{F}_n)$-martingale.
In the symmetric case, this martingale can be used to recover gambler's ruin probabilities.

Application : Martingales of SRW (exercise II.3)

Let us compute

$$
\begin{aligned}
&\mathbb{E}[(S_{n+1} - (n+1)(2p-1))^2 \mid \mathcal{F}_n] \\
= \; & p(S_n + 1 - (n+1)(2p-1))^2 + (1-p)(S_n - 1 - (n+1)(2p-1))^2 \\
= \; & p(S_n - n(2p-1) + 2(1-p))^2 + (1-p)(S_n - n(2p-1) - 2p)^2 \\
= \; & (S_n - n(2p-1))^2 + 4(1-p)p(S_n - n(2p-1)) \\
& \qquad -4p(1-p)(S_n - n(2p-1)) + 4p(1-p)^2 + 4p^2(1-p) \\
= \; & (S_n - n(2p-1))^2 + 4p(1-p).
\end{aligned}
$$

It follows from Theorem (2.2) that
$\left((S_n - n(2p-1))^2 - 4np(1-p)\right)_{n\in\mathbb{N}}$ is a $(\mathcal{F}_n)$-martingale.
In the symmetric case, this martingale can be used to compute the expected time of play in gambler's ruin.

Application : Exponential martingales of SRW (exercise II.3)

For $f(x) = \exp(\alpha x)$ we find

$$Pf(x) = (p\exp(\alpha) + (1-p)\exp(-\alpha))f(x),$$

that is, such f is an eigenfunction of P associated with eigenvalue $\lambda = pe^{\alpha} + (1-p)e^{-\alpha}$. It follows from Corollary (2.2.1) that for any $\alpha \in \mathbb{R}$,

$$\left(\frac{\exp(\alpha S_n)}{(pe^{\alpha} + (1-p)e^{-\alpha})^n}\right)_{n\in\mathbb{N}}$$

is a $(\mathcal{F}_n)$-martingale (it is refered to as an *exponential martingale* of the SRW).

Note that λ spans values $[2\sqrt{\frac{p}{(1-p)}}, +\infty)$. Now $\alpha_0 = \log\left(\frac{1-p}{p}\right)$ corresponds to $\lambda_0 = p\frac{1-p}{p} + (1-p)\frac{p}{1-p} = 1$, so that $\exp(\alpha_0 \cdot)$ is P-harmonic and $\left(\frac{1-p}{p}\right)^{S_n}$ is a $(\mathcal{F}_n)$-martingale. This last martingale can be used to recover gambler's ruin probabilities in the asymmetric case.

Martingales of Galton-Watson process (exercise II.5)

Assume that if ξ has the offspring distribution, then $\mathbb{E}[\xi] = m$, and that ζ is a minimal fixed point of the moment generating function of ξ (as we've seen previously, ζ also is the probability of extinction for the process started with a single one).
Then $\left(\frac{Z_n}{m^n}\right)_{n\geq 0}$, $(\zeta^{Z_n})_{n\geq 0}$ are both $(\mathcal{F}_n)$-martingales.
Note however, that this particular case rather shows a limitation of the general method. Indeed , it is here much easier to make direct proofs using $Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n,k}$ and computing the conditional expectations, rather than trying to express the kernel of Z, which in general does not have a nice expression.
Similar considerations hold for the Wright-Fisher model (see exercise II.6).

As we will see in the next paragraph, when X is irreducible on E finite, the only P-harmonic functions are the constants. So long for deducing useful martingales. Thankfully, we have the following result

Theorem (2.3)

Let $D \subsetneq E$, $T := \inf\{n \geq 0 : X_n \in D^c\}$, and assume $f : E \to \mathbb{R}$ is such that $f(X_{n\wedge T})$ is integrable for any $n \geq 0$ and that for some $\alpha \in \mathbb{R}^*$, $Pf(x) = \alpha f(x) \forall x \in D$, Then

$$\left(\frac{f(X_{n\wedge T})}{\alpha^{n\wedge T}}\right)_{n\geq 0} \quad \text{is a } (\mathcal{F}_n)\text{-martingale.}$$

Variables $\mathbb{1}_{\{T>n\}}, \frac{X_T}{\alpha^T}\mathbb{1}_{\{T\leq n\}}$ are both $\mathcal{F}_n$-measurable because T is a stopping time. Also $X_n \in D$ on $\{T > n\}$ so $Pf(X_n)\mathbb{1}_{\{T>n\}} = \alpha f(X_n)\mathbb{1}_{\{T>n\}}$. It follows

$$
\begin{aligned}
& \mathbb{E}\left[\frac{f(X_{(n+1)\wedge T})}{\alpha^{(n+1)\wedge T}} \mid \mathcal{F}_n\right] \\
= \; & \mathbb{E}\left[\frac{f(X_{n+1})}{\alpha^{n+1}}\mathbb{1}_{\{T>n\}} \mid \mathcal{F}_n\right] + \frac{f(X_T)}{\alpha^T}\mathbb{1}_{\{T\leq n\}} \\
= \; & \frac{Pf(X_n)\mathbb{1}_{\{T>n\}}}{\alpha^{n+1}} + \frac{f(X_T)}{\alpha^T}\mathbb{1}_{\{T\leq n\}} \\
= \; & \frac{f(X_n)}{\alpha^n}\mathbb{1}_{\{T>n\}} + \frac{f(X_T)}{\alpha^T}\mathbb{1}_{\{T\leq n\}} = \frac{f(X_{n\wedge T})}{\alpha^{n\wedge T}},
\end{aligned}
$$

and we are done.

A solution to $Pf = f$ on D, $f \equiv \phi$ on D^c is called a solution to the Dirichlet problem with boudary condition ϕ. As we shall see in next section, under irreducibility assumptions such solution exists and is unique.

Now Theorem (2.3) ensures that if f is f solves such Dirichlet problem, then $(f(X_{n\wedge T}))_{n\geq 0}$ is a martingale. If $T < \infty$ a.s. and Doob's optional stopping theorem applies (e.g. if f remains bounded, which will be the case if ϕ is itself bounded on $\partial D = \{y \in D^c : \exists x \in DP(x,y) > 0\}$), this ensures that for any $x \in D$,

$$f(x) = \mathbb{E}_x[\phi(X_T)].$$

For example, if for some $y_0 \in \partial D$ we set $\phi = \mathbb{1}_{y_0}$, then we can compute the probability to exit D through y_0 starting at any $x \in D$ by solving the corresponding Dirichlet problem (and in fact, this is exactly what we did to compute gambler's ruin probabilities).