Potentials

Assume X is Markov with state space E and kernel $P, D \subset E$ and $T := \inf\{n \geq 0 : X_n \in D^c\}$. For $g : D \to \mathbb{R}_+$ a *cost function* and $\phi : D^c \to \mathbb{R}_+$ a *boundary condition*, define the *potential* $u : E \to \overline{\mathbb{R}_+}$ by

$$u(x) := \mathbb{E}_x \left[\sum_{k=0}^{T-1} g(X_k) + \phi(X_T) \mathbb{1}_{\{T<\infty\}} \right], \ x \in E.$$

Remark : It is actually enough to define ϕ only on $\partial D = \{y \in D^c : \exists x \in D \ P(x,y) > 0\}$

Equation satisfied by a potential

Theorem (2.4)

(i) *Potential u satisfies*

$$(E_{g,\phi}) \qquad u(x) = \begin{cases} Pu(x) + g(x) & \text{if } x \in D \\ \phi(x) & \text{if } x \in D^c \end{cases} .$$

(ii) *If $T < \infty$ $\mathbb{P}_x$-a.s., $(E_{g,\phi})$ possesses at most one nonnegative solution bounded on D.*

(iii) *u is the nonnegative minimal solution of $(E_{g,\phi})$.*

Combining (ii) and (iii) above shows that when $T < \infty$, $\mathbb{P}_x$-a.s., either u is bounded and it is the unique nonnegative minimal solution of $(E_{g,\phi})$, either it is not, and there exists no such solution. On the other hand when $\exists x \in D : \mathbb{P}_x(T = \infty) > 0$, we may have several nonnegative bounded solutions of $(E_{g,\phi})$. Think, e.g. of $g \equiv 0$ on D, and $\phi \equiv 1$ on D^c. Then $u(x) = \mathbb{P}_x(T < \infty)$, but $v \equiv 1$ on E is also solution to $(E_{g,\phi})$.

Equation satisfied by a potential

Note that potentials and equation $(E_{g,\phi})$ depend linearly on g, ϕ. More precisely, if $g = \alpha_1 g_1 + \alpha_2 g_2, \phi = \alpha_1 \phi_1 + \alpha_2 \phi_2$, then $u = \alpha_1 u_1 + \alpha_2 u_2$ (where u_i is the potential associated with g_i, ϕ_i). Also $v = \alpha_1 v_1 + \alpha_2 v_2$ solves $(E_{g,\phi})$ provided v_i solves (E_{g_i,ϕ_i}). This allows to relax the assumption of nonnegativity of g, ϕ : a potential associated with general g, ϕ simply is defined as the difference between the potential associated with the positive parts of g, ϕ and the potential associated with the negative parts.

Proof of Theorem (2.4) (i)

If $x \in D^c$, then $T = 0$ and it is obvious that $u(x) = \phi(x)$.
Otherwise $x \in D$ so $T \geq 1$, and applying Markov at time 1 leads to

$$\begin{aligned} u(x) &= g(x) + \sum_{y \in E} P(x,y)\mathbb{E}_x\left[\sum_{k=1}^{T-1} g(X_k) + \phi(X_T)\mathbb{1}_{\{T<\infty\}} \mid X_1 = y\right] \\ &= g(x) + \sum_{y \in E} P(x,y)\mathbb{E}_y\left[\sum_{k=0}^{T-1} g(X_k) + \phi(X_T)\right], \\ &= g(x) + Pu(x) \end{aligned}$$

Proof of Theorem (2.4) (ii), (iii)

Assume v is a nonnegative solution of $(E_{g,\phi})$. Then, for $x_0 \in D$,

$$\begin{aligned} v(x_0) &= \sum_{x_1 \in D^c} g(x_0) + \phi(x_1) + \cdots + \\ &+ \sum_{x_1,\ldots,x_{n-1} \in D, x_n \in D^c} [g(x_0) + \cdots + g(x_{n-1}) + \phi(x_n)] \\ &+ \sum_{x_1,\ldots,x_n \in D} [g(x_1) + \cdots + g(x_{n-1}) + v(x_n)] \end{aligned}$$

Thus

$$
\begin{aligned}
v(x_0) &= \mathbb{E}_x \left[\left[\sum_{k=0}^{T-1} g(X_k) + \phi(X_T) \right] \mathbb{1}_{\{T \le n\}} \right] + \\
&\quad + \mathbb{E}_x \left[\left[\sum_{k=0}^{n-1} g(X_k) + v(X_n) \right] \mathbb{1}_{\{T > n\}} \right]
\end{aligned}
$$

The first term above converges to $u(x_0)$ as $n \to \infty$ by MCT. When v bounded on D and $T < \infty$ $\mathbb{P}_{x_0}$-a.s, the second goes to 0 as $n \to \infty$ by DCT, yielding (ii).
In any case the second term is always nonnegative, yielding $v(x_0) \ge u(x_0)$ and (iii).

Dirichlet problem : a sufficient condition for existence and unicity

Corollary (2.4.1)

Assume $D \subsetneq E$, X irreducible, recurrent, $g \equiv 0$ and ϕ bounded on ∂D. Then u is the unique bounded solution to the Dirichlet problem with boundary condition ϕ.

With the assumption on g, ϕ, it is clear that for any $x \in D, u(x) \leq \max_{y \in \partial D} \phi(y)$, so that u is bounded on D.
If X is assumed irreducible, recurrent, then for any $x \in D, y \in D^c$, $T_y < \infty$, $\mathbb{P}_x$-a.s., and thus $T < \infty$, $\mathbb{P}_x$-a.s. By Theorem 2.4(ii) we deduce that u is the unique bounded solution to $(E_{0,\phi})$.

Dirichlet problem : remarks

When $D = E$, by looking at the time-reversed chain (see exercises 11 and 12), one can prove that if X is irreducible, recurrent, the only harmonic functions on the whole of D are the constants.

On the other hand, as we have seen with the example of asymmetric SRW on $\mathbb{Z}$, these results do not hold anymore for a transient chain. Assume for example that $p > 1/2$, and take e.g. $D = \mathbb{N}^*$, $\phi(0) = 1$, then for any $\alpha \in [0, 1]$, the function u_α such that

$$u_\alpha(x) = \alpha \left(\frac{1-p}{p}\right)^x + (1-\alpha), x \in \mathbb{N}$$

is bounded, and does satisfy $u = Pu$ on D.

Sufficient condition for existence and unicity of bounded solution to $(E_{c,\phi})$

Corollary (2.4.2)

Assume D finite, ϕ bounded, and X irreducible. Then u is the unique bounded solution to $(E_{g,\phi})$.

Proof : Fix $y \in D^c$. Since X irreducible, for any $x \in D$, there exists $n(x)$ such that $P^{n(x)} =: p_x > 0$. Since D is finite, $\mathrm{lcm}(n(x), x \in D) =: N$ is finite, and $p := \inf\{p(x)^{N/n(x)}, x \in D\} > 0$. But then $T \leq T_y \leq NG$, with $G \sim \mathrm{Geom}(p)$, and therefore $\max_{x \in D} \mathbb{E}_x[T] < \infty$. In particular assumptions of Theorem 2.4 (ii) is satisfied, by Theorem 2.4 we only have to show that u is bounded. Now

$$||u||_\infty \leq ||g||_\infty \left(\max_{x \in D} \mathbb{E}_x[T] \right) + ||\phi||_\infty < \infty.$$

The case $\phi \equiv 0$, $D = E$ and Green functions

When $\phi \equiv 0$, $D = E$, and for some $y \in E$, $g = \mathbb{1}_{\{y\}}$, then $u = u_y$ is related to the *Green function*

$$G(x,y) := u_y(x) = \mathbb{E}_x\left[\sum_{n\geq 0} \mathbb{1}_{\{X_n=y\}}\right] = \sum_{n\geq 0} P^n(x,y)$$

Clearly $G(x,y) = +\infty$ iff $x \to y$ and y is recurrent.
Otherwise, by strong Markov at successive returns at y, the total numbers of visits at y is 0 with probability $\mathbb{P}_x(T_y = \infty)$, otherwise it is geometrically distributed with parameter $\mathbb{P}_y(T_y^+ = \infty)$. Thus $G(x,y) = \frac{\mathbb{P}_x(T_y<\infty)}{\mathbb{P}_y(T_y=+\infty)}$.
Finally, for $\phi \equiv 0$, $D = E$, and general g, by superposition we find that

$$u(x) = \sum_{y\in E} G(x,y)g(y).$$

The case $\phi \equiv 0$ and Green functions

This can be easily extended to $D \subsetneq E$:

$$G_D(x,y) = \mathbb{E}_x \left[\sum_{n=0}^{T-1} \mathbb{1}_{\{X_n = y\}} \right] = \frac{\mathbb{P}_x(T_y < T)}{\mathbb{P}_y(T < T_y^+)}$$

and for general g,

$$u(x) = \sum_{y \in D} G_D(x,y) g(y).$$

Discounted costs

Theorem (2.5)

Let $g : E \to \mathbb{R}_+$ be bounded, $\alpha \in (0,1)$. Then $u : E \to \mathbb{R}$ such that $u(x) = \mathbb{E}_x \left[\sum_{n \geq 0} \alpha^n g(X_n) \right]$ is the unique bounded solution to the equation $(\star)$ $u = \alpha P u + g$.

Proof : The fact that u solves $(\star)$ comes from applying Markov at time 1 as in the previous proof. Moreover, if g bounded, so is u because $||u||_\infty \leq \frac{||c||_\infty}{1-\alpha}$. Finally if v is bounded and solves $(\star)$ then $w = v - u$ is bounded and satisfies $w = \alpha P w$. But as P is contractive, it must be that $||w||_\infty \leq \alpha ||w||_\infty$, forcing $w \equiv 0$, as required.

Resolvent

What we just explained can be rewritten as follows : $|||P||| \leq 1$ so $|||\alpha P||| < 1$ and thus $Id - \alpha P$ is inversible. This guarantees the solution to

$$u = \alpha P u + g \;\Leftrightarrow\; (Id - \alpha P)u = g$$

is given by

$$u = \left(\sum_{n \geq 0} \alpha^n P^n \right) g = (Id - \alpha P)^{-1} g.$$

The operator $R_\alpha := \sum_{n \geq 0} \alpha^n P^n$ is called the *resolvent*.