Recurrence criterion for a reversible chain via effective resistance to infinity (exercise II.19)

Consider $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ infinite countable, locally finite graph, and let $x_0 \in \mathcal{V}$. Assume $(\mathcal{G}_n)_{n \geq 0}$ is a nondecreasing sequence of finite subgraphs of $\mathcal{G}$ with $\mathcal{G}_n \to \mathcal{G}$. Then, by Rayleigh, $\lim_{n\to\infty} \mathcal{R}(x_0 \leftrightarrow \mathcal{G}_n^c)$ exists (in $\overline{\mathbb{R}_+}$). Since for any n there exist n_1, n_2 such that $\mathcal{G}_{n_1} \subset B(x_0, n) \subset \mathcal{G}_{n_2}$, again by Rayleigh the above limit does not depend on the choice of $(\mathcal{G}_n)$, we denote it by $\mathcal{R}(x_0 \leftrightarrow \infty)$.
Using the theorem of the previous paragraph

$$\mathbb{P}_{x_0}(T_{B(x_0,n)^c} < T_{x_0}^+) = \frac{1}{c(x_0)\mathcal{R}(x_0 \leftrightarrow B(x_0, n)^c)} \underset{n\to\infty}{\longrightarrow} \frac{1}{c(x_0)\mathcal{R}(x_0 \leftrightarrow \infty)}$$

Since under $\mathbb{P}_{x_0}$, $T_{B(x_0,n)^c} \geq n$, the above LHS goes to $\mathbb{P}(T_{x_0}^+ = +\infty)$ as $n \to \infty$, so

Theorem (2.16)

$$x_0 \textit{ recurrent } \Leftrightarrow \mathcal{R}(x_0 \leftrightarrow \infty) = +\infty.$$

SRW on $\mathbb{Z}^d, d \leq 2$ is recurrent (exercise II.20)

Here $\mathcal{G}$ is the usual lattice graph on $\mathbb{Z}^d$. Let $\Pi_k := \{(x,y) \in \mathcal{E} : ||x||_\infty = k, ||y||_\infty = k+1$, so that, for $0 \leq k \leq n$, Π_k is a cut set between 0 and $B(0,n)^c$ (where the distance used for $B(0,n)$ is the infinite norm). By Nash-Williams,

$$\begin{aligned}\mathcal{R}(0 \leftrightarrow B(0,n)^c) &\geq \sum_{k=0}^{n} \frac{1}{\sum_{e \in \Pi_k} c(e)} \\ &= \sum_{k=0}^{n} \frac{1}{2d(2k+1)^{d-1}},\end{aligned}$$

and it follows that $\mathcal{R}(0 \leftrightarrow \infty) = +\infty$ for $d \leq 2$.
We conclude that SRW on $\mathbb{Z}^d, d = 1,2$ is recurrent.

SRW on $\mathbb{Z}^d$, $d \geq 3$ (exercises II.23,24)

Let θ be the unit flow from 0 to $B(0,n)^c$ (where, in this slide, we now use $||\cdot||_1$ to define $B(0,n)$) such that the incoming and outcoming flow at every node of $S(0,k)$ is $1/\#(S(0,k))$. The existence of such flow can be proven thanks to an urn model (see exercise 23). Now since $\#(S(0,k)) \sim C_d k^{d-1}$, it follows that

$$E(I_1) \leq E(\theta) \leq \sum_{k=1}^{n} \sum_{e \in S(0,k)} \frac{1}{\#(S(0,k))^2} \leq C'_d \sum_{k=1}^{n} \frac{1}{k^{d-1}},$$

and we conclude by Thomson that $R(0 \leftrightarrow \infty) < \infty$ when $d \geq 3$. We conclude that SRW is transient on $\mathbb{Z}^d$, $d \geq 3$.