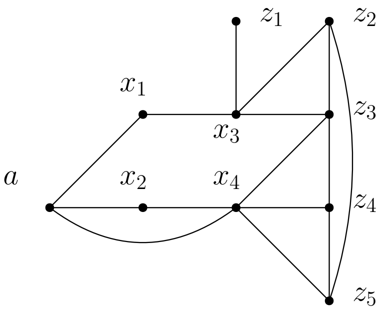
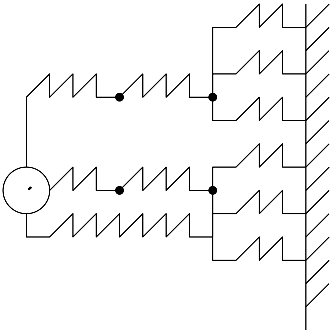
Context

Consider X an irreducible, reversible chain, The reversibility assumption means there exists an invariant measure μ such that for any $(x, y) \in E^2$ $\mu(x)P(x, y) = \mu(y)P(y, x)$.

As we have seen previously, setting $\mathcal{V} = E$, $\mathcal{E} = \{\{x, y\} : P(x, y) > 0\})$ and $c(x, y) = K\mu(x)P(x, y) = c(y, x)$ for some $K > 0$, the conductance model on $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with conductance function c has the same kernel as X.
This constant K can be taken in an arbitrary way. Beware that certain quantities defined below may depend on K, but it should always be the case that functionals of the chain X will not.

In practice, it may be sometimes useful (in particular when comparing two conductance models on the same set of edges) to consider a slightly looser setting : allow some pairs $\{x, y\}$ such that $P(x, y) = 0$ to be in the set $\mathcal{E}$ of edges, and set the conductance function at zero along these edges (in other words, having no edge present between two states is equivalent to putting an edge of infinite resistance between those states)





In what follows, we single out a *source* $a \in E$, a *sink* $Z \subsetneq E \setminus \{a\}$ and we denote by $D := E \setminus (Z \cup \{a\})$. As in the previous paragraph, we let $T := \inf\{n \geq 0 : X_n \in D^c\}$.

We always make the assumption that for any $x \in D$, $T < \infty$ $\mathbb{P}_x$-a.s. (note that this is e.g. the case if D finite, or if X is recurrent)

Electric potential

Definition (2.6)

For $\alpha > \beta$ two real numbers, let the *(electric) potential* fixed at α at a, and at β on Z be the unique function $V_{\alpha,\beta}$ such that

$$V_{\alpha,\beta}(a) = \alpha, \quad V_{\alpha,\beta}(z) = \beta \; \forall z \in Z,$$
$$V_{\alpha,\beta} \text{ harmonic on } D, \text{i.e.} PV_{\alpha,\beta}(x) = V_{\alpha,\beta}(x) \; \forall x \in D$$

Existence and unicity of $V_{\alpha,\beta}$ was established in the previous paragraph (Thm 2.4). These also guarantee the following easy lemma.

Lemma (2.6.1)

We have $V_{1,0}(x) = \mathbb{P}_x(T_a < T_Z), x \in E.$
For any $\alpha > \beta$, $V_{\alpha,\beta} = (\alpha - \beta)V_{1,0} + \beta.$

Current

Let $\vec{\mathcal{E}}$ the set of oriented edges of $\mathcal{G}$. For a function $\Phi : \vec{\mathcal{E}} \to \mathbb{R}$ and $\vec{e} = (x, y) \in \vec{E}$ we write $\Phi(\vec{e}) = \Phi(x, y)$

Definition (2.7)

To a potential $V = V_{\alpha,\beta}$ one associates the *(electric) current* from a to Z, $I = I_{\alpha,\beta} : \overrightarrow{\mathcal{E}} \to \mathbb{R}$ such that for any $\{x, y\} \in \mathcal{E}$,

$$I(x, y) = -I(y, x) = c(x, y)(V(x) - V(y)).$$

Note that $I_{\alpha+c,\beta+c} = I_{\alpha,\beta}$ that is, $I_{\alpha,\beta}$ depends only on α, β through the difference $\alpha - \beta$. More precisely for any $c \geq 0$ there is a unique current associated with all potentials $V_{\alpha,\beta}$ such that $\alpha - \beta = c$.

Properties of a current from a to Z

Lemma (2.7.1)

A current I from a to Z satisfies

(i) $V(x) - V(y) = I(x,y)r(x,y)$, *with* $r(x,y) = \frac{1}{c(x,y)}$.

(ii) *For any* $x \in D$, $\mathrm{div}_x(I) := \sum_{y:\{x,y\}\in\mathcal{E}} I(x,y) = 0$.

(iii) *If* $x_1 \sim x_2 \sim \cdots \sim x_n \sim x_{n+1} = x_1$ *is a cycle* $\sum_{k=1}^{n} r(x_k, x_{k+1})I(x_k, x_{k+1}) = 0$.

(iv) $\mathrm{div}_a(I) =: ||I|| > 0$

(i) above is *Ohm's law*, (ii) and (iii) are *Kirchoff's node and cycle laws*. The current I such that $||I|| = 1$ is called the *unit current* from a to Z. It is the one associated with potentials $V_{\alpha,\alpha-c}$ such that $c = \frac{1}{||I_{1,0}||}$

Proof of lemma (2.7.1)

(i) is the definition of the current I.
For any $x \in D$,

$$\begin{aligned}\sum_{y:y\sim x} I(x,y) &= \sum_{y:y\sim x} c(x,y)(V(x)-V(y)) \\ &= c(x)PV(x) - c(x)V(x) = 0,\end{aligned}$$

by the fact that V is harmonic on D, yielding (ii).
If $x_1 \sim \cdots \sim x_{n+1} = x_1$, we have by Ohm's law

$$\sum_{k=1}^{n} r(x_k, x_{k+1}) I(x_k, x_{k+1}) = \sum_{k=1}^{n} V(x_k) - V(x_{k+1}) = 0.$$

Finally, (iv) relies on the fact that V reaches its maximum at a by Lemma (2.6.1), so that

$$\sum_{y:y\sim a} I(x,y) = \sum_{y:y\sim a} c(a,y)(V(a)-V(y)) \geq 0.$$

Maximum principle for harmonic functions

We just used that the function V, harmonic on D has to reach its maximum on $D \cup \partial D$ somewhere on the boundary ∂D. This is known as *maximum principle for harmonic functions*, which can be easily proven independently.

Theorem (2.8)

If $f : E \to \mathbb{R}$ is harmonic on D and reaches its supremum at some element $x_0 \in D$. Then f is constant on the set of states which can be reached by the chain $(X_{n \wedge T})$ (also called the chain absorbed at D^c).

Proof of theorem (2.8) : Let $S = \{x \in E : f(x) = f(x_0)\}$. If $x \in S$ and $P(x,z) > 0$, we must have $z \in S$ because

$$||f||_\infty = f(x) = \sum_{y \sim x} P(x,y) f(y) \leq P(x,z) f(z) + (1 - P(x,z)) ||f||_\infty.$$

Probabilistic interpretation of current

Theorem (2.9)

Unit current I from a to Z can be expressed along a given oriented edge (x, y) as

$$I(x,y) = \mathbb{E}_a\left[\sum_{k=0}^{T_Z-1} \mathbb{1}_{\{X_k=x, X_{k+1}=y\}} - \mathbb{1}_{\{X_k=y, X_{k+1}=y\}}\right],$$

Proof : Observe first that RHS has divergence 1 at a, since in the trajectory from a up to T_Z oriented edges incident from a are exactly crossed once more than edges towards a. Using Markov at time k, the above RHS is

$$\begin{aligned} & G_{D\cup\{a\}}(a,x)P(x,y) - G_{D\cup\{a\}}(a,y)P(y,x) \\ = \quad & c(x,y)\left(\frac{G_{D\cup\{a\}}(a,x)}{c(x)} - \frac{G_{D\cup\{a\}}(a,y)}{c(y)}\right). \end{aligned}$$

Current as expected edge crossings : proof

Letting $W(x) := \frac{G_{D\cup\{a\}}(a,x)}{c(x)}$, we have found that the expected edge crossings is given by $c(x,y)(W(x) - W(y))$. To verify this defines a current (hence the unit current by our preliminary remark), it remains to check W is harmonic on D.

$$\begin{aligned}
PW(x) &= \sum_{y\sim x} P(x,y)W(y) \\
&= \sum_{y\sim x} \frac{c(x,y)}{c(x)} \frac{G_{D\cup\{a\}}(a,y)}{c(y)} \\
&= \frac{1}{c(x)} \sum_{y\sim x} P(y,x) G_{D\cup\{a\}}(a,y) \\
&= \frac{1}{c(x)} \sum_{y\sim x} \mathbb{E}_a \left[\sum_{k=0}^{T_Z-1} \mathbb{1}_{\{X_k=y, X_{k+1}=y\}} \right] \\
&= \frac{1}{c(x)} G_{D\cup\{a\}}(x).
\end{aligned}$$

Flows

Definition (2.10)

A function $\theta : \vec{\mathcal{E}} \to \mathbb{R}$ is a *flow* from a to Z iff

- θ is antisymmetric, that is $\theta((x,y)) = -\theta((y,x))$ for any $(x,y) \in \vec{\mathcal{E}}$.
- $\mathrm{div}_x(\theta) = 0$ for any $x \in D$.
- $\mathrm{div}_a(\theta) \geq 0$,
- $\theta(x,y) = 0$ for any $x, y \in Z$

Theorem (2.10)

Assume D finite, and that θ is a flow from a to Z satisfying Kirchoff's cycle law. Then θ is a current from a to Z.

Proof of Theorem (2.10)

For any $\vec{e} = (x, y)$ such that x and y are both in Z we have $\theta(\vec{e}) = I(\vec{e}) = 0$.

Thus we may as well identify all vertices in Z into a single one, say $\tilde{z}$, allowing for multiple edges between a vertex $x \in D \cup \{a\}$ and $\tilde{z}$. More precisely, the new set of vertices is $\tilde{E} = \{a\} \cup D \cup \{\tilde{z}\}$, and edges between any pair of vertices with at least one not initially in Z are kept with their original conductance (note that the new graph $\tilde{\mathcal{G}}$ is finite because D is finite and $\mathcal{G}$ was assumed locally finite).

Note also that θ remains, on $\tilde{\mathcal{G}}$, a flow from a to $\tilde{z}$, which still satisfies the cycle law (any part of the cycle inside Z can be simply forgotten as the flow along the corresponding edges is null).

Since $\sum_{x \in \tilde{E}} \mathrm{div}_x(\theta) = \sum_{x \in E} \sum_{y : y \sim x} \theta(x, y) = 0$ by antisymmetry of θ, the fact that $\mathrm{div}_x(\theta) = 0$ for any $x \in D$ implies that

$$\mathrm{div}_a(\theta) = -\mathrm{div}_{\tilde{z}}(\theta).$$

Proof of Theorem (2.10)

Now let I be a current from a to Z such that $||I|| = \mathrm{div}_a(\theta)$. As the corresponding potential is fixed on Z, I remains a current on $\tilde{\mathcal{G}}$, and by the same proof as for θ we must have

$$\mathrm{div}_a(I) = -\mathrm{div}_{\tilde{z}}(I).$$

Now $f = \theta - I$ remains a flow from a to $\tilde{z}$, and it satisfies the node law at every vertex of $\tilde{E}$, and it must also satisfy the cycle law because both I and θ do.

Assume by contradiction there exists $x_1, x_2 \in \tilde{E}$ such that $f(x_1, x_2) > 0$. By the node law at x_2 there must exist x_3 such that $f(x_2, x_3) > 0$, etc... Because $\tilde{E}$ is finite, there must exist $p < p + n$ such that $x_p = x_{p+n}$, but this would contradict the cycle law.

In the end $I = \theta$ on $\tilde{\mathcal{G}}$, so it remains true on $\mathcal{G}$ and θ indeed is a current.

Circuit reduction

Theorem (2.11)

One can perform the following changes in the circuit without changing potentials at the preserved nodes, nor current along the preserved edges

(i) *Two resistors in series with resistances r_1, r_2 can be replaced by a single resistor with resistance $r_1 + r_2$.*

(ii) *Two resistors in parallel with* conductances *$c_1.c_2$ can be replaced by a single resistor with* conductance *$c_1 + c_2$.*

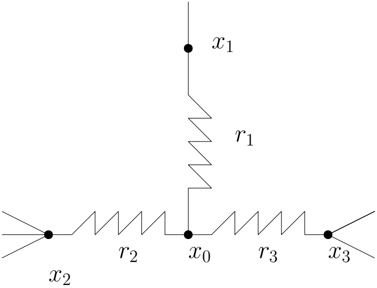
(iii) *Nodes with the same potential can be identified.*

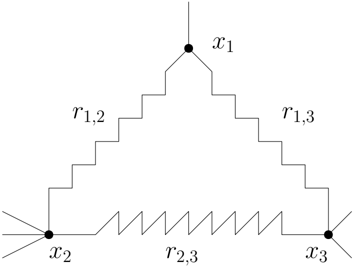
(iv) *Star-Triangle (or $Y - \Delta$) transformation (see the corresponding figure).*

Circuit reduction

(i) Ohm's and Kirchoff's law remain satisfied replacing e, e' with $\tilde{e}$, preserving potentials at the preserved extremeties of e, e' and setting $I(\tilde{e}) = I(e) = I(e')$. Everything else in the circuit is preserved. This transformation obviously does not affect node's law at the preserved nodes.
(ii) Replace e, e' with $\tilde{e}$. Preserve potentials at the preserved extremities, and set $I(\tilde{e}) = I(e) + I(e')$. Everything else in the circuit is preserved. This transformation obviously does not affect node's law at the preserved nodes.
(iii) If there exist edges between nodes with same potential, current has to be null along those edges, so we may as well supress them, and node's law at the newly created vertex follows from node's law at old vertices which are identified. Current and potentials are preserved elsewhere in the circuit.

Star-Triangle ($Y - \Delta$)





$$r_1 = \frac{r_{1,2} \times r_{1,3}}{r_{1,2} + r_{1,3} + r_{2,3}} \qquad c_{1,2} = \frac{c_1 \times c_2}{c_1 + c_2 + c_3}$$

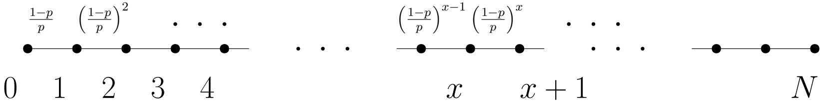
The expression for $c_{1,2}$ comes from checking the two circuits are equivalent when setting $V_{x_1} = V_{x_3} = 0$, $V_{x_2} = 1$. By the same method, one obtains similar expressions for $c_{1,3}, c_{2,3}$. General potentials can then be treated thanks to superposition principle We leave details as an exercise.

Gambler's ruin through circuit reduction (exercise II.17)

SRW on $\mathbb{Z}$ with parameter $p \in (0,1)$ exactly corresponds to the conductance model with $c(x,x+1) = r(x,x+1)^{-1} = \left(\frac{p}{1-p}\right)^x, x \in \mathbb{Z}$. As for gambler's ruin problem, we may as well consider the chain absorbed at $\{0, N\}$, so we may set $a = \{0\}$, $D = [|1, N-1|]$, $Z = \{N\}$. By Theorem 2.10, resistances in series add up, so, for $p = 1/2$, effective resistance between the origin and $x \in D$ is given by x, and that between x and N is given by $N - x$.

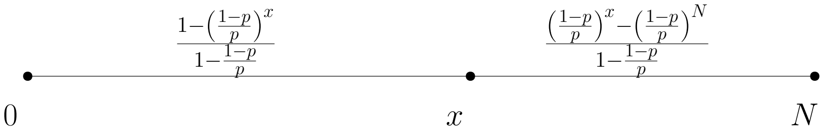
For $p \neq 1/2$, effective resistance between the origin and $x \in D$ is given, for $p = 1/2$, by $\frac{1-\left(\frac{1-p}{p}\right)^x}{1-\frac{1-p}{p}}$, that between x and N is given by $\frac{\left(\frac{1-p}{p}\right)^x - \left(\frac{1-p}{p}\right)^N}{1-\frac{1-p}{p}}$ (see also the figure in the next slide).

Gambler's ruin through circuit reduction (exercise II.17)



Quantities expressed above and below

correspond to *resistances* of edges



Gambler's ruin through circuit reduction (exercise II.17)

Now the chain on the reduced circuit is simply a three-state chain on $\{0, x, N\}$ absorbed at $\{0, N\}$, and its whole trajectory depends only on the first step. More precisely, the probability of ruin can be expressed as the *conductance* between 0 and x divided by the sum of the conductances from x, thus for $p = 1/2$,

$$\mathbb{P}_x(T_0 < T_N) = \frac{1}{x} \times \frac{1}{\frac{1}{x} + \frac{1}{N-x}} = \frac{N-x}{N}$$

For $p \neq 1/2$,

$$\begin{aligned}\mathbb{P}_x(T_0 < T_N) &= \frac{1}{1 - \left(\frac{1-p}{p}\right)^x} \frac{1}{\frac{1}{\left(\frac{1-p}{p}\right)^x - \left(\frac{1-p}{p}\right)^N} + \frac{1}{1 - \left(\frac{1-p}{p}\right)^x}} \\ &= \frac{\left(\frac{1-p}{p}\right)^x - \left(\frac{1-p}{p}\right)^N}{1 - \left(\frac{1-p}{p}\right)^N}\end{aligned}$$

Effective resistance from a to Z

Definition (2.12)

Let I a current from a to Z associated with potential V. The *effective resistance from a to Z* is

$$\mathcal{R}(a \leftrightarrow Z) := \frac{V(a) - V(Z)}{||I||}.$$

Since $||I||$ and $V(a) - V(Z)$ are both proportional to $\alpha - \beta$, the above quantity does not depend on the choice of α, β, hence the definition is well-posed.

Note however, that effective resistance does depend on the choice of the constant K such that $c(x,y) = K\mu(x)P(x,y) = c(y,x)$ (it is proportional to $1/K$, see exercice II.13).

Notice that $\mathcal{R}(a \leftrightarrow Z) = \frac{1}{||I_{1,0}||}$.

Also if I is the unit current and V_1 is the corresponding potential null on Z then $\mathcal{R}(a \leftrightarrow Z) = V_1(a)$.

Effective resistance

Theorem (2.12)

$$\mathbb{P}_a(T_Z < T_a^+) = \frac{1}{c(a)\mathcal{R}(a \leftrightarrow Z)}$$

Corollary (2.12.1)

$G_{D\cup\{a\}}(a,a) = c(a)\mathcal{R}(a \leftrightarrow Z)$.

Proof : Recall that for any $x \in D$, $V_{1,0}(x) = \mathbb{P}_x(T_a < T_Z)$. Thus

$$\begin{aligned} ||I_{1,0}|| &= \sum_{y:\{a,y\}\in\mathcal{E}} c(a,y)\,(1 - \mathbb{P}_y(T_a < T_Z)) \\ &= c(a)\sum_{y:y\sim a} P(a,y)\mathbb{P}_y(T_Z < T_a) = c(a)\mathbb{P}_a(T_Z < T_a^+) \end{aligned}$$

yielding the result of the theorem. Now, the number of visits at a before reaching Z is geometrically distributed with parameter $\mathbb{P}_a(T_Z < T_a^+)$, yielding the result of the corollary.

Hitting time, commute time and effective resistance

Theorem

(a) *Let* $a \in E, a \notin Z \subset E$, $D = E \setminus (\{a\} \cup Z)$, *and* V_1 *be the potential associated with the unit current from* a *to* Z. *Then* $\mathbb{E}_a[\tau_Z] = \sum_{x \in E} c(x) V_1(x)$.

(b) *For* $a, b \in E$, $\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c_{\mathcal{G}} \mathcal{R}(a \leftrightarrow b)$

The quantity $\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a]$ is refered to as the *commute time* between a and b.

Proof of (a) : Recall, from the proof of the probabilistic interpretation of the unit current, that

$$V_1(x) = \frac{G_{D \cup \{a\}}(a,x)}{c(x)} = \frac{\mathbb{E}_a\left[\sum_{k=0}^{\tau_Z - 1} \mathbb{1}_{\{X_k = x\}}\right]}{c(x)},$$

so that

$$\sum_{x \in E} c(x) V_1(x) = \sum_{x \in E} \mathbb{E}_a\left[\sum_{k=0}^{\tau_Z - 1} \mathbb{1}_{\{X_k = x\}}\right] = \mathbb{E}_a[\tau_Z]$$

Proof of (b) : When $Z = \{b\}$, V_1, the potential associated with the unit current from a to b, is the unique harmonic function on D taking value $\mathcal{R}(a \leftrightarrow b)$ at a and 0 at b.
Now $\widetilde{V_1} = \mathcal{R}(a \to b) - V_1$ remains harmonic on D, it takes value 0 at a and $\mathcal{R}(a \leftrightarrow b)$ at b, so it is the potential associated with the unit current from b to a.
Therefore, by (a),

$$\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = \sum_{x \in E} c(x)(V_1(x) + \widetilde{V_1}(x)) = \mathcal{R}(a \leftrightarrow b) \sum_{x \in E} c(x),$$

and we are done.

Energy

Definition

If θ is a flow from a to Z define the *energy* of θ as

$$E(\theta) := \sum_{e \in \mathcal{E}} r(e)\theta(e)^2.$$

When θ is a current, this corresponds physically to the power dissipated in the circuit.

Lemma (2.12.2)

The energy of the unit current I is $E(I) = \mathcal{R}(a \leftrightarrow Z)$. Moreover $E(I_{1,0}) = (\mathcal{R}(a \leftrightarrow Z))^{-1}$

Proof of Lemma (2.12.2) : Let V associated with unit current I, then $E(I) = \sum_{\{x,y\}\in\mathcal{E}} I(x,y)(V(x) - V(y))$ by Ohm's law. Since each edge has two orientations and $I(x,y)(V(x) - V(y)) = I(y,x)(V(y) - V(x))$ we find

$$\begin{aligned} E(I) &= \frac{1}{2}\sum_{x\in E}\sum_{y\in E} I(x,y)(V(x) - V(y)) \\ &= \sum_{x\in E} V(x) \sum_{y\in E} I(x,y) \end{aligned}$$

where we used antisymmetry of I once again at the last line. Now $V(x) = 0$ for any $x \in Z$, while for any $x \in D$, $\sum_{y\in E} I(x,y) = 0$. It follows that

$$E(I) = V(a)\mathrm{div}_a(I) = V(a) = \mathcal{R}(a \leftrightarrow Z).$$

Finally, as we have seen previously, $I_{0,1} = \frac{I}{\mathcal{R}(a\leftrightarrow)}$ yielding the second result of the lemma.

Theorem (2.13 — Thomson's principle)

Assume D is finite (and recall $\mathcal{G}$ is locally finite). Then

$$\mathcal{R}(a \leftrightarrow Z) = \inf\{E(\theta) : \theta \text{ unit flow from } a \text{ to } Z\},$$

and this infimum is reached for $\theta = I$, the unit current.

Proof : By our assumptions, the set S of edges with at least one extremity in $D \cup \{a\}$ is finite, hence flows from a to Z form a subset of $\mathbb{R}^S$. Pick $M \geq E(I)$, unit flows with energy bounded by M form a non-empty compact set K, hence the infimum in the RHS above is reached at some unit flow $\theta_0 \in K$.

Now consider n edges (each with at least one extremity in $D \cup \{a\}$) $\vec{e_1}, ..., \vec{e_n}$ forming a cycle and define the flow $\gamma : \gamma(\vec{e_i}) = 1, 1 \leq i \leq n$ (γ being null on edges outside the cycle). For any $\varepsilon \in \mathbb{R}$, we must have $E(\theta_0 + \varepsilon\gamma) \geq E(\theta_0)$. However

$$0 \leq E(\theta_0 + \varepsilon\gamma) - E(\theta_0) = 2\varepsilon \sum_{i=1}^{n} \theta_0(\vec{e_i}) r(e_i) + O(\varepsilon^2)$$

and so it must be that $\sum_{i=1}^{n} r(e_i)\theta_0(\vec{e_i}) = 0$. Since the reasoning is valid for any cycle, it follows that θ_0 satisfies the cycle law, by Theorem (2.9) it must be the unit current.

Theorem (2.14 — Rayleigh's principle)

Consider two conductance models X, X' on the same graph $\mathcal{G}$, with respective conductance functions c, c'. Assume $c'(e) \leq c(e) \leftrightarrow r(e) \leq r'(e)$ for any $e \in \mathcal{E}$. Then $\mathcal{R}(a \leftrightarrow Z) \leq \mathcal{R}'(a \leftrightarrow Z)$.

Proof : Clearly for any unit flow θ from a to Z, $E(\theta) \leq E'(\theta)$, now apply Thomson's principle.

A consequence of Rayleigh's principle is that removing an edge between any two vertices can only increase effective resistance between a and Z.

Similarly, adding an edge can only decrease effective resistance.

Nash-Williams

Definition

The subset of edges $\Pi \subset \mathcal{E}$ is a *cutting set* between a and Z iff any path from a to some $z \in Z$ must contain at least one edge in Π.

Theorem (2.15 — Nash-Williams)

Assume $\{\Pi_k\}_{1\le k\le K}$ are disjoint cutting sets between a and Z. Then

$$\mathcal{R}(a \leftrightarrow Z) \ge \sum_{k=1}^{K} \frac{1}{\sum_{e\in\Pi_k} c(e)}.$$

Nash-Williams : proof

Let θ be a unit flow from a to Z. Because Π_k is a cutting set from a to Z it must be that $1 \leq \sum_{e \in \Pi_k} |\theta(e)| \leq \left(\sum_{e \in \Pi_k} |\theta(e)|\right)^2$. By Cauchy-Schwarz

$$1 \leq \left(\sum_{e \in \Pi_k} c(e)\right) \times \left(\sum_{e \in \Pi_k} r(e)\theta(e)^2\right),$$

But then because the $\{\Pi_k, 1 \leq k \leq K\}$ are disjoint

$$E(\theta) \geq \sum_{k=1}^{K} \frac{1}{\sum_{e \in \Pi_k} c(e)},$$

and we conclude thanks to Thomson's principle.