Definitions

Consider X an irreducible Markov chain on a finite state space E of size n, and denote by π the unique invariant probability of X.

Definition (3.2)

For $t \in \mathbb{N}$ define

$$d(t) := \max_{x \in E} ||P^t(x, \cdot) - \pi||_{TV}$$

$$\overline{d}(t) := \max_{x \in E, y \in E} ||P^t(x, \cdot) - P^t(y, \cdot)||_{TV}.$$

The mixing time at level ε is defined as

$$t_{\text{mix}}(\varepsilon) = \inf\{t \in \mathbb{N} : d(t) \leq \varepsilon\}.$$

We'll have a particular interest in

$$t_{\text{mix}} := t_{\text{mix}}(1/4).$$

Theorem (3.2.1)

$d(t) \leq \overline{d}(t) \leq 2d(t).$

Proof : The second inequality follows trivially from the triangle inequality. As for the first, since π is invariant,

$$\begin{aligned} d(t) &= \max_{x \in E} \max_{A \subset E} \left| P^t(x, A) - \pi(A) \right| \\ &= \max_{x \in E} \max_{A \subset E} \left| \sum_{y \in E} \pi(y)(P^t(x, A) - P^t(y, A)) \right| \\ &\leq \max_{x \in E} \max_{A \subset E} \sum_{y \in E} \pi(y) \left| P^t(x, A) - P^t(y, A) \right| \leq \overline{d}(t) \end{aligned}$$

Theorem (3.2.2)

$\overline{d}$ is submultiplicative, that is, for any $s, t \in \mathbb{N}$,

$$\overline{d}(t+s) \leq \overline{d}(t)\overline{d}(s).$$

Proof : Fix $x, y \in E$ and assume (X_s, Y_s) realize an optimal coupling of $P^s(x,\cdot)$, $P^s(y,\cdot)$ and denote by $\mathbb{P}_{x,y}$ their joint law (so under $\mathbb{P}_{x,y}$, $Pt(X_s,\cdot)$ has law $P^{t+s}(x,\cdot)$, $P^t(Y_s,\cdot)$ has law $P^{t+s}(y,\cdot)$, and $\mathbb{P}_{x,y}(X_s \neq Y_s) = \overline{d}(s)$). Then

$$\begin{aligned}
& ||P^{t+s}(x,\cdot) - P^{t+s}(y,\cdot)||_{TV} \\
= \; & \frac{1}{2}\sum_{w\in E} \left|\mathbb{E}_{x,y}\left[P^t(X_s,w) - P^t(Y_s,w)\right]\right| \\
\leq \; & \frac{1}{2}\sum_{w\in E} \mathbb{E}_{x,y}\left[\left|P^t(X_s,w) - P^t(Y_s,w)\right| \mathbb{1}_{\{X_s\neq Y_s\}}\right] \\
\leq \; & \overline{d}(t)\mathbb{P}_{x,y}(X_s \neq Y_s) = \overline{d}(t)\overline{d}(s),
\end{aligned}$$

where for the inequality at the last line, we used from its definition that $\overline{d}(t) \geq \frac{1}{2}\sum_{w\in E}|P^t(X_s,w) - P^t(Y_s,w)|$.

Theorem (3.2.3)

For any $\varepsilon \leq 1/2$*, we have*

$$t_{\mathrm{mix}}(\varepsilon) \leq \lceil \log_2 \left(\frac{1}{\varepsilon}\right) \rceil t_{\mathrm{mix}}.$$

Proof : By the above results, for $\ell \in \mathbb{N}^*$,

$$\begin{aligned} d(\ell t_{\mathrm{mix}}) &\leq \overline{d}(\ell t_{\mathrm{mix}}) \\ &\leq \overline{d}(t_{\mathrm{mix}})^{\ell} \\ &\leq (2d(t_{\mathrm{mix}}))^{\ell} \leq \frac{1}{2^{\ell}}, \end{aligned}$$

thus if $\ell \geq \log_2\left(\frac{1}{\varepsilon}\right)$, $d(\ell t_{\mathrm{mix}}) \leq \varepsilon$, as required.