General considerations on the spectrum of the transition matrix P

We assume in this paragraph that E is finite, $\#E = n$. Note that any ordering of E allows to express the transition kernel P of X as a $n \times n$ stochastic matrix.

Lemma (3.a)

(i) *If λ is an eigenvalue of P then $|\lambda| \leq 1$.*

(ii) *If P is irreducible then* $\ker(P - Id) = \mathrm{Vect}\left\{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\right\}$

(iii) *If P is irreducible and aperiodic then -1 is not an eigenvalue of P.*

General considerations on the spectrum of the transition matrix P

Proof of Lemma (3.a) (i) : Let $f : E \to \mathbb{R}$, we have

$$|Pf(x)| = \left| \sum_{y \in E} P(x,y) f(y) \right| \leq ||f||_\infty \sum_{y \in E} P(x,y) = ||f||_\infty,$$

thus $||Pf||_\infty \leq ||f||_\infty$. If λ is an eigenvalue of P associated with eigenvalue f we have $||Pf||_\infty = |\lambda| ||f||_\infty$, and thus $|\lambda| \leq 1$.

General considerations on the spectrum of the transition matrix P

Proof of Lemma (3.a) (ii) : An eigenfunction f associated with eigenvalue 1 is simply a function f that is harmonic on the whole of E. By the maximum principle for harmonic functions Theorem (2.8) we conclude that f must be constant on E.
Note that this result also implies unicity of invariant distribution : it is indeed the case that
$\dim(\ker(P - Id)) = 1 \Rightarrow \dim(\ker(P^T - Id)) = 1.$

General considerations on the spectrum of the transition matrix P

Proof of Lemma (3.a) (iii) : By exercise I.11.2 since P is irreducible, aperiodic and E is finite there must exists $r \in \mathbb{N}^*$ such that for any $k \geq r$, $P^k(x, y) > 0$ for any $x, y \in E$. Now assume f is an eigenfunction associated with eigenvalue -1, it must exist $x \in E$ such that $f(x) \neq 0$. Without loss of generality (even if it means taking $-f$ instead of f) we may assume $f(x) > 0$. Then $P^{2r}f(x) = \sum_{y \in E} P^{2r}(x, y)f(y)$ implies, since P^{2r} is stochastic, that f has to be positive on the whole of E. But then $P^{2r+1}f(x) = -f(x)$ brings a contradiction.
Note : We could also have used Perron-Frobenius theorem to prove the three assertions of the Lemma.

Spectral decomposition for reversible chain

Theorem (3.b)

Assume P is irreducible, reversible with respect to its unique invariant probability π.

(i) *The prehilbertian space $(\mathbb{R}^E, \langle\cdot,\cdot\rangle_\pi)$ possesses an orthonormal basis of real-valued eigenfunctions $(f_j, 1 \leq j \leq n)$ associated with real eigenvalues $1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq -1$.*

(ii) *For any $t \in \mathbb{N}, (x,y) \in E^2$,*

$$\frac{P^t(x,y)}{\pi(y)} = \sum_{j=1}^{n} f_j(x) f_j(y) \lambda_j^t = 1 + \sum_{j=2}^{n} f_j(x) f_j(y) \lambda_j^t.$$

Spectral decomposition for reversible chain

Proof of Theorem (3.b) (i) : Since $\pi(x) > 0, x \in E$ it is straightforward that $\langle f, g\rangle_\pi := \sum_{x\in E} \pi(x) f(x) g(x)$ defines a scalar product on $\mathbb{R}^E$. Now if X is reversible with respect to π,

$$A(x,y) := \sqrt{\frac{\pi(x)}{\pi(y)}} P(x,y), (x,y) \in E^2$$

defines a symmetric operator on $\mathbb{R}^E$ (in other words, any ordering of E makes A a $n \times n$ symmetric matrix). By well-known linear algebra results for symmetric matrices, eigenvalues of A are real and there exists $\{\phi_j, j = 1, ..., n\}$ an orthonormal basis (for the usual scalar product) of real-valued eigenfunctions associated with these eigenvalues sorted in decreasing order. Now setting $f_j(x) := \frac{\phi_j(x)}{\sqrt{\pi(x)}}, x \in E$, we find

$$\langle f_i, f_j\rangle_\pi = \sum_{x\in E} \pi(x) f_i(x) f_j(x) = \sum_{x\in E} \phi_i(x)\phi_j(x) = \delta_{ij},$$

so $\{f_j, 1 \le j \le n\}$ is orthonormal for $\langle \cdot, \cdot\rangle_\pi$.

Spectral decomposition for reversible chain

It remains to check that

$$
\begin{aligned}
Pf_j(x) &= \sum_{y\in E} P(x,y) f_j(y) \\
&= \sum_{y\in E} \sqrt{\frac{\pi(y)}{\pi(x)}} A(x,y) \frac{\phi_j(y)}{\sqrt{\pi(y)}} \\
&= \sqrt{\frac{1}{\pi(x)}} A\phi_j(x) \\
&= \sqrt{\frac{1}{\pi(x)}} \lambda_j \phi_j(x) = \lambda_j f_j(x),
\end{aligned}
$$

as we wished. Now by Lemma (3.a) it must be that $\lambda_1 = 1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -1$. Note in addition that if X is aperiodic, we can moreover deduce that $\lambda_n > -1$.

Proof of Theorem (3.b) (ii) : Writing the decomposition of $\mathbb{1}_{\{y\}}$ along the orthonormal basis $(f_j, 1 \leq j \leq n)$, we have

$$\mathbb{1}_{\{y\}}(\cdot) = \sum_{j=1}^{n} \langle f, f_j \rangle_\pi f_j(\cdot) = \sum_{j=1}^{n} f_j(y)\pi(y)f_j(\cdot).$$

Now

$$\begin{aligned} P^t(x,y) = P^t \mathbb{1}_{\{y\}}(x) &= \sum_{j=1}^{n} f_j(y)\pi(y)P^t f_j(x) \\ &= \sum_{j=1}^{n} \lambda_j^t f_j(y)\pi(y)f_j(x) \end{aligned}$$

Now by Lemma (3.a) $f_1(x) = 1$ for any $x \in E$, so the first term in the sum above is $\pi(y)$, yielding the desired result.

Bounding $t_{\mathrm{mix}}(\varepsilon)$ using spectral decomposition

Theorem

Assume that the chain X is irreducible, reversible, aperiodic.
We define $|\lambda^*| = \max\{|\lambda_i|, i \geq 2\}$*, and the (absolute)* spectral gap $\gamma^* := 1 - |\lambda^*|$*, which is* > 0 *in this setting by Lemma 4.a.*
We also define the relaxation time $t_{\mathrm{relax}} = \frac{1}{\gamma^*}$*. and let* $\pi_{\min} = \min_{x \in E} \pi(x)$*.*
Then

$$|\lambda^*|^t \leq 2d(t) \leq \frac{|\lambda^*|^t}{\pi_{\min}},$$

and

$$(t_{\mathrm{relax}} - 1) \log\left(\frac{1}{2\varepsilon}\right) \leq t_{\mathrm{mix}}(\varepsilon) \leq \log\left(\frac{1}{2\varepsilon\pi_{\min}}\right) t_{\mathrm{relax}}.$$

Proof of upper bound for $t_{\mathrm{mix}}(\varepsilon)$ using spectral decomposition

We start by observing that

$$\begin{aligned} ||P^t(x,\cdot) - \pi||_{TV} &= \frac{1}{2}\sum_{y\in E} \pi(y) \left| \frac{P^t(x,y)}{\pi(y)} - 1 \right| \\ &\leq \frac{1}{2}\max_{y\in E} \left| \frac{P^t(x,y)}{\pi(y)} - 1 \right|. \end{aligned}$$

By spectral decomposition, then Cauchy-Schwarz we have

$$\begin{aligned} \left| \frac{P^t(x,y)}{\pi(y)} - 1 \right| &= \left| \sum_{j=2}^{n} \lambda_j^t f_j(x) f_j(y) \right| \\ &\leq \max_{j\geq 2} |\lambda_j|^t \left(\sum_{j=2}^{n} f_j^2(x) \right)^{1/2} \left(\sum_{j=2}^{n} f_j^2(x) \right)^{1/2} \end{aligned}$$

Proof of upper bound for $t_{\mathrm{mix}}(\varepsilon)$ using spectral decomposition

It remains to bound above $\left(\sum_{j=2}^{n} f_j^2(x)\right)^{1/2}$. For that, use orthonormality of the basis $\{f_j\}_{1\leq j\leq n}$ to get

$$\begin{aligned}
\pi(x) &= \langle \mathbb{1}_x, \mathbb{1}_x \rangle_\pi \\
&= \langle \sum_{j=1}^{n} \pi(x) f_j(x) f_j, \sum_{j=1}^{n} \pi(x) f_j(x) f_j \rangle_\pi \\
&= \sum_{j=1}^{n} \pi^2(x) f_j^2(x),
\end{aligned}$$

so that $\left(\sum_{j=2}^{n} f_j^2(x)\right)^{1/2} \leq \frac{1}{\sqrt{\pi(x)}}$ In the end, we have proven that $d(t) \leq \frac{|\lambda^*|^t}{\pi_{\min}}$, as required. The upper bound for $t_{\mathrm{mix}}(\varepsilon)$ then follows from its definition and the fact that $\max_{j\geq 2} |\lambda_j|^t \leq \exp(-\gamma^* t)$.

Proof of lower bound for $t_{\mathrm{mix}}(\varepsilon)$ using spectral decomposition

The starting point is that if $f : E \to \mathbb{R}$, then

$$|P^t(x,\cdot)f - \pi f| = \sum_{y \in E} |P^t(x,y) - \pi(y)||f(y)| \leq 2||f||_\infty d(t).$$

Now a good choice is to take $f = f_2$ or $f = f_n$ the eigenfunction associated to the eigenvalue λ such that $|\lambda| = \max_{j \geq 2} |\lambda_j|$ (the one realizing the spectral gap). In that case, not only $P^t f = \lambda^t f$, but moreover $\pi f = 0$ because f is orthogonal to the first (constant) eigenvector. Choosing x such that $|f(x)| = ||f||_\infty$, we get, as required,

$$|\lambda|^t = (1 - \gamma^*)^t \leq 2d(t).$$

The lower bound for $t_{\mathrm{mix}}(\varepsilon)$ then follows from its definition and elementary considerations, using in particular that $\log(|\lambda|^{-1}) \geq |\lambda|^{-1} - 1$.