Bounding mixing times with hitting times for a lazy reversible chain

Theorem (3.4)

Assume X with kernel P on a finite state space E is irreducible and reversible with respect to π.

(i) *For any $t \in \mathbb{N}$, $||P^t(x,\cdot) - \pi||_{TV}^2 \leq \frac{1}{4}\left[\frac{P^{2t}(x,x)}{\pi(x)} - 1\right]$,*

(ii) *Assume in addition that X is lazy, then*

$$t_{\mathrm{mix}} \leq \lceil 2 \max_{x\in E} \mathbb{E}_\pi[\tau_x] \rceil \leq 2 \max_{x,y\in E} \mathbb{E}_y[\tau_x] =: 2t_{\mathrm{hit}}.$$

Proof of Theorem (3.4) (i)

We have

$$\begin{aligned}
||P^t(x,\cdot)-\pi||_{TV}^2 &= \frac{1}{4}\left(\sum_{y\in E}\pi(y)\left|\frac{P^t(x,y)}{\pi(y)}-1\right|\right)^2 \\
&= \frac{1}{4}||\frac{P^t(x,\cdot)}{\pi}-1||_{1,\pi}^2 \\
&\leq \frac{1}{4}||\frac{P^t(x,\cdot)}{\pi}-1||_{2,\pi}^2 \\
&= \frac{1}{4}\left(\sum_{y\in E}\pi(y)\left|\frac{P^t(x,y)}{\pi(y)}-1\right|^2\right)^2
\end{aligned}$$

Proof of Theorem (3.4) (i)

Now using reversibility of the chain

$$\begin{aligned}
& \frac{1}{4}\left(\sum_{y\in E}\pi(y)\left|\frac{P^t(x,y)}{\pi(y)}-1\right|^2\right) \\
= & \frac{1}{4}\sum_{y\in E}\left[P^t(x,y)\frac{P^t(y,x)}{\pi(x)}-2P^t(x,y)+1\right] \\
= & \frac{1}{4}\sum_{y\in E}\left[\frac{P^{2t}(x,x)}{\pi(x)}-1\right]
\end{aligned}$$

yielding the result.

Proof of Theorem (3.4) (ii)

Lemma

$$\pi(x)\mathbb{E}_\pi[\tau_x] = \sum_{t\geq 0}\left[P^t(x,x) - \pi(x)\right]$$

Proof : If $\tau_x^{(m)} = \inf\{t \geq m : X_t = x\}$, we have, thanks to the characterization of invariant measures,

$$\begin{aligned}\sum_{t=0}^{m-1} P^t(x,x) &= \mathbb{E}_x\left[\sum_{t=0}^{\tau_x^{(m)}-1} \mathbb{1}_{\{X_t=x\}}\right] \\ &= \pi(x)\mathbb{E}_x[\tau_x^{(m)}] = \pi(x)[m + \mathbb{E}_{\mu_m}(\tau_x)],\end{aligned}$$

with $\mu_m = P^m(x,\cdot)$. It follows that $\sum_{t=0}^{m-1}(P^t(x,x) - \pi(x)) = \pi(x)\mathbb{E}_{\mu_m}[\tau_x]$, and it remains to let $m \to \infty$ ($\mu_m \to \pi$ by ergodicity of the chain) to deduce the announced result.

Proof of Theorem (3.4) (ii)

Lemma

For X reversible and lazy, for any $x \in E$ the application $t \to P^t(x,x)$ is nonincreasing

This follows from the spectral decomposition. Lazyness implies $P = (Id + Q)/2$ with Q reversible, so eigenvalues of P are $\lambda_1 = 1 > \lambda_2 \geq ... \geq \lambda_n \geq 0$. By the spectral decomposition theorem

$$\frac{P^t(x,x)}{\pi(x)} = 1 + \sum_{j=2}^{n} \lambda_j^t f_j(x)^2,$$

and the lemma follows.

Proof of Theorem (3.4) (ii)

Using the two lemmas, we have

$$\pi(x)\mathbb{E}_\pi[\tau_x] \geq \sum_{k=0}^{2t-1}(P^k(x,x)-\pi(x)) \geq 2t(P^{2t}(x,x)-\pi(x)),$$

so using (i),

$$\frac{\mathbb{E}_\pi[\tau_x]}{8t} \geq \frac{1}{4}\left[\frac{P^{2t}(x,x)}{\pi(x)}-1\right] \geq ||P^t(x\cdot)-\pi||_{TV}^2,$$

and we conclude.

Combining Theorem 3.4 with effective resistance result for commute time

Corollary (3.4.2)

Assume the lazy reversible chain X is described through a conductance model on $\mathcal{G}$ with conductance function c and let $\mathcal{R}_{\max} := \max_{x,y \in E} \mathcal{R}(x \leftrightarrow y)$. Then

$$t_{\mathrm{mix}} \le 2 c_{\mathcal{G}} \mathcal{R}_{\max}$$

Proof : By Theorem 2.12.2, we have

$$\mathbb{E}_x[T_y] + \mathbb{E}_y[T_x] = c_{\mathcal{G}} \mathcal{R}(x \leftrightarrow y),$$

Thus $t_{\mathrm{hit}} \le c_{\mathcal{G}} \mathcal{R}_{\max}$, and the conclusion follows from Theorem 3.4 (ii).

Corollary (3.4.2)

Assume the lazy reversible chain X is described through a conductance model on $\mathcal{G}$ with conductance function c and let $\mathcal{R}_{\max} := \max_{x,y \in E} \mathcal{R}(x \leftrightarrow y)$. Then

$$t_{\mathrm{mix}} \leq 2c_{\mathcal{G}}\mathcal{R}_{\max}$$

Remark 1 : If X is the lazy version of the chain Y, note that $t_{\mathrm{hit}}^X = 2t_{\mathrm{hit}}^Y$. This is consistent with Theorem 2.12.2, since $c_{\mathcal{G}}^X = 2c_{\mathcal{G}}^Y$, and effective resistances are unchanged.

Remark 2 : If x, y are such that $t_{\mathrm{hit}} = \mathbb{E}_x[T_y] = \mathbb{E}_y[T_x]$ (which is often the case when $\mathcal{G}$ has symmetries), then by the same proof, the bound of the corollary can be improved to $t_{\mathrm{mix}} \leq c_{\mathcal{G}}\mathcal{R}_{\max}$.