Definition : generator

Definition (4.1)

Let E at most countable. We say $Q : \begin{cases} E \times E \to \mathbb{R} \\ (x,y) \to q_{xy} \end{cases}$ is a generator iff for any $x \in E$,

(a) $q_{xx} := -q_x \in \mathbb{R}_-$

(b) for any $y \neq x$, $q_{xy} \in rr_+$,

(c) $\sum_{y \in E} q_{xy} = 0$.

Definition (4.2)

To a generator Q we associate the transition kernel Π such that for any $x \neq y$,

$$\Pi(x,y) := \begin{cases} \frac{q_{xy}}{q_x} \text{ if } q_x > 0 \\ 0 \text{ if } q_x = 0 \end{cases}$$

and for any $x \in E$,

$$\Pi(x,x) := \begin{cases} 0 \text{ if } q_x > 0 \\ 1 \text{ if } q_x = 0 \end{cases}$$

Definition : decomposition of trajectory for E-valued cadlag continuous-time process

Definition (4.3)

To a continuous-time cadlag E-valued process $(X_t, t \geq 0)$ we associate

(i) $Y_0, Y_1, \ldots$ the successive values taken by X in E.

(ii) $S_1, S_2, \ldots$ the (nonnegative) holding times spent by X at each of these values.

(iii) $J_1 = S_1, J_2 = S_1 + S_2, J_3 = J_2 + S_3, \ldots$ the successive jump times of X, so that, setting $J_0 := 0$, $X_t = Y_k$ for any $t \in [J_k, J_{k+1}[$.

(iv) $\zeta = \sum_{n \geq 1} S_n = \lim_{n \to \infty} J_n \in \overline{\mathbb{R}_+}$ is the explosion time of X.

Remarks on this decomposition

Remark 1 : It is possible that for some $n_0 \in \mathbb{N}^*$, $S_{n_0} = +\infty$, that is X jumps only finitely many times. Then X takes only finitely many successive values. Wether or not $(Y_k)_{k \geq n_0+1}, (S_k)_{k \geq n_0}$ are defined is irrelevant to describe the trajectory of X in that case.

Remark 2 : Whenever $\zeta < \infty$ the above only describes the trajectory of X up to time ζ. From now on, when we want to work with a process defined on the full time line, we shall, by default, consider the minimal version of the process. More precisely, we add a cemetary point $\dagger$ to the state space, and consider the minimal process $\tilde{X}$ (i.e. the process killed at time ζ) with values in $E \cup \{\dagger\}$, defined by

$$\tilde{X}_t = X_t, \ t < \zeta, \qquad \tilde{X}_t = \dagger, \ t \geq \zeta.$$

With a slight abuse we may forget the ˜ in further notation.

Definition : Continuous-time Markov chain

Definition (4.4)

We say X is (the minimal version of) a continuous-time Markov chain with generator Q, started at λ, and write X is Markov (λ, Q) iff

(i) $(Y_n)_{n\geq 0}$ is (discrete-time) Markov (λ, Π).

(ii) For any $n \in \mathbb{N}^*$, conditionally given $\{Y_0 = x_0, Y_1 = x_1, ..., Y_{n-1} = x_{n-1}\}$, $(S_1, ..., S_n)$ are independent exponential variables with respective parameters $q_{x_0}, ..., q_{x_{n-1}}$

(iii) $X_t = \dagger, \quad t \geq \zeta$

We write $\mathbb{P}_\lambda$ for the law of the above X, and use $\mathbb{P}_x$ as shorthand for $\mathbb{P}_{\delta_x}$.

Remark : The notation Markov(λ, R) may refer to a discrete or continuoust-time chain, which kind is determined by R being a kernel or a generator (these can not be confused : for any x,

An example on $E = \{1, 2, 3\}$

Consider the continuous-time chain on $\{1, 2, 3\}$ with generator

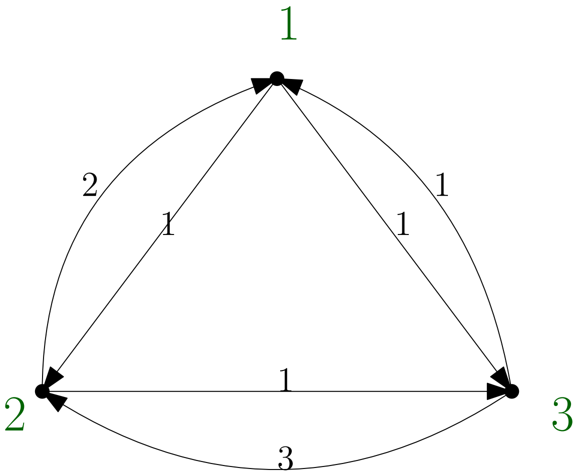
$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & 3 & -4 \end{pmatrix}$$

Note that the corresponding transition kernel is given by

$$\Pi = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/4 & 3/4 & 0 \end{pmatrix}$$

Of course Q is fully determined by $q_{xy}, x \neq y$, so we may represent the chain via the following diagramm (for any x, y such that $q_{xy} > 0$, we draw an arrow from x to y with the label q_{xy}).

An example on $E = \{1, 2, 3\}$



An example : description of a possible path

By definition, a trajectory can be decomposed into the successive values $(Y_0, Y_1, ...)$ and the corresponding holding times $(S_1, S_2, ...)$. If the chain starts at 3, say, then, since $q_3 = 4$, it waits there for $S_1 \sim \exp(4)$ before it jumps according to the kernel Π, that is, at time S_1, it goes to 1 with probability $1/4$, and otherwise to 2. Let's say it goes to 2 at time S_1. Then it waits there a time $S_2 \sim \exp(3)$ then it jumps according to Π, that is, to 1 with probability $2/3$ and otherwise to 3, etc...
It is in fact quite easy to express the probability of so-called cylinder events, e.g. :

$$\mathbb{P}_\mu(Y_0 = 3, Y_1 = 2, Y_2 = 1, S_1 > t_1, S_2 > t_2, S_3 > t_3)$$
$$= \mu(3)\Pi(3,2)\Pi(2,1)\exp(-q_3 t_1)\exp(-q_2 t_2)\exp(-q_1 t_3)$$

An easy example

As we will prove with much more generality in the following slides, the chain satisfies the simple Markov property.
For example, this means that given $\{X_s = 3\}$, the law of $(X_{s+t}, t \geq 0)$ is Markov(δ_3, Q). Indeed, for any $n \in \mathbb{N}$, given that the chain has jumped exactly n times before time s and is at state 3 at time s, it will remain at s for the time $\tilde{S}_{n+1} = S_{n+1} - (s - S_n)$. However, we are precisely conditioning on $\{S_{n+1} > s - S_n\}$, so the memoryless property of the exponential distribution guarantees that the law of $\tilde{S}_{n+1}$ remains exponential with parameter 4. Then,

- values taken after time s are given by $Y_n = 3, Y_{n+1}, Y_{n+2} \ldots$ which is discrete Markov (δ_3, Π).
- Conditionally given these values, further waiting times $\tilde{S}_{n+1}, S_{n+2}, S_{n+3}, \ldots$ are independent exponentials with respective parameters $4, q_{Y_{n+1}}, q_{Y_{n+2}}, \ldots$