Strong Markov property

Theorem (4.1)

Let X be Markov (λ, Q), $\mathcal{F}_t := \sigma(X_u, u \leq t)$, and assume T is a $(\mathcal{F}_t)_{t\geq 0}$ stopping-time. Let $\mathcal{F}_T = \{A \in \mathcal{F} : \ \forall t \geq 0 \ A \cap \{T \leq t\} \in \mathcal{F}_t\}$. Conditionally given $\{T < \infty, X_T = x\}$, $(X_{T+t}, t \geq 0)$ is Markov (δ_x, Q), independent of $\mathcal{F}_T$.

Remark 1 : Simple Markov property corresponds to the case when T is a deterministic time, say $T = s$. Then $\mathcal{F}_T = \mathcal{F}_s$, and the above ensures that conditionally given $\{X_s = x\}$, $(X_{s+t}, t \geq 0)$ is Markov (δ_x, Q), independent of $\mathcal{F}_s$.

Remark 2 : Often this property is stated with a conditioning on $\{T < \zeta, X_T = x\}$. Here it does not change anything because we have choosen to work with the minimal version of the process.

Proof of simple Markov property

Fix $s \geq 0$. The first key observation is that, thanks to the memoryless property of the exponential distribution, the law of $S_{n+1} - s$ given $\{Y_n = x, S_{n+1} > s\}$ is exactly an exponential distribution with parameter q_x.
It follows that

$$\mathbb{P}(S_{n+1} - s > t \mid J_n \leq s < J_{n+1}, Y_n = x) = \exp(-q_x t).$$

Moreover, on the event $\{J_n \leq s < J_{n+1}\}$, it is clear that the decomposition of the trajectory of $(\tilde{X}_t := X_{s+t}, t \geq 0)$ is given by $(\tilde{Y}_k = Y_{n+k}, k \in \mathbb{N})$, and $(\tilde{S}_1 = S_{n+1} - s, \tilde{S}_k = S_{n+k}, k \geq 2)$.
It follows we can prove the property at least for cylinder events : for $y_0, ..., y_n, \tilde{y}_0, ..., \tilde{y}_k$ states in E, and $t_1, ..., t_n, \tilde{t}_1, ..., \tilde{t}_k$ nonnegative reals, it is straightforward to deduce from the simple Markov property for the discrete-time chain Y, and the above observation, that

$$
\begin{aligned}
& \mathbb{P}_\lambda\big(\{Y_0 = y_0, ..., Y_n = y_n, S_1 > t_1, ..., S_n > t_n\} \\
& \qquad \cap \{J_n \leq s < J_n + 1\} \cap \{X_s = x\} \\
& \qquad \cap \{\tilde{Y}_0 = \tilde{y}_0, ..., \tilde{Y}_k = \tilde{y}_k, \tilde{S}_1 > \tilde{t}_1, ..., \tilde{S}_k > \tilde{t}_k\}\big) \\
= \ & \mathbb{P}_\lambda\big(\{Y_0 = y_0, ..., Y_n = y_n, S_1 > t_1, ..., S_n > t_n\} \\
& \qquad \cap \{J_n \leq s < J_n + 1\} \cap \{X_s = x\}\big) \\
& \qquad \times \mathbb{P}_x\left(\{Y_0 = \tilde{y}_0, ..., Y_k = \tilde{y}_k, S_1 > \tilde{t}_1, ..., S_k > \tilde{t}_k\}\right).
\end{aligned}
$$

Proof of simple Markov property

Since events of the form $\{Y_0 = y_0, ..., Y_n = y_n, S_1 > t_1, ..., S_n > t_n\}$ clearly generate $\mathcal{G}_n := \sigma(Y_0, ..., Y_n, S_1, ..., S_n)$, we deduce from the above that for any $A_n \in \mathcal{G}_n$,

$$
\begin{aligned}
& \mathbb{P}_\lambda\big(A_n \cap \{J_n \leq s < J_n + 1\} \cap \{X_s = x\} \\
& \qquad \cap \{\tilde{Y}_0 = \tilde{y}_0, ..., \tilde{Y}_k = \tilde{y}_k, \tilde{S}_1 > \tilde{t}_1, ..., \tilde{S}_k > \tilde{t}_k\}\big) \\
= \; & \mathbb{P}_\lambda\left(A_n \cap \{J_n \leq s < J_n + 1\} \cap \{X_s = x\}\right) \\
& \times \mathbb{P}_x\left(\{Y_0 = \tilde{y}_0, ..., Y_k = \tilde{y}_k, S_1 > \tilde{t}_1, ..., S_k > \tilde{t}_k\}\right).
\end{aligned}
$$

Proof of simple Markov property

The second key observation is that the trajectory of the chain up to time s *on the event* $\{J_n \leq s < J_{n+1}\}$ requires only the knowledge of $Y_0, \dots, Y_{n-1}, S_1, \dots, S_n$. More rigourously, we claim that if $A \in \mathcal{F}_s$, then $A \cap \{J_n \leq s < J_{n+1}\} = A_n \cap \{J_n \leq s < J_{n+1}\}$ for some $A_n \in \mathcal{G}_n$. Indeed

$$\mathcal{G}_s := \Big\{ A \in \mathcal{F}_s \ \forall n \in \mathbb{N} \ \exists A_n \in \mathcal{G}_n : \\ A \cap \{J_n \leq s < J_{n+1}\} = A_n \cap \{J_n \leq s < J_{n+1}\Big\}.$$

is a sub-sigma-algebra of $\mathcal{F}_s$, and for any n it contains events of the form
$\{Y_0 = y_0, \dots, Y_n = y_n, S_1 > t_1, \dots, S_n > t_n\} \cap \{J_n \leq s < J_n + 1\}$, which generate $\mathcal{F}_s$, hence $\mathcal{G}_s = \mathcal{F}_s$.

Proof of simple Markov property

We deduce from the above that for any $A \in \mathcal{F}_s$,

$$\begin{aligned} & \mathbb{P}_\lambda\big(A \cap \{J_n \leq s < J_n + 1\} \cap \{X_s = x\} \\ & \qquad \cap \{\tilde{Y}_0 = \tilde{y}_0, ..., \tilde{Y}_k = \tilde{y}_k, \tilde{S}_1 > \tilde{t}_1, ..., \tilde{S}_k > \tilde{t}_k\}\big) \\ = \ & \mathbb{P}_\lambda\left(A \cap \{J_n \leq s < J_n + 1\} \cap \{X_s = x\}\right) \\ & \times \mathbb{P}_x\left(\{Y_0 = \tilde{y}_0, ..., Y_k = \tilde{y}_k, S_1 > \tilde{t}_1, ..., S_k > \tilde{t}_k\}\right). \end{aligned}$$

By summing over $n \in \mathbb{N}$ we reach the desired conclusion :

$$\begin{aligned} & \mathbb{P}_\lambda\big(A \cap \{X_s = x\} \\ & \qquad \cap \{\tilde{Y}_0 = \tilde{y}_0, ..., \tilde{Y}_k = \tilde{y}_k, \tilde{S}_1 > \tilde{t}_1, ..., \tilde{S}_k > \tilde{t}_k\}\big) \\ = \ & \mathbb{P}_\lambda\left(A \cap \{X_s = x\}\right) \\ & \times \mathbb{P}_x\left(\{Y_0 = \tilde{y}_0, ..., Y_k = \tilde{y}_k, S_1 > \tilde{t}_1, ..., S_k > \tilde{t}_k\}\right). \end{aligned}$$

Proof of strong Markov property

Let T an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. We let $(\tilde{X}_t = X_{T+t}, t \geq 0)$, and let $(\tilde{Y}_k)_{k\geq 0}, (\tilde{S}_k)_{k\geq 1}$ its path decomposition. We can approximate T by a sequence of simpler stopping times, indeed

$$T_N := \sum_{K=1}^{N2^N} \frac{K}{2^N} \mathbb{1}_{\{\frac{K-1}{2^N} < T \leq \frac{K}{2^N}\}} + (N+1)\mathbb{1}_{\{T>N\}}$$

is such that, a.s., $T_N \to T$ as $N \to \infty$. Moreover, on $\{T < \infty\}$, observe that $(T_N)_{N\geq \lceil T \rceil}$ is nonincreasing,

We introduce $(\hat{X}_t = X_{T_N+t}, t \geq 0)$ and let $(\hat{Y}_k)_{k\geq 0}, (\hat{S}_k)_{k\geq 1}$ its path decomposition. Obviously all these quantities with a hat superscript a priori depend on N, even though we droped it from the notation.

Proof of strong Markov property

Let $A \in \mathcal{F}_T$, so that $A \cap \{T_N = \frac{K}{2^N}\} =: A_N \in \mathcal{F}_{\frac{K}{2^N}}$, and by simple Markov property at time $K2^{-N}$ we find that for any $k \in \mathbb{N}$, for any $y_0, ..., y_k$ in E, and for any $t_1, ..., t_k$ nonnegative reals, we have

$$\begin{aligned} & \mathbb{P}_\lambda\Big(A \cap \{T_N = \frac{K}{2^N}\} \cap \{X_{T_N} = x\} \\ & \qquad \cap \{\hat{Y}_0 = y_0, ..., \hat{Y}_k = y_k, \hat{S}_1 > t_1, ..., \hat{S}_k > t_k\}\Big) \\ = \; & \mathbb{P}_\lambda\left(A_N \cap \{X_{T_N} = x\}\right) \\ & \times \mathbb{P}_x\left(\{Y_0 = y_0, ..., Y_k = y_k, S_1 > t_1, ..., S_k > t_k\}\right). \end{aligned}$$

Summing over K from 1 to $N2^N$ yields

$$\begin{aligned} & \mathbb{P}_\lambda\Big(A \cap \{X_{T_N} = x\} \cap \{T \leq N\} \\ & \qquad \cap \{\hat{Y}_0 = y_0, ..., \hat{Y}_k = y_k, \hat{S}_1 > t_1, ..., \hat{S}_k > t_k\}\Big) \\ = \; & \mathbb{P}_\lambda\left(A \cap \{X_{T_N} = x\} \cap \{T \leq N\}\right) \\ & \times \mathbb{P}_x\left(\{Y_0 = y_0, ..., Y_k = y_k, S_1 > t_1, ..., S_k > t_k\}\right). \end{aligned}$$

Proof of strong Markov property

The process X is right-continuous, and (T_N) is eventually nonincreasing on $\{T < \infty\}$ so
$\{X_{T_N} = x, T \leq N\} \to \{X_T = x, T < \infty\}$ as $N \to \infty$.
By the same argument

$$\left\{X_{T_N} = x, T \leq N, \hat{Y}_0 = y_0, \dots, \hat{Y}_k = y_k, \hat{S}_1 > t_1, \dots, \hat{S}_k > t_k\right\}$$
$$\underset{N \to \infty}{\longrightarrow} \left\{T < \infty, X_T = x, \tilde{Y}_0 = y_0, \dots, \tilde{Y}_k = y_k, \tilde{S}_1 > t_1, \dots, \tilde{S}_k > t_k\right\}$$

Using dominated convergence on both sides of the previous equality then yields the desired result

$$\begin{aligned}
& \mathbb{P}_\lambda\big(A \cap \{T < \infty, X_T = x\} \\
& \qquad \cap \{\tilde{Y}_0 = y_0, \dots, \tilde{Y}_k = y_k, \tilde{S}_1 > t_1, \dots, \tilde{S}_k > t_k\}\big) \\
= \; & \mathbb{P}_\lambda\left(A \cap \{T < \infty, X_T = x\}\right) \\
& \times \mathbb{P}_x\left(\{Y_0 = y_0, \dots, Y_k = y_k, S_1 > t_1, \dots, S_k > t_k\}\right).
\end{aligned}$$