Definition

Definition (4.5)

A Poisson process with rate $\lambda > 0$ is the continuous-time Markov chain on $E = \mathbb{N}$ such that for any $k \in \mathbb{N}$, $q_k = q_{k,k+1} = \lambda$.

In other words, the successive values taken by X are deterministic : if the process is started at x_0 then $Y_n = x_0 + n$, $n \geq 0$, and all the randomness of the process is that of the sequence of holding times $(S_k, k \geq 1)$, a sequence of i.i.d exponential(λ).

Independent and stationary Increments

Theorem (4.2)

Assume X is a Poisson process with rate $\lambda > 0$. Then, for any $s \geq 0$, the process $(X_{t+s} - X_s, t \geq 0)$ remains a Poisson process with rate λ, started at 0, and is independent of $(X_u, 0 \leq u \leq s)$.

The above follows from applying Markov property at time s (along with translation invariance of Q).
Hence the process X has independent and stationary increments.
Indeed it follows from the theorem that for any $n \in \mathbb{N}, 0 \leq t_1 < t_2 < ... < t_n$, $(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ has independent coordinates, and moreover for any k, the law of $X_{t_k} - X_{t_{k-1}}$ is that of $X_{t_k - t_{k-1}}$. We are going to see now that in addition, $X_t \sim \mathrm{Poisson}(\lambda t)$, (which gives an alternate characterization of the law of the process).

Equivalent characterizations of Poisson process with rate λ

Theorem (4.3)

Let $\lambda > 0$ fixed, and X be a càdlàg increasing process taking values in $\mathbb{N}$, started at 0. The following assertions are equivalent :

(i) $Y_n = n$, $n \geq 0$, *and* $(S_k, k \geq 1)$ *are i.i.d exponential*(λ)

(ii) X *has independent increments and uniformly in* $t \geq 0$, $\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$, $\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$.

(iii) X *has independent and stationary increments, and* $X_t \sim \mathrm{Poisson}(\lambda t)$

Proof of characterization theorem

In what follows we write, for short $p_{ij}(t) := \mathbb{P}_i(X_t = j)$. Assume X satisfies (i). Observe that $p_{ii}(h) = \mathbb{P}_i(S_1 > h) = \exp(-\lambda h)$. It follows that $\mathbb{P}(X_{t+h} - X_t = 0) = \exp(-\lambda h)$, so indeed the first equation of (ii) is satisfied.
Also

$$\begin{aligned} p_{i(i+1)}(t) &= \mathbb{P}(S_1 \leq t, S_2 > t - S - 1) \\ &= \int_0^t \lambda \exp(-\lambda u) \int_{t-u}^{\infty} \lambda \exp(-\lambda v) dv du \\ &= \lambda t \exp(-\lambda t) \end{aligned}$$

By the previous theorem $\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h \exp(-\lambda h)$, and indeed the second equation of (ii) is satisfied.
Obviously the above implies that uniformly in t, $\mathbb{P}(X_{t+h} - X_t \geq 2) = o(h)$. Note however that the latter also is $\mathbb{P}(S_1 + S_2 \leq h) \leq Ch^2$ which recovers this result.

Proof of characterization theorem

Now assume X satisfies (ii), so that uniformly in t, $\mathbb{P}(X_{t+h} - X_t \notin \{0,1\}) = o(h)$. Thus, uniformly in t, we have $p_{00}(t+h) = p_{00}(t)(1 - \lambda h + o(h)$. This yields $p'_{00}(t) = -\lambda p_{00}(t)$, and $p_{00}(0) = 1$, so $p_{00}(t) = \exp(-\lambda t)$. By the same reasoning

$$p_{ij}(t) = p_{i(j-1)}(t)\lambda h + p_{ij}(t)(1 - \lambda h) + o(h)$$

so $p'_{ij}(t) = -\lambda p_{ij}(t) + \lambda p_{i(j-1)}(t)$, and of course $p_{ij}(0) = 0$ for any $i < j$. This can be rewritten

$$\frac{d}{dt}(\exp(\lambda t) p_{ij}(t)) = \lambda \exp(\lambda t) p_{i(j-1)}(t), \quad p_{ij}(0) = 0$$

hence by an easy induction, for $i < j$,

$$p_{ij}(t) = \exp(-\lambda t)\frac{(\lambda t)^{j-i}}{(j-i)!}.$$

Proof of characterization theorem

We have therefore established, assuming (ii), that $X_t \sim \text{Poisson}(\lambda t)$. Now, by assumption, X has independent increments, and $(X_{t+s} - X_s)_{t \geq 0}$ also satisfies (ii), by the same reasoning $X_{t+s} - X_s$ also follows a Poisson(λt) distribution, so (iii) follows.

To finish the proof, observe that (i) and (iii) both uniquely characterize the law of X, and we have proven $(i) \Rightarrow (iii)$ so it must be that $(iii) \Rightarrow (i)$.

An important remark

Observe that the system of ordinary differential equations which we have obtained for $\{p_{ij}(t), (i,j) \in E^2\}$ can be rewritten in the following condensed form

$$P'(t) = P(t)Q, \quad P(0) = Id.$$

Further properties of Poisson processes (exercise IV.7

Theorem (4.4)

Let I be a set of indices, assume it is at most countable.

(a) *Assume $((X_i(t), t \geq 0), i \in I)$ are independent Poisson processes with respective rates $\lambda_i, i \in I$ such that $\Lambda = \sum_{i \in I} \lambda_i < \infty$. Then $(X(t) := \sum X_i(t), t \geq 0)$ is a Poisson process of rate Λ.*

(b) *Assume X is a Poisson process of rate λ. Independently label c_j each jump of X so that $\mathbb{P}(c_j = i) = p_i, i \in I$. Then for any $i \in I$ $\{J_j : c_j = i\}$ are the jumps of a Poisson process X_i with rate $p_i\lambda$, and the $(X_i, i \in I)$ are independent.*

This result implies in particular the validity of Harris' graphical representation of Markov chains

Further properties of Poisson processes

Theorem (4.5)

Let X be a Poisson process with rate λ. Conditionally given $\{X_t = n\}$, the set of jump times $\{J_1, ..., J_n\}$ have the same law as $\{U_1, ..., U_n\}$ with $(U_k, 1 \leq k \leq n)$ are independent variables, uniformly distributed on $[0, t]$. Consequently $(J_1, ..., J_n)$ has the same distribution as the ordered statistics of an n-sample of such independent Unif$[0, t]$, with joint density

$$f_{(J_1,...,J_n)}(y_1, ..., y_n) = \frac{n!}{t^n} \mathbb{1}_{\{0 \leq y_1 \leq \cdots \leq y_n \leq t\}}.$$