## Theorem (4.6)

*Assume $(X_t, t \geq 0)$ is a càdlàg process taking values in a finite set $E$ (with associated $(Y_n)_{n\geq 0}, (S_n)_{n\geq 1}$). Assume $Q$ is a generator on $E$, with $\Pi$ the associated jump matrix. Define for $t \geq 0$, $P(t) := \exp(tQ) = \sum_{k\geq 0} \frac{t^k Q^k}{k!}$. Then, TFAAE*

(i) *Given $\{X_0 = x_0\}$, $(Y_n)_{n\geq 0}$ is Markov $(\delta_{x_0}, \Pi)$, and given $(Y_0, \ldots, Y_{n-1}) = (x_0, x_1, \ldots, x_{n-1})$, $(S_1, \ldots, S_n)$ are independent exponential with respective parameters $q_{x_0}, \ldots, q_{x_{n-1}}$.*

(ii) *For any $t \geq 0$, $h \geq 0$, $x \in E$, given $\{X_t = x\}$, $X_{t+h}$ is independent of $\mathcal{F}_t$ and uniformly in $t$,*

$$\mathbb{P}(X_{t+h} = y \mid X_t = x) = \mathbb{1}_{\{x=y\}} + q_{xy} h + o(h)$$

(iii) *For any $n \geq 0$, for any $0 \leq t_0 \leq t_1 \ldots \leq t_{n+1}$, $(x_0, \ldots, x_{n+1}) \in E^{n+2}$,*

$$\mathbb{P}(X_{t_{n+1}=x_{n+1} | X_{t_0}=x_0, \ldots, X_{t_n}=x_n}) = (P(t_{n+1} - t_n))_{x_n x_{n+1}} .$$

# Proof of characterization theorem

$(i)$ is just our definition of continuous-time Markov chain, so if we assume $X$ satisfies $(i)$, by section 4.2 it satisfies Markov property, so the first part of $(ii)$ is satisfied. In addition for $y \neq x$,

$$\begin{aligned}
\mathbb{P}_x(X_h = x) &\geq \mathbb{P}_x(S_1 > h) = \exp(-q_x h) = 1 - q_x h + o(h) \\
\mathbb{P}_x(X_h = y) &\geq \mathbb{P}_x(S_1 < h, Y_1 = y, S_2 > h) \\
&= (1 - \exp(-q_x h))\pi_{xy} \exp(-q_y h) = q_{xy} h + o(h)
\end{aligned}$$

Since $E$ is assumed finite, $\max_{(x,y)\in E^2} |q_{xy}| =: q_{\max} < \infty$, thus the $o(h)$ above are uniform in $t, x, y$. Since the above righthand sides sum up to $1 + o(h)$, we must have equalities everywhere, yielding the second part of $(ii)$.

# Proof of characterization theorem

If we assume $X$ satisfies $(ii)$, then by assumption

$$\mathbb{P}(X_{t_{n+1}=x_{n+1}|X_{t_0}=x_0,\dots,X_{t_n}=x_n}) = \mathbb{P}(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n),$$

and it remains for us to show that for any $s, t \geq 0, (x, y) \in E^2$, $\mathbb{P}(X_{s+t} = y \mid X_s = x) = P_{xy}(t)$.
Let us fix $s \geq 0$ and set $\tilde{P}^{(s)}_{xy}(t) = \mathbb{P}(X_{s+t} = y \mid X_s = x)$. By assumption

$$\tilde{P}^{(s)}_{xy}(t+h) = \tilde{P}^{(s)}_{xy}(t)(1 - q_y h + o(h)) + \sum_{z \neq y} \tilde{P}^{(s)}_{xz}(t)(q_{zy} h + o(h)).$$

Since those $o(h)$ are uniform in $t, s$ by assumption, we get

$$\frac{d}{dt}\tilde{P}^{(s)}_{xy}(t) = (\tilde{P}^{(s)}(t)Q)_{xy}, \quad \tilde{P}^{(s)}_{xy}(0) = \mathbb{1}_{\{x=y\}}.$$

Since $P$ is the unique solution to the above system of equations, this shows that for any $s \geq 0$, $\tilde{P}^{(s)} = P$, yielding $(iii)$.

# Proof of characterization theorem

Finally it is easy that ($iii$) uniquely characterizes the law of $X$ (it gives its finite dimensional distributions, the law of the process $X$ is their unique extension by Kolmogorov's theorem).
Of course ($i$) does as well, and $(i) \Rightarrow (iii)$ so it must be that $(iii) \Rightarrow (i)$.