Non explosion criteria

Theorem (4.7)

The irreducible chain X is non explosive as soon as one of the three following conditions holds

(i) *E is finite*

(ii) $q_{\max} := \max_{x \in E} q_x < \infty$

(iii) *the chain is recurrent*

If (i) holds then obviously (ii) holds as well.
If (ii) holds, whatever the trajectory of the chain, the parameters of the exponential variable $(S_n)_{n\geq 1}$ are no larger than $q_{\max}$, thus we can couple the sequence of holding times $(S_n)_{n\geq 1}$ with a sequence $(e_n)_{n\geq 1}$ of i.i.d exponential($q_{\max}$) in such a way that $S_n \geq e_n$ for any $n \geq 1$. Now $\sum_{n\geq 1} S_n \geq \sum_{n\geq 1} e_n$ which is a.s. infinite.
Finally if (iii) holds, the chain visits its starting point infinitely often, and the total holding time at that point is therefore an infinite sum of i.i.d. exponential variables, hence the total holding time at that point is already a.s. infinite.

Laplace transform of the explosion time ζ

Theorem (4.8)

Assume $\theta > 0$*, and let* $\psi_\theta(x) = \mathbb{E}_x[\exp(-\theta\zeta)]$*. Then*

(i) $|\psi_\theta(x)| \leq 1$ *for any* $x \in E$.

(ii) $Q\psi_\theta = \theta\psi_\theta$

Moreover if f_θ *is solution to* $Qf_\theta = \theta f_\theta$ *and satisfies* $|f_\theta(x)| \leq 1$ *for any* $x \in E$*, then* $f_\theta \leq \psi_\theta$.

Proof of Theorem 4.8

The fact that ψ_θ satisfies (i) is obvious. As for (ii), if $q_x = 0$, then $q_{xy} = 0$ for any $y \in E$, and $Q\psi_\theta(x) = 0$. But the explosion time is $+\infty$ under $\mathbb{P}_x$ since x is absorbing and we have
$Q\psi_\theta(x) = \theta\psi_\theta(x) = 0$.
Otherwise $J_1 \sim \exp(q_x)$. The Laplace transform of J_1 under $\mathbb{P}_x$ evaluated at θ is $\frac{q_x}{q_x+\theta}$. Using Markov property at J_1, we get

$$\psi_\theta(x) = \sum_{y\in E, y\neq x} \Pi_{xy} \frac{q_x}{q_x+\theta}\psi_\theta(y),$$

hence

$$\psi_\theta(x) = Q\psi_\theta(x) + \frac{q_x}{q_x+\theta}\psi_\theta(x),$$

which yields (ii).

Now assume f_θ solves $Qf_\theta = \theta f_\theta$ and satisfies $|f_\theta(x)| \leq 1$ for any $x \in E$. We are going to prove by induction that $f_\theta(y) \leq \mathbb{E}_y[\exp(-\theta J_n)]$ for any $y \in E$, and any $n \geq 0$ (using the convention $J_0 = 0$). The assertion is obvious by (ii) for $n = 0$. Assume it holds for a given $n \in \mathbb{N}$. For any x such that $q_x > 0$, by the same reasoning as above

$$\begin{aligned} \theta\mathbb{E}_x[\exp(-\theta J_{n+1}] &= \sum_{y \in E} q_{xy}\mathbb{E}_y[\exp(-\theta J_n)] \\ &\geq Qf_\theta(x) = \theta f_\theta(x), \end{aligned}$$

so the assertion holds for $n+1$. If $q_x = 0$, then $f_\theta(x) = \frac{1}{\theta}Qf_\theta(x) = 0$ so the assertion also holds for $n+1$. Thus by induction for any $y \in E, n \geq 0$, $f_\theta(y) \leq \mathbb{E}_y[\exp(-\theta J_n)]$, but the latter converges to $\psi_\theta(x)$ by dominated convergence, and we conclude that $f_\theta \leq \psi_\theta$.

Corollary

If X is non explosive then the only solution to $Qf_\theta = \theta f_\theta$ satisfying $|f_\theta(x)| \leq 1$ for any $x \in E$ is the null function.

Proof : Since X is non explosive we must have $\psi_\theta(x) = 0$ for any $x \in E$. Now observe that if f_θ is a solution to the above then $-f_\theta$ also is. Now by the previous theorem we have $f_\theta \leq \psi_\theta \equiv 0$, and $-f_\theta \leq \psi_\theta \equiv 0$, hence $f_\theta \equiv 0$.